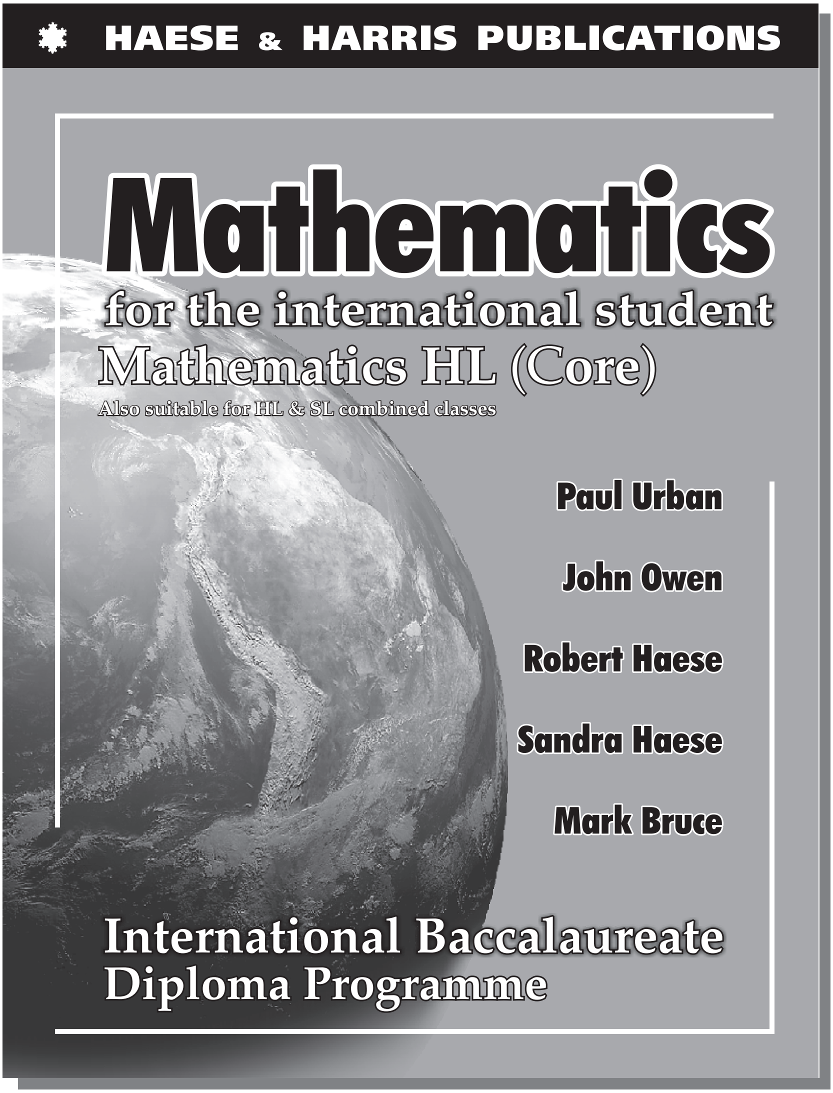
WORKED SOLUTIONS



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Mathematics

for the international student

Mathematics HL (Core)

Also suitable for HL & SL combined classes

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International Baccalaureate
Diploma Programme

Roger Dixon
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Robert Haese
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Haese & Harris Publications

MATHEMATICS FOR THE INTERNATIONAL STUDENT
Mathematics HL (Core) – WORKED SOLUTIONS
International Baccalaureate Diploma Programme

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FOREWORD

This book gives you fully worked solutions for every question in each chapter of the Haese & Harris Publications textbook **Mathematics HL (Core)** which is one of three textbooks in our series 'Mathematics for the International Student'. The other two textbooks are **Mathematics SL** and **Mathematical Studies SL**, and books of fully worked solutions are available for those textbooks also.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modeled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for **Mathematics HL (Core)** on our website. Please contact us if you have any additions to this list.

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Background knowledge

EXERCISE A

1 **a**
$$\begin{aligned}&\sqrt{3}\times\sqrt{5}\\&=\sqrt{3\times5}\\&=\sqrt{15}\end{aligned}$$

b
$$\begin{aligned}&(\sqrt{3})^2\\&=\sqrt{3}\times\sqrt{3}\\&=3\end{aligned}$$

c
$$\begin{aligned}&2\sqrt{2}\times\sqrt{2}\\&=2(\sqrt{2}\times\sqrt{2})\\&=2\times2\\&=4\end{aligned}$$

d
$$\begin{aligned}&3\sqrt{2}\times2\sqrt{2}\\&=(3\times2)(\sqrt{2}\times\sqrt{2})\\&=6\times2\\&=12\end{aligned}$$

e
$$\begin{aligned}&3\sqrt{7}\times2\sqrt{7}\\&=(3\times2)(\sqrt{7}\times\sqrt{7})\\&=6\times7\\&=42\end{aligned}$$

f
$$\begin{aligned}&\frac{\sqrt{12}}{\sqrt{2}}\\&=\sqrt{\frac{12}{2}}\\&=\sqrt{6}\end{aligned}$$

g
$$\begin{aligned}&\frac{\sqrt{12}}{\sqrt{6}}\\&=\sqrt{\frac{12}{6}}\\&=\sqrt{2}\end{aligned}$$

h
$$\begin{aligned}&\frac{\sqrt{18}}{\sqrt{3}}\\&=\sqrt{\frac{18}{3}}\\&=\sqrt{6}\end{aligned}$$

2 **a**
$$\begin{aligned}&2\sqrt{2}+3\sqrt{2}\\&=(2+3)\sqrt{2}\\&=5\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&2\sqrt{2}-3\sqrt{2}\\&=(2-3)\sqrt{2}\\&=-\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}&5\sqrt{5}-3\sqrt{5}\\&=(5-3)\sqrt{5}\\&=2\sqrt{5}\end{aligned}$$

d
$$\begin{aligned}&5\sqrt{5}+3\sqrt{5}\\&=(5+3)\sqrt{5}\\&=8\sqrt{5}\end{aligned}$$

e
$$\begin{aligned}&3\sqrt{5}-5\sqrt{5}\\&=(3-5)\sqrt{5}\\&=-2\sqrt{5}\end{aligned}$$

f
$$\begin{aligned}&7\sqrt{3}+2\sqrt{3}\\&=(7+2)\sqrt{3}\\&=9\sqrt{3}\end{aligned}$$

g
$$\begin{aligned}&9\sqrt{6}-12\sqrt{6}\\&=(9-12)\sqrt{6}\\&=-3\sqrt{6}\end{aligned}$$

h
$$\begin{aligned}&\sqrt{2}+\sqrt{2}+\sqrt{2}\\&=3\times\sqrt{2}\\&=3\sqrt{2}\end{aligned}$$

3 **a**
$$\begin{aligned}&\sqrt{8}\\&=\sqrt{4\times2}\\&=\sqrt{4}\times\sqrt{2}\\&=2\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&\sqrt{12}\\&=\sqrt{4\times3}\\&=\sqrt{4}\times\sqrt{3}\\&=2\sqrt{3}\end{aligned}$$

c
$$\begin{aligned}&\sqrt{20}\\&=\sqrt{4\times5}\\&=\sqrt{4}\times\sqrt{5}\\&=2\sqrt{5}\end{aligned}$$

d
$$\begin{aligned}&\sqrt{32}\\&=\sqrt{16\times2}\\&=\sqrt{16}\times\sqrt{2}\\&=4\sqrt{2}\end{aligned}$$

e
$$\begin{aligned}&\sqrt{27}\\&=\sqrt{9\times3}\\&=\sqrt{9}\times\sqrt{3}\\&=3\sqrt{3}\end{aligned}$$

f
$$\begin{aligned}&\sqrt{45}\\&=\sqrt{9\times5}\\&=\sqrt{9}\times\sqrt{5}\\&=3\sqrt{5}\end{aligned}$$

g
$$\begin{aligned}&\sqrt{48}\\&=\sqrt{16\times3}\\&=\sqrt{16}\times\sqrt{3}\\&=4\sqrt{3}\end{aligned}$$

h
$$\begin{aligned}&\sqrt{54}\\&=\sqrt{9\times6}\\&=\sqrt{9}\times\sqrt{6}\\&=3\sqrt{6}\end{aligned}$$

i
$$\begin{aligned}&\sqrt{50}\\&=\sqrt{25\times2}\\&=\sqrt{25}\times\sqrt{2}\\&=5\sqrt{2}\end{aligned}$$

j
$$\begin{aligned}&\sqrt{80}\\&=\sqrt{16\times5}\\&=\sqrt{16}\times\sqrt{5}\\&=4\sqrt{5}\end{aligned}$$

k
$$\begin{aligned}&\sqrt{96}\\&=\sqrt{16\times6}\\&=\sqrt{16}\times\sqrt{6}\\&=4\sqrt{6}\end{aligned}$$

l
$$\begin{aligned}&\sqrt{108}\\&=\sqrt{36\times3}\\&=\sqrt{36}\times\sqrt{3}\\&=6\sqrt{3}\end{aligned}$$

4 **a**
$$\begin{aligned}&4\sqrt{3}-\sqrt{12}\\&=4\sqrt{3}-\sqrt{4\times3}\\&=4\sqrt{3}-2\times\sqrt{3}\\&=4\sqrt{3}-2\sqrt{3}\\&=2\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}&3\sqrt{2}+\sqrt{50}\\&=3\sqrt{2}+\sqrt{25\times2}\\&=3\sqrt{2}+5\times\sqrt{2}\\&=3\sqrt{2}+5\sqrt{2}\\&=8\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}&3\sqrt{6}+\sqrt{24}\\&=3\sqrt{6}+\sqrt{4\times6}\\&=3\sqrt{6}+2\times\sqrt{6}\\&=3\sqrt{6}+2\sqrt{6}\\&=5\sqrt{6}\end{aligned}$$

d
$$\begin{aligned}&2\sqrt{27}+2\sqrt{12}\\&=2\sqrt{9\times3}+2\sqrt{4\times3}\\&=6\sqrt{3}+4\sqrt{3}\\&=10\sqrt{3}\end{aligned}$$

e
$$\begin{aligned}&\sqrt{75}-\sqrt{12}\\&=\sqrt{25\times3}-\sqrt{4\times3}\\&=5\sqrt{3}-2\sqrt{3}\\&=3\sqrt{3}\end{aligned}$$

f
$$\begin{aligned}&\sqrt{2}+\sqrt{8}-\sqrt{32}\\&=\sqrt{2}+\sqrt{4\times2}-\sqrt{16\times2}\\&=\sqrt{2}+2\sqrt{2}-4\sqrt{2}\\&=-\sqrt{2}\end{aligned}$$

5 **a** $\frac{1}{\sqrt{2}}$
$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2}}{2}$

b $\frac{6}{\sqrt{3}}$
$= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{6\sqrt{3}}{3}$
$= 2\sqrt{3}$

c $\frac{7}{\sqrt{2}}$
$= \frac{7}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{7\sqrt{2}}{2}$

d $\frac{10}{\sqrt{5}}$
$= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$
$= \frac{10\sqrt{5}}{5}$
$= 2\sqrt{5}$

e $\frac{10}{\sqrt{2}}$
$= \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{10\sqrt{2}}{2}$
$= 5\sqrt{2}$

f $\frac{18}{\sqrt{6}}$
$= \frac{18}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$
$= \frac{18\sqrt{6}}{6}$
$= 3\sqrt{6}$

g $\frac{12}{\sqrt{3}}$
$= \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{12\sqrt{3}}{3}$
$= 4\sqrt{3}$

h $\frac{5}{\sqrt{7}}$
$= \frac{5}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$
$= \frac{5\sqrt{7}}{7}$

i $\frac{14}{\sqrt{7}}$
$= \frac{14}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$
$= \frac{14\sqrt{7}}{7}$
$= 2\sqrt{7}$

j $\frac{2\sqrt{3}}{\sqrt{2}}$
$= \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{2\sqrt{6}}{2}$
$= \sqrt{6}$

EXERCISE B

1 **a** 259
$= 2.\overset{\curvearrowleft}{59} \times 10^2$
$= 2.59 \times 10^2$

b $259\,000$
$= 2.\overset{\curvearrowleft}{590\,00} \times 10^5$
$= 2.59 \times 10^5$

c 2.59
$= 2.59 \times 1$
$= 2.59 \times 10^0$

d 0.259
$= 0\overset{\curvearrowright}{2}.59 \div 10$
$= 2.59 \times 10^{-1}$

e $0.000\,259$
$= 0\overset{\curvearrowright}{0002}.59 \div 10^4$
$= 2.59 \times 10^{-4}$

f 40.7
$= 4.\overset{\curvearrowleft}{0}7 \times 10$
$= 4.07 \times 10^1$

g 4070
$= 4.\overset{\curvearrowleft}{070} \times 10^3$
$= 4.07 \times 10^3$

h 0.0407
$= 0\overset{\curvearrowright}{04}.07 \div 10^2$
$= 4.07 \times 10^{-2}$

i $407\,000$
$= 4.\overset{\curvearrowleft}{070\,00} \times 10^5$
$= 4.07 \times 10^5$

j $407\,000\,000$
$= 4.\overset{\curvearrowleft}{070\,000\,00} \times 10^8$
$= 4.07 \times 10^8$

k $0.000\,040\,7$
$= 0\overset{\curvearrowright}{00004}.07 \div 10^5$
$= 4.07 \times 10^{-5}$

2 **a** $149\,500\,000\,000$ m
$= 1.\overset{\curvearrowleft}{49\,500\,000\,000} \times 10^{11}$
$= 1.495 \times 10^{11}$ m

b 0.0003 mm
$= \overset{\curvearrowright}{0003}. \times 10^{-4}$
$= 3 \times 10^{-4}$ mm

c 0.001 mm
$= \overset{\curvearrowright}{001}. \times 10^{-3}$
$= 1 \times 10^{-3}$ mm

d 15 million degrees
$= 15\,000\,000$ ^{o}C
$= 1.500\,000\,0 \times 10^7$ ^{o}C
$= 1.5 \times 10^7$ ^{o}C

e $300\,000$ times
$= 3 \times 100\,000$
$= 3 \times 10^5$ times

3 **a** 4×10^3
$= 4 \times 1000$
$= 4000$

b 5×10^2
$= 5 \times 100$
$= 500$

c 2.1×10^3
$= 2.\overset{\curvearrowright}{100} \times 10^3$
$= 2100$

d 7.8×10^4
$= 7.\overset{\curvearrowright}{8000} \times 10^4$
$= 78\,000$

e 3.8×10^5
$= 3.\overset{\curvearrowright}{800\,00} \times 10^5$
$= 380\,000$

f 8.6×10^1
$= 8.\overset{\curvearrowright}{6} \times 10$
$= 86$

g 4.33×10^7
$= 4.\overset{\curvearrowright}{330\,000\,0} \times 10^7$
$= 43\,300\,000$

h 6×10^7
$= 6 \times 10\,000\,000$
$= 60\,000\,000$

4 **a** 4×10^{-3}
$= \overbrace{004}. \div 10^3$
$= 0.004$

b 5×10^{-2}
$= \overbrace{05}. \div 10^2$
$= 0.05$

c 2.1×10^{-3}
$= \overbrace{002}.1 \div 10^3$
$= 0.0021$

d 7.8×10^{-4}
$= \overbrace{0007}.8 \div 10^4$
$= 0.000\,78$

e 3.8×10^{-5}
$= \overbrace{00003}.8 \div 10^5$
$= 0.000\,038$

f 8.6×10^{-1}
$= \overbrace{8}.6 \div 10^1$
$= 0.86$

g 4.33×10^{-7}
$= \overbrace{0000004}.33 \div 10^7$
$= 0.000\,000\,433$

h 6×10^{-7}
$= \overbrace{0000006}. \div 10^7$
$= 0.000\,000\,6$

5 **a** 9×10^{-7} m
$= \overbrace{0000009}. \div 10^7$
$= 0.000\,000\,9$ m

b 6.130×10^9 people
$= 6.\overbrace{130\,000\,000} \times 10^9$
$= 6\,130\,000\,000$ people

c 1×10^5 light years
$= 1 \times 100\,000$
$= 100\,000$ light years

d 1×10^{-5} mm
$= \overbrace{00001}. \div 10^5$
$= 0.000\,01$ mm

6 **a**
$$\begin{aligned}&(3.42 \times 10^5) \times (4.8 \times 10^4)\\ &= (3.42 \times 4.8) \times (10^5 \times 10^4)\\ &= 16.416 \times 10^9\\ &= 1.6416 \times 10^{10}\\ &= 1.64 \times 10^{10} \quad \text{(2 d.p.)}\end{aligned}$$

b
$$\begin{aligned}&(6.42 \times 10^{-2})^2\\ &= (6.42)^2 \times (10^{-2})^2\\ &= 41.2164 \times 10^{-4}\\ &= 4.121\,64 \times 10^{-3}\\ &= 4.12 \times 10^{-3} \quad \text{(2 d.p.)}\end{aligned}$$

c
$$\begin{aligned}&\frac{3.16 \times 10^{-10}}{6 \times 10^7}\\ &= \frac{3.16}{6} \times \frac{10^{-10}}{10^7}\\ &= 0.52\bar{6} \times 10^{-17}\\ &= 5.2\bar{6} \times 10^{-18}\\ &= 5.27 \times 10^{-18} \quad \text{(2 d.p.)}\end{aligned}$$

d
$$\begin{aligned}&(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})\\ &= \frac{9.8 \times 10^{-4}}{7.2 \times 10^{-6}}\\ &= \frac{9.8}{7.2} \times \frac{10^{-4}}{10^{-6}}\\ &= 1.36\bar{1} \times 10^2\\ &= 1.36 \times 10^2 \quad \text{(2 d.p.)}\end{aligned}$$

e
$$\begin{aligned}&\frac{1}{3.8 \times 10^5}\\ &= \frac{1}{3.8} \times 10^{-5}\\ &= 2.63 \times 10^{-6} \quad \text{(2 d.p.)}\end{aligned}$$

f
$$\begin{aligned}&(1.2 \times 10^3)^3\\ &= (1.2)^3 \times (10^3)^3\\ &= 1.728 \times 10^9\\ &= 1.73 \times 10^9 \quad \text{(2 d.p.)}\end{aligned}$$

7 **a** 1 day = 24 hours
i.e., missile travels 5400×24
$= 129\,600$
$= 1.296 \times 10^5$
$\doteqdot 1.30 \times 10^5$ km

b 1 week = 7 days
= 7×24 hours
= 168 hours

i.e., missile travels 5400×168
$= 907\,200$
$= 9.072 \times 10^5$
$\doteqdot 9.07 \times 10^5$ km

c 2 years = 2×365.25 days
= 730.5 days
= 730.5×24 hours
= 17 532 hours

i.e., missile travels $5400 \times 17\,532$
$= 94\,672\,800$
$= 9.467\,28 \times 10^7$
$\doteqdot 9.47 \times 10^7$ km

8 **a** distance = speed × time

time = 1 minute = 60 seconds

so, light travels $(3 \times 10^8) \times 60$

$= 180 \times 10^8$

$= 1.80 \times 10^{10}$ m

b distance = speed × time

time = 1 day = 24 hours

$= 24 \times 60 \times 60$ seconds

$= 86\,400$ seconds

$= 8.64 \times 10^4$ seconds

i.e., light travels $(3 \times 10^8) \times (8.64 \times 10^4)$

$= 3 \times 8.64 \times 10^{12}$

$= 25.92 \times 10^{12}$

$\doteqdot 2.59 \times 10^{13}$ m

c distance = speed × time

time = 1 year = 365.25 days

$= 365.25 \times 8.64 \times 10^4$ sec {from **b**}

$= 3155.76 \times 10^4$

$\doteqdot 3.16 \times 10^7$ sec

i.e., light travels $(3 \times 10^8) \times (3.156 \times 10^7)$

$= 3 \times 3.156 \times 10^{15}$

$= 9.468 \times 10^{15}$

$\doteqdot 9.47 \times 10^{15}$ m

EXERCISE C

1 **a** $\{x : x > 5\}$ reads 'the set of all x such that x is greater than 5'

b $\{x : x \leqslant 3\}$ reads 'the set of all x such that x is less than or equal to 3'

c $\{y : 0 < y < 6\}$ reads 'the set of all y such that y lies between 0 and 6'

d $\{x : 2 \leqslant x \leqslant 4\}$ reads 'the set of all x such that x is greater than or equal to 2, but less than or equal to 4'

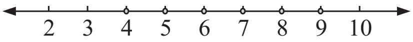
e $\{t : 1 < t < 5\}$ reads 'the set of all t such that t lies between 1 and 5'

f $\{n : n < 2 \text{ or } n \geqslant 6\}$ reads 'the set of all n such that n is less than 2 or greater than or equal to 6'

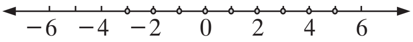
2 **a** $\{x : x > 2\}$ **b** $\{x : 1 < x \leqslant 5\}$ **c** $\{x : x \leqslant -2 \text{ or } x \geqslant 3\}$

d $\{x : x \in Z, -1 \leqslant x \leqslant 3\}$ **e** $\{x : x \in Z, 0 \leqslant x \leqslant 5\}$ **f** $\{x : x < 0\}$

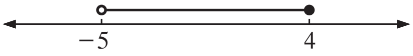
3 **a**



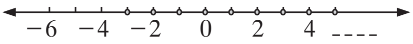
b



c



d



e

EXERCISE D

1 **a** $3x + 7x - 10$

$= 10x - 10$

b $3x + 7x - x$

$= 9x$

c $2x + 3x + 5y$

$= 5x + 5y$

d $8 - 6x - 2x$

$= 8 - 8x$

e $7ab + 5ba$

$= 7ab + 5ab$

$= 12ab$

f $3x^2 + 7x^3$

$= 3x^2 + 7x^3$

i.e., cannot be simplified

2 **a** $3(2x + 5) + 4(5 + 4x)$

$= 6x + 15 + 20 + 16x$

$= 22x + 35$

b $6 - 2(3x - 5)$

$= 6 - 6x + 10$

$= 16 - 6x$

c
$$\begin{aligned} &5(2a-3b)-6(a-2b)\\ &=10a-15b-6a+12b\\ &=4a-3b \end{aligned}$$

d
$$\begin{aligned} &3x(x^2-7x+3)-(1-2x-5x^2)\\ &=3x^3-21x^2+9x-1+2x+5x^2\\ &=3x^3-16x^2+11x-1 \end{aligned}$$

3 **a**
$$\begin{aligned} &2x(3x)^2\\ &=2x\times 9x^2\\ &=18x^3 \end{aligned}$$

b
$$\begin{aligned} &\frac{3a^2b^3}{9ab^4}\\ &=\frac{\cancel{3}\times\cancel{a}\times a\times\cancel{b}\times\cancel{b}\times\cancel{b}}{\cancel{3}\times 3\times\cancel{a}\times\cancel{b}\times\cancel{b}\times\cancel{b}\times b}\\ &=\frac{a}{3b} \end{aligned}$$

c
$$\begin{aligned} &\sqrt{16x^4}\\ &=\sqrt{16}\times\sqrt{x^4}\\ &=4\times\sqrt{(x^2)^2}\\ &=4x^2 \end{aligned}$$

d
$$\begin{aligned} &(2a^2)^3\times 3a^4\\ &=2^3\times(a^2)^3\times 3a^4\\ &=8\times a^6\times 3a^4\\ &=24a^{10} \end{aligned}$$

EXERCISE E

1 **a**
$$\begin{aligned} 2x+5&=25\\ \therefore\ 2x&=20\\ \therefore\ x&=10 \end{aligned}$$

b
$$\begin{aligned} 3x-7&>11\\ \therefore\ 3x&>18\\ \therefore\ x&>6 \end{aligned}$$

c
$$\begin{aligned} 5x+16&=20\\ \therefore\ 5x&=4\\ \therefore\ x&=\tfrac{4}{5} \end{aligned}$$

d
$$\begin{aligned} \frac{x}{3}-7&=10\\ \therefore\ \frac{x}{3}&=17\\ \therefore\ x&=51 \end{aligned}$$

e
$$\begin{aligned} 6x+11&<4x-9\\ \therefore\ 2x&<-20\\ \therefore\ x&<-10 \end{aligned}$$

f
$$\begin{aligned} \frac{3x-2}{5}&=8\\ \therefore\ 3x-2&=40\\ \therefore\ 3x&=42\\ \therefore\ x&=14 \end{aligned}$$

g
$$\begin{aligned} 1-2x&\geqslant 19\\ \therefore\ -2x&\geqslant 18\\ \therefore\ 2x&\leqslant -18\\ \therefore\ x&\leqslant -9 \end{aligned}$$

h
$$\begin{aligned} \tfrac{1}{2}x+1&=\tfrac{2}{3}x-2\\ \therefore\ \tfrac{3}{6}x-\tfrac{4}{6}x&=-3\\ \therefore\ -\tfrac{1}{6}x&=-3\\ \therefore\ x&=18 \end{aligned}$$

i $\dfrac{2}{3}-\dfrac{3x}{4}=\tfrac{1}{2}(2x-1)$

Multiplying each term by the LCD of 12 gives

$$\begin{aligned} \therefore\ 8-9x&=6(2x-1)\\ \therefore\ 8-9x&=12x-6\\ \therefore\ 14&=21x \quad\text{i.e., } x=\tfrac{2}{3} \end{aligned}$$

2 **a**
$$\begin{aligned} x+2y&=9 \quad ..... (1)\\ x-y&=3 \quad ..... (2) \end{aligned}$$

Multiplying (2) by 2 gives

$$\begin{aligned} x+2y&=9\\ 2x-2y&=6\\ \hline \therefore\ 3x\quad&=15 \quad \{\text{adding}\}\\ \therefore\ x&=5 \end{aligned}$$

Substituting $x=5$ into (2) gives

$$\begin{aligned} 5-y&=3\\ \therefore\ y&=2 \end{aligned}$$

$\therefore\ x=5$ and $y=2$

b
$$\begin{aligned} 2x+5y&=28 \quad ..... (1)\\ x-2y&=2 \quad ..... (2) \end{aligned}$$

Multiplying (2) by -2 gives

$$\begin{aligned} 2x+5y&=28\\ -2x+4y&=-4\\ \hline \therefore\ 9y&=24 \quad \{\text{adding}\}\\ \therefore\ y&=\tfrac{24}{9}=\tfrac{8}{3} \end{aligned}$$

Substituting $y=\frac{8}{3}$ into (2) gives

$x-2(\frac{8}{3})=2 \quad \therefore\ x-\frac{16}{3}=2$ and so $x=\frac{22}{3}$

$\therefore\ x=\frac{22}{3}$ and $y=\frac{8}{3}$

c
$$\begin{aligned} 7x+2y&=-4 \quad ..... (1)\\ 3x+4y&=14 \quad ..... (2) \end{aligned}$$

Multiplying (1) by -2 gives

$$\begin{aligned} -14x-4y&=8\\ 3x+4y&=14\\ \hline \therefore\ -11x\quad&=22 \quad \{\text{adding}\}\\ \therefore\ x&=-2 \end{aligned}$$

Substituting $x=-2$ into (2) gives

$$\begin{aligned} 3(-2)+4y&=14\\ \therefore\ -6+4y&=14\\ \therefore\ 4y&=20 \quad\text{and } \therefore\ y=5 \end{aligned}$$

$\therefore\ x=-2$ and $y=5$

d
$$\begin{aligned} 5x-4y&=27 \quad ..... (1)\\ 3x+2y&=9 \quad ..... (2) \end{aligned}$$

Multiplying (2) by 2 gives

$$\begin{aligned} 5x-4y&=27\\ 6x+4y&=18\\ \hline \therefore\ 11x\quad&=45 \quad \{\text{adding}\}\\ \therefore\ x&=\tfrac{45}{11} \end{aligned}$$

Substituting $x=\frac{45}{11}$ into (1) gives

$5(\frac{45}{11})-4y=27 \quad \therefore\ \frac{225}{11}-27=4y$

$\therefore\ 4y=-\frac{72}{11}$ and $\therefore\ y=-\frac{18}{11}$

$\therefore\ x=\frac{45}{11}$ and $y=-\frac{18}{11}$

e
$$\begin{aligned} x+2y &= 5 && \text{..... (1)}\\ 2x+4y &= 1 && \text{..... (2)}\end{aligned}$$

Multiplying (1) by -2 gives

$$\begin{aligned} -2x-4y &= -10\\ 2x+4y &= 1\end{aligned}$$

$$\therefore \quad 0 = -9 \qquad \{\text{adding}\}$$

which is absurd

$\therefore$ there are no solutions

f
$$\begin{aligned} \frac{x}{2}+\frac{y}{3} &= 5 && \text{..... (1)}\\ \frac{x}{3}+\frac{y}{4} &= 1 && \text{..... (2)}\end{aligned}$$

Multiplying (1) by 18 and (2) by -24 gives

$$\begin{aligned} 9x+6y &= 90 && \text{..... (3)}\\ -8x-6y &= -24\end{aligned}$$

$$\therefore \quad x = 66 \qquad \{\text{adding}\}$$

Substituting $x = 66$ into (3) gives

$$\begin{aligned} 9\times 66+6y &= 90\\ \therefore\ 6y &= 90-594 = -504\\ \therefore\ y &= -84\end{aligned}$$

$\therefore$ $x = 66$ and $y = -84$

EXERCISE F

1 **a**
$$\begin{aligned} &5-(-11)\\ &= 5+11\\ &= 16\end{aligned}$$

b
$$\begin{aligned} &|5|-|-11|\\ &= 5-11\\ &= -6\end{aligned}$$

c
$$\begin{aligned} &|5-(-11)|\\ &= |5+11|\\ &= |16|\\ &= 16\end{aligned}$$

d
$$\begin{aligned} &\left|(-2)^2+11(-2)\right|\\ &= |4-22|\\ &= |-18|\\ &= 18\end{aligned}$$

e
$$\begin{aligned} &|-6|-|-8|\\ &= 6-8\\ &= -2\end{aligned}$$

f
$$\begin{aligned} &|-6-(-8)|\\ &= |-6+8|\\ &= |2|\\ &= 2\end{aligned}$$

2 **a**
$$\begin{aligned} |a| &= |-2|\\ &= 2\end{aligned}$$

b
$$\begin{aligned} |b| &= |3|\\ &= 3\end{aligned}$$

c
$$\begin{aligned} |a|\,|b| &= |-2|\,|3|\\ &= 2\times 3\\ &= 6\end{aligned}$$

d
$$\begin{aligned} |ab| &= |-2\times 3|\\ &= |-6|\\ &= 6\end{aligned}$$

e
$$\begin{aligned} |a-b| &= |-2-3|\\ &= |-5|\\ &= 5\end{aligned}$$

f
$$\begin{aligned} |a|-|b| &= |-2|-|3|\\ &= 2-3\\ &= -1\end{aligned}$$

g
$$\begin{aligned} |a+b| &= |-2+3|\\ &= |1|\\ &= 1\end{aligned}$$

h
$$\begin{aligned} |a|+|b| &= |-2|+|3|\\ &= 2+3\\ &= 5\end{aligned}$$

i
$$\begin{aligned} |a|^2 &= |-2|^2\\ &= 2^2\\ &= 4\end{aligned}$$

j
$$a^2 = (-2)^2 = 4$$

k
$$\left|\frac{c}{a}\right| = \left|\frac{-4}{-2}\right| = |2| = 2$$

l
$$\frac{|c|}{|a|} = \frac{|-4|}{|-2|} = \frac{4}{2} = 2$$

3 **a**
$$\begin{aligned} |x| &= 3\\ \therefore\ x &= \pm 3\end{aligned}$$

b
$$|x| = -5$$

but $|x| \geqslant 0$ for all x

(property of modulus)

$\therefore$ no solution

c
$$\begin{aligned} |x| &= 0\\ \therefore\ x &= 0\end{aligned}$$

d
$$\begin{aligned} |x-1| &= 3\\ \therefore\ x-1 &= \pm 3\\ \therefore\ x &= 1\pm 3\\ \therefore\ x &= -2 \text{ or } 4\end{aligned}$$

e
$$\begin{aligned} |3-x| &= 4\\ \therefore\ 3-x &= \pm 4\\ \therefore\ -x &= -3\pm 4\\ \therefore\ x &= 3\mp 4\\ \therefore\ x &= -1 \text{ or } 7\end{aligned}$$

f
$$|x+5| = -1$$

but $|x+5| \geqslant 0$ for all x

(property of modulus)

$\therefore$ no solution

g
$$\begin{aligned} |3x-2| &= 1 \\ \therefore\ 3x-2 &= \pm 1 \\ \therefore\ 3x &= 2\pm 1 \\ \therefore\ 3x &= 3 \text{ or } 1 \\ \therefore\ x &= \tfrac{1}{3} \text{ or } 1 \end{aligned}$$

h
$$\begin{aligned} |3-2x| &= 3 \\ \therefore\ 3-2x &= \pm 3 \\ \therefore\ 2x &= 3\mp 3 \\ \therefore\ 2x &= 0 \text{ or } 6 \\ \therefore\ x &= 0 \text{ or } 3 \end{aligned}$$

i
$$\begin{aligned} |2-5x| &= 12 \\ \therefore\ 2-5x &= \pm 12 \\ \therefore\ 5x &= 2\mp 12 \\ \therefore\ 5x &= -10 \text{ or } 14 \\ \therefore\ x &= -2 \text{ or } \tfrac{14}{5} \end{aligned}$$

EXERCISE G

1 **a**
$$\begin{aligned} &(2x+3)(x+1) \\ &= 2x^2+2x+3x+3 \\ &= 2x^2+5x+3 \end{aligned}$$

b
$$\begin{aligned} &(3x+4)(x+2) \\ &= 3x^2+6x+4x+8 \\ &= 3x^2+10x+8 \end{aligned}$$

c
$$\begin{aligned} &(5x-2)(2x+1) \\ &= 10x^2+5x-4x-2 \\ &= 10x^2+x-2 \end{aligned}$$

d
$$\begin{aligned} &(x+2)(3x-5) \\ &= 3x^2-5x+6x-10 \\ &= 3x^2+x-10 \end{aligned}$$

e
$$\begin{aligned} &(7-2x)(2+3x) \\ &= 14+21x-4x-6x^2 \\ &= -6x^2+17x+14 \end{aligned}$$

f
$$\begin{aligned} &(1-3x)(5+2x) \\ &= 5+2x-15x-6x^2 \\ &= -6x^2-13x+5 \end{aligned}$$

g
$$\begin{aligned} &(3x+4)(5x-3) \\ &= 15x^2-9x+20x-12 \\ &= 15x^2+11x-12 \end{aligned}$$

h
$$\begin{aligned} &(1-3x)(2-5x) \\ &= 2-5x-6x+15x^2 \\ &= 15x^2-11x+2 \end{aligned}$$

i
$$\begin{aligned} &(7-x)(3-2x) \\ &= 21-14x-3x+2x^2 \\ &= 2x^2-17x+21 \end{aligned}$$

j
$$\begin{aligned} &(5-2x)(3-2x) \\ &= 15-10x-6x+4x^2 \\ &= 4x^2-16x+15 \end{aligned}$$

k
$$\begin{aligned} &-(x+1)(x+2) \\ &= -(x^2+2x+x+2) \\ &= -(x^2+3x+2) \\ &= -x^2-3x-2 \end{aligned}$$

l
$$\begin{aligned} &-2(x-1)(2x+3) \\ &= -2(2x^2+3x-2x-3) \\ &= -2(2x^2+x-3) \\ &= -4x^2-2x+6 \end{aligned}$$

2 **a**
$$\begin{aligned} &(x+6)(x-6) \\ &= x^2-6^2 \\ &= x^2-36 \end{aligned}$$

b
$$\begin{aligned} &(x+8)(x-8) \\ &= x^2-8^2 \\ &= x^2-64 \end{aligned}$$

c
$$\begin{aligned} &(2x-1)(2x+1) \\ &= (2x)^2-1^2 \\ &= 4x^2-1 \end{aligned}$$

d
$$\begin{aligned} &(3x-2)(3x+2) \\ &= (3x)^2-2^2 \\ &= 9x^2-4 \end{aligned}$$

e
$$\begin{aligned} &(4x+5)(4x-5) \\ &= (4x)^2-5^2 \\ &= 16x^2-25 \end{aligned}$$

f
$$\begin{aligned} &(5x-3)(5x+3) \\ &= (5x)^2-3^2 \\ &= 25x^2-9 \end{aligned}$$

g
$$\begin{aligned} &(3-x)(3+x) \\ &= 3^2-x^2 \\ &= 9-x^2 \end{aligned}$$

h
$$\begin{aligned} &(7-x)(7+x) \\ &= 7^2-x^2 \\ &= 49-x^2 \end{aligned}$$

i
$$\begin{aligned} &(7+2x)(7-2x) \\ &= 7^2-(2x)^2 \\ &= 49-4x^2 \end{aligned}$$

j
$$\begin{aligned} &(x+\sqrt{2})(x-\sqrt{2}) \\ &= x^2-(\sqrt{2})^2 \\ &= x^2-2 \end{aligned}$$

k
$$\begin{aligned} &(x+\sqrt{5})(x-\sqrt{5}) \\ &= x^2-(\sqrt{5})^2 \\ &= x^2-5 \end{aligned}$$

l
$$\begin{aligned} &(2x-\sqrt{3})(2x+\sqrt{3}) \\ &= (2x)^2-(\sqrt{3})^2 \\ &= 4x^2-3 \end{aligned}$$

3 **a**
$$\begin{aligned} &(x+5)^2 \\ &= x^2+2(x)(5)+5^2 \\ &= x^2+10x+25 \end{aligned}$$

b
$$\begin{aligned} &(x+7)^2 \\ &= x^2+2(x)(7)+7^2 \\ &= x^2+14x+49 \end{aligned}$$

c
$$\begin{aligned} &(x-2)^2 \\ &= x^2-2(x)(2)+2^2 \\ &= x^2-4x+4 \end{aligned}$$

d
$$\begin{aligned} &(x-6)^2 \\ &= x^2-2(x)(6)+6^2 \\ &= x^2-12x+36 \end{aligned}$$

e
$$\begin{aligned} &(3+x)^2 \\ &= 3^2+2(3)(x)+x^2 \\ &= x^2+6x+9 \end{aligned}$$

f
$$\begin{aligned} &(5+x)^2 \\ &= 5^2+2(5)(x)+x^2 \\ &= x^2+10x+25 \end{aligned}$$

g
$$\begin{aligned} &(11-x)^2 \\ &= 11^2-2(11)(x)+x^2 \\ &= x^2-22x+121 \end{aligned}$$

h
$$\begin{aligned} &(10-x)^2 \\ &= 10^2-2(10)(x)+x^2 \\ &= x^2-20x+100 \end{aligned}$$

i
$$\begin{aligned} &(2x+7)^2 \\ &= (2x)^2+2(2x)(7)+7^2 \\ &= 4x^2+28x+49 \end{aligned}$$

j
$$\begin{aligned} &(3x+2)^2 \\ &= (3x)^2+2(3x)(2)+2^2 \\ &= 9x^2+12x+4 \end{aligned}$$

k
$$\begin{aligned} &(5-2x)^2 \\ &= 5^2-2(5)(2x)+(2x)^2 \\ &= 4x^2-20x+25 \end{aligned}$$

l
$$\begin{aligned} &(7-3x)^2 \\ &= 7^2-2(7)(3x)+(3x)^2 \\ &= 9x^2-42x+49 \end{aligned}$$

4 **a**
$$\begin{aligned} y &= 2(x+2)(x+3) \\ &= 2(x^2+3x+2x+6) \\ &= 2(x^2+5x+6) \\ &= 2x^2+10x+12 \end{aligned}$$

b
$$\begin{aligned} y &= 3(x-1)^2+4 \\ &= 3(x^2-2(x)(1)+1^2)+4 \\ &= 3(x^2-2x+1)+4 \\ &= 3x^2-6x+3+4 \\ &= 3x^2-6x+7 \end{aligned}$$

c
$$\begin{aligned} y &= -(x+1)(x-7) \\ &= -(x^2-7x+x-7) \\ &= -(x^2-6x-7) \\ &= -x^2+6x+7 \end{aligned}$$

d
$$\begin{aligned} y &= -(x+2)^2-11 \\ &= -(x^2+2(x)(2)+2^2)-11 \\ &= -(x^2+4x+4)-11 \\ &= -x^2-4x-4-11 \\ &= -x^2-4x-15 \end{aligned}$$

e
$$\begin{aligned} y &= 4(x-1)(x-5) \\ &= 4(x^2-5x-x+5) \\ &= 4(x^2-6x+5) \\ &= 4x^2-24x+20 \end{aligned}$$

f
$$\begin{aligned} y &= -\tfrac{1}{2}(x+4)^2-6 \\ &= -\tfrac{1}{2}(x^2+2(x)(4)+4^2)-6 \\ &= -\tfrac{1}{2}(x^2+8x+16)-6 \\ &= -\tfrac{1}{2}x^2-4x-8-6 \\ &= -\tfrac{1}{2}x^2-4x-14 \end{aligned}$$

g
$$\begin{aligned} y &= -5(x-1)(x-6) \\ &= -5(x^2-6x-x+6) \\ &= -5(x^2-7x+6) \\ &= -5x^2+35x-30 \end{aligned}$$

h
$$\begin{aligned} y &= \tfrac{1}{2}(x+2)^2-6 \\ &= \tfrac{1}{2}(x^2+2(x)(2)+2^2)-6 \\ &= \tfrac{1}{2}(x^2+4x+4)-6 \\ &= \tfrac{1}{2}x^2+2x+2-6 \\ &= \tfrac{1}{2}x^2+2x-4 \end{aligned}$$

i
$$\begin{aligned} y &= -\tfrac{5}{2}(x-4)^2 \\ &= -\tfrac{5}{2}(x^2-2(x)(4)+4^2) \\ &= -\tfrac{5}{2}(x^2-8x+16) \\ &= -\tfrac{5}{2}x^2+20x-40 \end{aligned}$$

5 **a**
$$\begin{aligned} &1+2(x+3)^2 \\ &= 1+2(x^2+2(x)(3)+3^2) \\ &= 1+2(x^2+6x+9) \\ &= 1+2x^2+12x+18 \\ &= 2x^2+12x+19 \end{aligned}$$

b
$$\begin{aligned} &2+3(x-2)(x+3) \\ &= 2+3(x^2+3x-2x-6) \\ &= 2+3(x^2+x-6) \\ &= 2+3x^2+3x-18 \\ &= 3x^2+3x-16 \end{aligned}$$

c
$$\begin{aligned} &3-(3-x)^2 \\ &= 3-(9-2(3)(x)+x^2) \\ &= 3-(x^2-6x+9) \\ &= 3-x^2+6x-9 \\ &= -x^2+6x-6 \end{aligned}$$

d
$$\begin{aligned} &5-(x+5)(x-4) \\ &= 5-(x^2-4x+5x-20) \\ &= 5-(x^2+x-20) \\ &= 5-x^2-x+20 \\ &= -x^2-x+25 \end{aligned}$$

e
$$\begin{aligned} &1+2(4-x)^2 \\ &= 1+2(4^2-2(4)(x)+x^2) \\ &= 1+2(x^2-8x+16) \\ &= 1+2x^2-16x+32 \\ &= 2x^2-16x+33 \end{aligned}$$

f
$$\begin{aligned} &x^2-3x-(x+2)(x-2) \\ &= x^2-3x-(x^2-2^2) \\ &= x^2-3x-x^2+4 \\ &= -3x+4 \end{aligned}$$

g
$$\begin{aligned} &(x+2)^2-(x+1)(x-4) \\ &= x^2+2(x)(2)+2^2-(x^2-4x+x-4) \\ &= x^2+4x+4-(x^2-3x-4) \\ &= x^2+4x+4-x^2+3x+4 \\ &= 7x+8 \end{aligned}$$

h
$$\begin{aligned} &(2x+3)^2+3(x+1)^2 \\ &= (2x)^2+2(2x)(3)+3^2+3(x^2+2(x)(1)+1^2) \\ &= 4x^2+12x+9+3(x^2+2x+1) \\ &= 4x^2+12x+9+3x^2+6x+3 \\ &= 7x^2+18x+12 \end{aligned}$$

i
$$\begin{aligned}&x^2+3x-2(x-4)^2\\&=x^2+3x-2(x^2-2(x)(4)+4^2)\\&=x^2+3x-2(x^2-8x+16)\\&=x^2+3x-2x^2+16x-32\\&=-x^2+19x-32\end{aligned}$$

j
$$\begin{aligned}&(3x-2)^2-2(x+1)^2\\&=(3x)^2-2(3x)(2)+2^2-2(x^2+2(x)(1)+1^2)\\&=9x^2-12x+4-2(x^2+2x+1)\\&=9x^2-12x+4-2x^2-4x-2\\&=7x^2-16x+2\end{aligned}$$

EXERCISE H

1 **a**
$$\begin{aligned}&3x^2+9x\\&=3x(x+3)\end{aligned}$$

b
$$\begin{aligned}&2x^2+7x\\&=x(2x+7)\end{aligned}$$

c
$$\begin{aligned}&4x^2-10x\\&=2x(2x-5)\end{aligned}$$

d
$$\begin{aligned}&6x^2-15x\\&=3x(2x-5)\end{aligned}$$

e
$$\begin{aligned}&9x^2-25\\&=(3x)^2-5^2\\&=(3x+5)(3x-5)\end{aligned}$$

f
$$\begin{aligned}&16x^2-1\\&=(4x)^2-1^2\\&=(4x+1)(4x-1)\end{aligned}$$

g
$$\begin{aligned}&2x^2-8\\&=2(x^2-4)\\&=2(x^2-2^2)\\&=2(x+2)(x-2)\end{aligned}$$

h
$$\begin{aligned}&3x^2-9\\&=3(x^2-3)\\&=3(x^2-(\sqrt{3})^2)\\&=3(x+\sqrt{3})(x-\sqrt{3})\end{aligned}$$

i
$$\begin{aligned}&4x^2-20\\&=4(x^2-5)\\&=4(x^2-(\sqrt{5})^2)\\&=4(x+\sqrt{5})(x-\sqrt{5})\end{aligned}$$

j
$$\begin{aligned}&x^2-8x+16\\&=x^2-2(x)(4)+4^2\\&=(x-4)^2\end{aligned}$$

k
$$\begin{aligned}&x^2-10x+25\\&=x^2-2(x)(5)+5^2\\&=(x-5)^2\end{aligned}$$

l
$$\begin{aligned}&2x^2-8x+8\\&=2(x^2-4x+4)\\&=2(x^2-2(x)(2)+2^2)\\&=2(x-2)^2\end{aligned}$$

m
$$\begin{aligned}&16x^2+40x+25\\&=(4x)^2+2(4x)(5)+5^2\\&=(4x+5)^2\end{aligned}$$

n
$$\begin{aligned}&9x^2+12x+4\\&=(3x)^2+2(3x)(2)+2^2\\&=(3x+2)^2\end{aligned}$$

o
$$\begin{aligned}&x^2-22x+121\\&=x^2-2(x)(11)+11^2\\&=(x-11)^2\end{aligned}$$

2 **a**
$$\begin{aligned}&x^2+9x+8\\&=(x+1)(x+8)\\&\{\text{as sum}=9,\quad\text{product}=8\}\end{aligned}$$

b
$$\begin{aligned}&x^2+7x+12\\&=(x+3)(x+4)\\&\{\text{as sum}=7,\quad\text{product}=12\}\end{aligned}$$

c
$$\begin{aligned}&x^2-7x-18\\&=(x-9)(x+2)\\&\{\text{as sum}=-7,\quad\text{product}=-18\}\end{aligned}$$

d
$$\begin{aligned}&x^2+4x-21\\&=(x+7)(x-3)\\&\{\text{as sum}=4,\quad\text{product}=-21\}\end{aligned}$$

e
$$\begin{aligned}&x^2-9x+18\\&=(x-6)(x-3)\\&\{\text{as sum}=-9,\quad\text{product}=18\}\end{aligned}$$

f
$$\begin{aligned}&x^2+x-6\\&=(x+3)(x-2)\\&\{\text{as sum}=1,\quad\text{product}=-6\}\end{aligned}$$

g
$$\begin{aligned}&-x^2+x+2\\&=-(x^2-x-2)\\&=-(x-2)(x+1)\\&\{\text{as sum}=-1,\quad\text{product}=-2\}\end{aligned}$$

h
$$\begin{aligned}&3x^2-42x+99\\&=3(x^2-14x+33)\\&=3(x-3)(x-11)\\&\{\text{as sum}=-14,\quad\text{product}=33\}\end{aligned}$$

i
$$\begin{aligned}&-2x^2-4x-2\\&=-2(x^2+2x+1)\\&=-2(x^2+2(x)(1)+1^2)\\&=-2(x+1)^2\end{aligned}$$

j
$$\begin{aligned}&2x^2+6x-20\\&=2(x^2+3x-10)\\&=2(x+5)(x-2)\\&\{\text{as sum}=3,\quad\text{product}=-10\}\end{aligned}$$

k
$$\begin{aligned}&2x^2-10x-48\\&=2(x^2-5x-24)\\&=2(x-8)(x+3)\end{aligned}$$
{as sum = -5, product = -24}

l
$$\begin{aligned}&-2x^2+14x-12\\&=-2(x^2-7x+6)\\&=-2(x-6)(x-1)\end{aligned}$$
{as sum = -7, product = 6}

m
$$\begin{aligned}&-3x^2+6x-3\\&=-3(x^2-2x+1)\\&=-3(x^2-2(x)(1)+1^2)\\&=-3(x-1)^2\end{aligned}$$

n
$$\begin{aligned}&-x^2-2x-1\\&=-(x^2+2x+1)\\&=-(x^2+2(x)(1)+1^2)\\&=-(x+1)^2\end{aligned}$$

o
$$\begin{aligned}&-5x^2+10x+40\\&=-5(x^2-2x-8)\\&=-5(x-4)(x+2)\end{aligned}$$
{as sum = -2, prod. = -8}

3 **a**
$$\begin{aligned}&2x^2+5x-12\\&=2x^2+8x-3x-12\\&=2x(x+4)-3(x+4)\\&=(2x-3)(x+4)\end{aligned}$$
has $ac=2\times-12=-24$
Factors of -24 which add to 5 are 8 and -3.

b
$$\begin{aligned}&3x^2-5x-2\\&=3x^2-6x+x-2\\&=3x(x-2)+(x-2)\\&=(3x+1)(x-2)\end{aligned}$$
has $ac=3\times-2=-6$
Factors of -6 which add to -5 are -6 and 1.

c
$$\begin{aligned}&7x^2-9x+2\\&=7x^2-7x-2x+2\\&=7x(x-1)-2(x-1)\\&=(7x-2)(x-1)\end{aligned}$$
has $ac=7\times 2=14$
Factors of 14 which add to -9 are -7 and -2.

d
$$\begin{aligned}&6x^2-x-2\\&=6x^2+3x-4x-2\\&=3x(2x+1)-2(2x+1)\\&=(3x-2)(2x+1)\end{aligned}$$
has $ac=6\times-2=-12$
Factors of -12 which add to -1 are 3 and -4.

e
$$\begin{aligned}&4x^2-4x-3\\&=4x^2+2x-6x-3\\&=2x(2x+1)-3(2x+1)\\&=(2x-3)(2x+1)\end{aligned}$$
has $ac=4\times-3=-12$
Factors of -12 which add to -4 are -6 and 2.

f
$$\begin{aligned}&10x^2-x-3\\&=10x^2+5x-6x-3\\&=5x(2x+1)-3(2x+1)\\&=(5x-3)(2x+1)\end{aligned}$$
has $ac=10\times-3=-30$
Factors of -30 which add to -1 are -6 and 5.

g
$$\begin{aligned}&2x^2-11x-6\\&=2x^2-12x+x-6\\&=2x(x-6)+(x-6)\\&=(2x+1)(x-6)\end{aligned}$$
has $ac=2\times-6=-12$
Factors of -12 which add to -11 are -12 and 1.

h
$$\begin{aligned}&3x^2-5x-28\\&=3x^2-12x+7x-28\\&=3x(x-4)+7(x-4)\\&=(3x+7)(x-4)\end{aligned}$$
has $ac=3\times-28=-84$
Factors of -84 which add to -5 are -12 and 7.

i $8x^2+2x-3$ has $ac=8\times -3=-24$

$=8x^2-4x+6x-3$ Factors of -24 which add to 2 are 6 and -4.

$=4x(2x-1)+3(2x-1)$

$=(4x+3)(2x-1)$

j $10x^2-9x-9$ has $ac=10\times -9=-90$

$=10x^2-15x+6x-9$ Factors of -90 which add to -9 are -15 and 6.

$=5x(2x-3)+3(2x-3)$

$=(5x+3)(2x-3)$

k $3x^2+23x-8$ has $ac=3\times -8=-24$

$=3x^2-x+24x-8$ Factors of -24 which add to 23 are 24 and -1.

$=x(3x-1)+8(3x-1)$

$=(x+8)(3x-1)$

l $6x^2+7x+2$ has $ac=6\times 2=12$

$=6x^2+3x+4x+2$ Factors of 12 which add to 7 are 4 and 3.

$=3x(2x+1)+2(2x+1)$

$=(3x+2)(2x+1)$

m $-4x^2-2x+6$

$=-2(2x^2+x-3)$ has $ac=2\times -3=-6$

$=-2(2x^2-2x+3x-3)$ Factors of -6 which add to 1 are 3 and -2.

$=-2[2x(x-1)+3(x-1)]$

$=-2(2x+3)(x-1)$

n $12x^2-16x-3$ has $ac=12\times -3=-36$

$=12x^2-18x+2x-3$ Factors of -36 which add to -16 are -18 and 2.

$=6x(2x-3)+(2x-3)$

$=(6x+1)(2x-3)$

o $-6x^2-9x+42$

$=-3(2x^2+3x-14)$ has $ac=2\times -14=-28$

$=-3(2x^2-4x+7x-14)$ Factors of -28 which add to 3 are 7 and -4.

$=-3[2x(x-2)+7(x-2)]$

$=-3(2x+7)(x-2)$

p $21x-10-9x^2$

$=-(9x^2-21x+10)$ has $ac=9\times 10=90$

$=-(9x^2-6x-15x+10)$ Factors of 90 which add to -21 are -6 and -15.

$=-[3x(3x-2)-5(3x-2)]$

$=-(3x-5)(3x-2)$

q $8x^2-6x-27$ has $ac=8\times -27=-216$

$=8x^2+12x-18x-27$ Factors of -216 which add to -6 are -18 and 12.

$=4x(2x+3)-9(2x+3)$

$=(4x-9)(2x+3)$

r $12x^2+13x+3$ has $ac=12\times 3=36$

$=12x^2+4x+9x+3$ Factors of 36 which add to 13 are 9 and 4.

$=4x(3x+1)+3(3x+1)$

$=(4x+3)(3x+1)$

s $12x^2+20x+3$ has $ac = 12 \times 3 = 36$

$= 12x^2+2x+18x+3$ Factors of 36 which add to 20 are 2 and 18.

$= 2x(6x+1)+3(6x+1)$

$= (2x+3)(6x+1)$

t $15x^2-22x+8$ has $ac = 15 \times 8 = 120$

$= 15x^2-10x-12x+8$ Factors of 120 which add to -22 are -10 and -12.

$= 5x(3x-2)-4(3x-2)$

$= (5x-4)(3x-2)$

u $14x^2-11x-15$ has $ac = 14 \times -15 = -210$

$= 14x^2-21x+10x-15$ Factors of -210 which add to -11 are -21 and 10.

$= 7x(2x-3)+5(2x-3)$

$= (7x+5)(2x-3)$

4 **a** $3(x+4)+2(x+4)(x-1)$

$= (x+4)[3+2(x-1)]$

$= (x+4)(3+2x-2)$

$= (x+4)(2x+1)$

b $8(2-x)-3(x+1)(2-x)$

$= (2-x)[8-3(x+1)]$

$= (2-x)(8-3x-3)$

$= (2-x)(5-3x)$

c $6(x+2)^2+9(x+2)$

$= (x+2)[6(x+2)+9]$

$= (x+2)(6x+12+9)$

$= (x+2)(6x+21)$

$= (x+2) \times 3(2x+7)$

$= 3(x+2)(2x+7)$

d $4(x+5)+8(x+5)^2$

$= (x+5)[4+8(x+5)]$

$= (x+5)(4+8x+40)$

$= (x+5)(8x+44)$

$= (x+5) \times 4(2x+11)$

$= 4(x+5)(2x+11)$

e $(x+2)(x+3)-(x+3)(2-x)$

$= (x+3)[(x+2)-(2-x)]$

$= (x+3)(x+2-2+x)$

$= (x+3)(2x)$

$= 2x(x+3)$

f $(x+3)^2+2(x+3)-x(x+3)$

$= (x+3)[(x+3)+2-x]$

$= (x+3)(x+3+2-x)$

$= (x+3)(5)$

$= 5(x+3)$

g $5(x-2)-3(2-x)(x+7)$

$= 5(x-2)+3(x-2)(x+7)$

$= (x-2)[5+3(x+7)]$

$= (x-2)(5+3x+21)$

$= (x-2)(3x+26)$

h $3(1-x)+2(x+1)(x-1)$

$= -3(x-1)+2(x+1)(x-1)$

$= (x-1)[-3+2(x+1)]$

$= (x-1)(-3+2x+2)$

$= (x-1)(2x-1)$

5 **a** $(x+3)^2-16$

$= (x+3)^2-4^2$

$= (x+3+4)(x+3-4)$

$= (x+7)(x-1)$

b $4-(1-x)^2$

$= 2^2-(1-x)^2$

$= [2+(1-x)][2-(1-x)]$

$= (2+1-x)(2-1+x)$

$= (3-x)(x+1)$

c $(x+4)^2-(x-2)^2$

$= [(x+4)+(x-2)][(x+4)-(x-2)]$

$= (x+4+x-2)(x+4-x+2)$

$= (2x+2)(6)$

$= 2(x+1)(6)$

$= 12(x+1)$

d $16-4(x+2)^2$

$= 4[4-(x+2)^2]$

$= 4[2+(x+2)][2-(x+2)]$

$= 4(x+4)(-x)$

$= -4x(x+4)$

e
$$\begin{aligned}&(2x+3)^2-(x-1)^2\\&=[(2x+3)+(x-1)][(2x+3)-(x-1)]\\&=(2x+3+x-1)(2x+3-x+1)\\&=(3x+2)(x+4)\end{aligned}$$

g
$$\begin{aligned}&3x^2-3(x+2)^2\\&=3[x^2-(x+2)^2]\\&=3[x+(x+2)][x-(x+2)]\\&=3(x+x+2)(x-x-2)\\&=3(2x+2)(-2)\\&=-6(2x+2)\\&=-12(x+1)\end{aligned}$$

i
$$\begin{aligned}&12x^2-27(3+x)^2\\&=3[4x^2-9(3+x)^2]\\&=3[(2x)^2-3^2(3+x)^2]\\&=3[(2x)^2-(3(3+x))^2]\\&=3[2x+3(3+x)][2x-3(3+x)]\\&=3[(2x+9+3x)(2x-9-3x)]\\&=3(5x+9)(-x-9)\\&=-3(5x+9)(x+9)\end{aligned}$$

f
$$\begin{aligned}&(x+h)^2-x^2\\&=[(x+h)+x][(x+h)-x]\\&=(x+h+x)(x+h-x)\\&=(2x+h)(h)\\&=h(2x+h)\end{aligned}$$

h
$$\begin{aligned}&5x^2-20(2-x)^2\\&=5[x^2-4(2-x)^2]\\&=5[x^2-2^2(2-x)^2]\\&=5[x^2-(2(2-x))^2]\\&=5[x+2(2-x)][x-2(2-x)]\\&=5(x+4-2x)(x-4+2x)\\&=5(-x+4)(3x-4)\\&=-5(x-4)(3x-4)\end{aligned}$$

EXERCISE I

1 **a**
$$\begin{aligned}a+x&=b\\ \therefore\ x&=b-a\end{aligned}$$

b
$$\begin{aligned}ax&=b\\ \therefore\ x&=\frac{b}{a}\end{aligned}$$

c
$$\begin{aligned}2x+a&=d\\ \therefore\ 2x&=d-a\\ \therefore\ x&=\frac{d-a}{2}\end{aligned}$$

d
$$\begin{aligned}c+x&=t\\ \therefore\ x&=t-c\end{aligned}$$

e
$$\begin{aligned}5x+2y&=20\\ \therefore\ 5x&=20-2y\\ \therefore\ x&=\frac{20-2y}{5}\end{aligned}$$

f
$$\begin{aligned}2x+3y&=12\\ \therefore\ 2x&=12-3y\\ \therefore\ x&=\frac{12-3y}{2}\end{aligned}$$

g
$$\begin{aligned}7x+3y&=d\\ \therefore\ 7x&=d-3y\\ \therefore\ x&=\frac{d-3y}{7}\end{aligned}$$

h
$$\begin{aligned}ax+by&=c\\ \therefore\ ax&=c-by\\ \therefore\ x&=\frac{c-by}{a}\end{aligned}$$

i
$$\begin{aligned}y&=mx+c\\ \therefore\ mx&=y-c\\ \therefore\ x&=\frac{y-c}{m}\end{aligned}$$

2 **a**
$$\begin{aligned}az&=\frac{b}{c}\\ \therefore\ z&=\frac{b}{ac}\end{aligned}$$

b
$$\begin{aligned}\frac{a}{z}&=d\\ \therefore\ \frac{z}{a}&=\frac{1}{d}\\ \therefore\ z&=\frac{a}{d}\end{aligned}$$

c
$$\begin{aligned}\frac{3}{d}&=\frac{2}{z}\\ \therefore\ 3z&=2d\\ \therefore\ z&=\frac{2d}{3}\end{aligned}$$

3 **a**
$$\begin{aligned}F&=ma\\ \therefore\ a&=\frac{F}{m}\end{aligned}$$

b
$$\begin{aligned}C&=2\pi r\\ \therefore\ r&=\frac{C}{2\pi}\end{aligned}$$

c
$$\begin{aligned}V&=ldh\\ \therefore\ d&=\frac{V}{lh}\end{aligned}$$

d
$$\begin{aligned}A&=\frac{b}{K}\\ \therefore\ KA&=b\\ \therefore\ K&=\frac{b}{A}\end{aligned}$$

4 **a**
$$A = \pi r^2$$
$$\therefore \frac{A}{\pi} = r^2$$
$$\therefore r = \sqrt{\frac{A}{\pi}}$$

b
$$N = \frac{x^5}{a}$$
$$\therefore aN = x^5$$
$$\therefore x = \sqrt[5]{aN}$$

c
$$V = \tfrac{4}{3}\pi r^3$$
$$\therefore \tfrac{3}{4}V = \pi r^3$$
$$\therefore \frac{3V}{4\pi} = r^3$$
$$\therefore r = \sqrt[3]{\frac{3V}{4\pi}}$$

d
$$D = \frac{n}{x^3}$$
$$\therefore Dx^3 = n$$
$$\therefore x^3 = \frac{n}{D}$$
$$\therefore x = \sqrt[3]{\frac{n}{D}}$$

5 **a**
$$d = \frac{\sqrt{a}}{n}$$
$$\therefore dn = \sqrt{a}$$
$$\therefore a = (dn)^2$$
$$\therefore a = d^2n^2$$

b
$$T = \tfrac{1}{5}\sqrt{l}$$
$$\therefore 5T = \sqrt{l}$$
$$\therefore l = (5T)^2$$
$$\therefore l = 25T^2$$

c
$$c = \sqrt{a^2 - b^2}$$
$$\therefore c^2 = a^2 - b^2$$
$$\therefore a^2 = c^2 + b^2$$
$$\therefore a = \pm\sqrt{b^2 + c^2}$$

d
$$T = 2\pi\sqrt{\frac{l}{g}}$$
$$\therefore \frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$
$$\therefore \left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$$
$$\therefore \frac{T^2}{4\pi^2} = \frac{l}{g}$$
$$\therefore l = \frac{gT^2}{4\pi^2}$$

e
$$P = 2(a+b)$$
$$\therefore \frac{P}{2} = a + b$$
$$\therefore a = \frac{P}{2} - b$$

f
$$A = \pi r^2 + 2\pi rh$$
$$\therefore A - \pi r^2 = 2\pi rh$$
$$\therefore h = \frac{A - \pi r^2}{2\pi r}$$

g
$$I = \frac{E}{R+r}$$
$$\therefore E = I(R+r)$$
$$\therefore \frac{E}{I} = R + r$$
$$\therefore r = \frac{E}{I} - R$$

h
$$A = \frac{B}{p-q}$$
$$\therefore A(p-q) = B$$
$$\therefore p - q = \frac{B}{A}$$
$$\therefore q = p - \frac{B}{A}$$

6 **a**
$$k = \frac{d^2}{2ab}$$
$$\therefore d^2 = k(2ab)$$
$$\therefore \frac{d^2}{k} = 2ab$$
$$\therefore a = \frac{d^2}{2kb}$$

b When $k = 112$, $d = 24$, $b = 2$,
$$a = \frac{24^2}{2 \times 112 \times 2}$$
$$= \frac{576}{448}$$
$$\doteqdot 1.29$$

7 **a**
$$V = \tfrac{4}{3}\pi r^3$$
$$\therefore \tfrac{3}{4}V = \pi r^3$$
$$\therefore r^3 = \frac{3V}{4\pi}$$
$$\therefore r = \sqrt[3]{\frac{3V}{4\pi}}$$

b When $V = 40$,
$$r = \sqrt[3]{\frac{3 \times 40}{4\pi}}$$
$$\doteqdot 2.122 \text{ cm}$$

8 **a**
$$S = \tfrac{1}{2}at^2$$
$$\therefore t^2 = \frac{2S}{a}$$
$$\therefore t = \sqrt{\frac{2S}{a}}$$

b 10 m $\equiv 10 \times 100$ cm $= 1000$ cm
i.e., we need to find t when $a = 8$, $S = 1000$
$$t = \sqrt{\frac{2 \times 1000}{8}}$$
$$\therefore t \doteqdot 15.81 \text{ seconds}$$

9 **a** $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v} \quad \therefore \quad \dfrac{1}{f} - \dfrac{1}{u} = \dfrac{1}{v}$

$\therefore \quad \dfrac{u-f}{uf} = \dfrac{1}{v}$ and so $\therefore \quad v = \dfrac{uf}{u-f}$

b **i** When $u = 50, \quad f = 8$

so $v = \dfrac{50 \times 8}{50 - 8}$

$\therefore \quad v \doteqdot 9.52$ cm

ii When $u = 30, \quad f = 8$

so $v = \dfrac{30 \times 8}{30 - 8}$

$\therefore \quad v \doteqdot 10.9$ cm

10 **a**

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$\therefore \quad m\sqrt{1 - \left(\frac{v}{c}\right)^2} = m_0$$

$$\therefore \quad \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{m_0}{m}$$

$$\therefore \quad 1 - \left(\frac{v}{c}\right)^2 = \left(\frac{m_0}{m}\right)^2$$

$$\therefore \quad \left(\frac{v}{c}\right)^2 = 1 - \frac{{m_0}^2}{m^2}$$

$$\therefore \quad \frac{v}{c} = \sqrt{\frac{m^2 - {m_0}^2}{m^2}}$$

$$\therefore \quad v = \frac{c}{m}\sqrt{m^2 - {m_0}^2}$$

b When $m = 3m_0$,

$$\begin{aligned} v &= \frac{c}{3m_0}\sqrt{(3m_0)^2 - {m_0}^2} \\ &= \frac{c}{3m_0}\sqrt{9{m_0}^2 - {m_0}^2} \\ &= \frac{c}{3m_0}\sqrt{8{m_0}^2} \\ &= \frac{c}{3m_0}\sqrt{8}m_0 \\ &= \tfrac{\sqrt{8}}{3}\, c \end{aligned}$$

c When $m = 30m_0, \quad c = 3 \times 10^8$,

$$\begin{aligned} v &= \frac{3 \times 10^8}{30m_0}\sqrt{(30m_0)^2 - {m_0}^2} \\ &= \frac{30 \times 10^7}{30m_0}\sqrt{900{m_0}^2 - {m_0}^2} \\ &= \frac{10^7}{m_0}\sqrt{899{m_0}^2} \\ &= \frac{10^7}{m_0} \times \sqrt{899} \times m_0 \\ &= 10^7 \times \sqrt{899} \\ &\doteqdot 2.998 \times 10^8 \text{ m/s} \end{aligned}$$

EXERCISE J

1 **a**

$$\begin{aligned} & 3 + \frac{x}{5} \\ &= 3 \times \frac{5}{5} + \frac{x}{5} \\ &= \frac{15}{5} + \frac{x}{5} \\ &= \frac{x + 15}{5} \end{aligned}$$

b

$$\begin{aligned} & 1 + \frac{3}{x} \\ &= 1 \times \frac{x}{x} + \frac{3}{x} \\ &= \frac{x}{x} + \frac{3}{x} \\ &= \frac{x + 3}{x} \end{aligned}$$

c

$$\begin{aligned} & 3 + \frac{x - 2}{2} \\ &= 3 \times \frac{2}{2} + \frac{x - 2}{2} \\ &= \frac{6}{2} + \frac{x - 2}{2} \\ &= \frac{6 + x - 2}{2} \\ &= \frac{x + 4}{2} \end{aligned}$$

d $3-\dfrac{x-2}{4}$

$=3\times\dfrac{4}{4}-\dfrac{x-2}{4}$

$=\dfrac{12}{4}-\dfrac{x-2}{4}$

$=\dfrac{12-x+2}{4}$

$=\dfrac{14-x}{4}$

e $\dfrac{2+x}{3}+\dfrac{x-4}{5}$

$=\dfrac{2+x}{3}\times\dfrac{5}{5}+\dfrac{x-4}{5}\times\dfrac{3}{3}$

$=\dfrac{5(2+x)}{15}+\dfrac{3(x-4)}{15}$

$=\dfrac{10+5x+3x-12}{15}$

$=\dfrac{8x-2}{15}$

f $\dfrac{2x+5}{4}-\dfrac{x-1}{6}$

$=\dfrac{2x+5}{4}\times\dfrac{3}{3}-\dfrac{x-1}{6}\times\dfrac{2}{2}$

$=\dfrac{3(2x+5)}{12}-\dfrac{2(x-1)}{12}$

$=\dfrac{6x+15-2x+2}{12}$

$=\dfrac{4x+17}{12}$

2 **a** $1+\dfrac{3}{x+2}$

$=1\times\dfrac{x+2}{x+2}+\dfrac{3}{x+2}$

$=\dfrac{x+2+3}{x+2}$

$=\dfrac{x+5}{x+2}$

b $-2+\dfrac{3}{x-4}$

$=-2\times\dfrac{x-4}{x-4}+\dfrac{3}{x-4}$

$=\dfrac{-2(x-4)+3}{x-4}$

$=\dfrac{-2x+8+3}{x-4}$

$=\dfrac{11-2x}{x-4}$

c $-3-\dfrac{2}{x-1}$

$=-3\times\dfrac{x-1}{x-1}-\dfrac{2}{x-1}$

$=\dfrac{-3(x-1)-2}{x-1}$

$=\dfrac{-3x+3-2}{x-1}$

$=\dfrac{1-3x}{x-1}$

d $\dfrac{2x-1}{x+1}+3$

$=\dfrac{2x-1}{x+1}+3\times\dfrac{x+1}{x+1}$

$=\dfrac{2x-1+3(x+1)}{x+1}$

$=\dfrac{2x-1+3x+3}{x+1}$

$=\dfrac{5x+2}{x+1}$

e $3-\dfrac{x}{x+1}$

$=3\times\dfrac{x+1}{x+1}-\dfrac{x}{x+1}$

$=\dfrac{3(x+1)-x}{x+1}$

$=\dfrac{3x+3-x}{x+1}$

$=\dfrac{2x+3}{x+1}$

f $-1+\dfrac{4}{1-x}$

$=-1\times\dfrac{1-x}{1-x}+\dfrac{4}{1-x}$

$=\dfrac{-(1-x)+4}{1-x}$

$=\dfrac{x-1+4}{1-x}$

$=\dfrac{x+3}{1-x}$

3 **a** $\dfrac{3x}{2x-5}+\dfrac{2x+5}{x-2}$

$=\dfrac{3x}{2x-5}\times\dfrac{x-2}{x-2}+\dfrac{2x+5}{x-2}\times\dfrac{2x-5}{2x-5}$

$=\dfrac{3x(x-2)+(2x+5)(2x-5)}{(2x-5)(x-2)}$

$=\dfrac{3x^2-6x+(4x^2-5^2)}{(2x-5)(x-2)}$

$=\dfrac{3x^2-6x+4x^2-25}{(2x-5)(x-2)}$

$=\dfrac{7x^2-6x-25}{(2x-5)(x-2)}$

b $\dfrac{1}{x-2}-\dfrac{1}{x-3}$

$=\dfrac{1}{x-2}\times\dfrac{x-3}{x-3}-\dfrac{1}{x-3}\times\dfrac{x-2}{x-2}$

$=\dfrac{(x-3)-(x-2)}{(x-2)(x-3)}$

$=\dfrac{x-3-x+2}{(x-2)(x-3)}$

$=-\dfrac{1}{(x-2)(x-3)}$

c $\dfrac{5x}{x-4}+\dfrac{3x-2}{x+4}$

$=\dfrac{5x}{x-4}\times\dfrac{x+4}{x+4}+\dfrac{3x-2}{x+4}\times\dfrac{x-4}{x-4}$

$=\dfrac{5x(x+4)+(3x-2)(x-4)}{(x+4)(x-4)}$

$=\dfrac{5x^2+20x+3x^2-12x-2x+8}{(x+4)(x-4)}$

$=\dfrac{8x^2+6x+8}{(x+4)(x-4)}$

d $\dfrac{2x+1}{x-3}-\dfrac{x+4}{2x+1}$

$=\dfrac{2x+1}{x-3}\times\dfrac{2x+1}{2x+1}-\dfrac{x+4}{2x+1}\times\dfrac{x-3}{x-3}$

$=\dfrac{(2x+1)^2-(x+4)(x-3)}{(x-3)(2x+1)}$

$=\dfrac{(2x)^2+2(2x)(1)+1^2-(x^2-3x+4x-12)}{(x-3)(2x+1)}$

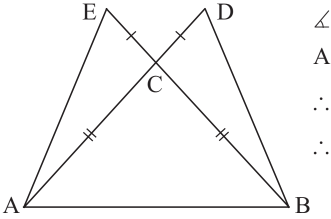
$=\dfrac{4x^2+4x+1-(x^2+x-12)}{(x-3)(2x+1)}$

$=\dfrac{4x^2+4x+1-x^2-x+12}{(x-3)(2x+1)}$

$=\dfrac{3x^2+3x+13}{(x-3)(2x+1)}$

EXERCISE K.1

1



$\measuredangle$ECA = $\measuredangle$DCB {opposite angles}

Also, EC = CD and AC = BC {given}

∴ Δs AEC and BDC are congruent (SAS)

∴ AE = BD

2 MP = NP {given}

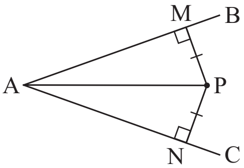
AP is common to both ΔAMP and ΔANP

∴ Δs AMP and ANP are congruent (RHS)

∴ $\measuredangle$MAP = $\measuredangle$NAP {corresponding angles}

i.e., AP bisects $\measuredangle$BAC

∴ P lies on the bisector of $\measuredangle$BAC



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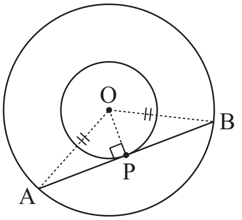


OA = OB {both radii of circle}

$\measuredangle$OPA = $\measuredangle$OPB = 90^o

{since AB is a tangent to inner circle}

OP is common to both ΔAOP and ΔBOP

∴ Δs AOP and BOP are congruent (RHS)

∴ AP = BP {corresponding sides}

∴ P is the midpoint of AB

EXERCISE K.2

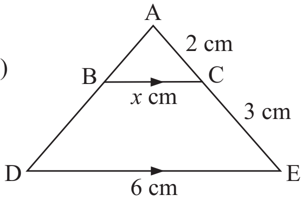
1 **a** $\measuredangle$ABC = $\measuredangle$ADE {corresponding angles}

$\measuredangle$ACB = $\measuredangle$AED {corresponding angles}

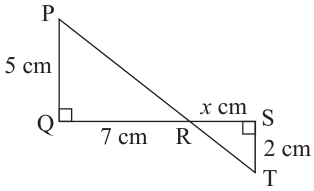
i.e., Δs ABC and ADE are equiangular (AAA) and hence similar.

∴ $\dfrac{x}{2}=\dfrac{6}{2+3}=\dfrac{6}{5}$

∴ $x=\frac{12}{5}=2.4$



b



$\measuredangle PRQ = \measuredangle TRS$ {opposite angles}

$\therefore$ Δs PQR and TSR are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{5}{7} = \frac{2}{x}$$

$$\therefore \quad 5x = 14$$

$$\therefore \quad x = \tfrac{14}{5} = 2.8$$

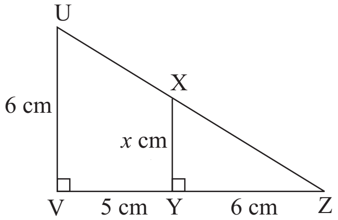
c $\measuredangle VUZ = \measuredangle YXZ$ {corresponding angles}

$\therefore$ Δs UVZ and XYZ are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{6} = \frac{6}{5+6}$$

$$\therefore \quad \frac{x}{6} = \frac{6}{11}$$

$$\therefore \quad x = \tfrac{36}{11} = 3\tfrac{3}{11}$$



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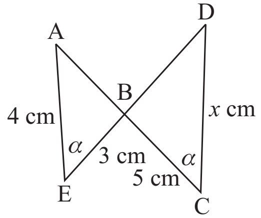


$\measuredangle ABE = \measuredangle DBC$ {opposite angles}

$\therefore$ Δs ABE and DBC are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{5} = \frac{4}{3}$$

$$\therefore \quad x = \tfrac{20}{3} = 6\tfrac{2}{3}$$

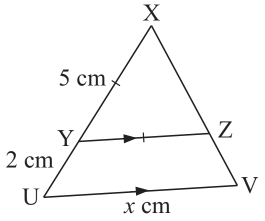
e $\measuredangle XYZ = \measuredangle XUV$ {corresponding angles}

$\measuredangle XZY = \measuredangle XVU$ {corresponding angles}

$\therefore$ Δs XYZ and XUV are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{2+5} = \frac{5}{5}$$

$$\therefore \quad \frac{x}{7} = 1 \quad \text{and so} \quad x = 7$$



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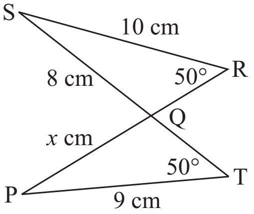


$\measuredangle SQR = \measuredangle PQT$ {opposite angles}

$\therefore$ Δs SQR and PQT are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{9} = \frac{8}{10}$$

$$\therefore \quad x = \tfrac{72}{10} = 7.2$$

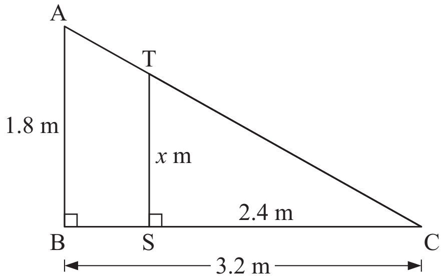
2 $\measuredangle BAC = \measuredangle STC$ {corresponding angles}

$\therefore$ Δs ABC and TSC are equiangular (AAA) and hence similar.

$$\therefore \quad \frac{x}{2.4} = \frac{1.8}{3.2}$$

$$\therefore \quad x = \frac{2.4 \times 1.8}{3.2} = 1.35$$

i.e., the son is 1.35 m tall



EXERCISE L

1 **a**
$$\begin{aligned}\text{AB} &= \sqrt{(x_\text{B}-x_\text{A})^2+(y_\text{B}-y_\text{A})^2}\\ &= \sqrt{(4-1)^2+(5-3)^2}\\ &= \sqrt{9+4}\\ &= \sqrt{13}\text{ units}\end{aligned}$$

b
$$\begin{aligned}\text{OC} &= \sqrt{(x_\text{C}-x_\text{O})^2+(y_\text{C}-y_\text{O})^2}\\ &= \sqrt{(3-0)^2+(-5-0)^2}\\ &= \sqrt{9+25}\\ &= \sqrt{34}\text{ units}\end{aligned}$$

c
$$\begin{aligned}\text{PQ} &= \sqrt{(x_\text{Q}-x_\text{P})^2+(y_\text{Q}-y_\text{P})^2}\\ &= \sqrt{(1-5)^2+(4-2)^2}\\ &= \sqrt{16+4}\\ &= \sqrt{20}\text{ units}\end{aligned}$$

d
$$\begin{aligned}\text{ST} &= \sqrt{(x_\text{T}-x_\text{S})^2+(y_\text{T}-y_\text{S})^2}\\ &= \sqrt{(-1-0)^2+(0--3)^2}\\ &= \sqrt{1+9}\\ &= \sqrt{10}\text{ units}\end{aligned}$$

2 **a** $x_\text{M} = \dfrac{x_\text{A}+x_\text{B}}{2} = \dfrac{3+1}{2} = 2$ and $y_\text{M} = \dfrac{y_\text{A}+y_\text{B}}{2} = \dfrac{6+0}{2} = 3$

$\therefore$ the midpoint is at $(2, 3)$

b $x_\text{M} = \dfrac{x_\text{A}+x_\text{B}}{2} = \dfrac{5-1}{2} = 2$ and $y_\text{M} = \dfrac{y_\text{A}+y_\text{B}}{2} = \dfrac{2-4}{2} = -1$

$\therefore$ the midpoint is at $(2, -1)$

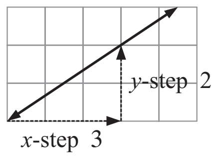
c $x_\text{M} = \dfrac{x_\text{A}+x_\text{B}}{2} = \dfrac{7+0}{2} = \frac{7}{2}$ and $y_\text{M} = \dfrac{y_\text{A}+y_\text{B}}{2} = \dfrac{0+3}{2} = \frac{3}{2}$

$\therefore$ the midpoint is at $(\frac{7}{2}, \frac{3}{2})$

d $x_\text{M} = \dfrac{x_\text{A}+x_\text{B}}{2} = \dfrac{5-1}{2} = 2$ and $y_\text{M} = \dfrac{y_\text{A}+y_\text{B}}{2} = \dfrac{-2-3}{2} = -\frac{5}{2}$

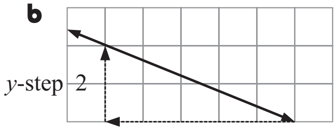
$\therefore$ the midpoint is at $(2, -\frac{5}{2})$

3 **a**



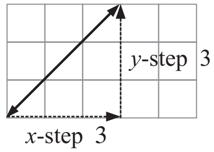
$$\begin{aligned}\text{slope} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \tfrac{2}{3}\end{aligned}$$

b



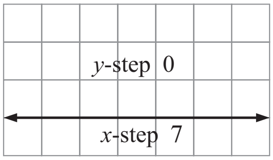
$$\begin{aligned}\text{slope} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \tfrac{2}{-5}\\ &= -\tfrac{2}{5}\end{aligned}$$

c



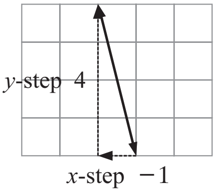
$$\begin{aligned}\text{slope} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \tfrac{3}{3}\\ &= 1\end{aligned}$$

d



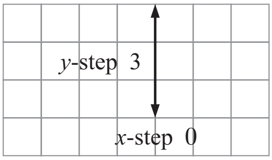
$$\begin{aligned}\text{slope} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \tfrac{0}{7}\\ &= 0\end{aligned}$$

e



$$\begin{aligned}\text{slope} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \tfrac{4}{-1}\\ &= -4\end{aligned}$$

f



$$\begin{aligned}\text{slope} &= \frac{y\text{-step}}{x\text{-step}}\\ &= \tfrac{3}{0}\end{aligned}$$

which is undefined

4 **a**
$$\begin{aligned}\text{gradient} &= \frac{y_2-y_1}{x_2-x_1}\\ &= \frac{7-3}{4-2}\\ &= 2\end{aligned}$$

b
$$\begin{aligned}\text{gradient} &= \frac{y_2-y_1}{x_2-x_1}\\ &= \frac{8-2}{5-3}\\ &= 3\end{aligned}$$

c
$$\begin{aligned}\text{gradient} &= \frac{y_2-y_1}{x_2-x_1}\\ &= \frac{5-2}{-1-(-1)}\\ &= \tfrac{3}{0}\end{aligned}$$

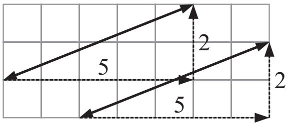
which is undefined

d $$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-3)}{-1-4} = \tfrac{0}{-5} = 0$$

e $$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{-1-0} = -4$$

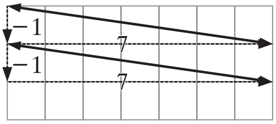
f $$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-1)}{-1-3} = \tfrac{-1}{-4} = \tfrac{1}{4}$$

5 **a**



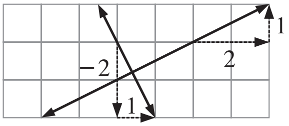
The lines have the same slope ($\frac{2}{5}$), so are parallel.

b



The lines have the same slope ($-\frac{1}{7}$), so are parallel.

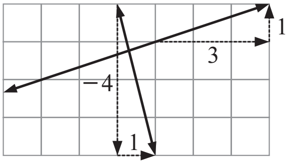
c



The two lines have slopes $\frac{1}{2}$ and $\frac{-2}{1}$, so the product of the slopes is -1.

Hence the lines are perpendicular.

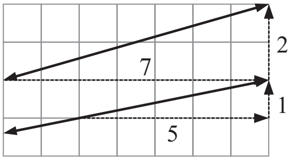
d



The two lines have slopes $\frac{1}{3}$ and $\frac{-4}{1}$, so the product of the slopes is $-\frac{4}{3}$.

Hence the lines are neither parallel nor perpendicular.

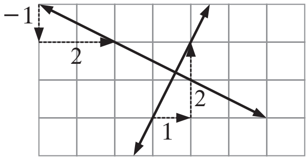
e



The two lines have slopes $\frac{2}{7}$ and $\frac{1}{5}$, so the product of the slopes is $\frac{2}{35}$.

Hence they are neither parallel nor perpendicular.

f



The two lines have slopes $\frac{-1}{2}$ and $\frac{2}{1}$, so the product of the slopes is -1.

Hence the lines are perpendicular.

6 **a** $\text{gradient} = -\dfrac{1}{(\frac{3}{4})} = -\frac{4}{3}$ **b** $\text{gradient} = -\dfrac{1}{(\frac{11}{3})} = -\frac{3}{11}$ **c** $\text{gradient} = -\frac{1}{4}$

d $\text{gradient} = -\dfrac{1}{(-\frac{1}{3})} = 3$ **e** $\text{gradient} = -\frac{1}{(-5)} = \frac{1}{5}$ **f** $\text{gradient} = -\frac{1}{0}$

which is undefined

7 **a** The equation of the line is

$$\begin{aligned} \frac{y-1}{x-4} &= 2 \\ \therefore\ y-1 &= 2(x-4) \\ \therefore\ y-1 &= 2x-8 \\ \therefore\ y &= 2x-7 \end{aligned}$$

b The equation of the line is

$$\begin{aligned} \frac{y-2}{x-1} &= -2 \\ \therefore\ y-2 &= -2(x-1) \\ \therefore\ y-2 &= -2x+2 \\ \therefore\ y &= -2x+4 \end{aligned}$$

c The equation of the line is

$$\begin{aligned}\frac{y-0}{x-5} &= 3\\ \therefore\ y &= 3(x-5)\\ \therefore\ y &= 3x-15\end{aligned}$$

d The equation of the line is

$$\begin{aligned}\frac{y-7}{x-(-1)} &= -3\\ \therefore\ y-7 &= -3(x+1)\\ \therefore\ y-7 &= -3x-3\\ \therefore\ y &= -3x+4\end{aligned}$$

e The equation of the line is

$$\begin{aligned}\frac{y-5}{x-1} &= -4\\ \therefore\ y-5 &= -4(x-1)\\ \therefore\ y-5 &= -4x+4\\ \therefore\ y &= -4x+9\end{aligned}$$

f The equation of the line is

$$\begin{aligned}\frac{y-7}{x-2} &= 1\\ \therefore\ y-7 &= x-2\\ \therefore\ y &= x+5\end{aligned}$$

8 **a** The equation of the line is

$$\begin{aligned}\frac{y-1}{x-2} &= \frac{3}{2}\\ \therefore\ 2(y-1) &= 3(x-2)\\ \therefore\ 2y-2 &= 3x-6\\ \therefore\ 3x-2y &= 4\end{aligned}$$

b The equation of the line is

$$\begin{aligned}\frac{y-4}{x-1} &= -\frac{3}{2}\\ \therefore\ -2(y-4) &= 3(x-1)\\ \therefore\ -2y+8 &= 3x-3\\ \therefore\ 3x+2y &= 11\end{aligned}$$

c The equation of the line is

$$\begin{aligned}\frac{y-0}{x-4} &= \frac{1}{3}\\ \therefore\ 3y &= x-4\\ \therefore\ x-3y &= 4\end{aligned}$$

d The equation of the line is

$$\begin{aligned}\frac{y-6}{x-0} &= -4\\ \therefore\ y-6 &= -4x\\ \therefore\ 4x+y &= 6\end{aligned}$$

e The equation of the line is

$$\begin{aligned}\frac{y-(-3)}{x-(-1)} &= 3\\ \therefore\ y+3 &= 3(x+1)\\ \therefore\ y+3 &= 3x+3\\ \therefore\ 3x-y &= 0\end{aligned}$$

f The equation of the line is

$$\begin{aligned}\frac{y-(-2)}{x-4} &= -\frac{4}{9}\\ \therefore\ -9(y+2) &= 4(x-4)\\ \therefore\ -9y-18 &= 4x-16\\ \therefore\ 4x+9y &= -2\end{aligned}$$

9 **a** The slope of the line is $\frac{2-1}{3-0} = \frac{1}{3}$ $\therefore$ its equation is $\frac{y-1}{x-0} = \frac{1}{3}$

$$\begin{aligned}\therefore\ 3(y-1) &= x\\ \therefore\ 3y-3 &= x\\ \therefore\ x-3y &= -3\end{aligned}$$

b The slope of the line is $\frac{-1-4}{0-1} = 5$ $\therefore$ its equation is $\frac{y-(-1)}{x-0} = 5$

$$\begin{aligned}\therefore\ y+1 &= 5x\\ \therefore\ 5x-y &= 1\end{aligned}$$

c The slope of the line is $\frac{-4-(-1)}{-1-2} = \frac{-3}{-3} = 1$ $\therefore$ its equation is $\frac{y-(-1)}{x-2} = 1$

$$\begin{aligned}\therefore\ y+1 &= x-2\\ \therefore\ x-y &= 3\end{aligned}$$

d The slope of the line is $\frac{2-(-2)}{5-0} = \frac{4}{5}$ $\therefore$ its equation is $\frac{y-(-2)}{x-0} = \frac{4}{5}$

$$\begin{aligned}\therefore\ 5(y+2) &= 4x\\ \therefore\ 5y+10 &= 4x\\ \therefore\ 4x-5y &= 10\end{aligned}$$

e The slope of the line is $\dfrac{2-0}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$ $\therefore$ its equation is $\dfrac{y-0}{x-(-1)} = \dfrac{1}{2}$

$$\therefore\ 2y = x+1$$
$$\therefore\ x-2y = -1$$

f The slope of the line is $\dfrac{-3-(-1)}{2-(-1)} = \dfrac{-2}{3}$ $\therefore$ its equation is $\dfrac{y-(-1)}{x-(-1)} = -\dfrac{2}{3}$

$$\therefore\ -3(y+1) = 2(x+1)$$
$$\therefore\ -3y-3 = 2x+2$$
$$\therefore\ 2x+3y = -5$$

10 **a** Since both points on the line have y-coordinate -2, it must be horizontal.
$\therefore$ its equation is $y=-2$.

b Since both points on the line have x-coordinate 6, it must be vertical.
$\therefore$ its equation is $x=6$.

c Since both points on the line have x-coordinate -3, it must be vertical.
$\therefore$ its equation is $x=-3$.

11 **a** $2x-3y = 6$
$\therefore\ 3y = 2x-6$
$\therefore\ y = \frac{2}{3}x-2$

$\therefore$ slope $= \frac{2}{3}$, y-intercept $=-2$

When $y=0$, $2x=6$
$\therefore$ x-intercept $=3$

b $4x+5y = 20$
$\therefore\ 5y = -4x+20$
$\therefore\ y = -\frac{4}{5}x+4$

$\therefore$ slope $=-\frac{4}{5}$, y-intercept $=4$

When $y=0$, $4x=20$
$\therefore$ x-intercept $=5$

c $y=-2x+5$
$\therefore$ slope $=-2$, y-intercept $=5$
When $y=0$, $-2x+5 = 0$ $\therefore$ $x = \frac{5}{2}$
i.e., the x-intercept $= \frac{5}{2}$

d $x=8$
$\therefore$ the line is vertical
$\therefore$ slope is undefined,
no y-intercept, x-intercept $=8$

e $y=5$
$\therefore$ the line is horizontal
$\therefore$ slope $=0$,
y-intercept $=5$, no x-intercept

f $x+y = 11$
$\therefore\ y = 11-x$
$\therefore$ slope $=-1$, y-intercept $=11$
When $y=0$, $x=11$
$\therefore$ x-intercept $=11$

g $4x+y = 8$
$\therefore\ y = -4x+8$
$\therefore$ slope $=-4$, y-intercept $=8$
When $y=0$, $4x=8$
$\therefore$ x-intercept $=2$

h $x-3y = 12$
$\therefore\ 3y = x-12$
$\therefore\ y = \frac{1}{3}x-4$
$\therefore$ slope $=\frac{1}{3}$, y-intercept $=-4$
When $y=0$, $x=12$
$\therefore$ x-intercept $=12$

Summary of results:

	Equation	*Slope*	*x-int.*	*y-int.*
a	$2x-3y=6$	$\frac{2}{3}$	3	-2
b	$4x+5y=20$	$-\frac{4}{5}$	5	4
c	$y=-2x+5$	-2	$\frac{5}{2}$	5
d	$x=8$	undefined	8	none

	Equation	*Slope*	*x-int.*	*y-int*
e	$y=5$	0	none	5
f	$x+y=11$	-1	11	11
g	$4x+y=8$	-4	2	8
h	$x-3y=12$	$\frac{1}{3}$	12	-4

12 **a** Substituting (3, 4) into $3x - 2y = 1$ gives $3(3) - 2(4) = 1$

i.e., $1 = 1$ which is true

$\therefore$ (3, 4) lies on the line

b Substituting (−2, 5) into $5x + 3y = -5$ gives $5(-2) + 3(5) = -5$

i.e., $5 = -5$ which is false

$\therefore$ (−2, 5) does not lie on the line

c Substituting $(6, -\frac{1}{2})$ into $3x - 8y = 22$ gives $3(6) - 8(-\frac{1}{2}) = 22$

i.e., $22 = 22$ which is true

$\therefore$ $(6, -\frac{1}{2})$ lies on the line

13 **a** For $x + 2y = 8$,

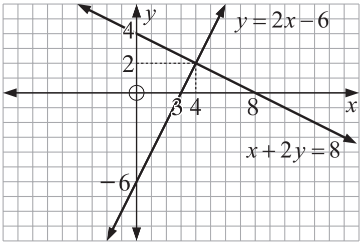
when $x = 0$, $y = 4$
when $y = 0$, $x = 8$

x	0	8
y	4	0

For $y = 2x - 6$,

when $x = 0$, $y = -6$
when $y = 0$, $x = 3$

x	0	3
y	−6	0



The lines meet at (4, 2).

b For $y = -3x - 3$,

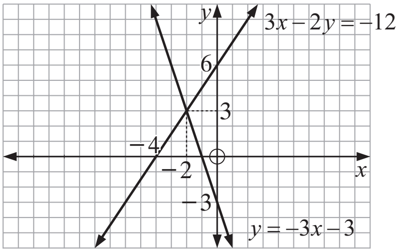
when $x = 0$, $y = -3$
when $y = 0$, $x = -1$

x	0	−1
y	−3	0

For $3x - 2y = -12$,

when $x = 0$, $y = 6$
when $y = 0$, $x = -4$

x	0	−4
y	6	0



The lines meet at (−2, 3).

c For $3x + y = -3$,

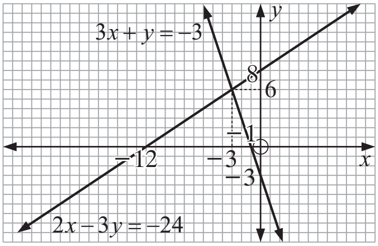
when $x = 0$, $y = -3$
when $y = 0$, $x = -1$

x	0	−1
y	−3	0

For $2x - 3y = -24$,

when $x = 0$, $y = 8$
when $y = 0$, $x = -12$

x	0	−12
y	8	0



The lines meet at (−3, 6).

d For $2x - 3y = 8$,

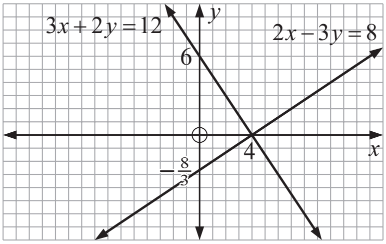
when $x = 0$, $y = -\frac{8}{3}$
when $y = 0$, $x = 4$

x	0	4
y	$-\frac{8}{3}$	0

For $3x + 2y = 12$,

when $x = 0$, $y = 6$
when $y = 0$, $x = 4$

x	0	4
y	6	0



The lines meet at (4, 0).

e For $x + 3y = 10,$

when $x = 0, \quad y = \frac{10}{3}$

when $y = 0, \quad x = 10$

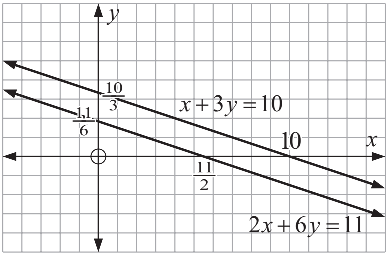
x	0	10
y	$\frac{10}{3}$	0

For $2x + 6y = 11,$

when $x = 0, \quad y = \frac{11}{6}$

when $y = 0, \quad x = \frac{11}{2}$

x	0	$\frac{11}{2}$
y	$\frac{11}{6}$	0



The lines are parallel, so never meet.

f For $5x + 3y = 10,$

when $x = 0, \quad y = \frac{10}{3}$

when $y = 0, \quad x = 2$

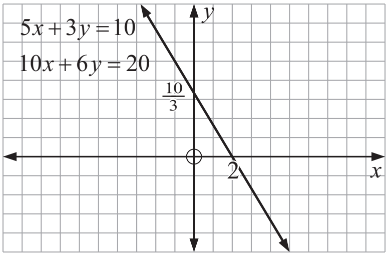
x	0	2
y	$\frac{10}{3}$	0

For $10x + 6y = 20,$

when $x = 0, \quad y = \frac{20}{6} = \frac{10}{3}$

when $y = 0, \quad x = 2$

x	0	2
y	$\frac{10}{3}$	0



The lines are coincident.

14 **a** Since the line is horizontal, its equation is $y = -4$.

b Since the line is vertical, its equation is $x = 5$.

c Since the line is vertical, its equation is $x = -1$.

d Since the line is horizontal, its equation is $y = 2$.

e The x-axis corresponds to $y = 0$.

f The y-axis corresponds to $x = 0$.

15 **a** The line has equation $\dfrac{y-4}{x-(-1)} = \dfrac{3}{4}$

$$\therefore \ 4(y-4) = 3(x+1)$$
$$\therefore \ 4y - 16 = 3x + 3$$
$$\therefore \ 3x - 4y = -19$$

b The line has slope $= \dfrac{0-(-5)}{7-2} = \dfrac{5}{5} = 1 \quad \therefore$ its equation is $\dfrac{y-0}{x-7} = 1$

$$\therefore \ y = x - 7$$
$$\therefore \ x - y = 7$$

c $y = 3x - 2$ has slope 3, so this line has slope 3 also.

It passes through (0, 0), so its equation is $\dfrac{y-0}{x-0} = 3$ i.e., $y = 3x$

d Now a line parallel to $2x + 3y = 8$ has equation $2x + 3y = k$, where k is a constant.

Since $(-1, 7)$ lies on the line, $2(-1) + 3(7) = k$

$$\therefore \ k = 19$$

$\therefore$ the line is $2x + 3y = 19$

e $y = -2x + 5$ has slope $-2 \quad \therefore$ lines perpendicular to it have slope $\frac{1}{2}$.

But this line must pass through $(3, -1)$. $\quad \therefore \ \dfrac{y-(-1)}{x-3} = \dfrac{1}{2}$

$$\therefore \ 2(y+1) = x - 3$$
$$\therefore \ 2y + 2 = x - 3$$
$$\therefore \ x - 2y = 5$$

f If $3x - y = 11$, then $y = 3x - 11$

$\therefore$ this line has slope 3 $\therefore$ lines perpendicular to it have slope $-\frac{1}{3}$.

But this line must pass through $(-2, 5)$.

$$\begin{aligned} \therefore\ \frac{y-5}{x-(-2)} &= -\frac{1}{3} \\ \therefore\ -3(y-5) &= x+2 \\ \therefore\ -3y+15 &= x+2 \\ \therefore\ x+3y &= 13 \end{aligned}$$

16 **a** Keach Avenue passes through $(5, 11)$ and $(13, 12)$

$\therefore$ its line has slope $\dfrac{12-11}{13-5} = \dfrac{1}{8}$

$\therefore$ its equation is $\dfrac{y-12}{x-13} = \dfrac{1}{8}$

$$\begin{aligned} \therefore\ 8(y-12) &= x-13 \\ \therefore\ 8y-96 &= x-13 \\ \therefore\ x-8y &= -83 \end{aligned}$$

b Peacock Street is perpendicular to Keach Avenue, so its line has slope -8.

But this line also passes through $(3, 17)$.

$$\begin{aligned} \therefore\ \frac{y-17}{x-3} &= -8 \\ \therefore\ y-17 &= -8(x-3) \\ \therefore\ y-17 &= -8x+24 \\ \therefore\ 8x+y &= 41 \end{aligned}$$

c Diagonal Road runs from $(5, 11)$ to $(7, 20)$, but ends at these points.

Hence there is a restricted domain $5 \leqslant x \leqslant 7$.

Its slope is $\dfrac{20-11}{7-5} = \dfrac{9}{2}$

$\therefore$ its equation is $\dfrac{y-11}{x-5} = \dfrac{9}{2}$

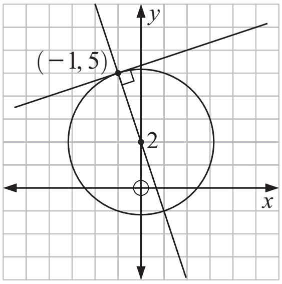
$$\begin{aligned} \therefore\ 2(y-11) &= 9(x-5) \\ \therefore\ 2y-22 &= 9x-45 \\ \therefore\ 9x-2y &= 23, \quad 5 \leqslant x \leqslant 7 \end{aligned}$$

d Plunkit Street has $x = 8$, so it meets Keach Avenue when

$$\begin{aligned} 8-8y &= -83 \quad \{\text{using } \mathbf{a}\} \\ \therefore\ 8y &= 91 \\ \therefore\ y &= \tfrac{91}{8} \end{aligned}$$

$\therefore$ they intersect at $(8, \frac{91}{8})$

17 **a**



The line through $(-1, 5)$ and $(0, 2)$

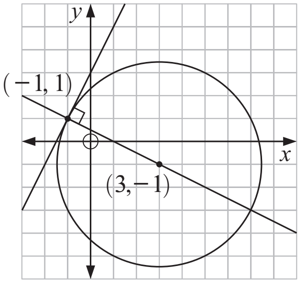
has slope $= \dfrac{5-2}{-1-0} = -3$

$\therefore$ the slope of the tangent is $\frac{1}{3}$

$\therefore$ its equation is $\dfrac{y-5}{x-(-1)} = \dfrac{1}{3}$

$$\begin{aligned} \therefore\ 3(y-5) &= x+1 \\ \therefore\ 3y-15 &= x+1 \\ \therefore\ x-3y &= -16 \end{aligned}$$

b



The line through $(-1,\ 1)$ and $(3,\ -1)$

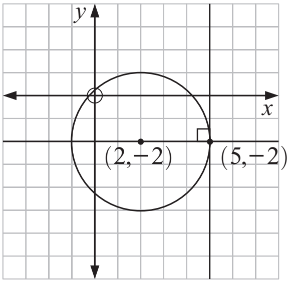
has slope $= \dfrac{-1-1}{3-(-1)} = -\dfrac{1}{2}$

$\therefore$ the slope of the tangent is 2

$\therefore$ its equation is $\dfrac{y-1}{x-(-1)} = 2$

$$\therefore\ y-1 = 2(x+1)$$
$$\therefore\ y-1 = 2x+2$$
$$\therefore\ 2x-y = -3$$

c



The line through $(2,\ -2)$ and $(5,\ -2)$

has slope $= \dfrac{-2-(-2)}{5-2} = 0$

i.e., it is horizontal

$\therefore$ the tangent must be vertical

$\therefore$ its equation is $x = 5$

18 Suppose the outlet is at $\mathrm{T}(x,\ 8)$.

Then $\mathrm{AT} = \mathrm{BT}$, so $\sqrt{(x-5)^2+(8-5)^2} = \sqrt{(x-7)^2+(8-10)^2}$

$$\therefore\ (x-5)^2+(8-5)^2 = (x-7)^2+(8-10)^2$$
$$\therefore\ x^2-10x+25+9 = x^2-14x+49+4$$
$$\therefore\ -10x+34 = -14x+53$$
$$\therefore\ 4x = 19$$
$$\therefore\ x = \tfrac{19}{4} \qquad \therefore\ \text{the outlet is at } (4\tfrac{3}{4},\ 8)$$

19 **a** Suppose C is at $(x,\ 7)$

Then $\mathrm{AC} = \mathrm{BC}$, so $\sqrt{(x-2)^2+(7-3)^2} = \sqrt{(x-5)^2+(7-4)^2}$

$$\therefore\ (x-2)^2+(7-3)^2 = (x-5)^2+(7-4)^2$$
$$\therefore\ x^2-4x+4+16 = x^2-10x+25+9$$
$$\therefore\ -4x+20 = -10x+34$$
$$\therefore\ 6x = 14$$
$$\therefore\ x = \tfrac{7}{3} \qquad \therefore\ \text{the pumping station is at } (\tfrac{7}{3},\ 7)$$

b The length of each pipe $= \sqrt{(x-2)^2+(7-3)^2}$

$$= \sqrt{(\tfrac{7}{3}-2)^2+16}$$
$$= \sqrt{16\tfrac{1}{9}} \qquad \therefore\ \text{the total length } = 2\sqrt{16\tfrac{1}{9}} \doteqdot 8.03 \text{ km}$$

c Now CD is horizontal, so BD is vertical $\therefore$ the x-coordinate of D is 5

$\therefore$ D is at $(5,\ 7)$

Then $\mathrm{AB}+\mathrm{BD} = \sqrt{(5-2)^2+(4-3)^2} + \sqrt{(5-5)^2+(7-4)^2}$

$$= \sqrt{9+1} + \sqrt{0+9}$$
$$= \sqrt{10}+3$$
$$\doteqdot 6.162 \text{ km}$$

Hence yes, it would be much cheaper.

20 Suppose Jason's girlfriend lives at G$(x, 8)$

$$\begin{aligned} \text{Then}\quad \text{GJ} = \sqrt{(x-4)^2+(8-1)^2} &= 11.73 \\ \therefore\quad (x-4)^2+49 &= 11.73^2 \\ \therefore\quad (x-4)^2 &= 88.5929 \\ \therefore\quad x-4 &\doteqdot \pm 9.412 \\ \therefore\quad x &\doteqdot -5.412 \text{ or } 13.412 \end{aligned}$$

i.e., Jason's girlfriend lives at $(-5.412, 8)$ or $(13.41, 8)$

21 If the centre is C(a, b) then PC is always r units.

$$\begin{aligned} \therefore\quad \sqrt{(x-a)^2+(y-b)^2} &= r \quad \{\text{distance formula}\} \\ \therefore\quad (x-a)^2+(y-b)^2 &= r^2 \end{aligned}$$

22 **a** $(x-4)^2+(y-3)^2 = 5^2$

i.e., $(x-4)^2+(y-3)^2 = 25$

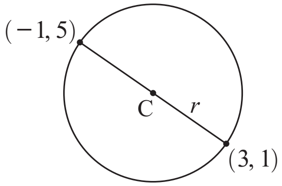
b $(x--1)^2+(y-5)^2 = 2^2$

i.e., $(x+1)^2+(y-5)^2 = 4$

c $(x-0)^2+(y-0)^2 = 10^2$

i.e., $x^2+y^2 = 100$

d



C is $\left(\frac{-1+3}{2}, \frac{5+1}{2}\right)$ i.e., $(1, 3)$

$$\begin{aligned} \text{and}\quad r &= \sqrt{(3-1)^2+(1-3)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \end{aligned}$$

$\therefore$ equation is $(x-1)^2+(y-3)^2 = (\sqrt{8})^2$

i.e., $(x-1)^2+(y-3)^2 = 8$

23 **a** The centre is $(1, 3)$ and radius $\sqrt{4} = 2$ units.

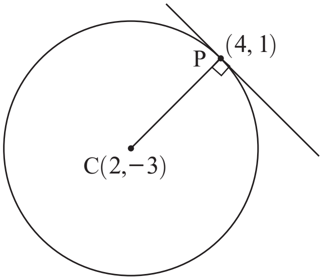
b The centre is $(0, -2)$ and radius $\sqrt{16} = 4$ units.

c The centre is $(0, 0)$ and radius $\sqrt{7}$ units.

24 **a** $(x-2)^2+(y+3)^2 = 20$ has centre $(2, -3)$ and radius $\sqrt{20}$ units.

b If $x = 4$ and $y = 1$,

$$\begin{aligned} \text{LHS} &= (4-2)^2+(1+3)^2 \\ &= 2^2+4^2 \\ &= 20 \\ &= \text{RHS} \quad \checkmark \end{aligned}$$

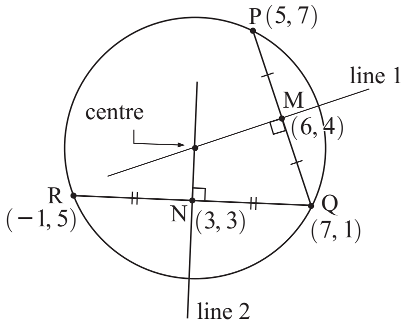


c Slope of radius CP is $\dfrac{1--3}{4-2} = \frac{4}{2} = \frac{2}{1}$

$\therefore$ slope of tangent is $-\frac{1}{2}$

$$\begin{aligned} \therefore\quad \text{equation is}\quad \frac{y-1}{x-4} &= -\tfrac{1}{2} \\ \text{i.e.,}\quad 2y-2 &= -x+4 \\ \text{i.e.,}\quad x+2y &= 6 \end{aligned}$$

25



M is $\left(\dfrac{5+7}{2}, \dfrac{7+1}{2}\right)$ i.e., (6, 4)

N is $\left(\dfrac{7+-1}{2}, \dfrac{1+5}{2}\right)$ i.e., (3, 3)

Slope of PQ $= \dfrac{1-7}{7-5} = -\frac{6}{2} = -3$

$\therefore$ slope of line 1 $= \frac{1}{3}$ {negative reciprocal}

$\therefore$ equation of line 1 is $\dfrac{y-4}{x-6} = \frac{1}{3}$

i.e., $y - 4 = \frac{1}{3}x - 2$

i.e., $y = \frac{1}{3}x + 2$ ...... (1)

Slope of QR $= \dfrac{5-1}{-1-7} = \frac{4}{-8} = -\frac{1}{2}$

$\therefore$ slope of line 2 is 2 {negative reciprocal}

$\therefore$ equation of line 2 is $\dfrac{y-3}{x-3} = 2$

i.e., $y - 3 = 2x - 6$

i.e., $y = 2x - 3$ ...... (2)

We now solve (1) and (2) simultaneously

Now $\frac{1}{3}x + 2 = 2x - 3$

$\therefore\ x + 6 = 6x - 9$

$\therefore\ 15 = 5x$

$\therefore\ x = 3$ and so $y = 2(3) - 3 = 3$

$\therefore$ the centre is at (3, 3).

Chapter 1

FUNCTIONS

EXERCISE 1A

1 **a** (1, 3), (2, 4), (3, 5), (4, 6) is a function since no two ordered pairs have the same x-coordinate.

b (1, 3), (3, 2), (1, 7), (−1, 4) is not a function since two of the ordered pairs, (1, 3) and (1, 7), have the same x-coordinate of 1.

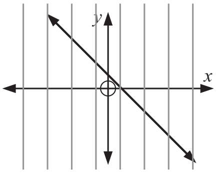
c (2, −1), (2, 0), (2, 3), (2, 11) is not a function since each ordered pair has the same x-coordinate of 2.

d (7, 6), (5, 6), (3, 6), (−4, 6) is a function since no two ordered pairs have the same x-coordinate.

e (0, 0), (1, 0), (3, 0), (5, 0) is a function since no two ordered pairs have the same x-coordinate.

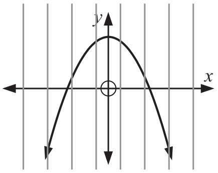
f (0, 0), (0, −2), (0, 2), (0, 4) is not a function since each ordered pair has the same x-coordinate of 0.

2 **a**



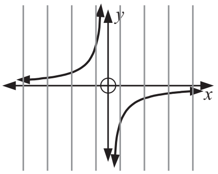
i.e., each line cuts the graph no more than once $\therefore$ it is a function

b



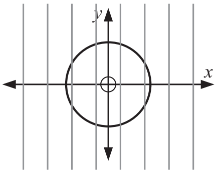
i.e., each line cuts the graph no more than once $\therefore$ it is a function

c



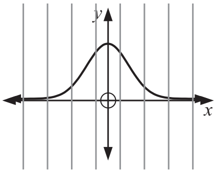
i.e., each line cuts the graph no more than once $\therefore$ it is a function

d



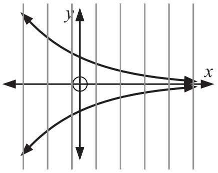
i.e., the lines cut the graph more than once $\therefore$ it is not a function

e



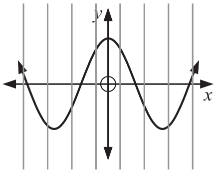
i.e., each line cuts the graph no more than once $\therefore$ it is a function

f



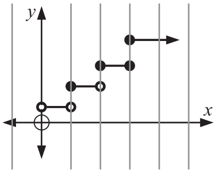
i.e., the lines cut the graph more than once $\therefore$ it is not a function

g



i.e., each line cuts the graph no more than once $\therefore$ it is a function

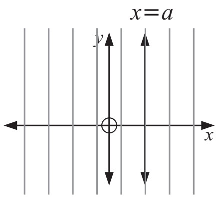
h



i.e., one line cuts the graph more than once $\therefore$ it is not a function

3 The graph of a straight line is not a function if the graph is a vertical line, i.e., $x = a$ for all a.

The vertical line through $x = a$ cuts the graph at every point $\therefore$ it is not a function.



4 $x^2 + y^2 = 9$ is the equation of a circle, centre (0, 0) and radius 3.

Now
$$\begin{aligned} x^2 + y^2 &= 9 \\ \therefore\ y^2 &= 9 - x^2 \\ \therefore\ y &= \pm\sqrt{9 - x^2} \end{aligned}$$
Hence y has two real values for any value of x where $-3 < x < 3$.

EXERCISE 1B

1 **a** Domain is $\{x\colon\ -1 < x \leqslant 5\}$
Range is $\{y\colon\ 1 < y \leqslant 3\}$

b Domain is $\{x\colon\ x \neq 2\}$
Range is $\{y\colon\ y \neq -1\}$

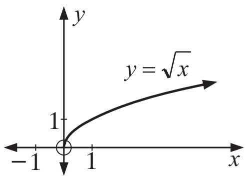
c Domain is $\{x\colon\ x \in \mathcal{R}\}$
Range is $\{y\colon\ 0 < y \leqslant 2\}$

d Domain is $\{x\colon\ x \in \mathcal{R}\}$
Range is $\{y\colon\ y \geqslant -1\}$

e Domain is $\{x\colon\ x \geqslant -4\}$
Range is $\{y\colon\ y \geqslant -3\}$

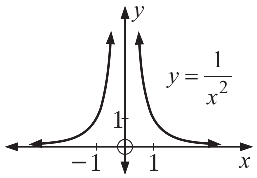
f Domain is $\{x\colon\ x \neq \pm 2\}$
Range is $\{y\colon\ y \leqslant -1 \text{ or } y > 0\}$

2 **a**



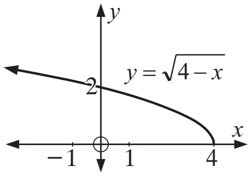
Domain is $\{x\colon\ x \geqslant 0\}$
Range is $\{y\colon\ y \geqslant 0\}$

b



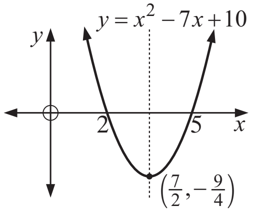
Domain is $\{x\colon\ x \neq 0\}$
Range is $\{y\colon\ y > 0\}$

c



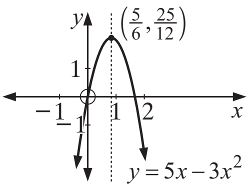
Domain is $\{x\colon\ x \leqslant 4\}$
Range is $\{y\colon\ y \geqslant 0\}$

d



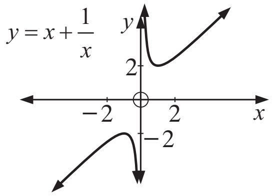
Domain is $\{x\colon\ x \in \mathcal{R}\}$
Range is $\{y\colon\ y \geqslant -2\frac{1}{4}\}$

e



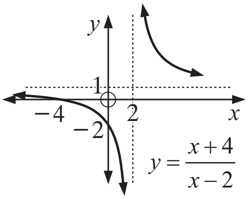
Domain is $\{x\colon\ x \in \mathcal{R}\}$
Range is $\{y\colon\ y \leqslant \frac{25}{12}\}$

f



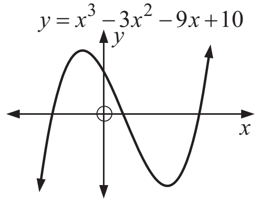
Domain is $\{x\colon\ x \neq 0\}$
Range is $\{y\colon\ y \leqslant -2$
or $y \geqslant 2\}$

g



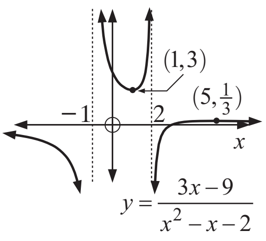
Domain is $\{x\colon\ x \neq 2\}$
Range is $\{y\colon\ y \neq 1\}$

h



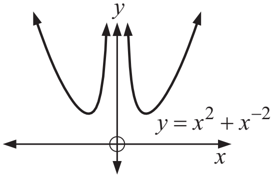
Domain is $\{x\colon\ x \in \mathcal{R}\}$
Range is $\{y\colon\ y \in \mathcal{R}\}$

i



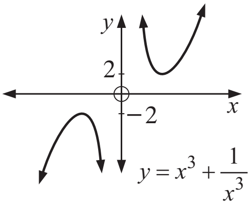
Domain is $\{x\colon\ x \neq -1,$
$x \neq 2\}$
Range is $\{y\colon\ y \leqslant \frac{1}{3}$
or $y \geqslant 3\}$

j



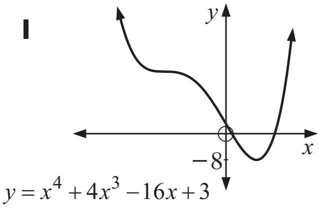
Domain is $\{x\colon\ x \neq 0\}$
Range is $\{y\colon\ y \geqslant 2\}$

k



Domain is $\{x\colon\ x \neq 0\}$
Range is $\{y\colon\ y \leqslant -2$
or $y \geqslant 2\}$

l



Domain is $\{x\colon\ x \in \mathcal{R}\}$
Range is $\{y\colon\ y \geqslant -8\}$

EXERCISE 1C

1 **a** $$\begin{aligned} f(0) &= 3(0)+2 \\ &= 2 \end{aligned}$$

b $$\begin{aligned} f(2) &= 3(2)+2 \\ &= 8 \end{aligned}$$

c $$\begin{aligned} f(-1) &= 3(-1)+2 \\ &= -1 \end{aligned}$$

d $$\begin{aligned} f(-5) &= 3(-5)+2 \\ &= -13 \end{aligned}$$

e $$\begin{aligned} f(-\tfrac{1}{3}) &= 3(-\tfrac{1}{3})+2 \\ &= 1 \end{aligned}$$

2 **a** $$\begin{aligned} f(0) &= 3(0)-0^2+2 \\ &= 2 \end{aligned}$$

b $$\begin{aligned} f(3) &= 3(3)-3^2+2 \\ &= 9-9+2 \\ &= 2 \end{aligned}$$

c $$\begin{aligned} f(-3) &= 3(-3)-(-3)^2+2 \\ &= -9-9+2 \\ &= -16 \end{aligned}$$

d $$\begin{aligned} f(-7) &= 3(-7)-(-7)^2+2 \\ &= -21-49+2 \\ &= -68 \end{aligned}$$

e $$\begin{aligned} f(\tfrac{3}{2}) &= 3(\tfrac{3}{2})-(\tfrac{3}{2})^2+2 \\ &= \tfrac{9}{2}-\tfrac{9}{4}+2 \\ &= \tfrac{17}{4} \end{aligned}$$

3 **a** $f(a)=7-3a$

b $$\begin{aligned} f(-a) &= 7-3(-a) \\ &= 7+3a \end{aligned}$$

c $$\begin{aligned} f(a+3) &= 7-3(a+3) \\ &= 7-3a-9 \\ &= -3a-2 \end{aligned}$$

d $$\begin{aligned} f(b-1) &= 7-3(b-1) \\ &= 7-3b+3 \\ &= 10-3b \end{aligned}$$

e $$\begin{aligned} f(x+2) &= 7-3(x+2) \\ &= 7-3x-6 \\ &= 1-3x \end{aligned}$$

4 **a** $$\begin{aligned} &F(x+4) \\ &= 2(x+4)^2+3(x+4)-1 \\ &= 2(x^2+8x+16)+3x+12-1 \\ &= 2x^2+16x+32+3x+11 \\ &= 2x^2+19x+43 \end{aligned}$$

b $$\begin{aligned} &F(2-x) \\ &= 2(2-x)^2+3(2-x)-1 \\ &= 2(4-4x+x^2)+6-3x-1 \\ &= 8-8x+2x^2+5-3x \\ &= 2x^2-11x+13 \end{aligned}$$

c $$\begin{aligned} &F(-x) \\ &= 2(-x)^2+3(-x)-1 \\ &= 2x^2-3x-1 \end{aligned}$$

d $$\begin{aligned} &F(x^2) \\ &= 2(x^2)^2+3(x^2)-1 \\ &= 2x^4+3x^2-1 \end{aligned}$$

e $$\begin{aligned} &F(x^2-1) \\ &= 2(x^2-1)^2+3(x^2-1)-1 \\ &= 2(x^4-2x^2+1)+3x^2-3-1 \\ &= 2x^4-4x^2+2+3x^2-4 \\ &= 2x^4-x^2-2 \end{aligned}$$

5 **a** **i** $$\begin{aligned} G(2) &= \frac{2(2)+3}{2-4} \\ &= \tfrac{7}{-2} \\ &= -\tfrac{7}{2} \end{aligned}$$

ii $$\begin{aligned} G(0) &= \frac{2(0)+3}{0-4} \\ &= \tfrac{3}{-4} \\ &= -\tfrac{3}{4} \end{aligned}$$

iii $$\begin{aligned} G(-\tfrac{1}{2}) &= \frac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4} \\ &= \frac{-1+3}{-\frac{9}{2}} \\ &= \frac{2}{(-\frac{9}{2})} \\ &= -\tfrac{4}{9} \end{aligned}$$

b $G(x)=\dfrac{2x+3}{x-4}$ is undefined when $x-4=0$

i.e., when $x=4$

So, when $x=4$, $G(x)$ does not exist.

c $$G(x+2)=\frac{2(x+2)+3}{(x+2)-4}=\frac{2x+4+3}{x+2-4}=\frac{2x+7}{x-2}$$

d $G(x)=-3$ i.e., $\dfrac{2x+3}{x-4}=-3$

$$\begin{aligned} \therefore\ 2x+3 &= -3(x-4) \\ \therefore\ 2x+3 &= -3x+12 \\ \therefore\ 5x &= 9 \quad \text{and so} \quad x=\tfrac{9}{5} \end{aligned}$$

6 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

7 **a** $V(4) = 9650 - 860(4)$
$= 9650 - 3440$
$= 6210$

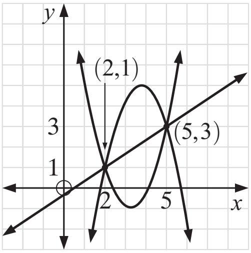
i.e., the value of the photocopier 4 years after purchase is 6210 Yen.

b If $V(t) = 5780$,
then $9650 - 860t = 5780$
$\therefore\ 860t = 3870$
$\therefore\ t = 4.5$

i.e., the value of the photocopier $4\frac{1}{2}$ years after purchase is 5780 Yen.

c Original purchase price is when $t = 0$,
i.e., $V(0) = 9650 - 860(0)$
$= 9650$ i.e., the original purchase price was 9650 Yen.

8



First sketch the linear function which passes through the two points (2, 1) and (5, 3).

Then sketch two quadratic functions which also pass through the two points.

9 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

i.e., $a(2) + b = 1$ and $a(-3) + b = 11$
$\therefore\ 2a + b = 1$ $\therefore\ -3a + b = 11$
$\therefore\ b = 1 - 2a$ ..... (1) $\therefore\ b = 11 + 3a$ ..... (2)

Solving (1) and (2) simultaneously, $1 - 2a = 11 + 3a$
$\therefore\ 5a = -10$
$\therefore\ a = -2$

Substituting $a = -2$ into (1) gives $b = 1 - 2(-2) = 5$ i.e., $a = -2$, $b = 5$

Hence $f(x) = -2x + 5$

10 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$

i.e., $a(0)^2 + b(0) + c = -4$
$\therefore\ c = -4$

Also, $a(1)^2 + b(1) + c = -2$ and $a(2)^2 + b(2) + c = 6$
$\therefore\ a + b + c = -2$ and $\therefore\ 4a + 2b + c = 6$

Substituting $c = -4$ into both equations gives

$a + b + (-4) = -2$ and $4a + 2b + (-4) = 6$
$\therefore\ a + b = 2$ $\therefore\ 4a + 2b = 10$ ..... (2)
$\therefore\ a = 2 - b$ ..... (1)

Substituting (1) into (2) gives $4(2 - b) + 2b = 10$ $\therefore\ 8 - 4b + 2b = 10$
$\therefore\ -2b = 2$
$\therefore\ b = -1$

Now, substituting $b = -1$ into (1) gives $a = 2 - (-1) = 3$ i.e., $a = 3$, $b = -1$, $c = -4$

EXERCISE 1D

1 **a**
$$\begin{aligned}&(f\circ g)(x)\\&=f(g(x))\\&=f(1-x)\\&=2(1-x)+3\\&=2-2x+3\\&=5-2x\end{aligned}$$

b
$$\begin{aligned}&(g\circ f)(x)\\&=g(f(x))\\&=g(2x+3)\\&=1-(2x+3)\\&=1-2x-3\\&=-2x-2\end{aligned}$$

c
$$\begin{aligned}&(f\circ g)(-3)\\&=f(g(-3))\\&=f(1-(-3))\\&=f(4)\\&=2(4)+3\\&=11\end{aligned}$$

2
$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f(2-x)\\&=(2-x)^2\end{aligned}$$

Domain is $\{x\colon\ x\in\mathcal{R}\}$
Range is $\{y\colon\ y\geqslant 0\}$

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g(x^2)\\&=2-x^2\end{aligned}$$

Domain is $\{x\colon\ x\in\mathcal{R}\}$
Range is $\{y\colon\ y\geqslant 2\}$

3 **a**
$$\begin{aligned}&(f\circ g)(x)\\&=f(g(x))\\&=f(3-x)\\&=(3-x)^2+1\\&=9-6x+x^2+1\\&=x^2-6x+10\end{aligned}$$

b
$$\begin{aligned}&(g\circ f)(x)\\&=g(f(x))\\&=g(x^2+1)\\&=3-(x^2+1)\\&=3-x^2-1\\&=-x^2+2\end{aligned}$$

c
$$\begin{aligned}(g\circ f)(x)&=f(x)\\\text{i.e., }\ -x^2+2&=f(x)\quad\{\text{from } \mathbf{b}\}\\\text{i.e., }\ -x^2+2&=x^2+1\\\therefore\ 2x^2&=1\\\therefore\ x^2&=\tfrac{1}{2}\\\therefore\ x&=\pm\tfrac{1}{\sqrt{2}}\end{aligned}$$

4 **a** $ax+b=cx+d$ is true for all x {given}

Let $x=0$,
$$\begin{aligned}\therefore\ a(0)+b&=c(0)+d\\\therefore\ b&=d\quad .....\ (1)\end{aligned}$$

Let $x=1$,
$$\begin{aligned}\therefore\ a(1)+b&=c(1)+d\\\therefore\ a+b&=c+d\\\text{but }\ b&=d\quad(\text{from }(1))\\\therefore\ a+d&=c+d\\\therefore\ a&=c\end{aligned}$$

b
$$\begin{aligned}(f\circ g)(x)&=x\quad\text{for all } x\quad\{\text{given}\}\\\therefore\ f(g(x))&=x\\\therefore\ f(ax+b)&=x\\\therefore\ 2(ax+b)+3&=x\\\therefore\ 2ax+2b+3&=x\quad\text{for all } x\end{aligned}$$

When $x=0$,
$$\begin{aligned}2a(0)+2b+3&=0\\\therefore\ 2b&=-3\\\therefore\ b&=-\tfrac{3}{2}\end{aligned}$$

When $x=1$,
$$\begin{aligned}2a(1)+2b+3&=1\\\therefore\ 2a+2b&=-2\\\therefore\ 2a&=-2-2(-\tfrac{3}{2})=1\\\therefore\ a&=\tfrac{1}{2}\end{aligned}$$

i.e., $a=\frac{1}{2}$ and $b=-\frac{3}{2}$ as required.

c
$$\begin{aligned}\text{If }\ (g\circ f)(x)&=x\\\text{then }\ g(f(x))&=x\\\text{i.e., }\ g(2x+3)&=x\\\text{i.e., }\ a(2x+3)+b&=x\\\therefore\ 2ax+3a+b&=x\end{aligned}$$

When $x=0$,
$$\begin{aligned}2a(0)+3a+b&=0\\\therefore\ 3a&=-b\\\therefore\ a&=-\tfrac{1}{3}b\quad .....\ (1)\end{aligned}$$

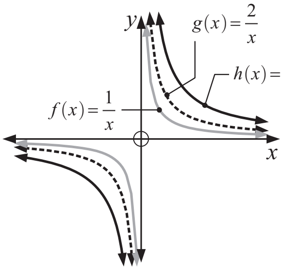
When $x=1$,
$$\begin{aligned}2a(1)+3a+b&=1\\\therefore\ 2a+3a+b&=1\\\therefore\ 5a+b&=1\\\therefore\ 5(-\tfrac{1}{3}b)+b&=1\quad(\text{from }(1))\\\therefore\ -\tfrac{2}{3}b&=1\\\therefore\ b&=-\tfrac{3}{2}\end{aligned}$$

Substituting $b=-\frac{3}{2}$ into (1) gives
$$a=-\tfrac{1}{3}(-\tfrac{3}{2})=\tfrac{1}{2}$$
i.e., $a=\frac{1}{2}$ and $b=-\frac{3}{2}$ as required.

$\therefore$ the result in **b** is also true if $(g\circ f)(x)=x$ for all x.

EXERCISE 1E

1

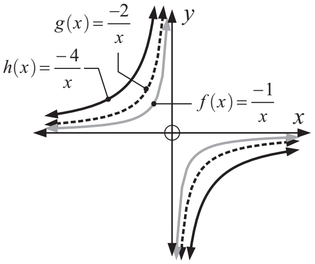


$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x- and y-axes.

The graphs all lie in the 1st and 3rd quadrants.

The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x)=\dfrac{1}{x}$ is closer to the axes than $g(x)=\dfrac{2}{x}$ for corresponding values of x, and $g(x)=\dfrac{2}{x}$ is closer to the axes than $h(x)=\dfrac{4}{x}$.

2



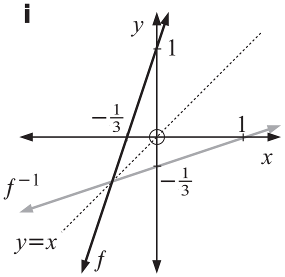
$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x- and y-axes.

The graphs all lie in the 2nd and 4th quadrants.

The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x)=-\dfrac{1}{x}$ is closer to the axes than $g(x)=-\dfrac{2}{x}$ for corresponding values of x, and $g(x)=-\dfrac{2}{x}$ is closer to the axes than $h(x)=-\dfrac{4}{x}$.

EXERCISE 1F

1 **a** **i**



ii $f(x)$ passes through (0, 1) and $(-\frac{1}{3}, 0)$

$\therefore$ $f^{-1}(x)$ passes through (1, 0) and $(0, -\frac{1}{3})$

$f^{-1}(x)$ has slope $\dfrac{-\frac{1}{3}-0}{0-1}=\dfrac{-\frac{1}{3}}{-1}=\dfrac{1}{3}$

So, its equation is $\dfrac{y-0}{x-1}=\dfrac{1}{3}$

i.e., $y=\dfrac{x-1}{3}$

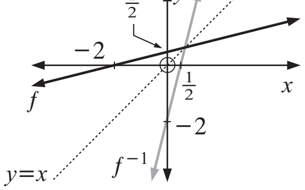
i.e., $f^{-1}(x)=\dfrac{x-1}{3}$

iii

$$\begin{aligned} f \text{ is } \quad y &= 3x+1 \\ \text{so } f^{-1} \text{ is } \quad x &= 3y+1 \\ \therefore \quad x-1 &= 3y \\ \therefore \quad y &= \frac{x-1}{3} \end{aligned}$$

i.e., $f^{-1}(x)=\dfrac{x-1}{3}$

b **i**



ii $f(x)$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$

$\therefore$ $f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$

$f^{-1}(x)$ has slope $\dfrac{-2-0}{0-\frac{1}{2}}=\dfrac{-2}{-\frac{1}{2}}=4$

So, its equation is $\dfrac{y-0}{x-\frac{1}{2}}=4$

i.e., $y=4x-2$

i.e., $f^{-1}(x)=4x-2$

iii

$$\begin{aligned} f \text{ is } \quad y &= \frac{x+2}{4} \\ \text{so } f^{-1} \text{ is } \quad x &= \frac{y+2}{4} \end{aligned}$$

$$\begin{aligned} \therefore \quad 4x &= y+2 \\ \therefore \quad y &= 4x-2 \\ \text{i.e.,} \quad f^{-1}(x) &= 4x-2 \end{aligned}$$

2 **a** **i** f is $y = 2x + 5$

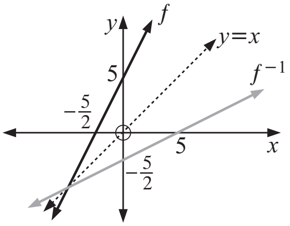
so f^{-1} is $x = 2y + 5$

$\therefore\ x - 5 = 2y$

$\therefore\ y = \dfrac{x-5}{2}$

i.e., $f^{-1}(x) = \dfrac{x-5}{2}$

ii



$f(x)$ passes through $(0, 5)$ and $(-\frac{5}{2}, 0)$

$\therefore\ f^{-1}(x)$ passes through $(5, 0)$ and $(0, -\frac{5}{2})$

iii $(f^{-1} \circ f)(x)$

$= f^{-1}(2x + 5)$

$\dfrac{2x + 5 - 5}{2}$

$= \dfrac{2x}{2}$

$= x$

and

$(f \circ f^{-1})(x)$

$= f(f^{-1}(x))$

$= f\left(\dfrac{x-5}{2}\right)$

$= 2\left(\dfrac{x-5}{2}\right) + 5$

$= x - 5 + 5$

$= x$

b **i** f is $y = \dfrac{3-2x}{4}$

so f^{-1} is $x = \dfrac{3-2y}{4}$

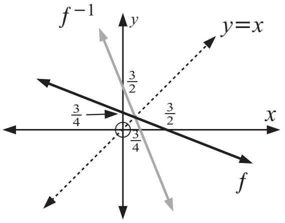
$\therefore\ 4x = 3 - 2y$

$\therefore\ 4x - 3 = -2y$

$\therefore\ y = -2x + \frac{3}{2}$

i.e., $f^{-1}(x) = -2x + \frac{3}{2}$

ii



$f(x)$ passes through $(0, \frac{3}{4})$ and $(\frac{3}{2}, 0)$

$\therefore\ f^{-1}(x)$ passes through $(\frac{3}{4}, 0)$ and $(0, \frac{3}{2})$

iii $(f^{-1} \circ f)(x)$

$= f^{-1}(f(x))$

$= f^{-1}\left(\dfrac{3-2x}{4}\right)$

$= -2\left(\dfrac{3-2x}{4}\right) + \frac{3}{2}$

$= \dfrac{3-2x}{-2} + \frac{3}{2}$

$= -\frac{3}{2} + x + \frac{3}{2}$

$= x$

and

$(f \circ f^{-1})(x)$

$= f(f^{-1}(x))$

$= f\left(-2x + \frac{3}{2}\right)$

$= \dfrac{3 - 2(-2x + \frac{3}{2})}{4}$

$= \dfrac{3 + 4x - 3}{4}$

$= \dfrac{4x}{4}$

$= x$

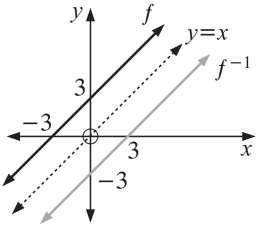
c **i** f is $y = x + 3$

so f^{-1} is $x = y + 3$

$\therefore\ y = x - 3$

i.e., $f^{-1}(x) = x - 3$

ii



$f(x)$ passes through $(0, 3)$ and $(-3, 0)$

$\therefore\ f^{-1}(x)$ passes through $(3, 0)$ and $(0, -3)$

iii $(f^{-1} \circ f)(x)$

$= f^{-1}(f(x))$

$= f^{-1}(x + 3)$

$= (x + 3) - 3$

$= x$

and

$(f \circ f^{-1})(x)$

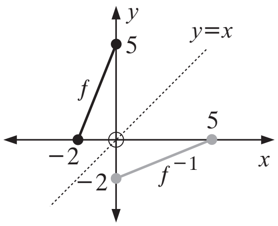
$= f(f^{-1}(x))$

$= f(x - 3)$

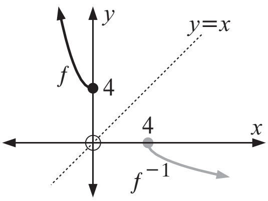
$= (x - 3) + 3$

$= x$

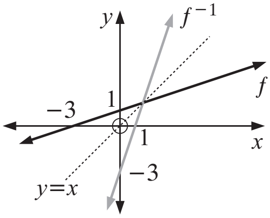
3 **a**



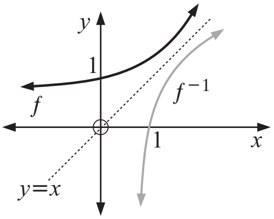
b



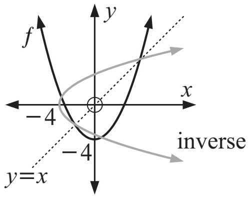
c



d



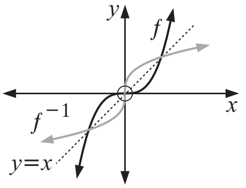
4 **a**



b Using the 'horizontal line test', f does not have an inverse function as a horizontal line through $y = x^2 - 4$ cuts it more than once.

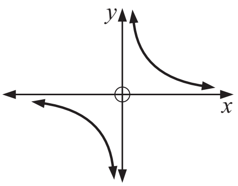
c For $x \geqslant 0$, any horizontal line cuts it only once, i.e., f does have an inverse function for $x \geqslant 0$.

5



EXERCISE 1G

1 **a** $f(x) = \frac{1}{x}$ has graph



No vertical line cuts the graph more than once, so it is a function.

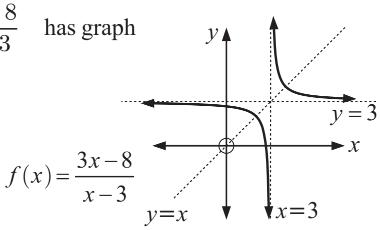
No horizontal line cuts the graph more than once.

Hence, $f(x) = \frac{1}{x}$, $x \neq 0$ has an inverse function.

b $f(x) = \frac{1}{x}$ i.e., $y = \frac{1}{x}$ has an inverse function $x = \frac{1}{y}$ and so $y = \frac{1}{x}$

So, $f^{-1}(x) = \frac{1}{x}$ It is a self-inverse function.

2 **a** $f(x) = \frac{3x-8}{x-3}$ has graph



The vertical line test shows it to be a function.
The horizontal line test shows it has an inverse function.
Symmetry about $y = x$ shows it is a self-inverse function.

b $f(x)=\dfrac{3x-8}{x-3}$ i.e., $y=\dfrac{3x-8}{x-3}$ has an inverse function $x=\dfrac{3y-8}{y-3}$

$$\begin{aligned}\therefore\quad x(y-3)&=3y-8\\ \therefore\quad xy-3x&=3y-8\\ \therefore\quad y(x-3)&=3x-8\\ \therefore\quad y&=\frac{3x-8}{x-3}\end{aligned}$$

i.e., $f(x)=f^{-1}(x)$ $\therefore$ it is a self-inverse function.

3 **a** If $y=f(x)$ has an inverse function, then the inverse function must also be a function. Thus, it must satisfy the 'vertical line test', i.e., no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the 'horizontal line test'. Thus, the 'horizontal line test' is a valid test for the existence of an inverse function.

b **i** This graph satisfies the 'horizontal line test' and therefore has an inverse function.

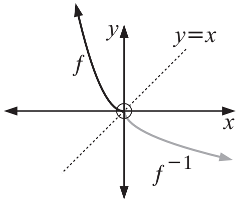
ii, **iii** These graphs both fail the 'horizontal line test' so neither of these have inverse functions.

c **ii** Domain $\{x\colon\ x\geqslant 1\}$ (or $\{x\colon\ x\leqslant 1\}$) **iii** Domain $\{x\colon\ x\geqslant 1\}$ (or $\{x\colon\ x\leqslant -2\}$)

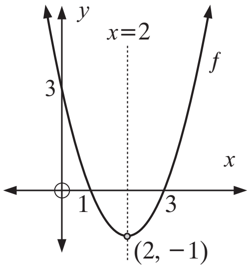
4 **a**

$$\begin{aligned}f\ \text{ is }\ y&=x^2,\quad x\leqslant 0\\ \text{so }\ f^{-1}\ \text{ is }\ x&=y^2,\quad y\leqslant 0\\ \therefore\quad y&=-\sqrt{x}\\ \text{i.e.,}\quad f^{-1}(x)&=-\sqrt{x}\end{aligned}$$

b



5 **a**



$f: x\to x^2-4x+3$ satisfies the 'vertical line test' so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

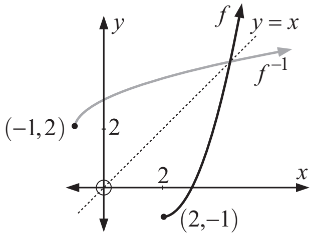
b For $x\geqslant 2$, all horizontal lines cut the graph no more than once. Therefore f has an inverse function for $x\geqslant 2$.

c

$$\begin{aligned}f\ \text{ is }\ y&=x^2-4x+3,\quad x\geqslant 2\\ \text{so }\ f^{-1}\ \text{ is }\ x&=y^2-4y+3,\quad y\geqslant 2\\ \text{i.e.,}\quad x&=(y-2)^2-4+3,\quad y\geqslant 2\\ &=(y-2)^2-1,\quad y\geqslant 2\\ \therefore\quad x+1&=(y-2)^2,\quad y\geqslant 2\\ \therefore\quad y-2&=\sqrt{x+1},\quad y\geqslant 2\\ \therefore\quad y&=2+\sqrt{1+x},\quad y\geqslant 2\\ \text{i.e.,}\quad f^{-1}(x)&=2+\sqrt{1+x}\\ &\text{as required}\end{aligned}$$

d **i** domain of f is $\{x\colon\ x\geqslant 2\}$, range is $\{y\colon\ y\geqslant -1\}$

ii domain of f^{-1} is $\{x\colon\ x\geqslant -1\}$, range is $\{y\colon\ y\geqslant 2\}$



e $f\circ f^{-1}=f(f^{-1})$

$$\begin{aligned}&=\left(2+\sqrt{1+x}\right)^2-4\left(2+\sqrt{1+x}\right)+3\\ &=4+4\sqrt{1+x}+1+x-8-4\sqrt{1+x}+3\\ &=x\end{aligned}$$

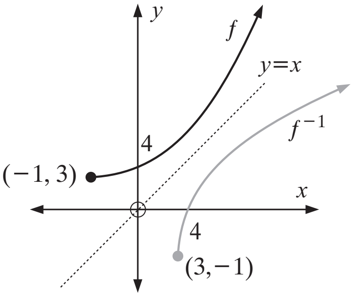
$f^{-1}\circ f=f^{-1}(f)$

$$\begin{aligned}&=2+\sqrt{1+x^2-4x+3}\\ &=2+\sqrt{(x-2)^2}\\ &=2+x-2\\ &=x\end{aligned}$$

6 **a**

$$\begin{aligned} f \text{ is } \quad y &= (x+1)^2+3, \quad x \geqslant -1 \\ \text{so } f^{-1} \text{ is } \quad x &= (y+1)^2+3, \quad y \geqslant -1 \\ \text{i.e., } \quad x-3 &= (y+1)^2, \quad y \geqslant -1 \\ \therefore \quad y+1 &= \sqrt{x-3}, \quad y \geqslant -1, \quad x \geqslant 3 \\ \therefore \quad y &= \sqrt{x-3}-1, \quad y \geqslant -1, \quad x \geqslant 3 \end{aligned}$$

b



c **i** Domain $\{x\colon\ x \geqslant -1\}$, Range $\{y\colon\ y \geqslant 3\}$

ii Domain $\{x\colon\ x \geqslant 3\}$, Range $\{y\colon\ y \geqslant -1\}$

7 **a**

$$\begin{aligned} g \text{ is } \quad y &= \frac{8-x}{2} \\ \text{so } g^{-1} \text{ is } \quad x &= \frac{8-y}{2} \\ \therefore \quad 2x &= 8-y \\ \therefore \quad y &= 8-2x \\ \text{i.e., } \quad g^{-1}(x) &= 8-2x \\ \text{Now } \quad g^{-1}(-1) &= 8-2(-1) = 10 \end{aligned}$$

b

$$\begin{aligned} (f \circ g^{-1})(x) &= 9 \\ \therefore \quad f(g^{-1}(x)) &= 9 \\ \therefore \quad f(8-2x) &= 9 \\ \therefore \quad 2(8-2x)+5 &= 9 \\ \therefore \quad 16-4x+5 &= 9 \\ \therefore \quad -4x &= -12 \\ \therefore \quad x &= 3 \end{aligned}$$

8 **a** **i**

$$\begin{aligned} f \text{ is } \quad y &= 5^x \\ \text{so, } \quad f(2) &= 5^2 \\ &= 25 \end{aligned}$$

ii

$$\begin{aligned} g \text{ is } \quad y &= \sqrt{x} \\ \text{so } g^{-1} \text{ is } \quad x &= \sqrt{y} \end{aligned}$$

$$\begin{aligned} \therefore \quad y &= x^2 \\ \text{i.e., } \quad g^{-1}(x) &= x^2, \quad x \geqslant 0 \\ \therefore \quad g^{-1}(4) &= 4^2 \\ \therefore \quad g^{-1}(4) &= 16 \end{aligned}$$

b

$$\begin{aligned} (g^{-1} \circ f)(x) &= 25 \\ \therefore \quad g^{-1}(f(x)) &= 25 \\ \therefore \quad g^{-1}(5^x) &= 25 \\ \therefore \quad (5^x)^2 &= 25 \quad \{\text{as } g^{-1}(x) = x^2, \ x \geqslant 0\} \end{aligned}$$

$$\begin{aligned} \text{and so } \quad 5^{2x} &= 5^2 \\ \therefore \quad 2x &= 2 \\ \therefore \quad x &= 1 \end{aligned}$$

9 Show: $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$

$$\begin{aligned} f \text{ is } \quad y &= 2x \\ \text{so } f^{-1} \text{ is } \quad x &= 2y \\ \therefore \quad y &= \frac{x}{2} \\ \text{i.e., } \quad f^{-1}(x) &= \frac{x}{2} \end{aligned}$$

$$\begin{aligned} g \text{ is } \quad y &= 4x-3 \\ \text{so } g^{-1} \text{ is } \quad x &= 4y-3 \\ \therefore \quad 4y &= x+3 \\ \therefore \quad y &= \frac{x+3}{4} \\ \text{i.e., } \quad g^{-1}(x) &= \frac{x+3}{4} \end{aligned}$$

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x) \\ &= 4(2x)-3 \\ \text{i.e., } \quad (g \circ f)(x) &= 8x-3 \\ \text{i.e., } \quad g \circ f \text{ is } \quad y &= 8x-3 \\ \text{so } (g \circ f)^{-1} \text{ is } \quad x &= 8y-3 \\ \therefore \quad y &= \frac{x+3}{8} \\ \text{i.e., } \quad (g \circ f)^{-1}(x) &= \frac{x+3}{8} \end{aligned}$$

$$\begin{aligned} \text{Now, } \quad (f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x)) \\ &= f^{-1}\left(\frac{x+3}{4}\right) \\ &= \frac{\left(\frac{x+3}{4}\right)}{2} \\ \therefore \quad (f^{-1} \circ g^{-1})(x) &= \frac{x+3}{8} = (g \circ f)^{-1}(x) \quad \text{as required} \end{aligned}$$

10 **a**
$$\begin{aligned} f \text{ is } \quad y &= 2x \\ \text{so } f^{-1} \text{ is } \quad x &= 2y \\ \therefore \quad y &= \frac{x}{2} \\ \text{i.e., } \quad f^{-1}(x) &= \frac{x}{2} \neq 2x \\ \text{So, } \quad f^{-1}(x) &\neq f(x) \end{aligned}$$

b
$$\begin{aligned} f \text{ is } \quad y &= x \\ \text{so } f^{-1} \text{ is } \quad x &= y \\ \therefore \quad y &= x \\ \text{i.e., } \quad f^{-1}(x) &= x \\ \text{So, } \quad f^{-1}(x) &= f(x) \end{aligned}$$

c
$$\begin{aligned} f \text{ is } \quad y &= -x \\ \text{so } f^{-1} \text{ is } \quad x &= -y \\ \therefore \quad y &= -x \\ \text{i.e., } \quad f^{-1}(x) &= -x \\ \text{So, } \quad f^{-1}(x) &= f(x) \end{aligned}$$

d
$$\begin{aligned} f \text{ is } \quad y &= \frac{2}{x} \\ \text{so } f^{-1} \text{ is } \quad x &= \frac{2}{y} \\ \therefore \quad y &= \frac{2}{x} \\ \text{i.e., } \quad f^{-1}(x) &= \frac{2}{x} \\ \text{So, } \quad f^{-1}(x) &= f(x) \end{aligned}$$

e
$$\begin{aligned} f \text{ is } \quad y &= -\frac{6}{x} \\ \text{so } f^{-1} \text{ is } \quad x &= -\frac{6}{y} \\ \therefore \quad y &= -\frac{6}{x} \\ \text{i.e., } \quad f^{-1}(x) &= -\frac{6}{x} \\ \text{So, } \quad f^{-1}(x) &= f(x) \end{aligned}$$

i.e., $f^{-1}(x) = f(x)$ is true for parts **b**, **c**, **d** and **e**.

11 **a**
$$\begin{aligned} f \text{ is } \quad y &= 3x+1 \\ \text{so } f^{-1} \text{ is } \quad x &= 3y+1 \\ \therefore \quad y &= \frac{x-1}{3} \qquad \text{i.e., } \quad f^{-1}(x) = \frac{x-1}{3} \end{aligned}$$

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{x-1}{3}\right) \\ &= 3\left(\frac{x-1}{3}\right)+1 \\ &= x-1+1 \\ \text{i.e., } \quad (f \circ f^{-1})(x) &= x \end{aligned}$$

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(3x+1) \\ &= \frac{3x+1-1}{3} \\ &= \frac{3x}{3} \\ \text{i.e., } \quad (f^{-1} \circ f)(x) &= x = (f \circ f^{-1})(x) \quad \text{as required} \end{aligned}$$

b
$$f \text{ is } \quad y = \frac{x+3}{4}$$

$$\begin{aligned} \text{so } f^{-1} \text{ is } \quad x &= \frac{y+3}{4} \\ \therefore \quad 4x &= y+3 \\ \therefore \quad y &= 4x-3 \\ \text{i.e., } \quad f^{-1}(x) &= 4x-3 \end{aligned}$$

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f(4x-3) \\ &= \frac{4x-3+3}{4} \\ &= \frac{4x}{4} \\ \text{i.e., } \quad (f \circ f^{-1})(x) &= x \end{aligned}$$

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(\frac{x+3}{4}\right) \\ &= 4\left(\frac{x+3}{4}\right)-3 \\ &= x+3-3 \\ \text{i.e., } \quad (f^{-1} \circ f)(x) &= x = (f \circ f^{-1})(x) \quad \text{as required} \end{aligned}$$

c f is $y=\sqrt{x}$ for $x \geqslant 0$ so f^{-1} is $x = \sqrt{y}$

$\therefore \; y = x^2$

i.e., $f^{-1}(x) = x^2$ for $x \geqslant 0$

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f(x^2) \\ &= \sqrt{x^2}\end{aligned}$$

i.e., $(f \circ f^{-1})(x) = x$

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(\sqrt{x}) \\ &= (\sqrt{x})^2\end{aligned}$$

i.e., $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$ as required

12 **a** $f(x)$ passes through $\mathrm{A}(x, f(x))$, so $f^{-1}(x)$ passes through $\mathrm{B}(f(x), x)$

b Substitute the coordinates of $\mathrm{B}(f(x), x)$ into $y = f^{-1}(x)$:

i.e., $x = f^{-1}(f(x)) = (f^{-1} \circ f)(x)$

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$,

so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.

Substitute the coordinates of $\mathrm{A}(f^{-1}(x), x)$ into $y = f(x)$:

i.e., $x = f(f^{-1}(x))$

i.e., $f(f^{-1}(x)) = x$ as required

REVIEW SET 1A

1 **a** $$\begin{aligned}f(x) &= 2x - x^2 \\ f(2) &= 2(2) - 2^2 \\ &= 0\end{aligned}$$

b $$\begin{aligned}f(-3) &= 2(-3) - (-3)^2 \\ &= -6 - 9 \\ &= -15\end{aligned}$$

c $$\begin{aligned}f(-\tfrac{1}{2}) &= 2(-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 \\ &= -1 - \tfrac{1}{4} \\ &= -\tfrac{5}{4}\end{aligned}$$

2 **a** **i** range is $\{y\colon\; y \geqslant -5\}$, domain is $\{x\colon\; x \in \mathcal{R}\}$

ii x-intercepts are -1 and 5; y-intercept is $-\frac{25}{9}$

iii The graph passes the 'vertical line test' so is therefore a function.

iv No, as it fails the horizontal line test.

b **i** range is $\{y\colon\; y = 1 \text{ or } -3\}$, domain is $\{x\colon\; x \in \mathcal{R}\}$

ii there are no x-intercepts; y-intercept is 1

iii The graph passes the 'vertical line test' so is therefore a function.

iv No, as it fails the horizontal line test.

3 **a** domain is $\{x\colon\; x \geqslant -2\}$, range is $\{y\colon\; 1 \leqslant y < 3\}$

b domain is $\{x\colon\; x \in \mathcal{R}\}$, range is $\{y\colon\; y = -1,\ 1 \text{ or } 2\}$

4 **a** $$\begin{aligned}h(x) &= 7 - 3x \\ h(2x-1) &= 7 - 3(2x-1) \\ &= 7 - 6x + 3 \\ &= 10 - 6x\end{aligned}$$

b $$\begin{aligned}h(2x-1) &= -2 \\ \therefore\; 7 - 3(2x-1) &= -2 \\ \therefore\; 7 - 6x + 3 &= -2 \\ \therefore\; -6x &= -12 \\ \therefore\; x &= 2\end{aligned}$$

5 $f(x) = ax^2 + bx + c$, where $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$

When $f(0) = 5$,

$$\begin{aligned}5 &= a(0)^2 + b(0) + c \\ \therefore\; 5 &= c \\ \therefore\; c &= 5 \quad \text{..... (1)}\end{aligned}$$

When $f(-2) = 21$,

$$\begin{aligned}21 &= a(-2)^2 + b(-2) + c \\ &= 4a - 2b + c \\ &= 4a - 2b + 5 \quad \{\text{using (1)}\} \\ \therefore\; 4a - 2b &= 16 \\ \therefore\; 2a - b &= 8 \quad \text{and so} \quad b = 2a - 8 \quad \text{..... (2)}\end{aligned}$$

When $f(3) = -4$,

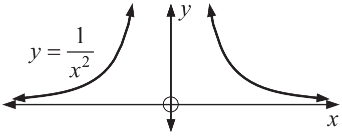
$$\begin{aligned} -4 &= a(3)^2 + b(3) + c \\ \therefore\quad -4 &= 9a + 3b + c \\ \therefore\quad -4 &= 9a + 3b + 5 \quad \{\text{using (1)}\} \\ \therefore\quad -4 &= 9a + 3(2a-8) + 5 \quad \{\text{using (2)}\} \\ \therefore\quad -9 &= 9a + 6a - 24 \\ \therefore\quad 15 &= 15a \quad \text{and so} \quad a = 1 \end{aligned}$$

Now, substituting $a = 1$ into (2) gives $b = 2(1) - 8 = -6$

i.e., $a = 1$, $b = -6$, $c = 5$

6 **a** $f(x) = \dfrac{1}{x^2}$ is meaningless when $x^2 = 0$
i.e., when $x = 0$

b



c domain of $f(x)$ is $\{x\colon\ x \neq 0\}$
range of $f(x)$ is $\{y\colon\ y > 0\}$

7 **a**
$$\begin{aligned} f(g(x)) &= f(x^2+2) \\ &= 2(x^2+2) - 3 \\ &= 2x^2 + 4 - 3 \\ &= 2x^2 + 1 \end{aligned}$$

b
$$\begin{aligned} g(f(x)) &= g(2x-3) \\ &= (2x-3)^2 + 2 \\ &= 4x^2 - 12x + 9 + 2 \\ &= 4x^2 - 12x + 11 \end{aligned}$$

8 **a** **i**
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x}) \\ &= 1 - 2\sqrt{x} \end{aligned}$$

ii
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(1-2x) \\ &= \sqrt{1-2x} \end{aligned}$$

b The domain of $f \circ g$ is $\{x\colon\ x \geqslant 0\}$ for $\sqrt{x}$ to be defined.
The range of $f \circ g$ is $\{y\colon\ y \leqslant 1\}$.
The domain of $g \circ f$ is obtained by noticing that $1 - 2x$ must be $\geqslant 0$.

$\therefore\quad 2x \leqslant 1$
i.e., $x \leqslant \frac{1}{2}$
i.e., $\{x\colon\ x \leqslant \frac{1}{2}\}$
i.e., $x \in \,]-\infty, \frac{1}{2}]$

The range of $g \circ f$ is $\{y\colon\ y \geqslant 0\}$ i.e., $y \in [0, \infty[$

9 **a**
$$\begin{aligned} f(g(x)) &= \sqrt{1-x^2} \\ &= f(1-x^2) \end{aligned}$$
i.e., $f(x) = \sqrt{x}$,
$g(x) = 1 - x^2$

b
$$g(f(x)) = \left(\frac{x-2}{x+1}\right)^2 = g\left(\frac{x-2}{x+1}\right)$$
i.e., $g(x) = x^2$,
$f(x) = \dfrac{x-2}{x+1}$

REVIEW SET 1B

1 $g(x) = x^2 - 3x$

a
$$\begin{aligned} g(x+1) &= (x+1)^2 - 3(x+1) \\ &= x^2 + 2x + 1 - 3x - 3 \\ &= x^2 - x - 2 \end{aligned}$$

b
$$\begin{aligned} g(x^2-2) &= (x^2-2)^2 - 3(x^2-2) \\ &= x^4 - 4x^2 + 4 - 3x^2 + 6 \\ &= x^4 - 7x^2 + 10 \end{aligned}$$

2 **a**

$$\begin{aligned} f(x) &= 7-4x \\ \text{i.e.,}\quad y &= 7-4x \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 7-4y \\ \therefore\quad y &= \frac{7-x}{4} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{7-x}{4} \end{aligned}$$

b

$$\begin{aligned} f(x) &= \frac{3+2x}{5} \\ \text{i.e.,}\quad y &= \frac{3+2x}{5} \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= \frac{3+2y}{5} \\ \therefore\quad 5x &= 3+2y \\ \therefore\quad y &= \frac{5x-3}{2} \quad \text{i.e.,}\quad f^{-1}(x) = \frac{5x-3}{2} \end{aligned}$$

3 **a** $y=(x-1)(x-5)$

i.e., x-intercepts are $x=1$ and 5

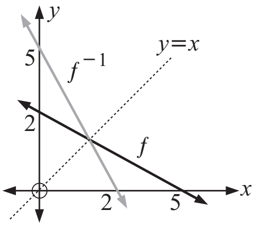
$\therefore$ vertex is at $x=3$, $y=(3-1)(3-5)=2\times(-2)=-4$

i.e., vertex is at $(3, -4)$

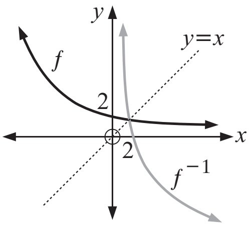
domain is $\{x\colon\ x\in\mathcal{R}\}$, range is $\{y\colon\ y\geqslant -4\}$ i.e., $y\in[-4, \infty[$

b From the graph, domain is $\{x\colon\ x\neq 0,\ 2\}$, range is $\{y\colon\ y\leqslant -1 \text{ or } y>0\}$

4 **a**



b



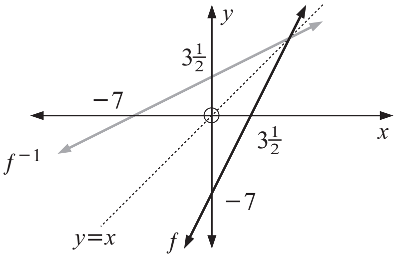
5 **a**

$$\begin{aligned} f(x) &= 4x+2 \\ \text{i.e.,}\quad y &= 4x+2 \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 4y+2 \\ \therefore\quad y &= \frac{x-2}{4} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x-2}{4} \end{aligned}$$

b

$$\begin{aligned} f(x) = \frac{3-5x}{4} \quad \text{i.e.,}\quad y &= \frac{3-5x}{4} \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= \frac{3-5y}{4} \\ \therefore\quad 4x &= 3-5y \\ \therefore\quad y &= \frac{3-4x}{5} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{3-4x}{5} \end{aligned}$$

6 **a**



b

$$\begin{aligned} f(x) &= 2x-7 \\ \text{i.e.,}\quad y &= 2x-7 \\ \text{so}\quad f^{-1}(x)\ \text{is}\quad x &= 2y-7 \\ \therefore\quad y &= \frac{x+7}{2} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x+7}{2} \end{aligned}$$

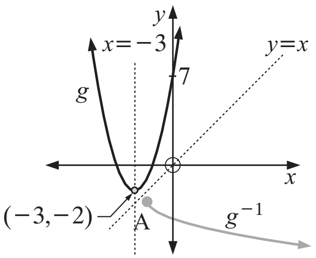
c $f\circ f^{-1}$ and $f^{-1}\circ f$

$$\begin{aligned} &f\circ f^{-1} \\ &= f\left(f^{-1}(x)\right) \\ &= f\left(\frac{x+7}{2}\right) \\ &= 2\left(\frac{x+7}{2}\right)-7 \\ &= x+7-7 \\ &= x \end{aligned}$$

$$\begin{aligned} &f^{-1}\circ f \\ &= f^{-1}\left(f(x)\right) \\ &= f^{-1}(2x-7) \\ &= \frac{2x-7+7}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

So $f\circ f^{-1}=f^{-1}\circ f=e$

7 **a**, **d**



b If $x \leqslant -3$, we have the graph to the left of $x = -3$, and any horizontal line through the graph cuts it no more than once.
Therefore it has an inverse function.

c
$$\begin{aligned} g(x) &= x^2+6x+7, \quad x \leqslant -3 \\ \text{i.e.,}\quad y &= x^2+6x+7, \quad x \leqslant -3 \\ \text{so}\quad g^{-1}(x) \text{ is } \quad x &= y^2+6y+7, \quad y \leqslant -3 \\ &= (y+3)^2-9+7 \\ \therefore\quad x+2 &= (y+3)^2 \\ \therefore\quad y+3 &= \pm\sqrt{x+2} \\ \therefore\quad y &= -3\pm\sqrt{x+2} \\ \text{but } y \leqslant -3, \text{ so}\quad y &= -3-\sqrt{x+2} \end{aligned}$$

e The range of g is $\{y\colon\ y \geqslant -2\}$, so the domain of g^{-1} is $\{x\colon\ x \geqslant -2\}$
and the range of g^{-1} is $\{y\colon\ y \leqslant -3\}$

8 **a**
$$\begin{aligned} h(x) &= (x-4)^2+3, \quad x \geqslant 4 \\ \text{i.e.,}\quad y &= (x-4)^2+3, \quad x \geqslant 4 \\ \text{so}\quad h^{-1}(x) \text{ is }\quad x &= (y-4)^2+3, \quad y \geqslant 4 \\ \therefore\quad x-3 &= (y-4)^2 \\ \therefore\quad y-4 &= \pm\sqrt{x-3} \\ \therefore\quad y &= 4\pm\sqrt{x-3} \\ \text{but } y \geqslant 4, \quad \text{so}\quad y &= 4+\sqrt{x-3} \\ \text{i.e.,}\quad h^{-1}(x) &= 4+\sqrt{x-3}, \quad x \geqslant 3 \end{aligned}$$

b
$$\begin{aligned} & h\circ h^{-1} \\ &= h\left(h^{-1}(x)\right) \\ &= h(4+\sqrt{x-3}) \\ &= (4+\sqrt{x-3}-4)^2+3 \\ &= (\sqrt{x-3})^2+3 \\ &= x-3+3 \\ &= x \end{aligned}$$

$$\begin{aligned} & h^{-1}\circ h \\ &= h^{-1}(h(x)) \\ &= h^{-1}\left((x-4)^2+3\right) \\ &= 4+\sqrt{(x-4)^2+3-3} \\ &= 4+\sqrt{(x-4)^2} \\ &= 4+x-4 \quad \text{as}\quad x \geqslant 4 \\ &= x \end{aligned}$$

9
$$\begin{aligned} f(x) &= 3x+6 \\ \text{i.e.,}\quad y &= 3x+6 \\ \text{so}\quad f^{-1}(x) \text{ is }\quad x &= 3y+6 \\ \therefore\quad y &= \frac{x-6}{3} \\ \text{i.e.,}\quad f^{-1}(x) &= \frac{x-6}{3} \end{aligned}$$

$$\begin{aligned} h(x) &= \frac{x}{3} \\ \text{i.e.,}\quad y &= \frac{x}{3} \\ \text{so}\quad h^{-1}(x) \text{ is }\quad x &= \frac{y}{3} \\ \therefore\quad y &= 3x \\ \text{i.e.,}\quad h^{-1}(x) &= 3x \end{aligned}$$

$$\begin{aligned} \text{Now}\quad (f^{-1}\circ h^{-1})(x) &= f^{-1}(h^{-1}(x)) \\ &= f^{-1}(3x) \\ &= \frac{3x-6}{3} \\ &= x-2 \end{aligned}$$

$$\begin{aligned} (h\circ f)(x) &= h(f(x)) \\ &= h(3x+6) \\ &= \frac{3x+6}{3} \\ \text{i.e.,}\quad y &= x+2 \\ \text{i.e.,}\quad (h\circ f)^{-1}(x) \text{ is }\quad x &= y+2 \\ \therefore\quad y &= x-2 \\ \text{i.e.,}\quad (h\circ f)^{-1}(x) &= x-2 \end{aligned}$$

i.e., $(f^{-1}\circ h^{-1})(x) = (h\circ f)^{-1}(x)$ as required

Chapter 2

SEQUENCES AND SERIES

EXERCISE 2A

1 **a** 4, 13, 22, 31, ... **b** 45, 39, 33, 27, ... **c** 2, 6, 18, 54, ... **d** 96, 48, 24, 12, ...

2 **a** The sequence starts at 8 and each term is 8 more than the previous term. The next two terms are 40 and 48.

b The sequence starts at 2 and each term is 3 more than the previous term. The next two terms are 14 and 17.

c The sequence starts at 36 and each term is 5 less than the previous term. The next two terms are 16 and 11.

d The sequence starts at 96 and each term is 7 less than the previous term. The next two terms are 68 and 61.

e The sequence starts at 1 and each term is 4 times the previous term. The next two terms are 256 and 1024.

f The sequence starts at 2 and each term is 3 times the previous term. The next two terms are 162 and 486.

g The sequence starts at 480 and each term is half the previous term. The next two terms are 30 and 15.

h The sequence starts at 243 and each term is one third of the previous term. The next two terms are 3 and 1.

i The sequence starts at 50 000 and each term is one fifth of the previous term. The next two terms are 80 and 16.

3 **a** Each term is the square of the number of the term. The next three terms are 25, 36 and 49.

b Each term is the cube of the number of the term. The next three terms are 125, 216 and 343.

c Each term is $n \times (n+1)$ where n is the number of the term. The next three terms are 30, 42 and 56.

EXERCISE 2B

1 **a** $\{2n\}$ generates the sequence 2, 4, 6, 8, 10, ... (letting $n = 1, 2, 3, 4, 5, ...$)

b $\{2n+2\}$ generates the sequence 4, 6, 8, 10, 12, ... (letting $n = 1, 2, 3, 4, 5, ...$)

c $\{2n-1)$ generates the sequence 1, 3, 5, 7, 9, ... (letting $n = 1, 2, 3, 4, 5, ...$)

d $\{2n-3\}$ generates the sequence -1, 1, 3, 5, 7, ... (letting $n = 1, 2, 3, 4, 5, ...$)

e $\{2n+3\}$ generates the sequence 5, 7, 9, 11, 13, ... (letting $n = 1, 2, 3, 4, 5, ...$)

f $\{2n+11\}$ generates the sequence 13, 15, 17, 19, 21, ... (letting $n = 1, 2, 3, 4, 5, ...$)

g $\{3n+1\}$ generates the sequence 4, 7, 10, 13, 16, ... (letting $n = 1, 2, 3, 4, 5, ...$)

h $\{4n-3\}$ generates the sequence 1, 5, 9, 13, 17, ... (letting $n = 1, 2, 3, 4, 5, ...$)

2 **a** $\{2^n\}$ generates the sequence 2, 4, 8, 16, 32, ... (letting $n = 1, 2, 3, 4, 5, ...$)

b $\{3 \times 2^n\}$ generates the sequence 6, 12, 24, 48, 96, ... (letting $n = 1, 2, 3, 4, 5, ...$)

c $\{6 \times (\frac{1}{2})^n\}$ generates the sequence $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$, ... (letting $n = 1, 2, 3, 4, 5, ...$)

d $\{(-2)^n\}$ generates the sequence $-2, 4, -8, 16, -32$, ... (letting $n = 1, 2, 3, 4, 5, ...$)

3 $\{15-(-2)^n\}$ generates the sequence with first five terms:

$t_1 = 15-(-2)^1 = 17$, $t_2 = 15-(-2)^2 = 11$, $t_3 = 15-(-2)^3 = 23$,
$t_4 = 15-(-2)^4 = -1$, $t_5 = 15-(-2)^5 = 47$

EXERCISE 2C

1 **a**
$$\begin{aligned} 17-6 &= 11\\ 28-17 &= 11\\ 39-28 &= 11\\ 50-39 &= 11 \end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by 11.

$\therefore$ the sequence is arithmetic with $u_1 = 6$, $d = 11$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d\\ &= 6 + (n-1)11 \end{aligned}$$
i.e., $u_n = 11n - 5$

c
$$\begin{aligned} u_{50} &= 11(50) - 5\\ &= 545 \end{aligned}$$

d Let $u_n = 325 = 11n - 5$
$$\begin{aligned} \therefore\ 330 &= 11n\\ \therefore\ n &= 30 \end{aligned}$$
So, 325 is a member, i.e., u_{30}.

e Let $u_n = 761 = 11n - 5$
$$\begin{aligned} \therefore\ 766 &= 11n\\ \therefore\ n &= 69\tfrac{7}{11}, \end{aligned}$$
but n is an integer, so 761 is not a member of the sequence.

2 **a**
$$\begin{aligned} 83-87 &= -4\\ 79-83 &= -4\\ 75-79 &= -4 \end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by -4.

$\therefore$ the sequence is arithmetic with $u_1 = 87$, $d = -4$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d\\ &= 87 + (n-1)(-4)\\ &= 87 - 4n + 4 \end{aligned}$$
i.e., $u_n = 91 - 4n$

c
$$\begin{aligned} u_{40} &= 91 - 4(40)\\ &= 91 - 160\\ &= -69 \end{aligned}$$

d Let $u_n = -143 = 91 - 4n$
$$\begin{aligned} \therefore\ 4n &= 234\\ \therefore\ n &= 58\tfrac{1}{2} \end{aligned}$$
but n is an integer, so -143 is not a member of the sequence.

3 **a** $u_n = 3n - 2$ $\qquad u_1 = 3(1) - 2 = 1$ $\qquad u_{n+1} = 3(n+1) - 2 = 3n + 1$

$$\begin{aligned} u_{n+1} - u_n &= (3n+1) - (3n-2)\\ &= 3, \quad \text{a constant} \end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by 3.

$\therefore$ the sequence is arithmetic with $u_1 = 1$ and $d = 3$.

b $u_1 = 1$, $d = 3$

c $u_{57} = 3(57) - 2 = 169$

d Let $u_n = 450 = 3n - 2$ i.e., $3n = 452$ and so $n = 150\frac{2}{3}$

So, try the two values on either side of $n = 150\frac{2}{3}$, i.e., for $n = 150$ and $n = 151$:

$$\begin{aligned} u_{150} &= 3(150) - 2 \\ &= 448 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u_{151} &= 3(151) - 2\\ &= 451 \end{aligned}$$

i.e., $u_{151} = 451$ is the least term which is greater than 450.

4 **a** $u_n = \dfrac{71-7n}{2} = 35\frac{1}{2} - \frac{7}{2}n \qquad u_1 = \dfrac{71-7(1)}{2} = 32$

$$u_{n+1} = \frac{71-7(n+1)}{2} = \frac{71-7n-7}{2} = \frac{64-7n}{2} = 32 - \tfrac{7}{2}n$$

$$u_{n+1} - u_n = (32 - \tfrac{7}{2}n) - (35\tfrac{1}{2} - \tfrac{7}{2}n) = -\tfrac{7}{2} \quad \text{a constant}$$

So, assuming that the pattern continues, consecutive terms differ by $-\frac{7}{2}$.

$\therefore$ the sequence is arithmetic with $u_1 = 32$, $d = -\frac{7}{2}$.

b $u_1 = 32$, $d = -\frac{7}{2}$

c $u_{75} = \dfrac{71-7(75)}{2} = -227$

d Let $u_n = -200 = \dfrac{71-7n}{2}$ i.e., $-400 = 71 - 7n$ $\therefore\ 7n = 471$

$\therefore\ n = 67\frac{2}{7}$

So, try the two values on either side of $n = 67\frac{2}{7}$, i.e., for $n = 67$ and $n = 68$:

$$u_{67} = \frac{71-7(67)}{2} = -199 \quad \text{and} \quad u_{68} = \frac{71-7(68)}{2} = -202\tfrac{1}{2}$$

i.e., terms of the sequence are less than -200 for $n \geqslant 68$.

5 **a** The terms are consecutive,

$\therefore\ k - 32 = 3 - k$

{equating common differences}

$\therefore\ 2k = 35$ and so $k = 17\frac{1}{2}$

b The terms are consecutive,

$\therefore\ (2k+1) - (k+1) = 13 - (2k+1)$

$\therefore\ k = 12 - 2k$

$\therefore\ 3k = 12$ and so $k = 4$

c The terms are consecutive,

$k - 5 = k^2 - 8 - k$

{equating common differences}

$\therefore\ k^2 - 2k - 3 = 0$

$\therefore\ (k-3)(k+1) = 0$

$\therefore\ k = -1$ or 3

6 **a** $u_7 = 41$ $\therefore\ u_1 + 6d = 41$ .... (1)

$u_{13} = 77$ $\therefore\ u_1 + 12d = 77$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 6d &= -41 \\ u_1 + 12d &= 77 \\ \hline \therefore\ 6d &= 36 \\ \therefore\ d &= 6 \end{aligned}$$

{adding the equations}

So in (1), $u_1 + 6(6) = 41$

$\therefore\ u_1 + 36 = 41$

$\therefore\ u_1 = 5$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 5 + (n-1)6$

$\therefore\ u_n = 6n - 1$

b $u_5 = -2$ $\therefore\ u_1 + 4d = -2$ .... (1)

$u_{12} = -12\frac{1}{2}$ $\therefore\ u_1 + 11d = -12\frac{1}{2}$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 4d &= 2 \\ u_1 + 11d &= -12\tfrac{1}{2} \\ \hline \therefore\ 7d &= -10\tfrac{1}{2} \\ \therefore\ d &= -\tfrac{3}{2} \end{aligned}$$

{adding the equations}

So in (1), $u_1 + 4(-\frac{3}{2}) = -2$

$\therefore\ u_1 = 4$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 4 + (n-1)(-\frac{3}{2})$

$\therefore\ u_n = -\frac{3}{2}n + \frac{11}{2}$

c $u_7 = 1$ $\therefore\ u_1 + 6d = 1$ .... (1)

$u_{15} = -39$ $\therefore\ u_1 + 14d = -39$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 6d &= -1 \\ u_1 + 14d &= -39 \\ \hline \therefore\ 8d &= -40 \\ \therefore\ d &= -5 \end{aligned}$$

{adding the equations}

So in (1), $u_1 + 6(-5) = 1$

$\therefore\ u_1 - 30 = 1$

$\therefore\ u_1 = 31$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = 31 + (n-1)(-5)$

$\therefore\ u_n = 31 - 5n + 5$

$\therefore\ u_n = -5n + 36$

d $u_{11} = -16$ $\therefore\ u_1 + 10d = -16$ .... (1)

$u_8 = -11\frac{1}{2}$ $\therefore\ u_1 + 7d = -11\frac{1}{2}$ .... (2)

Solving simultaneously,

$$\begin{aligned} -u_1 - 10d &= 16 \\ u_1 + 7d &= -11\tfrac{1}{2} \\ \hline \therefore\ -3d &= 4\tfrac{1}{2} \\ \therefore\ d &= -\tfrac{3}{2} \end{aligned}$$

{adding the equations}

So in (1), $u_1 + 10(-\frac{3}{2}) = -16$

$\therefore\ u_1 - 15 = -16$

$\therefore\ u_1 = -1$

Now $u_n = u_1 + (n-1)d$

$\therefore\ u_n = -1 + (n-1)(-\frac{3}{2})$

$\therefore\ u_n = -\frac{3}{2}n + \frac{1}{2}$

7 **a** Let the numbers be $5,\ 5+d,\ 5+2d,\ 5+3d,\ 10.$

Then $5+4d = 10$

$\therefore\ 4d = 5$

$\therefore\ d = \frac{5}{4} = 1\frac{1}{4}$

So the numbers are $5,\ 6\frac{1}{4},\ 7\frac{1}{2},\ 8\frac{3}{4},\ 10.$

b Let the numbers be $-1,\ -1+d,\ -1+2d,\ -1+3d,\ -1+4d,\ -1+5d,\ -1+6d,\ 32.$

Then $-1+7d = 32$

$\therefore\ 7d = 33$

$\therefore\ d = \frac{33}{7} = 4\frac{5}{7}$

So the numbers are $-1,\ 3\frac{5}{7},\ 8\frac{3}{7},\ 13\frac{1}{7},\ 17\frac{6}{7},\ 22\frac{4}{7},\ 27\frac{2}{7},\ 32.$

8 **a** $u_1 = 36,\quad 35\frac{1}{3} - 36 = -\frac{2}{3}$

$34\frac{2}{3} - 35\frac{1}{3} = -\frac{2}{3},\quad$ so $\ d = -\frac{2}{3}$

b $u_n = u_1 + (n-1)d$

$\therefore\ -30 = 36 + (n-1)(-\frac{2}{3})\quad$ {letting $u_n = -30$, the last term of the sequence}

$\therefore\ -66 = -\frac{2}{3}n + \frac{2}{3}$

$\therefore\ \frac{2}{3}n = 66\frac{2}{3}$

$\therefore\ n = 100\quad$ i.e., the sequence has 100 terms

9 $u_1 = 23,\quad 36-23 = 13$

$49-36 = 13$

$62-49 = 13,\quad$ so $\ d = 13$

$\therefore\ u_n = u_1 + (n-1)d$

i.e., $u_n = 23 + (n-1)13$

$= 23 + 13n - 13$

$\therefore\ u_n = 13n + 10$

Let $u_n = 100\,000 = 13n + 10$

$\therefore\ 99\,990 = 13n$

$\therefore\ n = 7691\frac{7}{13}$

So, try the two values on either side of $n = 7691\frac{7}{13}$, i.e., for $n = 7691$ and $n = 7692$:

i.e., $u_{7691} = 13(7691) + 10 = 99\,993$ and $u_{7692} = 13(7692) + 10 = 100\,006$

i.e., the first term to exceed 100 000 is $u_{7692} = 100\,006.$

EXERCISE 2D

1 **a** $\frac{6}{2} = 3\ \therefore\ r = 3,\ u_1 = 2\quad \therefore\ b = 6\times 3 = 18$ and $c = 18\times 3 = 54$

b $\frac{5}{10} = \frac{1}{2}\ \therefore\ r = \frac{1}{2},\ u_1 = 10\quad \therefore\ b = 5\times\frac{1}{2} = 2\frac{1}{2}$ and $c = 2\frac{1}{2}\times\frac{1}{2} = 1\frac{1}{4}$

c $\frac{-6}{12} = -\frac{1}{2}\ \therefore\ r = -\frac{1}{2},\ u_1 = 12\quad \therefore\ b = -6\times -\frac{1}{2} = 3$ and $c = 3\times -\frac{1}{2} = -1\frac{1}{2}$

2 **a** $\frac{10}{5} = \frac{20}{10} = \frac{40}{20} = 2$

So, assuming the pattern continues, consecutive terms have a common ratio of 2.

$\therefore$ the sequence is geometric with $u_1 = 5$ and $r = 2.$

b

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ \therefore\quad u_n &= 5 \times 2^{n-1} \\ \text{so}\quad u_{15} &= 5 \times 2^{14} \\ &= 81\,920 \end{aligned}$$

3 **a**

$$\begin{aligned} \frac{-6}{12} &= -\tfrac{1}{2} \\ \frac{3}{-6} &= -\tfrac{1}{2} \\ \frac{-1.5}{3} &= \frac{(-\frac{3}{2})}{3} = -\tfrac{1}{2} \end{aligned}$$

So, assuming the pattern continues, consecutive terms have a common ratio of $-\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 12$ and $r = -\frac{1}{2}$.

b

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ \therefore\quad u_n &= 12 \times \left(-\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

$$\begin{aligned} \text{so}\quad u_{13} &= 12 \times \left(-\tfrac{1}{2}\right)^{13-1} \\ &= 12 \times \left(-\tfrac{1}{2}\right)^{12} \\ &= 12 \times \tfrac{1}{4096} \\ &= 3 \times \tfrac{1}{1024} \\ &= \tfrac{3}{1024} \end{aligned}$$

4

$$\begin{aligned} \frac{-6}{8} &= -\tfrac{3}{4} \\ \frac{4.5}{-6} &= -\frac{(\frac{9}{2})}{6} = -\tfrac{3}{4} \\ \frac{-3.375}{4.5} &= \frac{(-\frac{27}{8})}{(\frac{9}{2})} = -\tfrac{3}{4} \end{aligned}$$

So, assuming the pattern continues, consecutive terms have a common ratio of $-\frac{3}{4}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = -\frac{3}{4}$.

$u_n = u_1 r^{n-1} = 8 \times (-\frac{3}{4})^{n-1}$ i.e., $u_{10} = 8 \times (-\frac{3}{4})^9 \doteq -0.600\,677\,49$

5

$$\begin{aligned} \frac{4\sqrt{2}}{8} &= \frac{\sqrt{2}}{2} = \tfrac{1}{\sqrt{2}} \\ \frac{4}{4\sqrt{2}} &= \tfrac{1}{\sqrt{2}} \\ \frac{2\sqrt{2}}{4} &= \frac{\sqrt{2}}{2} = \tfrac{1}{\sqrt{2}} \end{aligned}$$

So, assuming the pattern continues, consecutive terms have a common ratio of $\frac{1}{\sqrt{2}}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{\sqrt{2}}$.

$$\begin{aligned} u_n &= u_1 r^{n-1} = 8\left(\tfrac{1}{\sqrt{2}}\right)^{n-1} = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1} = 2^3 \times 2^{-\frac{1}{2}n+\frac{1}{2}} \\ \text{i.e.,}\quad u_n &= 2^{\frac{7}{2}-\frac{n}{2}} \end{aligned}$$

6 **a** Since the terms are geometric,

$$\frac{k}{7} = \frac{28}{k} \qquad \begin{aligned} \therefore\quad k^2 &= 196 \\ \therefore\quad k &= \pm 14 \end{aligned}$$

b Since the terms are geometric,

$$\begin{aligned} \frac{3k}{k} &= \frac{20-k}{3k} = 3 \\ \therefore\quad 20-k &= 9k \\ \therefore\quad 20 &= 10k \\ \therefore\quad k &= 2 \end{aligned}$$

c Since the terms are geometric,

$$\begin{aligned} \frac{k+8}{k} &= \frac{9k}{k+8} \\ \therefore\quad (k+8)^2 &= 9k^2 \\ \therefore\quad k^2+16k+64 &= 9k^2 \\ \therefore\quad 8k^2-16k-64 &= 0 \\ \therefore\quad k^2-2k-8 &= 0 \\ \therefore\quad (k+2)(k-4) &= 0 \quad \text{and so} \quad k = -2 \text{ or } 4 \end{aligned}$$

7 **a** $u_4 = 24 \quad \therefore\ u_1 \times r^3 = 24 \quad .... (1)$

$u_7 = 192 \quad \therefore\ u_1 \times r^6 = 192 \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^3} = \dfrac{192}{24} \quad \{(2) \div (1)\}$

$\therefore\ r^3 = 8 \qquad \therefore\ r = 2$

So in (1), $u_1 \times 2^3 = 24$

$\therefore\ u_1 \times 8 = 24$

$\therefore\ u_1 = 3$

$\therefore\ u_n = 3 \times 2^{n-1}$

b $u_3 = 8 \quad \therefore\ u_1 \times r^2 = 8 \quad .... (1)$

$u_6 = -1 \quad \therefore\ u_1 \times r^5 = -1 \quad .... (2)$

So, $\dfrac{u_1 r^5}{u_1 r^2} = -\dfrac{1}{8} \quad \{(2) \div (1)\}$

$\therefore\ r^3 = -\frac{1}{8} \qquad \therefore\ r = -\frac{1}{2}$

So in (1), $u_1 \times (-\frac{1}{2})^2 = 8$

$\therefore\ u_1 \times \frac{1}{4} = 8$

$\therefore\ u_1 = 32$

$\therefore\ u_n = 32 \times (-\frac{1}{2})^{n-1}$

c $u_7 = 24 \quad \therefore\ u_1 \times r^6 = 24 \quad .... (1)$

$u_{15} = 384 \quad \therefore\ u_1 \times r^{14} = 384 \quad .... (2)$

So, $\dfrac{u_1 r^{14}}{u_1 r^6} = \dfrac{384}{24} \quad \{(2) \div (1)\}$

$\therefore\ r^8 = 16 \qquad \therefore\ r = \sqrt{2}$

So in (1), $u_1 \times (\sqrt{2})^6 = 24$

$\therefore\ u_1 \times 8 = 24$

$\therefore\ u_1 = 3$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = 3 \times (\sqrt{2})^{n-1}$

d $u_3 = 5 \quad \therefore\ u_1 \times r^2 = 5 \quad .... (1)$

$u_7 = \frac{5}{4} \quad \therefore\ u_1 \times r^6 = \frac{5}{4} \quad .... (2)$

So, $\dfrac{u_1 r^6}{u_1 r^2} = \dfrac{(\frac{5}{4})}{5} \quad \{(2) \div (1)\}$

$\therefore\ r^4 = \frac{1}{4} \qquad \therefore\ r = \frac{1}{\sqrt{2}}$

So in (1), $u_1 \times (\frac{1}{\sqrt{2}})^2 = 5$

$\therefore\ u_1 \times \frac{1}{2} = 5$

$\therefore\ u_1 = 10$

Now $u_n = u_1 r^{n-1}$

$\therefore\ u_n = 10 \times (\frac{1}{\sqrt{2}})^{n-1}$

$\therefore\ u_n = 10 \times (\sqrt{2})^{1-n}$

8 **a** 2, 6, 18, 54, ... i.e., $u_1 = 2$ and $r = 3$

$u_n = u_1 r^{n-1} \qquad \therefore\ u_n = 2 \times 3^{n-1}$

Let $u_n = 10\,000 = 2 \times 3^{n-1}$ i.e., $5000 = 3^{n-1}$

$\therefore\ n \doteqdot 8.7527$ {using technology}

So, try the two values on either side of $n = 8.7527$, i.e., for $n = 8$ and $n = 9$:

$u_8 = 2 \times 3^7 = 4374$ and $u_9 = 2 \times 3^8 = 13\,122$

i.e., the first term to exceed 10 000 is $u_9 = 13\,122$.

b $4, 4\sqrt{3}, 12, 12\sqrt{3}$, ... i.e., $u_1 = 4$ and $r = \sqrt{3}$

$u_n = u_1 r^{n-1} \qquad \therefore\ u_n = 4 \times \left(\sqrt{3}\right)^{n-1}$

Let $u_n = 4800$ i.e., $1200 = \left(\sqrt{3}\right)^{n-1}$

$\therefore\ n \doteqdot 13.91$ {using technology}

So, try the two values on either side of $n \doteqdot 13.91$, i.e., for $n = 13$ and $n = 14$:

$u_{13} = 4 \times \left(\sqrt{3}\right)^{12} = 2916$ and $u_{14} = 4 \times \left(\sqrt{3}\right)^{13} \doteqdot 5050.66$

i.e., the first term to exceed 4800 is $u_{14} \doteqdot 5050.66$.

c 12, 6, 3, 1.5, ... i.e., $u_1 = 12$ and $r = \frac{1}{2}$

$u_n = u_1 r^{n-1} \qquad \therefore\ u_n = 12 \times (\frac{1}{2})^{n-1}$

Let $0.0001 = u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$

i.e., $0.000\,008\bar{3} = \left(\frac{1}{2}\right)^{n-1}$

$\therefore\ n \doteqdot 17.87$ {using technology}

So, try the two values on either side of $n \doteqdot 17.87$, i.e., for $n = 17$ and $n = 18$:

$u_{17} = 12 \times \left(\frac{1}{2}\right)^{16} \doteqdot 0.000\,1831$ and $u_{18} = 12 \times \left(\frac{1}{2}\right)^{17} \doteqdot 0.000\,091\,55$

i.e., the first term of the sequence which is less than 0.0001 is $u_{18} \doteqdot 0.000\,091\,55$.

9 **a** $u_{n+1} = u_1 \times r^n$, where $u_1 = 3000$, $r = 1.1$, $n = 3$

i.e., $u_4 = 3000 \times (1.1)^3$
$= 3993$

So it amounts to \$3993.

b Interest = amount after 3 years − initial amount
= \$3993 – \$3000
= \$993

10 $u_{n+1} = u_1 \times r^n$, where $u_1 = 20\,000$, $r = 1.12$, $n = 4$

i.e., $u_5 = 20\,000 \times (1.12)^4$
$= 31\,470.39$

Interest = 31 470.39 – 20 000 Euro
= 11 470.39 Euro

11 **a** $u_{n+1} = u_1 \times r^n$, where $u_1 = 30\,000$, $r = 1.1$, $n = 4$

i.e., $u_5 = 30\,000 \times (1.1)^4$
$= 43\,923$, i.e., the investment amounts to 43 923 Yen.

b Interest = amount after 4 years − initial amount
= 43 923 – 30 000 Yen
= 13 923 Yen

12 $u_{n+1} = u_1 \times r^n$, where $u_1 = 80\,000$, $r = 1.09$, $n = 3$

i.e., $u_4 = 80\,000 \times (1.09)^3$
$= 103\,602.32$

Interest = amount after 3 years − initial amount
= \$103 602.32 – \$80 000
= \$23 602.32

13 $u_{n+1} = u_1 \times r^n$, where $u_1 = 100\,000$, $r = 1 + \dfrac{0.08}{2} = 1.04$, $n = 10$

i.e., $u_{11} = 100\,000 \times (1.04)^{10}$
$= 148\,024.43$, i.e., it amounts to 148 024.43 Yen.

14 $u_{n+1} = u_1 \times r^n$, where $u_1 = 45\,000$, $r = 1 + \dfrac{0.075}{4} = 1.018\,75$,
$n = 9$ (21 months = 7 'quarters')

i.e., $u_{10} = 45\,000 \times (1.018\,75)^7$
$= 51\,249.06$

i.e., it amounts to £51 249.06

15 $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 20\,000$, $r = 1.075$, $n = 4$

$\therefore\ 20\,000 = u_1 \times (1.075)^4$

$\therefore\ u_1 = 14\,976.01$ So, $\$14\,976.01$ should be invested now.

16 $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 15\,000$, $r = 1.055$, $n = \frac{60}{12} = 5$

$\therefore\ 15\,000 = u_1 \times (1.055)^5$

$\therefore\ u_1 = 11\,477.02$ So, the initial investment required is £ 11 477.02.

17 $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 25\,000$, $r = 1 + \dfrac{0.08}{4} = 1.02$, $n = 3 \times 4 = 12$

$\therefore\ 25\,000 = u_1 \times (1.02)^{12}$

$\therefore\ u_1 = 19\,712.33$ i.e., should invest 19 712.33 Euro now.

18 $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 40\,000$, $r = 1 + \dfrac{0.09}{12} = 1.0075$, $n = 8 \times 12 = 96$

$\therefore\ 40\,000 = u_1 \times (1.0075)^{96}$

$\therefore\ u_1 = 19\,522.47$, i.e., initial investment should be 19 522.47 Yen.

19 $u_{n+1} = u_1 \times r^n$, where $u_1 = 500$, $r = 1.12$

a **i** $u_{11} = 500 \times (1.12)^{10}$
$\doteqdot 1552.92$
$\doteqdot 1550$ ants

ii $u_{21} = 500 \times (1.12)^{20}$
$\doteqdot 4823.15$
$\doteqdot 4820$ ants

b For the population to reach 2000, $u_{n+1} = 500 \times (1.12)^n = 2000$

$\therefore\ (1.12)^n = 4$

$\therefore\ n \doteqdot 12.23$ {using technology}

i.e., it will take approximately 12.2 weeks.

20 $u_{n+1} = u_1 \times r^n$, where $u_1 = 555$, $r = 0.955$

a $u_{16} = 555 \times (0.955)^{15}$
$\doteqdot 278.19$ i.e., the population is approximately 278 animals in the year 2000.

b For the population to have declined to 50,

$u_{n+1} = 555 \times (0.955)^n = 50$

$\therefore\ (0.955)^n = 0.0\overline{900}$

$\therefore\ n \doteqdot 52.3$ {using technology}

So, after approximately 52 years the population is 50, i.e., in the year 2037.

EXERCISE 2E.1

1 **a** **i** 3, 11, 19, 27, ... so $u_1 = 3$, $d = 8$, i.e., $u_n = 3 + (n-1)8$
$= 8n - 5$

i.e., $S_n = 3 + 11 + 19 + 27 + \ldots\ldots + (8n - 5)$

ii $S_5 = 3 + 11 + 19 + 27 + 35 = 95$

b **i** 42, 37, 32, 27, ... so $u_1 = 42$, $d = -5$, i.e., $u_n = 42 + (n-1)(-5)$
$= 47 - 5n$

i.e., $S_n = 42 + 37 + 32 + 27 + \ldots\ldots + (47 - 5n)$

ii $S_5 = 42 + 37 + 32 + 27 + 22 = 160$

c **i** 12, 6, 3, $1\frac{1}{2}$, ... so $u_1 = 12$, $r = \frac{1}{2}$, i.e., $u_n = 12 \times (\frac{1}{2})^{n-1}$

i.e., $S_n = 12 + 6 + 3 + 1\frac{1}{2} + \ldots\ldots + 12(\frac{1}{2})^{n-1}$

ii $S_5 = 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4} = 23\frac{1}{4}$

d **i** $2,\ 3,\ 4\frac{1}{2},\ 6\frac{3}{4},\ \ldots\ldots$ so $u_1 = 2,\quad r = \frac{3}{2},$ i.e., $u_n = 2 \times (\frac{3}{2})^{n-1}$

i.e., $S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + \ldots\ldots + 2(\frac{3}{2})^{n-1}$

ii $S_5 = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8} = 26\frac{3}{8}$

e **i** $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots\ldots$ so $u_1 = 1,\quad r = \frac{1}{2},$ i.e., $u_n = 1 \times (\frac{1}{2})^{n-1}$

$$= 2^{1-n}$$

i.e., $S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots + 2^{1-n}$

ii $S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$

f **i** $1,\ 8,\ 27,\ 64,\ \ldots\ldots$

i.e., $S_n = 1 + 8 + 27 + 64 + \ldots\ldots + n^3$ $\{$since $1 = 1^3,\ 8 = 2^3,\ 27 = 3^3,\ 64 = 4^3\}$

ii $S_5 = 1 + 8 + 27 + 64 + 125 = 225$

EXERCISE 2E.2

1 **a** The series is arithmetic with

$u_1 = 3,\quad d = 4,\quad n = 20$

$$\begin{aligned} S_n &= \frac{n}{2}\,(2u_1 + (n-1)d) \\ \text{So,}\quad S_{20} &= \tfrac{20}{2}\,(2 \times 3 + 19 \times 4) \\ &= 10(6 + 76) \\ &= 820 \end{aligned}$$

b The series is arithmetic with

$u_1 = \frac{1}{2},\quad d = 2\frac{1}{2},\quad n = 50$

$$\begin{aligned} S_n &= \frac{n}{2}\,(2u_1 + (n-1)d) \\ \text{So,}\quad S_{50} &= \tfrac{50}{2}\left(2 \times \tfrac{1}{2} + 49 \times \tfrac{5}{2}\right) \\ &= 25(1 + 122\tfrac{1}{2}) \\ &= 3087\tfrac{1}{2} \end{aligned}$$

c The series is arithmetic with

$u_1 = 100,\quad d = -7,\quad n = 40$

$$\begin{aligned} S_n &= \frac{n}{2}\,(2u_1 + (n-1)d) \\ \text{So,}\quad S_{40} &= \tfrac{40}{2}\,(2 \times 100 + 39 \times (-7)) \\ &= 20(200 - 273) \\ &= -1460 \end{aligned}$$

d The series is arithmetic with

$u_1 = 50,\quad d = -1\frac{1}{2},\quad n = 80$

$$\begin{aligned} S_n &= \frac{n}{2}\,(2u_1 + (n-1)d) \\ \text{So,}\quad S_{80} &= \tfrac{80}{2}\left(2 \times 50 + 79 \times (-\tfrac{3}{2})\right) \\ &= 40(100 - \tfrac{237}{2}) \\ &= -740 \end{aligned}$$

2 **a** The series is arithmetic with

$u_1 = 5,\quad d = 3,\quad u_n = 101$

$$\begin{aligned} \text{Since}\quad u_n &= 101, \\ \text{then}\quad u_1 + (n-1)d &= 101 \\ \therefore\quad 5 + 3(n-1) &= 101 \\ \therefore\quad 5 + 3n - 3 &= 101 \\ \therefore\quad 3n - 3 &= 96 \\ \therefore\quad 3n &= 99 \\ \therefore\quad n &= 33 \end{aligned}$$

$$\begin{aligned} \text{So,}\quad S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{33}{2}(5 + 101) \\ &= 1749 \end{aligned}$$

b The series is arithmetic with

$u_1 = 50,\quad d = -\frac{1}{2},\quad u_n = -20$

$$\begin{aligned} \text{Since}\quad u_n &= -20, \\ \text{then}\quad u_1 + (n-1)d &= -20 \\ \therefore\quad 50 + (-\tfrac{1}{2})(n-1) &= -20 \\ \therefore\quad -\tfrac{1}{2}n + \tfrac{1}{2} &= -70 \\ \therefore\quad -\tfrac{1}{2}n &= -\tfrac{141}{2} \\ \therefore\quad n &= 141 \end{aligned}$$

$$\begin{aligned} \text{So,}\quad S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{141}{2}(50 + (-20)) \\ &= 2115 \end{aligned}$$

c The series is arithmetic with

$u_1 = 8, \quad d = 2\frac{1}{2}, \quad u_n = 83$

$$\begin{aligned} \text{Since} \quad u_n &= 83, \\ \text{then} \quad u_1 + (n-1)d &= 83 \\ \therefore \quad 8 + \tfrac{5}{2}(n-1) &= 83 \\ \therefore \quad \tfrac{5}{2}n - \tfrac{5}{2} &= 75 \\ \therefore \quad \tfrac{5}{2}n &= \tfrac{155}{2} \\ \therefore \quad n &= 31 \end{aligned}$$

$$\begin{aligned} \text{So,} \quad S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{31}{2}(8 + 83) \\ &= 1410\tfrac{1}{2} \end{aligned}$$

3 $u_1 = 5, \quad n = 7,$
$u_n = 53$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{7}{2}(5 + 53) \\ &= 203 \end{aligned}$$

4 $u_1 = 6, \quad n = 11,$
$u_n = -27$

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \tfrac{11}{2}(6 + (-27)) \\ &= -115\tfrac{1}{2} \end{aligned}$$

5 The total number of bricks can be expressed as an arithmetic series:

$1 + 2 + 3 + 4 + \ldots\ldots + n$

We know that the total number of bricks is 171, i.e., $S_n = 171$.

Also, $u_1 = 1, \quad d = 1$ and we need to find n, the number of members (layers) of the series.

$$\begin{aligned} S_n = \frac{n}{2}(2u_1 + (n-1)d) &= 171 \\ \therefore \quad \frac{n}{2}(2 \times 1 + (n-1) \times 1) &= 171 \\ \therefore \quad n(2 + n - 1) &= 342 \\ \therefore \quad n(n+1) &= 342 \end{aligned}$$

$$\begin{aligned} \therefore \quad n^2 + n - 342 &= 0 \\ \therefore \quad (n-18)(n+19) &= 0 \\ \therefore \quad n &= -19 \text{ or } 18 \\ \text{but } n > 0, \quad \therefore \quad n &= 18 \end{aligned}$$

So, there are 18 layers placed.

6 The total number of seats can be expressed as an arithmetic series:

$22 + 23 + 24 + \ldots\ldots + u_n$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore \quad S_n &= \frac{n}{2}(2 \times 22 + 1(n-1)) \\ &= \frac{n}{2}(44 + n - 1) \quad \therefore \quad S_n = \frac{n}{2}(n + 43) \quad \text{which is the total number of seats in } n \text{ rows.} \end{aligned}$$

a
$$\begin{aligned} \text{Number of seats in row 44} &= \text{total no. of seats in every row} - \text{no. of seats in 43 rows} \\ &= S_{44} - S_{43} \\ &= \tfrac{44}{2}(44 + 43) - \tfrac{43}{2}(43 + 43) \\ &= 1914 - 1849 \\ &= 65 \end{aligned}$$

b Number of seats in a section $= S_{44} = 1914$ (from **a**)

c Number of seats in 25 sections $= S_{44} \times 25 = 1914 \times 25 = 47\,850$

7 **a** The first 50 multiples of 11 can be expressed as an arithmetic series:

$11 + 22 + 33 + \ldots\ldots + u_{50}$ where $u_1 = 11, \quad d = 11, \quad n = 50$

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$$\begin{aligned} \therefore \quad S_{50} &= \tfrac{50}{2}(2 \times 11 + 11(50 - 1)) \\ &= 25(22 + 539) \\ &= 14\,025 \end{aligned}$$

b The multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$7 + 14 + 21 + 28 + ... + u_n$ where $u_1 = 7$, $d = 7$

To find u_n, we need to find the largest multiple of 7 less than 1000, i.e., $\frac{1000}{7} \doteqdot 142.9$, so $u_n = 142 \times 7 = 994$

$$\begin{aligned} \text{Now} \quad u_n &= u_1 + (n-1)d \\ \therefore \quad 994 &= 7 + 7(n-1) \\ \therefore \quad 987 &= 7n - 7 \\ \therefore \quad 7n &= 994 \\ \therefore \quad n &= 142 \end{aligned}$$

So, $S_{142} = \frac{142}{2}(7 + 994) = 71\,071$

c The integers between 1 and 100 which are not divisible by 3 can be expressed as:

1, 2, 4, 5, 7, 8, ..., 100 where $u_1 = 1$, $u_n = 100$.

Alternatively, these integers can be expressed as two separate arithmetic series:

i.e., $S_1 = 1 + 4 + 7 + ... + 97 + 100$ where $u_1 = 1$, $d = 3$, $u_n = 100$

and $S_2 = 2 + 5 + 8 + ... + 95 + 98$ where $u_1 = 2$, $d = 3$, $u_n = 98$

Now for S_1,
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore \quad 100 &= 1 + 3(n-1) \\ \therefore \quad 99 &= 3n - 3 \\ \therefore \quad 3n &= 102 \\ \therefore \quad n &= 34 \end{aligned}$$

and for S_2,
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore \quad 98 &= 2 + 3(n-1) \\ \therefore \quad 96 &= 3n - 3 \\ \therefore \quad 3n &= 99 \\ \therefore \quad n &= 33 \end{aligned}$$

i.e., $S_1 = \frac{34}{2}(1 + 100) = 1717$ and $S_2 = \frac{33}{2}(2 + 98) = 1650$

i.e., total sum $= S_1 + S_2 = 1717 + 1650 = 3367$

8 We need to show that $1 + 2 + 3 + 4 + ... + n = \dfrac{n(n+1)}{2}$

The sum of the first n positive integers can be expressed as an arithmetic series:

$1 + 2 + 3 + 4 + ... + n$, where $u_1 = 1$, $d = 1$, $u_n = n$.

So the sum of the series is
$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{n}{2}(1 + n) \end{aligned}$$
i.e., $S_n = \dfrac{n(n+1)}{2}$ as required.

9 The series of odd numbers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ...$ where $u_1 = 1$, $d = 2$

a Now $u_n = u_1 + (n-1)d = 1 + 2(n-1)$

i.e., $u_n = 2n - 1$

b We need to show that S_n is n^2.

The sum of the first n odd numbers can be expressed as an arithmetic series:

$1 + 3 + 5 + 7 + ...(2n-1)$ {using $u_n = 2n - 1$ from **a**}

$$\begin{aligned} \text{So,} \quad S_n &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{n}{2}\,(1 + (2n-1)) \\ &= \frac{n}{2}(2n) \end{aligned}$$

i.e., $S_n = n^2$ as required

c
$$\begin{aligned} S_1 &= \tfrac{1}{2}(1+1) = 1 = 1^2 = n^2 \quad \text{for} \quad n = 1 \quad \checkmark \\ S_2 &= \tfrac{2}{2}(1+3) = 4 = 2^2 = n^2 \quad \text{for} \quad n = 2 \quad \checkmark \\ S_3 &= \tfrac{3}{2}(1+5) = 9 = 3^2 = n^2 \quad \text{for} \quad n = 3 \quad \checkmark \\ S_4 &= \tfrac{4}{2}(1+7) = 16 = 4^2 = n^2 \quad \text{for} \quad n = 4 \quad \checkmark \end{aligned}$$

10 $u_6 = 21$, $S_{17} = 0$; so we need to find u_1 and u_2

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ \therefore\ S_{17} &= \tfrac{17}{2}(2u_1 + 16d) = 0 \\ \therefore\ u_1 + 8d &= 0 \\ \therefore\ u_1 &= -8d \quad ......\ (1) \end{aligned}$$

$$\begin{aligned} \text{Also,}\quad u_n &= u_1 + (n-1)d \\ \therefore\ u_6 &= u_1 + 5d \\ \therefore\ 21 &= -8d + 5d \quad \{\text{using (1)}\} \\ \therefore\ 21 &= -3d \\ \therefore\ d &= -7 \\ \text{and}\quad u_1 &= -8(-7) = 56 \\ \text{so}\quad u_2 &= 56 - 7 = 49 \end{aligned}$$

i.e., the first two terms are 56 and 49.

11 Let the three consecutive terms be $x-d$, x and $x+d$.

$$\begin{aligned} \text{Now, sum of terms} &= 12 \\ \text{i.e.,}\quad (x-d) + x + (x+d) &= 12 \\ \therefore\ 3x &= 12 \\ \therefore\ x &= 4 \end{aligned}$$

So the terms are $4-d$, 4, $4+d$

$$\begin{aligned} \text{Also, product of terms} &= -80 \\ \text{i.e.,}\quad (4-d)4(4+d) &= -80 \\ \therefore\ 4(4^2 - d^2) &= -80 \\ \therefore\ 16 - d^2 &= -20 \\ \therefore\ d^2 &= 36 \\ \therefore\ d &= \pm 6 \end{aligned}$$

So, the three terms could be $4-6$, 4, $4+6$, i.e., -2, 4, 10

or $4-(-6)$, 4, $4+(-6)$, i.e., 10, 4, -2

12 Let the five consecutive terms be $n-2d$, $n-d$, n, $n+d$, $n+2d$.

$$\begin{aligned} \text{Now, sum of terms} &= 40 \\ \text{i.e.,}\quad (n-2d) + (n-d) + n + (n+d) + (n+2d) &= 40 \\ \therefore\ 5n &= 40 \\ \therefore\ n &= 8 \end{aligned}$$

So the terms are $8-2d$, $8-d$, 8, $8+d$, $8+2d$

$$\begin{aligned} \text{Also, product of middle and end terms} = 8 \times (8-2d) \times (8+2d) &= 224 \\ \therefore\ 8(8^2 - 4d^2) &= 224 \\ \therefore\ 64 - 4d^2 &= 28 \\ \therefore\ 4d^2 &= 36 \\ \therefore\ d^2 &= 9 \\ \therefore\ d &= \pm 3 \end{aligned}$$

So, the five terms could be $8-2(3)$, $8-3$, 8, $8+3$, $8+2(3)$, i.e., 2, 5, 8, 11, 14

or $8-2(-3)$, $8-(-3)$, 8, $8+(-3)$, $8+2(-3)$, i.e., 14, 11, 8, 5, 2

EXERCISE 2E.3

1 **a** The series is geometric with

$u_1 = 12$, $r = \frac{1}{2}$, $n = 10$

$$\begin{aligned} \text{Now}\quad S_n &= \frac{u_1(1-r^n)}{1-r} \\ \text{i.e.,}\quad S_{10} &= \frac{12\left(1 - (\frac{1}{2})^{10}\right)}{1 - \frac{1}{2}} \\ &\doteqdot 23.9766 \end{aligned}$$

So, $S_{10} \doteqdot 24.0$

b The series is geometric with

$u_1 = \sqrt{7}$, $r = \sqrt{7}$, $n = 12$

$$\begin{aligned} \text{Now}\quad S_n &= \frac{u_1(r^n - 1)}{r-1} \\ \text{i.e.,}\quad S_{12} &= \frac{\sqrt{7}\left((\sqrt{7})^{12} - 1\right)}{\sqrt{7} - 1} \\ &\doteqdot 189\,134 \end{aligned}$$

c The series is geometric with

$u_1 = 6, \quad r = -\frac{1}{2}, \quad n = 15$

$$\begin{aligned} \text{Now} \quad S_n &= \frac{u_1(1-r^n)}{1-r} \\ \text{i.e.,} \quad S_{15} &= \frac{6\left(1-(-\frac{1}{2})^{15}\right)}{1-(-\frac{1}{2})} \\ &\doteqdot 4.000 \end{aligned}$$

d The series is geometric with

$u_1 = 1, \quad r = -\frac{1}{\sqrt{2}}, \quad n = 20$

$$\begin{aligned} \text{Now} \quad S_n &= \frac{u_1(1-r^n)}{1-r} \\ \text{i.e.,} \quad S_{20} &= \frac{1\left(1-(-\frac{1}{\sqrt{2}})^{20}\right)}{1-(-\frac{1}{\sqrt{2}})} \\ &\doteqdot 0.5852 \end{aligned}$$

2 **a** The series is geometric with

$u_1 = \sqrt{3}, \quad r = \sqrt{3}$

$$\begin{aligned} S_n &= \frac{u_1(r^n-1)}{r-1} \\ &= \frac{\sqrt{3}\left((\sqrt{3})^n-1\right)}{\sqrt{3}-1} \times \left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) \\ &= \frac{\left(3+\sqrt{3}\right)\left((\sqrt{3})^n-1\right)}{3-1} \\ &= \frac{3+\sqrt{3}}{2}\left((\sqrt{3})^n-1\right) \end{aligned}$$

b The series is geometric with

$u_1 = 12, \quad r = \frac{1}{2}$

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{12\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} \\ &= 24\left(1-(\tfrac{1}{2})^n\right) \end{aligned}$$

c The series is geometric with

$u_1 = 0.9, \quad r = 0.1$

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{0.9\,(1-(0.1)^n)}{1-0.1} \\ &= 1-(0.1)^n \end{aligned}$$

d The series is geometric with

$u_1 = 20, \quad r = -\frac{1}{2}$

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{20\left(1-(-\frac{1}{2})^n\right)}{1-(-\frac{1}{2})} \\ &= \frac{20\left(1-(-\frac{1}{2})^n\right)}{(\frac{3}{2})} \\ &= \tfrac{40}{3}\left(1-(-\tfrac{1}{2})^n\right) \end{aligned}$$

3 **a**

$$\begin{aligned} A_2 &= A_1 \times 1.06 + 2000 \\ &= (A_0 \times 1.06 + 2000) \times 1.06 + 2000 \\ &= (2000 \times 1.06 + 2000) \times 1.06 + 2000 \\ \text{i.e.,} \quad A_2 &= 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2 \quad \text{as required} \end{aligned}$$

b

$$\begin{aligned} A_3 &= A_2 \times 1.06 + 2000 \\ &= \left[2000 + 2000 \times 1.06 + 2000 \times (1.06)^2\right] \times 1.06 + 2000 \quad \{\text{from } \mathbf{a}\} \\ \text{i.e.,} \quad A_3 &= 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3\right] \quad \text{as required} \end{aligned}$$

c

$$\begin{aligned} A_9 &= 2000\left[1 + 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7 \right. \\ &\qquad \left. + (1.06)^8 + (1.06)^9\right] \\ \text{i.e.,} \quad A_9 &= 26\,361.59 \end{aligned}$$

i.e., total bank balance after 10 years is $26 361.59

4 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$,
$S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$

c
$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r}, \quad \text{where} \quad u_1 = \tfrac{1}{2}, \quad r = \tfrac{1}{2} \\ &= \frac{\frac{1}{2}\left(1-(\frac{1}{2})^n\right)}{1-\frac{1}{2}} \\ \text{i.e.,} \quad S_n &= 1 - (\tfrac{1}{2})^n = 1 - \frac{1}{2^n} = \frac{2^n-1}{2^n} \end{aligned}$$

d As n gets very large, i.e., as $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$, so $S_n \to 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions.
As $n \to \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches one whole unit.

5 **a**
$$\begin{aligned} \text{Total time of motion} &= 1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1) \\ &\quad + (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + \ldots\ldots \\ &= 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + \ldots\ldots \\ &= 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + \ldots\ldots \quad \text{as required} \end{aligned}$$

b
$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r}, \quad \text{where} \quad u_1 = 2(0.9), \quad r = 0.9, \quad `n\text{'} = n-1 \\ \text{i.e.,} \quad S_n &= \frac{2(0.9)\left(1-(0.9)^{n-1}\right)}{1-0.9} + 1 \\ &= \frac{1.8\left(1-(0.9)^{n-1}\right)}{0.1} + 1 \\ \text{i.e.,} \quad S_n &= 1 + 18\left(1-(0.9)^{n-1}\right) \end{aligned}$$

(since the term u_1, used in calculating S_n, is the second term of the series, not the first)

c For the ball to come to rest, n must approach infinity,
thus $(0.9)^{n-1} \to 0$ so $\left(1-(0.9)^{n-1}\right) \to 1$ (from below)
So, $S_n \to 1 + 18(1)$, i.e., $S_n \to 19$
i.e., it takes 19 seconds for the ball to come to rest.

6 **a** **i** $u_1 = \frac{3}{10}$ **ii** $r = \dfrac{(\frac{3}{100})}{(\frac{3}{10})} = \frac{1}{10}$

b We need to show that $0.\bar{3} = \frac{1}{3}$.

Now $0.\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots\ldots$

So, let $S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots\ldots$

Since $n \to \infty$, then $S_\infty = \dfrac{u_1}{1-r} = \dfrac{\frac{3}{10}}{1-(\frac{1}{10})} = \frac{1}{3}$

i.e., $0.\bar{3} = \frac{1}{3}$ as required

7 **a** $0.\overline{4} = 0.444444....$

$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + ...$

which is a geometric series with $a = 0.4$, $r = 0.1$

$$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{0.4}{1-0.1} = \frac{0.4}{0.9} = \tfrac{4}{9}$$

So, $0.\overline{4} = \frac{4}{9}$

b $0.\overline{16} = 0.161616....$

$= \frac{16}{10^2} + \frac{16}{10^4} + \frac{16}{10^6} + ...$

which is a geometric series with $a = 0.16$, $r = 0.01$

$$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{0.16}{0.99} = \frac{16}{99}$$

So, $0.\overline{16} = \frac{16}{99}$

c $0.\overline{312} = 0.312\,312\,312\,....$

$= \frac{312}{10^3} + \frac{312}{10^6} + \frac{312}{10^9} + ...$

which is a geometric series with $a = 0.312$, $r = 0.001$

$$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{0.312}{0.999} = \frac{312}{999} = \frac{104}{333}$$

So, $0.\overline{312} = \frac{104}{333}$

8 Checking **4 d**:

$$S_\infty = \frac{u_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

i.e., $S_\infty = 1$ ✓

Checking **5 c**:

$$S_\infty = \frac{u_1}{1-r} = \frac{2(0.9)}{1-0.9} + 1$$

{since 1st term '1' is not part of the series}

i.e., $S_\infty = 19$ ✓

EXERCISE 2F

1 **a** $\sum_{r=1}^{4} (3r-5) = -2+1+4+7 = 10$

b $\sum_{r=1}^{5} (11-2r) = 9+7+5+3+1 = 25$

c $\sum_{r=1}^{7} r(r+1) = 2+6+12+20+30+42+56 = 168$

d $\sum_{i=1}^{5} 10 \times 2^{i-1} = 10+20+40+80+160 = 310$

2 $u_n = 3n-1$

$$u_1 + u_2 + u_3 + ... + u_{20} = 2+5+8+11+14+17+20+23+26+29+32+35 + 38+41+44+47+50+53+56+59$$

This series is arithmetic with $u_1 = 2$, $d = 3$ and $n = 20$

so its sum is $\frac{n}{2}[2u_1 + (n-1)d] = 10[4 + 19 \times 3] = 610$

3 **a** $\sum_{r=1}^{10} (2r+5) = 7+9+11+......+21+23+25$

This series is arithmetic with $u_1 = 7$, $d = 2$ and $n = 10$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{10}{2}[14 + 9 \times 2] = 160$

b $\sum_{r=1}^{15} (r-50) = (-49)+(-48)+(-47)+......+(-37)+(-36)+(-35)$

This series is arithmetic with $u_1 = -49$, $d = 1$ and $n = 15$.

$\therefore$ sum $= \frac{n}{2}[2u_1 + (n-1)d] = \frac{15}{2}[-98 + 14 \times 1] = -630$

c $\sum\limits_{r=1}^{20} \left(\frac{r+3}{2}\right) = 2 + \frac{5}{2} + 3 + \text{......} + \frac{21}{2} + 11 + \frac{23}{2}$

This series is arithmetic with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 20$.

$\therefore$ sum $= \dfrac{n}{2}[2u_1 + (n-1)d] = \frac{20}{2}[4 + 19 \times \frac{1}{2}] = 135$

4 **a** $\sum\limits_{r=1}^{10} \left(3 \times 2^{r-1}\right) = 3 + 6 + 12 + \text{......} + 384 + 768 + 1536$

This series is geometric with $u_1 = 3$, $r = 2$ and $n = 10$.

$\therefore$ sum $= \dfrac{u_1(r^n - 1)}{r-1} = \dfrac{3(2^{10}-1)}{1} = 3069$

b $\sum\limits_{r=1}^{12} \left(\frac{1}{2}\right)^{r-2} = 2 + 1 + \frac{1}{2} + \text{......} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 12$.

$\therefore$ sum $= \dfrac{u_1(1-r^n)}{1-r} = \dfrac{2\left(1-(\frac{1}{2})^{12}\right)}{\frac{1}{2}} = 4\left(1-\left(\frac{1}{2}\right)^{12}\right)$

$\therefore$ sum $\doteqdot 3.999$

c $\sum\limits_{r=1}^{25} (6 \times (-2)^r) = -12 + 24 + (-48) + \text{....} + 100\,663\,296 + (-201\,326\,592)$

This series is geometric with $u_1 = -12$, $r = -2$ and $n = 25$.

$\therefore$ sum $= \dfrac{u_1(1-r^n)}{1-r} = \dfrac{-12\left(1-(-2)^{25}\right)}{1+2} = -4\left(1-(-2)^{25}\right)$

$\therefore$ sum $= -134\,217\,732$

5 **a** $\sum\limits_{k=1}^{5} k(k+1)(k+2) = 6 + 24 + 60 + 120 + 210 = 420$

Note: This series is neither arithmetic nor geometric so the sum is found by adding the 5 terms.

b $\sum\limits_{k=6}^{12} \left(100(1.2)^{k-3}\right) = 172.8 + 207.36 + 248.832 + 298.5984 + 358.318\,08 + 429.981\,696 + 515.978\,0352$

This series is geometric with $u_1 = 172.8$, $r = 1.2$ and $n = 7$

$\therefore$ sum $= \dfrac{u_1(r^n-1)}{r-1} = \dfrac{172.8\left((1.2)^7 - 1\right)}{0.2} \doteqdot 2231.87$

6 **a** The LHS is arithmetic with $u_1 = 5$, $d = 2$, "n" $= n$

$$\begin{aligned}
\text{Now } \frac{n}{2}[2u_1 + (n-1)d] &= 1517 \\
\therefore \frac{n}{2}[10 + 2(n-1)] &= 1517 \\
\therefore \frac{n}{2}[2n+8] &= 1517 \\
\therefore n(n+4) - 1517 &= 0 \\
\therefore n^2 + 4n - 1517 &= 0 \quad \text{where } n > 1 \\
\text{and } n &= 37 \quad \{\text{using technology}\}
\end{aligned}$$

b The LHS is geometric with $u_1 = 2$, $r = 3$, "n" $= n$

$$\begin{aligned} \text{Now} \quad u_1\left(\frac{r^n-1}{r-1}\right) &= 177\,146 \\ \therefore \quad 2\left(\frac{3^n-1}{3-1}\right) &= 177\,146 \\ \therefore \quad 3^n - 1 &= 177\,146 \\ \therefore \quad 3^n &= 177\,147 \\ \therefore \quad n &= 11 \quad \{\text{using technology}\} \end{aligned}$$

EXERCISE 2G

1 11, 14, 17, 20, ... is arithmetic with $u_1 = 11$, $d = 3$

$$\begin{aligned} \text{and} \quad S_n &= \frac{n}{2}\left[2u_1 + (n-1)d\right] \\ &= \frac{n}{2}\left[22 + 3(n-1)\right] \\ &= \frac{n}{2}\left[3n + 19\right] \end{aligned}$$

Suppose $S_n = 2000$ $\quad \therefore \quad \frac{n}{2}[3n+19] = 2000$

$\therefore \quad 3n^2 + 19n = 4000$

Using technology, listing terms of $3n^2 + 19n$, $n = 33$ given $S_n = 3894$
$n = 34$ given $S_n = 4114$

$\therefore$ would sell the 2000th in week 34.

2 $u_1 = \$2795$, $r = 0.98$, $u_n = 500$

$$\begin{aligned} \text{Now} \quad u_n &= u_1 \times r^{n-1} \\ \therefore \quad 500 &= 2795 \times (0.98)^{n-1} \\ \therefore \quad (0.98)^{n-1} &= 0.178\,89 \end{aligned}$$

Using technology, listing the terms of $(0.98)^{n-1}$, $n \doteqdot 86$

when $n = 86$, $u_{86} = 2795 \times (0.98)^{85} \doteqdot 501.88$ (after 85 months)

when $u = 87$, $u_{87} = 2795 \times (0.98)^{86} \doteqdot 491.84$ (after 86 months)

So, it is \$500 during the 85th month.

3 $u_2 = u_1 r = 6$ and so $u_1 = \frac{6}{r}$

$$\begin{aligned} S_3 = u_1 + u_1 r + u_1 r^2 &= -14 \\ \therefore \quad \frac{6}{r} + 6 + 6r &= -14 \\ \therefore \quad \frac{6}{r} + 20 + 6r &= 0 \\ \therefore \quad 6 + 20r + 6r^2 &= 0 \\ \therefore \quad 3r^2 + 10r + 3 &= 0 \\ \therefore \quad (3r+1)(r+3) &= 0 \\ \therefore \quad r &= -\tfrac{1}{3} \text{ or } -3 \end{aligned}$$

when $r = -\frac{1}{3}$, $u_1 = -18$

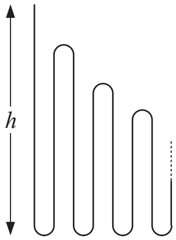
when $r = -3$, $u_1 = -2$

and as $u_4 = u_1 r^3$ then

$u_4 = -18 \times \left(-\frac{1}{3}\right)^3$ or $-2 \times (-3)^3$

i.e., $u_4 = \frac{2}{3}$ or 54

4



Total distance fallen

$= h + 2\left(\frac{3}{4}\right)h + 2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)h + ...$

$= h + 2\left(\frac{3}{4}\right)h\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ...\right]$

$= h + \frac{3}{2}h\left(\frac{1}{1-\frac{3}{4}}\right) \quad \left\{\text{as } |r| = \left|\frac{3}{4}\right| < 1 \text{ and } S_\infty = \frac{u_1}{1-r}\right\}$

$= h + \frac{3}{2}h(4)$

$= 7h$

But $7h = 490 \quad \therefore \quad h = 70$

$\therefore$ the table top is 70 cm above the floor.

5 $u_1 = 1$

$$\begin{aligned} u_2 &= u_1 + d && \{\text{for arithmetic}\} \\ \text{and} \quad u_2 &= u_1 r && \{\text{for geometric}\} \\ \therefore \quad u_1 + d &= u_1 r && \text{i.e., } 1 + d = r \quad ...... (1) \\ \text{Now} \quad u_{14} &= u_1 + 13d && \{\text{for arithmetic}\} \\ \text{and} \quad u_3 &= u_1 r^2 && \{\text{for geometric}\} \\ \therefore \quad u_1 + 13d &= 3u_1 r^2 && \text{i.e., } 1 + 13d = 3r^2 \quad ...... (2) \end{aligned}$$

From (1), $d = r - 1$ $\therefore$ in (2)

$$\begin{aligned} 1 + 13(r-1) &= 3r^2 \\ \therefore \quad 1 + 13r - 13 &= 3r^2 \\ \therefore \quad 3r^2 - 13r + 12 &= 0 \\ \therefore \quad (3r-4)(r-3) &= 0 \\ \therefore \quad r &= \tfrac{4}{3} \text{ or } 3 \end{aligned}$$

When $r = \frac{4}{3}$, $d = \frac{1}{3}$: $u_{20} = u_1 + 19d = 1 + 19(\frac{1}{3}) = \frac{22}{3} = 7\frac{1}{3}$

$u_{20} = u_1 r^{19} = 1 \times \left(\frac{4}{3}\right)^{19} = \left(\frac{4}{3}\right)^{19}$

When $r = 3$, $d = 2$: $u_{20} = u_1 + 19d = 1 + 19(2) = 39$

$u_{20} = u_1 r^{19} = 1 \times 3^{19} = 3^{19}$

So the 20th terms are $7\frac{1}{3}$ for arithmetic and $\left(\frac{4}{3}\right)^{19}$ for geometric or

39 for arithmetic and 3^{19} for geometric.

6

$$\begin{aligned} \sum_{r=1}^{\infty}\left(\frac{3x}{2}\right)^{r-1} &= \left(\frac{3x}{2}\right)^0 + \left(\frac{3x}{2}\right)^1 + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + ... \\ &= 1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \left(\frac{3x}{2}\right)^3 + ... \\ &= \frac{u_1}{1-r} \qquad \{\text{as it converges to 4 and is geometric}\} \\ &= \frac{1}{1-\frac{3x}{2}} \\ &= \frac{2}{2-3x} \end{aligned}$$

$\therefore \quad \frac{2}{2-3x} = 4$ and so $2 - 3x = \frac{1}{2}$

$\therefore \quad 3x = 1\frac{1}{2}$

$\therefore \quad x = \frac{1}{2}$

7 **a** $S_n = \dfrac{n(3n+11)}{2} \quad \therefore \quad S_1 = u_1 = \dfrac{1(14)}{2} = 7$

and $S_2 = u_1 + u_2 = \dfrac{2(17)}{2} = 17 \quad \therefore \quad u_1 = 7$ and $u_2 = 10$

b $u_1 = 7$ and $d = 3 \quad \therefore \quad u_{20} = u_1 + 19d = 7 + 19 \times 3 = 64$

$\therefore$ the twentieth term is 64.

8 **a**
$$\begin{aligned} A_3 &= A_2 \times 1.03 - R \\ &= \left(\$8000 \times (1.03)^2 - 1.03R - R\right) \times 1.03 - R \\ &= \$8000 \times (1.03)^3 - (1.03)^2R - (1.03)R - R \end{aligned}$$

b $A_8 = \$8000 \times (1.03)^8 - (1.03)^7R - (1.03)^6R - (1.05)^5R - \text{......} - 1.03R - R$

We want A_8 to have a value of 0

$$\therefore \quad R\left(1 + 1.03 + (1.03)^2 + (1.03)^3 + \text{......} + (1.03)^7\right) = \$8000 \times (1.03)^8$$

$$\therefore \quad R\left(1\left[\frac{(1.03)^8 - 1}{1.03 - 1}\right]\right) = \$8000 \times (1.03)^8$$

$$\therefore \quad R = \frac{\$8000 \times (1.03)^8 \times 0.03}{(1.03)^8 - 1} \doteqdot \$1139.65$$

c Notice in **b** that $\$8000 = P$ and $(1.03)^8 = \left(1 + \frac{3}{100}\right)^8 = \left(1 + \frac{r}{100}\right)^m$

$$0.03 = \tfrac{3}{100} = \frac{r}{100}$$

and $(1.03)^8 - 1 = \left(1 + \dfrac{r}{100}\right)^m - 1$

So, in the general case $R = \dfrac{P \times \left(1 + \frac{r}{100}\right)^m \times \frac{r}{100}}{\left(1 + \frac{r}{100}\right)^m - 1}$

REVIEW SET 2A

1 **a** $u_n = 3^{n-2} \quad \therefore \quad u_1 = 3^{-1} = \frac{1}{3}, \quad u_2 = 3^0 = 1, \quad u_3 = 3^1 = 3, \quad u_4 = 3^2 = 9$

b $u_n = \dfrac{3n+2}{n+3} \quad \therefore \quad u_1 = \frac{5}{4}, \quad u_2 = \frac{8}{5}, \quad u_3 = \frac{11}{6}, \quad u_4 = \frac{14}{7} = 2$

c $u_n = 2^n - (-3)^n$

$u_1 = 2 - (-3) = 5, \quad u_2 = 4 - 9 = -5, \quad u_3 = 8 - (-27) = 35, \quad u_4 = 16 - 81 = -65$

2 $u_n = 68 - 5n$

a
$$\begin{aligned} u_{n+1} - u_n &= 68 - 5(n+1) - [68 - 5n] \\ &= 68 - 5n - 5 - 68 + 5n \\ &= -5 \quad \text{for all } n \end{aligned}$$

b $u_1 = 68 - 5(1) = 63, \qquad d = -5$

c $u_{37} = 68 - 5(37) = -117$

d Let $u_n = -200$, and we need to find n.

$$\begin{aligned} u_n = 68 - 5n &= -200 \\ \therefore \quad 5n &= 268 \\ \therefore \quad n &= 53\tfrac{3}{5} \end{aligned}$$

So, try the two values on either side of $n = 53\frac{3}{5}$, i.e., for $n = 53$ and $n = 54$:

$$\begin{aligned} u_{53} &= 68 - 5(53) \\ &= -197 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u_{54} &= 68 - 5(54) \\ &= -202 \end{aligned}$$

i.e., the first term of the sequence less than -200 is $u_{54} = -202$.

3 **a** 3, 12, 48, 192, ...

$$\frac{12}{3} = 4 \qquad \frac{48}{12} = 4 \qquad \frac{192}{48} = 4$$

So, assuming the pattern continues, consecutive terms have a common ratio of 4.

$\therefore$ the sequence is geometric with $u_1 = 3$ and $r = 4$.

b

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ \therefore\ u_n &= 3 \times 4^{n-1} \\ \text{i.e.,}\ u_9 &= 3 \times 4^8 = 196\,608 \end{aligned}$$

4 Since the terms are consecutive,

$$\begin{aligned} \text{then}\quad (k-2) - 3k &= k + 7 - (k-2) \quad \{\text{equating common differences}\} \\ \therefore\ k - 2 - 3k &= k + 7 - k + 2 \\ \therefore\ -2 - 2k &= 9 \\ \therefore\ 2k &= -11 \\ \therefore\ k &= -\tfrac{11}{2} \end{aligned}$$

5

$$\begin{aligned} u_7 &= 31 & \therefore\ u_1 + 6d &= 31 & \text{..... (1)} \\ u_{15} &= -17 & \therefore\ u_1 + 14d &= -17 & \text{..... (2)} \end{aligned}$$

Solving simultaneously,

$$\begin{aligned} -u_1 - 6d &= -31 \\ u_1 + 14d &= -17 \\ \hline \therefore\ 8d &= -48 \quad \{\text{adding the equations}\} \\ \therefore\ d &= -6 \end{aligned}$$

$$\begin{aligned} \text{So in (1),}\quad u_1 + 6(-6) &= 31 \\ \therefore\ u_1 - 36 &= 31 \\ \therefore\ u_1 &= 67 \end{aligned}$$

$$\begin{aligned} \text{Now}\quad u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 67 + (n-1)(-6) \\ \therefore\ u_n &= 67 - 6n + 6 \\ \therefore\ u_n &= -6n + 73 \\ \text{So}\quad u_{34} &= -6(34) + 73 = -131 \end{aligned}$$

6 $u_n = 6\left(\frac{1}{2}\right)^{n-1}$

a $\dfrac{u_{n+1}}{u_n} = \dfrac{6\left(\frac{1}{2}\right)^{n+1-1}}{6\left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2}$ for all n

$\therefore$ $\{u_n\}$ is a geometric sequence.

b $u_1 = 6$,
$r = \frac{1}{2}$

c

$$\begin{aligned} u_{16} &= 6\left(\tfrac{1}{2}\right)^{15} \\ &= 0.000\,183 \end{aligned}$$

7 28, 23, 18, 13, ...

$$\begin{aligned} 23 - 28 &= -5 \\ 18 - 23 &= -5 \\ 13 - 18 &= -5 \end{aligned}$$

So, assuming that the pattern continues, consecutive terms differ by -5.

$\therefore$ the sequence is arithmetic with $u_1 = 28$, $d = -5$.

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 28 + (n-1)(-5) \\ &= 28 - 5n + 5 \\ &= -5n + 33 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}\left(2u_1 + (n-1)d\right) \\ &= \frac{n}{2}\left(2 \times 28 + (n-1)(-5)\right) \\ &= \frac{n}{2}(56 - 5n + 5) \\ \text{i.e.,}\quad S_n &= \frac{n}{2}(61 - 5n) \end{aligned}$$

8 Since the terms are geometric, then $\dfrac{k}{4} = \dfrac{k^2-1}{k}$

$$\begin{aligned} \therefore\ k^2 &= 4(k^2-1) \\ \therefore\ k^2 &= 4k^2-4 \\ \therefore\ 3k^2 &= 4 \\ \therefore\ k^2 &= \tfrac{4}{3} \end{aligned} \qquad \begin{aligned} \therefore\ k &= \pm\tfrac{2}{\sqrt{3}} \\ \therefore\ k &= \pm\tfrac{2\sqrt{3}}{3} \end{aligned}$$

9

$$\begin{aligned} u_6 &= \tfrac{16}{3} \quad \therefore\ u_1 \times r^5 = \tfrac{16}{3} \quad \text{..... (1)} \\ u_{10} &= \tfrac{256}{3} \quad \therefore\ u_1 \times r^9 = \tfrac{256}{3} \quad \text{..... (2)} \end{aligned}$$

So $\dfrac{u_1r^9}{u_1r^5} = \dfrac{\left(\frac{256}{3}\right)}{\left(\frac{16}{3}\right)}$ $\{(2) \div (1)\}$

$$\begin{aligned} \therefore\ r^4 &= 16 \\ \therefore\ r &= \pm 2 \end{aligned}$$

Substituting $r = 2$ into (1) gives

$$\begin{aligned} u_1 \times 2^5 &= \tfrac{16}{3} \\ \therefore\ u_1 \times 32 &= \tfrac{16}{3} \\ \therefore\ u_1 &= \tfrac{1}{6} \end{aligned}$$

Substituting $r = -2$ into (1) gives

$$\begin{aligned} u_1 \times (-2)^5 &= \tfrac{16}{3} \\ \therefore\ u_1 \times (-32) &= \tfrac{16}{3} \\ \therefore\ u_1 &= -\tfrac{1}{6} \end{aligned}$$

Now $u_n = u_1r^{n-1}$ $\quad\therefore\ u_n = \frac{1}{6} \times 2^{n-1}$ *or* $-\frac{1}{6} \times 2^{n-1}$

REVIEW SET 2B

1 **a** $24, 23\frac{1}{4}, 22\frac{1}{2}, ..., -36$ i.e., $u_1 = 24$, $u_n = -36$, and we need to find n.

The sequence is arithmetic with $d = -\frac{3}{4}$.

$$\begin{aligned} \text{Now}\quad u_n &= u_1 + (n-1)d \\ \therefore\ -36 &= 24 + (n-1)(-\tfrac{3}{4}) \\ \therefore\ -60 &= -\tfrac{3}{4}n + \tfrac{3}{4} \\ \therefore\ \tfrac{3}{4}n &= \tfrac{243}{4} \\ \therefore\ n &= 81 \end{aligned}$$

i.e., there are 81 terms in the sequence.

b

$$\begin{aligned} u_{35} &= 24 + (35-1)(-\tfrac{3}{4}) \\ &= 24 - \tfrac{102}{4} \\ &= -\tfrac{3}{2} \end{aligned}$$

c

$$\begin{aligned} S_n &= \frac{n}{2}(u_1 + u_n) \\ \text{i.e.,}\quad S_{81} &= \tfrac{81}{2}\,(24 + (-36)) \\ &= -486 \end{aligned}$$

2 Let the numbers be $23,\ 23+d,\ 23+2d,\ 23+3d,\ 23+4d,\ 23+5d,\ 23+6d,\ 9$

$$\begin{aligned} \text{Then}\quad 23 + 7d &= 9 \\ \text{i.e.,}\quad 7d &= -14 \\ \therefore\ d &= -2 \end{aligned}$$

i.e., the numbers are 23, 21, 19, 17, 15, 13, 11, 9.

3 **a** 86, 83, 80, 77, ... i.e., the sequence is arithmetic with $u_1 = 86$, $d = -3$

$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \text{i.e.,}\quad u_n &= 86 + (n-1)(-3) = 86 - 3n + 3 \\ \text{i.e.,}\quad u_n &= 89 - 3n \end{aligned}$$

b $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}, ...$ which can also be written as $\frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}, ...$

i.e., the numerator starts at 3 and increases by 2 each time, whilst the denominator starts at 4 and increases by 1 each time.

i.e., the nth term is $\dfrac{2n+1}{n+3}$, i.e., $u_n = \dfrac{2n+1}{n+3}$

c 100, 90, 81, 72.9, ... i.e., the sequence is geometric with $u_1 = 100$, $r = \frac{90}{100} = 0.9$

$$u_n = u_1 r^{n-1}$$

i.e., $u_n = 100(0.9)^{n-1}$

4 **a** $\displaystyle\sum_{r=1}^{7} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

$= 1 + 4 + 9 + 16 + 25 + 36 + 49$

b $\displaystyle\sum_{r=1}^{8} \frac{r+3}{r+2} = \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10}$

5 **a** $4 + 11 + 18 + 25 + \ldots\ldots$

i.e., the series is arithmetic with $u_1 = 4$, $d = 7$, $u_r = u_1 + (r-1)d$

$= 4 + 7(r-1)$

$= 7r - 3$

i.e., $\displaystyle\sum_{r=1}^{n} (7r-3)$

b $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\ldots$ i.e., the series is geometric with $u_1 = \frac{1}{4}$, $R = \frac{1}{2}$,

$u_r = u_1 R^{r-1} = \frac{1}{4} \times \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^{r+1}$

i.e., $\displaystyle\sum_{r=1}^{n} \left(\tfrac{1}{2}\right)^{r+1}$

6 **a** $3 + 9 + 15 + 21 + \ldots\ldots$

i.e., the series is arithmetic with

$u_1 = 3$, $d = 6$, $n = 23$

Now $S_n = \dfrac{n}{2}\,(2u_1 + (n-1)d)$

i.e., $S_{23} = \frac{23}{2}\,(2 \times 3 + 6(23-1))$

$\therefore$ $S_{23} = \frac{23}{2}(6 + 132)$

$= 1587$

b $24 + 12 + 6 + 3 + \ldots\ldots$

i.e., the series is geometric with

$u_1 = 24$, $r = \frac{1}{2}$, $n = 12$

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

i.e., $S_{12} = \dfrac{24\left(1 - (\frac{1}{2})^{12}\right)}{1 - \frac{1}{2}}$

$= 48\left(1 - (\tfrac{1}{2})^{12}\right)$

i.e., $S_{12} \doteqdot 47.99$

7 **a** $\displaystyle\sum_{r=1}^{8} \left(\frac{31-3r}{2}\right) = 14 + 12\tfrac{1}{2} + 11 + 9\tfrac{1}{2} + 8 + 6\tfrac{1}{2} + 5 + 3\tfrac{1}{2}$

This series is arithmetic with $u_1 = 14$, $d = -1\frac{1}{2}$ and $n = 8$.

$\therefore$ the sum is $\frac{8}{2}\left[28 + 7(-1\tfrac{1}{2})\right] = 70$

b $\displaystyle\sum_{r=1}^{15} 50(0.8)^{r-1} \doteqdot 50 + 40 + 32 + \ldots\ldots + 3.436 + 2.749 + 2.199$

This series is geometric with $u_1 = 50$, $r = 0.8$ and $n = 15$.

$\therefore$ the sum is $\dfrac{50\left[1 - (0.8)^{15}\right]}{1 - 0.8} \doteqdot 241.2$

8 5, 10, 20, 40, ... i.e., the sequence is geometric with $u_1 = 5$, $r = 2$

$u_n = u_1 r^{n-1} = 5 \times 2^{n-1}$

Let $u_n = 10\,000 = 5 \times 2^{n-1}$

$\therefore\ 2000 = 2^{n-1}$

$\therefore\ n \doteqdot 11.97$ {using technology}

So, try the two values on either side of $n \doteqdot 11.97$, i.e., for $n = 11$ and $n = 12$:

$$u_{11} = 5 \times 2^{10} = 5120 \qquad \text{and} \qquad u_{12} = 5 \times 2^{11} = 10\,240$$

i.e., the first term to exceed 10 000 is $u_{12} = 10\,240$.

9 **a** $u_6 = u_1 \times r^5$ is the amount after 5 years, where $r = 1.07$

$= 6000 \times (1.07)^5$

i.e., $u_6 = 8415.31$, i.e., the value of the investment will be 8415.31 Euro

b If interest is compounded quarterly, then $r = 1 + \dfrac{0.07}{4} = 1.0175$

and $n = 5 \times 4 = 20$

$u_{21} = u_1 \times r^{20}$

$= 6000 \times (1.0175)^{20}$

i.e., $u_{21} = 8488.67$, i.e., the value of the investment will be 8488.67 Euro

c If interest is compounded monthly, then $r = 1 + \dfrac{0.07}{12} = 1.005\,8\bar{3}$

and $n = 5 \times 12 = 60$

$u_{61} = u_1 \times r^{60}$

$= 6000 \times (1.005\,8\bar{3})^{60}$

i.e., $u_{61} = 8505.75$, i.e., the value of the investment will be 8505.75 Euro

REVIEW SET 2C

1 $u_6 = 24 \quad \therefore\ u_1 \times r^5 = 24$ ...... (1)

$u_{11} = 768 \quad \therefore\ u_1 \times r^{10} = 768$ ...... (2)

So $\dfrac{u_1 r^{10}}{u_1 r^5} = \dfrac{768}{24}$ {(2) ÷ (1)}

$\therefore\ r^5 = 32$

$\therefore\ r = 2$

Substituting $r = 2$ into (1) gives $u_1 \times 2^5 = 24 \quad \therefore\ u_1 = \dfrac{24}{32} = \dfrac{3}{4}$

$u_n = u_1 r^{n-1} = \left(\frac{3}{4}\right) 2^{n-1}$

a $u_{17} = \left(\frac{3}{4}\right) 2^{17-1}$

$= 49\,152$

b $S_n = \dfrac{u_1(r^n - 1)}{r - 1}$

$= \dfrac{\frac{3}{4}(2^n - 1)}{2 - 1}$

$= \frac{3}{4}(2^n - 1)$

$\therefore\ S_{15} = \frac{3}{4}(2^{15} - 1)$

$= 24\,575.25$

2 $11+16+21+26+\ldots\ldots$ i.e., the series is arithmetic with $u_1=11,\quad d=5$

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1+(n-1)d) = \frac{n}{2}(2\times 11+5(n-1)) \\ &= \frac{n}{2}(22+5n-5) \\ &= \frac{n}{2}(5n+17) \end{aligned}$$

Let $S_n=450$ and we need to find n,

i.e., $S_n=\frac{n}{2}(5n+17) = 450$

$\therefore\ \frac{5}{2}n^2+\frac{17}{2}n-450 = 0$

$\therefore\ 5n^2+17n-900 = 0$

$\therefore\ n \doteqdot -15.2,\ 11.8$ {using technology}

but $n>0$, $\therefore\ n \doteqdot 11.8$

So, try the two values on either side of $n \doteqdot 11.8$, i.e., for $n=11$ and $n=12$:

$S_{11} = \frac{11}{2}(5(11)+17) = 396$ and $S_{12} = \frac{12}{2}(5(12)+17) = 462$

i.e., 12 terms of the series are required to exceed a sum of 450.

3 $24,\ 8,\ \frac{8}{3},\ \frac{8}{9},\ \ldots\ldots$ i.e., the sequence is geometric with $u_1=24,\quad r=\frac{1}{3}$

$$\begin{aligned} u_n &= u_1r^{n-1} \\ &= 24\left(\tfrac{1}{3}\right)^{n-1} \end{aligned}$$

Let $u_n=0.001$ and we need to find n.

i.e., $u_n=24\left(\frac{1}{3}\right)^{n-1} = 0.001$

$\therefore\ \left(\frac{1}{3}\right)^{n-1} = \dfrac{0.001}{24}$

$\therefore\ n \doteqdot 10.18$ {using technology}

So, try the two values on either side of $n \doteqdot 10.18$, i.e., for $n=10$ and $n=11$:

$u_{10} = 24\left(\frac{1}{3}\right)^{9} = 0.001\,22$ and $u_{11} = 24\left(\frac{1}{3}\right)^{10} = 0.000\,406$

i.e., $u_{11} \doteqdot 0.000\,406$ is the first term of the sequence which is less than 0.001.

4 **a** $128,\ 64,\ 32,\ 16,\ \ldots\ldots,\ \frac{1}{512}$

i.e., the sequence is geometric with

$u_1=128,\quad r=\frac{1}{2},\quad u_n=\frac{1}{512}$

$$\begin{aligned} u_n &= u_1r^{n-1} \\ &= 128\left(\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

i.e., $\frac{1}{512} = 128\left(\frac{1}{2}\right)^{n-1}$

$\therefore\ \left(\frac{1}{2}\right)^{n-1} = \dfrac{1}{65\,536}$

$\therefore\ n = 17$ {using technology}

i.e., there are 17 terms in the sequence.

b $S_n = \dfrac{u_1(1-r^n)}{1-r}$

i.e., $S_{17} = \dfrac{128\left(1-\left(\frac{1}{2}\right)^{17}\right)}{1-\frac{1}{2}}$

$= 255.998$

i.e., $S_{17} \doteqdot 256$

5 **a** $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \dfrac{0.0825}{2} = 1.041\,25$, $n = 5 \times 2 = 10$

i.e., $u_{n+1} = 12\,500 \times (1.041\,25)^{10}$

$= 18\,726.65$, i.e., the value of the investment is $\$18\,726.65$

b $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \dfrac{0.0825}{12} = 1.006\,875$, $n = 5 \times 12 = 60$

i.e., $u_{n+1} = 12\,500 \times (1.006\,875)^{60}$

$= 18\,855.74$, i.e., the value of the investment is $\$18\,855.74$

6 $u_{n+1} = u_1 \times r^n$ where $u_{n+1} = 20\,000$, $r = 1 + \dfrac{0.09}{12} = 1.0075$, $n = 4 \times 12 = 48$

i.e., $20\,000 = u_1 \times (1.0075)^{48}$

i.e., $u_1 = 13\,972.28$, i.e., $\$13\,972.28$ should be invested

7 **a** $u_{n+1} = u_1 \times r^n$, where $u_1 = 3000$, $r = 1.05$, $n = 3$

i.e., $u_{n+1} = 3000 \times (1.05)^3$

$= 3472.875$, i.e., approximately 3470 koalas

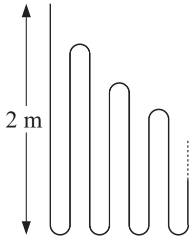
b $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $u_{n+1} = 5000$, $r = 1.05$

i.e., $5000 = 3000 \times (1.05)^n$

i.e., $n \doteqdot 10.47$

i.e., after 10.47 years the population will exceed 5000, i.e., in the year 2008.

8



Total distance travelled

$= 2 + 2 \times 0.8 \times 2 + 2 \times (0.8)^2 \times 2 + 2 \times (0.8)^3 \times 2 + ...$

$= 2 + 4 \times 0.8 \times \left[1 + 0.8 + (0.8)^2 + (0.8)^3 + ...\right]$

$= 2 + 3.2 \times \left[\dfrac{1}{1-0.8}\right]$ as $r = 0.8$, $|r| < 1$ $\therefore$ converges to $\dfrac{u_1}{1-r}$

$= 2 + \dfrac{3.2}{0.2}$

$= 2 + 16$

$= 18$ metres

9 **a** $\displaystyle\sum_{r=1}^{\infty} 50\,(2x-1)^{r-1}$ is a geometric series with $r = 2x - 1$ and converges if $-1 < r < 1$

$\therefore$ $-1 < 2x - 1 < 1$

$\therefore$ $0 < 2x < 2$

$\therefore$ $0 < x < 1$

b When $x = 0.3$, $2x - 1 = 0.6 - 1 = -0.4$

and $\displaystyle\sum_{r=1}^{\infty} 50\,(2x-1)^{r-1} = 50(-0.4)^0 + 50(-0.4)^1 + 50(-0.4)^2 + ...$ which is geometric

with $u_1 = 50$, $r = -0.4$

Now as $0 < 0.3 < 1$, the series converges and $S_\infty = \dfrac{u_1}{1-r} = \dfrac{50}{1+0.4} = \dfrac{50}{\frac{7}{5}} = 35\frac{5}{7}$

Chapter 3

EXPONENTS

EXERCISE 3A

1 **a** $2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,$ $2^5 = 32,\ 2^6 = 64$

b $3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81$

c $5^1 = 5,\ 5^2 = 25,\ 5^3 = 125,\ 5^4 = 625$

d $7^1 = 7,\ 7^2 = 49,\ 7^3 = 343$

EXERCISE 3B

1 **a**
$$\begin{aligned} &(-1)^3 \\ &= (-1) \times (-1) \times (-1) \\ &= 1 \times (-1) \\ &= -1 \end{aligned}$$

b
$$\begin{aligned} &(-1)^4 \\ &= (-1)^3 \times (-1) \\ &= (-1) \times (-1) \\ &= 1 \end{aligned}$$

c
$$\begin{aligned} &(-1)^{12} \\ &= 1 \end{aligned}$$

d
$$\begin{aligned} &(-1)^{17} \\ &= -1 \end{aligned}$$

e
$$\begin{aligned} &(-1)^6 \\ &= 1 \end{aligned}$$

f
$$\begin{aligned} &-1^6 \\ &= -(1^6) \\ &= -1 \end{aligned}$$

g
$$\begin{aligned} &-(-1)^6 \\ &= -(1) \\ &= -1 \end{aligned}$$

h
$$\begin{aligned} &(-2)^3 \\ &= (-2) \times (-2) \times (-2) \\ &= 4 \times (-2) \\ &= -8 \end{aligned}$$

i
$$\begin{aligned} &-2^3 \\ &= -(2^3) \\ &= -8 \end{aligned}$$

j
$$\begin{aligned} &-(-2)^3 \\ &= -(-8) \\ &= 8 \end{aligned}$$

k
$$\begin{aligned} &-(-5)^2 \\ &= -(25) \\ &= -25 \end{aligned}$$

l
$$\begin{aligned} &-(-5)^3 \\ &= -(-125) \\ &= 125 \end{aligned}$$

2 **a** 512 **b** -3125 **c** -243 **d** 16 807 **e** 512 **f** 6561 **g** -6561
h 5.117 264 691 **i** $-0.764\,479\,956$ **j** $-20.361\,584\,96$

3 **a** $0.\overline{142\,857}$ **b** $0.\overline{142\,857}$ **c** $0.\overline{1}$ **d** $0.\overline{1}$ **e** 0.015 625 **f** 0.015 625
g 1 **h** 1

We notice that $7^{-1} = \dfrac{1}{7^1},\quad 3^{-2} = \dfrac{1}{3^2},\quad 4^{-3} = \dfrac{1}{4^3}$ and $a^0 = 1$ for $a > 0$

4 $3^{33} = \underbrace{3^4 \times 3^4 \times 3^4 \times .... \times 3^4}_{\text{8 of these}} \times 3^1$

But $3^4 = 81$ i.e., ends in a 1

$\therefore\ \underbrace{3^4 \times 3^4 \times 3^4 \times .... \times 3^4}_{\text{8 of these}}$ ends in a 1

$\therefore\ 3^{33}$ ends in a 3

5 $7^1 = 7,\quad 7^2 = 49,\quad 7^3 = 343,\quad 7^4 = 2401,\quad 7^5 = 16\,807$

Now $7^{77} = \underbrace{7^4 \times 7^4 \times 7^4 \times .... \times 7^4}_{\substack{\text{19 of these} \\ \text{So, ends in a 1.}}} \times 7^1$

$\therefore\ 7^{77}$ ends in $1 \times 7 = 7.$

EXERCISE 3C

1 a $$\begin{aligned}&7^3 \times 7^2\\ &= 7^{3+2}\\ &= 7^5\end{aligned}$$

b $$\begin{aligned}&5^4 \times 5^3\\ &= 5^{4+3}\\ &= 5^7\end{aligned}$$

c $$\begin{aligned}&a^7 \times a^2\\ &= a^{7+2}\\ &= a^9\end{aligned}$$

d $$\begin{aligned}&a^4 \times a\\ &= a^4 \times a^1\\ &= a^{4+1}\\ &= a^5\end{aligned}$$

e $$\begin{aligned}&b^8 \times b^5\\ &= b^{8+5}\\ &= b^{13}\end{aligned}$$

f $$\begin{aligned}&a^3 \times a^n\\ &= a^{3+n}\end{aligned}$$

g $$\begin{aligned}&b^7 \times b^m\\ &= b^{7+m}\end{aligned}$$

h $$\begin{aligned}&m^4 \times m^2 \times m^3\\ &= m^{4+2+3}\\ &= m^9\end{aligned}$$

2 a $$\begin{aligned}&\frac{5^9}{5^2}\\ &= 5^{9-2}\\ &= 5^7\end{aligned}$$

b $$\begin{aligned}&\frac{11^{13}}{11^9}\\ &= 11^{13-9}\\ &= 11^4\end{aligned}$$

c $$\begin{aligned}&7^7 \div 7^4\\ &= 7^{7-4}\\ &= 7^3\end{aligned}$$

d $$\begin{aligned}&\frac{a^6}{a^2}\\ &= a^{6-2}\\ &= a^4\end{aligned}$$

e $$\begin{aligned}&\frac{b^{10}}{b^7}\\ &= b^{10-7}\\ &= b^3\end{aligned}$$

f $$\begin{aligned}&\frac{p^5}{p^m}\\ &= p^{5-m}\end{aligned}$$

g $$\begin{aligned}&\frac{y^a}{y^5}\\ &= y^{a-5}\end{aligned}$$

h $$\begin{aligned}&b^{2x} \div b\\ &= b^{2x} \div b^1\\ &= b^{2x-1}\end{aligned}$$

3 a $$\begin{aligned}&(3^2)^4\\ &= 3^{2\times 4}\\ &= 3^8\end{aligned}$$

b $$\begin{aligned}&(5^3)^5\\ &= 5^{3\times 5}\\ &= 5^{15}\end{aligned}$$

c $$\begin{aligned}&(2^4)^7\\ &= 2^{4\times 7}\\ &= 2^{28}\end{aligned}$$

d $$\begin{aligned}&(a^5)^2\\ &= a^{5\times 2}\\ &= a^{10}\end{aligned}$$

e $$\begin{aligned}&(p^4)^5\\ &= p^{4\times 5}\\ &= p^{20}\end{aligned}$$

f $$\begin{aligned}&(b^5)^n\\ &= b^{5\times n}\\ &= b^{5n}\end{aligned}$$

g $$\begin{aligned}&(x^y)^3\\ &= x^{y\times 3}\\ &= x^{3y}\end{aligned}$$

h $$\begin{aligned}&(a^{2x})^5\\ &= a^{2x\times 5}\\ &= a^{10x}\end{aligned}$$

4 a $$\begin{aligned}&8\\ &= 2^3\end{aligned}$$

b $$\begin{aligned}&25\\ &= 5^2\end{aligned}$$

c $$\begin{aligned}&27\\ &= 3^3\end{aligned}$$

d $$\begin{aligned}&4^3\\ &= (2^2)^3\\ &= 2^6\end{aligned}$$

e $$\begin{aligned}&9^2\\ &= (3^2)^2\\ &= 3^4\end{aligned}$$

f $$\begin{aligned}&3^a \times 9\\ &= 3^a \times 3^2\\ &= 3^{a+2}\end{aligned}$$

g $$\begin{aligned}&5^t \div 5\\ &= 5^t \div 5^1\\ &= 5^{t-1}\end{aligned}$$

h $$\begin{aligned}&3^n \times 9^n\\ &= 3^n \times (3^2)^n\\ &= 3^{n+2n}\\ &= 3^{3n}\end{aligned}$$

i $$\begin{aligned}&\frac{16}{2^x}\\ &= \frac{2^4}{2^x}\\ &= 2^{4-x}\end{aligned}$$

j $$\begin{aligned}&\frac{3^{x+1}}{3^{x-1}}\\ &= 3^{(x+1)-(x-1)}\\ &= 3^{x+1-x+1}\\ &= 3^2\end{aligned}$$

k $$\begin{aligned}&(5^4)^{x-1}\\ &= 5^{4(x-1)}\\ &= 5^{4x-4}\end{aligned}$$

l $$\begin{aligned}&2^x \times 2^{2-x}\\ &= 2^{x+2-x}\\ &= 2^2\end{aligned}$$

m $$\begin{aligned}&\frac{2^y}{4^x}\\ &= \frac{2^y}{(2^2)^x}\\ &= \frac{2^y}{2^{2x}}\\ &= 2^{y-2x}\end{aligned}$$

n $$\begin{aligned}&\frac{4^y}{8^x}\\ &= \frac{(2^2)^y}{(2^3)^x}\\ &= \frac{2^{2y}}{2^{3x}}\\ &= 2^{2y-3x}\end{aligned}$$

o $$\begin{aligned}&\frac{3^{x+1}}{3^{1-x}}\\ &= 3^{(x+1)-(1-x)}\\ &= 3^{x+1-1+x}\\ &= 3^{2x}\end{aligned}$$

p $$\begin{aligned}&\frac{2^t \times 4^t}{8^{t-1}}\\ &= \frac{2^t \times (2^2)^t}{(2^3)^{t-1}}\\ &= \frac{2^t \times 2^{2t}}{2^{3t-3}}\\ &= 2^{3t-(3t-3)}\\ &= 2^3\end{aligned}$$

5

a $(ab)^3 = a^3b^3$

b $(ac)^4 = a^4c^4$

c $(bc)^5 = b^5c^5$

d $(abc)^3 = a^3b^3c^3$

e $(2a)^4 = 2^4a^4 = 16a^4$

f $(5b)^2 = 5^2b^2 = 25b^2$

g $(3n)^4 = 3^4n^4 = 81n^4$

h $(2bc)^3 = 2^3b^3c^3 = 8b^3c^3$

i $(4ab)^3 = 4^3a^3b^3 = 64a^3b^3$

j $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$

k $\left(\dfrac{m}{n}\right)^4 = \dfrac{m^4}{n^4}$

l $\left(\dfrac{2c}{d}\right)^5 = \dfrac{2^5c^5}{d^5}$ or $\dfrac{32c^5}{d^5}$

6

a $(2b^4)^3 = 2^3b^{12} = 8b^{12}$

b $\left(\dfrac{3}{x^2y}\right)^2 = \dfrac{3^2}{x^4y^2} = \dfrac{9}{x^4y^2}$

c $(5a^4b)^2 = 5^2a^8b^2 = 25a^8b^2$

d $\left(\dfrac{m^3}{2n^2}\right)^4 = \dfrac{m^{12}}{2^4n^8} = \dfrac{m^{12}}{16n^8}$

e $\left(\dfrac{3a^3}{b^5}\right)^3 = \dfrac{3^3a^9}{b^{15}} = \dfrac{27a^9}{b^{15}}$

f $(2m^3n^2)^5 = 2^5m^{15}n^{10} = 32m^{15}n^{10}$

g $\left(\dfrac{4a^4}{b^2}\right)^2 = \dfrac{4^2a^8}{b^4} = \dfrac{16a^8}{b^4}$

h $(5x^2y^3)^3 = 5^3x^6y^9 = 125x^6y^9$

i $(-2a)^2 = (-2)^2a^2 = 4a^2$

j $(-6b^2)^2 = (-6)^2b^4 = 36b^4$

k $(-2a)^3 = (-2)^3a^3 = -8a^3$

l $(-3m^2n^2)^3 = (-3)^3m^6n^6 = -27m^6n^6$

m $(-2ab^4)^4 = (-2)^4a^4b^{16} = 16a^4b^{16}$

n $\left(\dfrac{-2a^2}{b^2}\right)^3 = \dfrac{(-2)^3a^6}{b^6} = -\dfrac{8a^6}{b^6}$

o $\left(\dfrac{-4a^3}{b}\right)^2 = \dfrac{(-4)^2a^6}{b^2} = \dfrac{16a^6}{b^2}$

p $\left(\dfrac{-3p^2}{q^3}\right)^2 = \dfrac{(-3)^2p^4}{q^6} = \dfrac{9p^4}{q^6}$

7

a $\dfrac{a^3}{a} = \dfrac{a^3}{a^1} = a^{3-1}$ or a^2

b $4b^2 \times 2b^3 = 8b^{2+3} = 8b^5$

c $\dfrac{m^5n^4}{m^2n^3} = m^{5-2}n^{4-3} = m^3n^1 = m^3n$

d $\dfrac{14a^7}{2a^2} = \frac{14}{2}a^{7-2} = 7a^5$

e $\dfrac{12a^2b^3}{3ab} = \frac{12}{3}a^{2-1}b^{3-1} = 4ab^2$

f $\dfrac{18m^7a^3}{4m^4a^3} = \frac{18}{4}m^{7-4}a^{3-3} = \frac{9}{2}m^3a^0 = \frac{9}{2}m^3$ or $\dfrac{9m^3}{2}$

g $10hk^3 \times 4h^4 = 40 \times h^{1+4}k^3 = 40h^5k^3$

h $\dfrac{m^{11}}{(m^2)^8} = \dfrac{m^{11}}{m^{16}} = m^{11-16} = m^{-5}$ or $\dfrac{1}{m^5}$

i $\dfrac{p^2 \times p^7}{(p^3)^2} = \dfrac{p^{2+7}}{p^6} = \dfrac{p^9}{p^6} = p^{9-6} = p^3$

8

a $5^0 = 1$

b $3^{-1} = \tfrac{1}{3}$

c $6^{-1} = \tfrac{1}{6}$

d $8^0 = 1$

e $2^2 = 4$

f $2^{-2} = \dfrac{1}{2^2} = \tfrac{1}{4}$

g $2^3 = 8$

h $2^{-3} = \dfrac{1}{2^3} = \tfrac{1}{8}$

i $5^2 = 25$

j $5^{-2} = \dfrac{1}{5^2} = \tfrac{1}{25}$

k $10^2 = 100$

l $10^{-2} = \dfrac{1}{10^2} = \tfrac{1}{100}$

9

a $(\tfrac{2}{3})^0 = 1$

b $\dfrac{4^3}{4^3} = 1$

c
$3y^0$
$= 3 \times 1$
$= 3$

d
$(3y)^0$
$= 1$

e
2×3^0
$= 2 \times 1$
$= 2$

f $6^0 = 1$

g
$\dfrac{5^2}{5^4}$
$= 5^{2-4}$
$= 5^{-2}$
$= \dfrac{1}{5^2}$ or $\tfrac{1}{25}$

h
$\dfrac{2^{10}}{2^{15}}$
$= 2^{10-15}$
$= 2^{-5}$
$= \dfrac{1}{2^5}$ or $\tfrac{1}{32}$

i
$(\tfrac{1}{3})^{-1}$
$= (\tfrac{3}{1})^1$
$= 3$

j
$(\tfrac{2}{5})^{-1}$
$= (\tfrac{5}{2})^1$
$= 2\tfrac{1}{2}$

k
$(\tfrac{4}{3})^{-1}$
$= (\tfrac{3}{4})^1$
$= \tfrac{3}{4}$

l
$(\tfrac{1}{12})^{-1}$
$= (\tfrac{12}{1})^1$
$= 12$

m
$(\tfrac{2}{3})^{-2}$
$= (\tfrac{3}{2})^2$
$= \tfrac{9}{4}$ or $2\tfrac{1}{4}$

n
$5^0 - 5^{-1}$
$= 1 - \tfrac{1}{5}$
$= \tfrac{4}{5}$

o
$7^{-1} + 7^0$
$= \tfrac{1}{7} + 1$
$= 1\tfrac{1}{7}$ or $\tfrac{8}{7}$

p
$2^0 + 2^1 + 2^{-1}$
$= 1 + 2 + \tfrac{1}{2}$
$= 3\tfrac{1}{2}$ or $\tfrac{7}{2}$

10

a
$(2a)^{-1}$
$= \dfrac{1}{2a}$

b
$2a^{-1}$
$= \dfrac{2}{1} \times \dfrac{1}{a}$
$= \dfrac{2}{a}$

c
$3b^{-1}$
$= \dfrac{3}{1} \times \dfrac{1}{b}$
$= \dfrac{3}{b}$

d
$(3b)^{-1}$
$= \dfrac{1}{3b}$

e
$\left(\dfrac{2}{b}\right)^{-2}$
$= \left(\dfrac{b}{2}\right)^2$
$= \dfrac{b^2}{4}$

f
$(2b)^{-2}$
$= \dfrac{1}{(2b)^2}$
$= \dfrac{1}{4b^2}$

g
$(3n)^{-2}$
$= \dfrac{1}{(3n)^2}$
$= \dfrac{1}{9n^2}$

h
$(3n^{-2})^{-1}$
$= 3^{-1}n^2$
$= \tfrac{1}{3}n^2$
$= \dfrac{n^2}{3}$

i
ab^{-1}
$= \dfrac{a}{1} \times \dfrac{1}{b}$
$= \dfrac{a}{b}$

j
$(ab)^{-1}$
$= \dfrac{1}{ab}$

k
ab^{-2}
$= a \times \dfrac{1}{b^2}$
$= \dfrac{a}{b^2}$

l
$(ab)^{-2}$
$= \dfrac{1}{(ab)^2}$
$= \dfrac{1}{a^2b^2}$

m
$(2ab)^{-1}$
$= \dfrac{1}{2ab}$

n
$2(ab)^{-1}$
$= \dfrac{2}{1} \times \dfrac{1}{ab}$
$= \dfrac{2}{ab}$

o
$2ab^{-1}$
$= \dfrac{2a}{1} \times \dfrac{1}{b}$
$= \dfrac{2a}{b}$

p
$\dfrac{(ab)^2}{b^{-1}}$
$= \dfrac{a^2b^2}{b^{-1}}$
$= a^2b^{2-(-1)}$
$= a^2b^3$

11 **a** $\frac{1}{3} = 3^{-1}$

b $\frac{1}{2} = 2^{-1}$

c $\frac{1}{5} = 5^{-1}$

d $\frac{1}{4}$
$= \frac{1}{2^2}$
$= 2^{-2}$

e $\frac{1}{27}$
$= \frac{1}{3^3}$
$= 3^{-3}$

f $\frac{1}{25}$
$= \frac{1}{5^2}$
$= 5^{-2}$

g $\frac{1}{8^x}$
$= \frac{1}{(2^3)^x}$
$= \frac{1}{2^{3x}}$
$= 2^{-3x}$

h $\frac{1}{16^y}$
$= \frac{1}{(2^4)^y}$
$= \frac{1}{2^{4y}}$
$= 2^{-4y}$

i $\frac{1}{81^a}$
$= \frac{1}{(3^4)^a}$
$= \frac{1}{3^{4a}}$
$= 3^{-4a}$

j $\frac{9}{3^4}$
$= \frac{3^2}{3^4}$
$= 3^{2-4}$
$= 3^{-2}$

k 25×5^{-4}
$= 5^2 \times 5^{-4}$
$= 5^{2+(-4)}$
$= 5^{-2}$

l $\frac{5^{-1}}{5^2}$
$= 5^{-1-2}$
$= 5^{-3}$

m $2 \div 2^{-3}$
$= 2^1 \div 2^{-3}$
$= 2^{1-(-3)}$
$= 2^4$

n 1
$= 2^0$ or 3^0 or 5^0

o 6^{-3}
$= (2 \times 3)^{-3}$
$= 2^{-3} \times 3^{-3}$

p 4×10^2
$= 2^2 \times (2 \times 5)^2$
$= 2^2 \times 2^2 \times 5^2$
$= 2^4 \times 5^2$

12 1 less day, i.e., 25 days

13 **a** $5^3 = 21 + 23 + 25 + 27 + 29$

b $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$

c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

14 We notice that 175 and 75 have a common factor of 25.

So 2^{175} and 5^{75}
$= (2^7)^{25}$ $= (5^3)^{25}$
$= (128)^{25}$ $= (125)^{25}$ $\therefore$ as $125 < 128$, 5^{75} is smaller.

EXERCISE 3D

1 **a** $\sqrt[5]{2}$
$= 2^{\frac{1}{5}}$

b $\frac{1}{\sqrt[5]{2}}$
$= \frac{1}{2^{\frac{1}{5}}}$
$= 2^{-\frac{1}{5}}$

c $2\sqrt{2}$
$= 2^1 \times 2^{\frac{1}{2}}$
$= 2^{\frac{3}{2}}$

d $4\sqrt{2}$
$= 2^2 \times 2^{\frac{1}{2}}$
$= 2^{\frac{5}{2}}$

e $\frac{1}{\sqrt[3]{2}}$
$= \frac{1}{2^{\frac{1}{3}}}$
$= 2^{-\frac{1}{3}}$

f $2 \times \sqrt[3]{2}$
$= 2^1 \times 2^{\frac{1}{3}}$
$= 2^{\frac{4}{3}}$

g $\frac{4}{\sqrt{2}}$
$= \frac{2^2}{2^{\frac{1}{2}}}$
$= 2^{2-\frac{1}{2}}$
$= 2^{\frac{3}{2}}$

h $(\sqrt{2})^3$
$= \left(2^{\frac{1}{2}}\right)^3$
$= 2^{\frac{1}{2} \times 3}$
$= 2^{\frac{3}{2}}$

i $\frac{1}{\sqrt[3]{16}}$
$= \frac{1}{(2^4)^{\frac{1}{3}}}$
$= \frac{1}{2^{\frac{4}{3}}}$
$= 2^{-\frac{4}{3}}$

j $\frac{1}{\sqrt{8}}$
$= \frac{1}{(2^3)^{\frac{1}{2}}}$
$= \frac{1}{2^{\frac{3}{2}}}$
$= 2^{-\frac{3}{2}}$

2 **a** $\sqrt[3]{3} = 3^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt[3]{3}} = \dfrac{1}{3^{\frac{1}{3}}} = 3^{-\frac{1}{3}}$

c $\sqrt[4]{3} = 3^{\frac{1}{4}}$

d $3\sqrt{3} = 3^1 \times 3^{\frac{1}{2}} = 3^{1\frac{1}{2}} = 3^{\frac{3}{2}}$

e $\dfrac{1}{9\sqrt{3}} = \dfrac{1}{3^2 3^{\frac{1}{2}}} = \dfrac{1}{3^{2\frac{1}{2}}} = 3^{-\frac{5}{2}}$

3 **a** $\sqrt[3]{7} = 7^{\frac{1}{3}}$

b $\sqrt[4]{27} = \left(3^3\right)^{\frac{1}{4}} = 3^{\frac{3}{4}}$

c $\sqrt[5]{16} = \left(2^4\right)^{\frac{1}{5}} = 2^{\frac{4}{5}}$

d $\sqrt[3]{32} = \left(2^5\right)^{\frac{1}{3}} = 2^{\frac{5}{3}}$

e $\sqrt[7]{49} = \left(7^2\right)^{\frac{1}{7}} = 7^{\frac{2}{7}}$

f $\dfrac{1}{\sqrt[3]{7}} = \dfrac{1}{7^{\frac{1}{3}}} = 7^{-\frac{1}{3}}$

g $\dfrac{1}{\sqrt[4]{27}} = \dfrac{1}{(3^3)^{\frac{1}{4}}} = \dfrac{1}{3^{\frac{3}{4}}} = 3^{-\frac{3}{4}}$

h $\dfrac{1}{\sqrt[5]{16}} = \dfrac{1}{(2^4)^{\frac{1}{5}}} = \dfrac{1}{2^{\frac{4}{5}}} = 2^{-\frac{4}{5}}$

i $\dfrac{1}{\sqrt[3]{32}} = \dfrac{1}{(2^5)^{\frac{1}{3}}} = \dfrac{1}{2^{\frac{5}{3}}} = 2^{-\frac{5}{3}}$

j $\dfrac{1}{\sqrt[7]{49}} = \dfrac{1}{(7^2)^{\frac{1}{7}}} = \dfrac{1}{7^{\frac{2}{7}}} = 7^{-\frac{2}{7}}$

4 **a** 2.280 **b** 1.834 **c** 0.794 **d** 0.435

5 **a** 3 **b** 1.682 **c** 1.933 **d** 0.523

6 **a** $4^{\frac{3}{2}} = \left(2^2\right)^{\frac{3}{2}} = 2^3 = 8$

b $8^{\frac{5}{3}} = \left(2^3\right)^{\frac{5}{3}} = 2^5 = 32$

c $16^{\frac{3}{4}} = \left(2^4\right)^{\frac{3}{4}} = 2^3 = 8$

d $25^{\frac{3}{2}} = \left(5^2\right)^{\frac{3}{2}} = 5^3 = 125$

e $32^{\frac{2}{5}} = \left(2^5\right)^{\frac{2}{5}} = 2^2 = 4$

f $4^{-\frac{1}{2}} = \left(2^2\right)^{-\frac{1}{2}} = 2^{-1} = \frac{1}{2}$

g $9^{-\frac{3}{2}} = \left(3^2\right)^{-\frac{3}{2}} = 3^{-3} = \dfrac{1}{3^3} = \frac{1}{27}$

h $8^{-\frac{4}{3}} = \left(2^3\right)^{-\frac{4}{3}} = 2^{-4} = \dfrac{1}{2^4} = \frac{1}{16}$

i $27^{-\frac{4}{3}} = \left(3^3\right)^{-\frac{4}{3}} = 3^{-4} = \dfrac{1}{3^4} = \frac{1}{81}$

j $125^{-\frac{2}{3}} = \left(5^3\right)^{-\frac{2}{3}} = 5^{-2} = \dfrac{1}{5^2} = \frac{1}{25}$

EXERCISE 3E

1 **a** $x^2(x^3 + 2x^2 + 1) = x^5 + 2x^4 + x^2$

b $2^x(2^x + 1) = 2^{2x} + 2^x$

c $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) = x^1 + x^0 = x + 1$

d $e^x(e^x + 2) = e^{2x} + 2e^x$

e $3^x(2 - 3^{-x}) = 2 \times 3^x - 3^0 = 2 \times 3^x - 1$

f $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}) = x^2 + 2x^1 + 3x^0 = x^2 + 2x + 3$

g
$$\begin{aligned}&2^{-x}(2^x+5)\\&=2^0+5\times2^{-x}\\&=1+5\times2^{-x}\end{aligned}$$

h
$$\begin{aligned}&5^{-x}(5^{2x}+5^x)\\&=5^{-x+2x}+5^{-x+x}\\&=5^x+5^0\\&=5^x+1\end{aligned}$$

i
$$\begin{aligned}&x^{-\frac{1}{2}}(x^2+x^1+x^{\frac{1}{2}})\\&=x^{1\frac{1}{2}}+x^{\frac{1}{2}}+x^0\\&=x^{\frac{3}{2}}+x^{\frac{1}{2}}+1\end{aligned}$$

2 **a**
$$\begin{aligned}&(2^x+1)(2^x+3)\\&=2^{2x}+3\times2^x+2^x+3\\&=2^{2x}+4\times2^x+3\\&=4^x+2^2\times2^x+3\\&=4^x+2^{2+x}+3\end{aligned}$$

b
$$\begin{aligned}&(3^x+2)(3^x+5)\\&=3^{2x}+5\times3^x+2\times3^x+10\\&=9^x+7\times3^x+10\end{aligned}$$

c
$$\begin{aligned}&(5^x-2)(5^x-4)\\&=5^{2x}-4\times5^x-2\times5^x+8\\&=25^x-6\times5^x+8\end{aligned}$$

d
$$\begin{aligned}&(2^x+3)^2\\&=(2^x)^2+2\times2^x\times3+3^2\\&=2^{2x}+3\times2^{1+x}+9\\&=4^x+3\times2^{x+1}+9\end{aligned}$$

e
$$\begin{aligned}&(3^x-1)^2\\&=(3^x)^2-2\times3^x+1\\&=3^{2x}-2\times3^x+1\\&=9^x-2\times3^x+1\end{aligned}$$

f
$$\begin{aligned}&(4^x+7)^2\\&=(4^x)^2+2\times4^x\times7+7^2\\&=4^{2x}+7\times2\times2^{2x}+49\\&=2^{4x}+7\times2^{2x+1}+49\end{aligned}$$

g
$$\begin{aligned}&(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)\\&=(x^{\frac{1}{2}})^2-2^2\\&=x-4\end{aligned}$$

h
$$\begin{aligned}&(2^x+3)(2^x-3)\\&=(2^x)^2-3^2\\&=2^{2x}-9\end{aligned}$$

i
$$\begin{aligned}&(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})\\&=x^1-x^{-1}\\&=x-\frac{1}{x}\end{aligned}$$

j
$$\begin{aligned}&\left(x+\frac{2}{x}\right)^2\\&=x^2+2\times x\times\frac{2}{x}+\frac{4}{x^2}\\&=x^2+4+\frac{4}{x^2}\end{aligned}$$

k
$$\begin{aligned}&(e^x-e^{-x})^2\\&=(e^x)^2-2e^xe^{-x}+e^{-2x}\\&=e^{2x}-2+e^{-2x}\end{aligned}$$

l
$$\begin{aligned}&(5-2^{-x})^2\\&=5^2-2\times5\times2^{-x}+(2^{-x})^2\\&=25-5\times2^{1-x}+2^{-2x}\end{aligned}$$

EXERCISE 3F

1 **a**
$$\begin{aligned}2^x&=2\\\therefore\ 2^x&=2^1\\\therefore\ x&=1\end{aligned}$$

b
$$\begin{aligned}2^x&=4\\\therefore\ 2^x&=2^2\\\therefore\ x&=2\end{aligned}$$

c
$$\begin{aligned}3^x&=27\\\therefore\ 3^x&=3^3\\\therefore\ x&=3\end{aligned}$$

d
$$\begin{aligned}2^x&=1\\\therefore\ 2^x&=2^0\\\therefore\ x&=0\end{aligned}$$

e
$$\begin{aligned}2^x&=\tfrac{1}{2}\\\therefore\ 2^x&=2^{-1}\\\therefore\ x&=-1\end{aligned}$$

f
$$\begin{aligned}3^x&=\tfrac{1}{3}\\\therefore\ 3^x&=3^{-1}\\\therefore\ x&=-1\end{aligned}$$

g
$$\begin{aligned}2^x&=\tfrac{1}{8}\\\therefore\ 2^x&=2^{-3}\\\therefore\ x&=-3\end{aligned}$$

h
$$\begin{aligned}2^{x+1}&=8\\\therefore\ 2^{x+1}&=2^3\\\therefore\ x+1&=3\\\therefore\ x&=2\end{aligned}$$

i
$$\begin{aligned}2^{x-2}&=\tfrac{1}{4}\\\therefore\ 2^{x-2}&=2^{-2}\\\therefore\ x-2&=-2\\\therefore\ x&=0\end{aligned}$$

j
$$\begin{aligned}3^{x+1}&=\tfrac{1}{27}\\\therefore\ 3^{x+1}&=3^{-3}\\\therefore\ x+1&=-3\\\therefore\ x&=-4\end{aligned}$$

k
$$\begin{aligned}2^{x+1}&=64\\\therefore\ 2^{x+1}&=2^6\\\therefore\ x+1&=6\\\therefore\ x&=5\end{aligned}$$

l
$$\begin{aligned}2^{1-2x}&=\tfrac{1}{2}\\\therefore\ 2^{1-2x}&=2^{-1}\\\therefore\ 1-2x&=-1\\\therefore\ -2x&=-2\\\therefore\ x&=1\end{aligned}$$

2 **a**
$$\begin{aligned} 4^x &= 32 \\ \therefore\ 2^{2x} &= 2^5 \\ \therefore\ 2x &= 5 \\ \therefore\ x &= \tfrac{5}{2} \end{aligned}$$

b
$$\begin{aligned} 8^x &= \tfrac{1}{4} \\ \therefore\ 2^{3x} &= 2^{-2} \\ \therefore\ 3x &= -2 \\ \therefore\ x &= -\tfrac{2}{3} \end{aligned}$$

c
$$\begin{aligned} 9^x &= \tfrac{1}{3} \\ \therefore\ 3^{2x} &= 3^{-1} \\ \therefore\ 2x &= -1 \\ \therefore\ x &= -\tfrac{1}{2} \end{aligned}$$

d
$$\begin{aligned} 49^x &= \tfrac{1}{7} \\ \therefore\ 7^{2x} &= 7^{-1} \\ \therefore\ 2x &= -1 \\ \therefore\ x &= -\tfrac{1}{2} \end{aligned}$$

e
$$\begin{aligned} 4^x &= \tfrac{1}{8} \\ \therefore\ 2^{2x} &= 2^{-3} \\ \therefore\ 2x &= -3 \\ \therefore\ x &= -\tfrac{3}{2} \end{aligned}$$

f
$$\begin{aligned} 25^x &= \tfrac{1}{5} \\ \therefore\ 5^{2x} &= 5^{-1} \\ \therefore\ 2x &= -1 \\ \therefore\ x &= -\tfrac{1}{2} \end{aligned}$$

g
$$\begin{aligned} 8^{x+2} &= 32 \\ \therefore\ 2^{3(x+2)} &= 2^5 \\ \therefore\ 3x+6 &= 5 \\ \therefore\ 3x &= -1 \\ \therefore\ x &= -\tfrac{1}{3} \end{aligned}$$

h
$$\begin{aligned} 8^{1-x} &= \tfrac{1}{4} \\ \therefore\ 2^{3(1-x)} &= 2^{-2} \\ \therefore\ 3-3x &= -2 \\ \therefore\ -3x &= -5 \\ \therefore\ x &= \tfrac{5}{3} \end{aligned}$$

i
$$\begin{aligned} 4^{2x-1} &= \tfrac{1}{2} \\ \therefore\ 2^{2(2x-1)} &= 2^{-1} \\ \therefore\ 4x-2 &= -1 \\ \therefore\ 4x &= 1 \\ \therefore\ x &= \tfrac{1}{4} \end{aligned}$$

j
$$\begin{aligned} 9^{x-3} &= 3 \\ \therefore\ 3^{2(x-3)} &= 3^1 \\ \therefore\ 2x-6 &= 1 \\ \therefore\ 2x &= 7 \\ \therefore\ x &= \tfrac{7}{2} \end{aligned}$$

k
$$\begin{aligned} \left(\tfrac{1}{2}\right)^{x+1} &= 2 \\ \therefore\ \left(2^{-1}\right)^{x+1} &= 2^1 \\ \therefore\ -x-1 &= 1 \\ \therefore\ -x &= 2 \\ \therefore\ x &= -2 \end{aligned}$$

l
$$\begin{aligned} \left(\tfrac{1}{3}\right)^{x+2} &= 9 \\ \therefore\ \left(3^{-1}\right)^{x+2} &= 3^2 \\ \therefore\ -x-2 &= 2 \\ \therefore\ -x &= 4 \\ \therefore\ x &= -4 \end{aligned}$$

m
$$\begin{aligned} 4^x &= 8^{-x} \\ \therefore\ 2^{2x} &= \left(2^3\right)^{-x} \\ \therefore\ 2x &= -3x \\ \therefore\ 5x &= 0 \\ \therefore\ x &= 0 \end{aligned}$$

n
$$\begin{aligned} \left(\tfrac{1}{4}\right)^{1-x} &= 8 \\ \therefore\ \left(2^{-2}\right)^{1-x} &= 2^3 \\ \therefore\ -2+2x &= 3 \\ \therefore\ 2x &= 5 \\ \therefore\ x &= \tfrac{5}{2} \end{aligned}$$

o
$$\begin{aligned} \left(\tfrac{1}{7}\right)^{x} &= 49 \\ \therefore\ \left(7^{-1}\right)^{x} &= 7^2 \\ \therefore\ -x &= 2 \\ \therefore\ x &= -2 \end{aligned}$$

p
$$\begin{aligned} \left(\tfrac{1}{2}\right)^{x+1} &= 32 \\ \therefore\ \left(2^{-1}\right)^{x+1} &= 2^5 \\ \therefore\ -x-1 &= 5 \\ \therefore\ -x &= 6 \\ \therefore\ x &= -6 \end{aligned}$$

3 **a**
$$\begin{aligned} 4^{2x+1} &= 8^{1-x} \\ \therefore\ \left(2^2\right)^{2x+1} &= \left(2^3\right)^{1-x} \\ \therefore\ 4x+2 &= 3-3x \\ \therefore\ 7x &= 1 \\ \therefore\ x &= \tfrac{1}{7} \end{aligned}$$

b
$$\begin{aligned} 9^{2-x} &= \left(\tfrac{1}{3}\right)^{2x+1} \\ \therefore\ \left(3^2\right)^{2-x} &= \left(3^{-1}\right)^{2x+1} \\ \therefore\ 4-2x &= -2x-1 \\ \therefore\ 4 &= -1 \end{aligned}$$

So, no solutions exist.

c
$$\begin{aligned} 2^x \times 8^{1-x} &= \tfrac{1}{4} \\ \therefore\ 2^x \times \left(2^3\right)^{1-x} &= 2^{-2} \\ \therefore\ x+3-3x &= -2 \\ \therefore\ -2x &= -5 \\ \therefore\ x &= \tfrac{5}{2} \end{aligned}$$

4 **a**
$$\begin{aligned} 3 \times 2^x &= 24 \\ \therefore\ 2^x &= 8 \\ \therefore\ 2^x &= 2^3 \\ \therefore\ x &= 3 \end{aligned}$$

b
$$\begin{aligned} 7 \times 2^x &= 56 \\ \therefore\ 2^x &= 8 \\ \therefore\ 2^x &= 2^3 \\ \therefore\ x &= 3 \end{aligned}$$

c
$$\begin{aligned} 3 \times 2^{x+1} &= 24 \\ \therefore\ 2^{x+1} &= 8 \\ \therefore\ 2^{x+1} &= 2^3 \\ \therefore\ x+1 &= 3 \\ \therefore\ x &= 2 \end{aligned}$$

d $12 \times 3^{-x} = \frac{4}{3}$

$\therefore\ 3^{-x} = \frac{4}{3} \div 12$

$\therefore\ 3^{-x} = \frac{4}{3} \times \frac{1}{12}$

$\therefore\ 3^{-x} = \frac{1}{9}$

$\therefore\ 3^{-x} = 3^{-2}$

$\therefore\ x = 2$

e $4 \times \left(\frac{1}{3}\right)^x = 36$

$\therefore\ \left(\frac{1}{3}\right)^x = 9$

$\therefore\ \left(3^{-1}\right)^x = 3^2$

$\therefore\ 3^{-x} = 3^2$

$\therefore\ -x = 2$

$\therefore\ x = -2$

f $5 \times \left(\frac{1}{2}\right)^x = 20$

$\therefore\ \left(\frac{1}{2}\right)^x = 4$

$\therefore\ \left(2^{-1}\right)^x = 2^2$

$\therefore\ -x = 2$

$\therefore\ x = -2$

EXERCISE 3G

1 **a** When $x = \frac{1}{2}$, $y = 2^{\frac{1}{2}}$

from point A, $y \doteqdot 1.4$

$\therefore\ 2^{\frac{1}{2}} \doteqdot 1.4$

b When $x = 0.8$, $y = 2^{0.8}$

from point B, $y \doteqdot 1.7$

$\therefore\ 2^{0.8} \doteqdot 1.7$

c When $x = 1.5$, $y = 2^{1.5}$

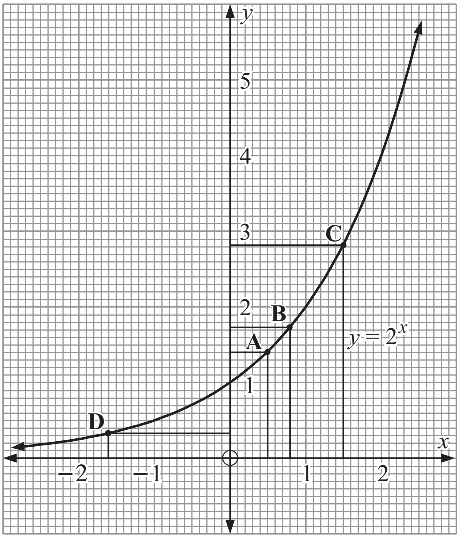
from point C, $y \doteqdot 2.8$

$\therefore\ 2^{1.5} \doteqdot 2.8$

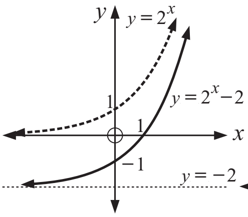
d When $x = -1.6$, $y = 2^{-1.6}$

from point D, $y \doteqdot 0.3$

$\therefore\ 2^{-1.6} \doteqdot 0.3$

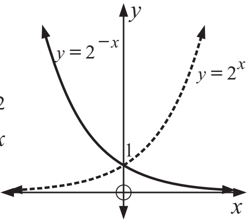


2 **a**



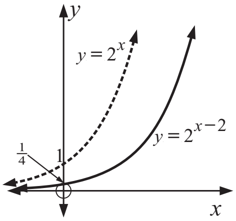
a vertical translation of 2 units downwards
$y = -2$ is a HA

b



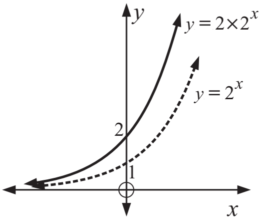
a reflection in the y-axis

c



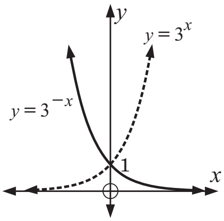
a vertical dilation of factor $\frac{1}{4}$

d



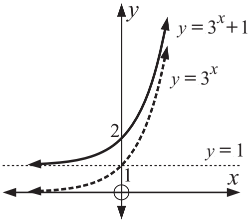
a vertical dilation of factor 2

4 **a**



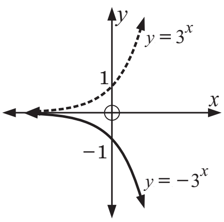
a reflection in the y-axis

b



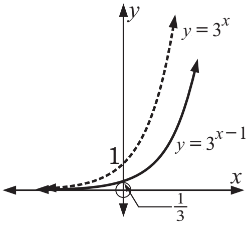
a vertical translation of 1 unit
$y = 1$ is the HA

c



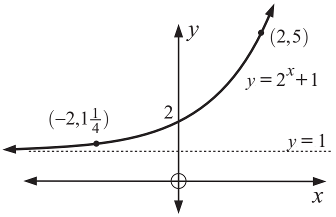
a reflection in the x-axis

d



a vertical dilation of factor $\frac{1}{3}$

5 **a**



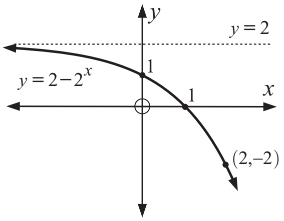
a vertical translation of 1 unit

Horizontal asymptote is $y = 1$.

When $x = 2$, $y = 4 + 1$
$= 5$

When $x = -2$, $y = \frac{1}{4} + 1$
$= 1\frac{1}{4}$

b



The horizontal asymptote is $y = 2$.

When $x = 0$, $y = 2 - 2^0 = 2 - 1 = 1$

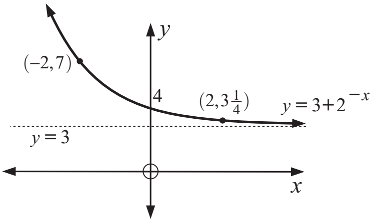
$\therefore$ y-intercept is 1

When $x = 1$, $y = 2 - 2 = 0$

When $x = 2$, $y = 2 - 4 = -2$

When $x = -2$, $y = 2 - \frac{1}{4} = 1\frac{3}{4}$

c



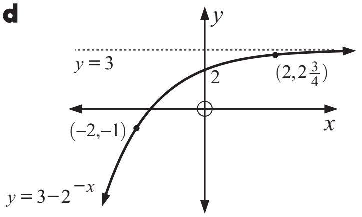
The HA is $y = 3$.

When $x = 0$, $y = 3 + 1 = 4$

When $x = 2$, $y = 3 + \frac{1}{4} = 3\frac{1}{4}$

When $x = -2$, $y = 3 + 2^2 = 7$

d



The HA is $y = 3$.

When $x = 0$, $y = 3 - 1 = 2$

When $x = 2$, $y = 3 - \frac{1}{4} = 2\frac{3}{4}$

When $x = -2$, $y = 3 - 4 = -1$

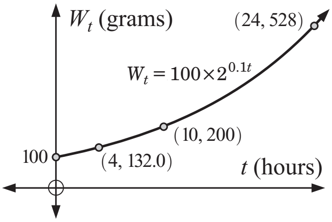
EXERCISE 3H

1 **a** When $t = 0$, $W_0 = 100$ grams

b **i** When $t = 4$,
$W_4 = 100 \times 2^{0.1\times 4}$
$= 100 \times 2^{0.4}$
$\doteqdot 132$ grams

ii When $t = 10$,
$W_{10} = 100 \times 2^1$
$= 200$ grams

iii When $t = 24$,
$W_{24} = 100 \times 2^{0.1\times 24}$
$= 100 \times 2^{2.4}$
$\doteqdot 528$ grams



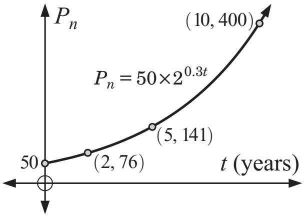
2 **a** $P_0 = 50$ (the initial population)

b **i** When $t = 2$,
$P_2 = 50 \times 2^{0.3\times 2}$
$= 50 \times 2^{0.6}$
$\doteqdot 75.785....$

So, the expected population is 76 possums.

ii When $t = 5$,
$P_5 = 50 \times 2^{0.3\times 5}$
$= 50 \times 2^{1.5}$
$\doteqdot 141.421....$

So, the expected population is 141 possums.



iii When $t = 10$, $P_{10} = 50 \times 2^{0.3\times 10}$
$= 50 \times 2^3$
$= 400$ So, the expected population is 400 possums.

3 **a** When $t = 0$

$$V_0 = V_0 \times 2^0 = V_0$$

So, the speed is V_0.

b When $t = 20$

$$V_{20} = V_0 \times 2^{0.05\times 20} = V_0 \times 2^1 = 2V_0$$

So, the speed is $2V_0$.

c V_0 becomes $2V_0$
a 100% increase.

d
$$\left(\frac{V_{50} - V_{20}}{V_{20}}\right) \times 100\%$$
$$= \left(\frac{V_0 \times 2^{2.5} - V_0 \times 2^1}{V_0 \times 2^1}\right) \times 100\%$$
$$= \left(\frac{2^{2.5} - 2^1}{2^1}\right) \times 100\%$$
$$\doteqdot 183\%$$

This expression is the percentage increase in the speed from a speed at 20^{o}C increased to the speed at 50^{o}C.

$V_{50} - V_{20}$ is the increase in speed.

4 **a** $B_0 = 6$ pairs $= 12$ bears

b At year 2018, $t = 20$

$$\therefore\ B_{20} = 12 \times 2^{0.18\times 20} = 12 \times 2^{3.6} \doteqdot 145.508.... \doteqdot 146 \text{ bears}$$

c At year 2008, $t = 10$

$$\therefore\ \%\text{ increase} = \left(\frac{B_{20} - B_{10}}{B_{10}}\right) \times 100\%$$
$$= \left(\frac{12 \times 2^{3.6} - 12 \times 2^{1.8}}{12 \times 2^{1.8}}\right) \times 100\%$$
$$= \left(\frac{2^{3.6} - 2^{1.8}}{2^{1.8}}\right) \times 100\%$$
$$\doteqdot 248\%$$

EXERCISE 3I

1 $W_t = 250 \times (0.998)^t$ grams

a
$$W_0 = 250 \times (0.998)^0 = 250 \times 1 = 250 \text{ grams}$$

b **i** When $t = 400$

$$W_{400} = 250 \times (0.998)^{400} \doteqdot 112 \text{ grams}$$

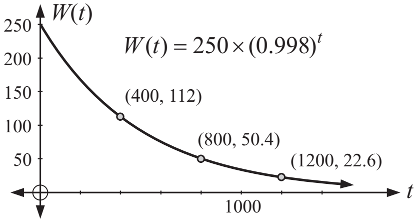
ii When $t = 800$

$$W_{800} = 250 \times (0.998)^{800} \doteqdot 50.4 \text{ grams}$$

iii When $t = 1200$

$$W_{1200} = 250 \times (0.998)^{1200} \doteqdot 22.6 \text{ grams}$$

c



d When $W(t) = 125$

$$250 \times (0.998)^t = 125$$
$$\therefore\ (0.998)^t = 0.5$$
$$\therefore\ t \doteqdot 346.2 \quad \{\text{technology}\}$$

i.e., it takes approximately 346 years

2 $R = 8000 \times (0.837)^t$

1 When $t = 0$, $R = 8000 \times (0.837)^0$
$= 8000 \times 1$
$= 8000$ rabbits

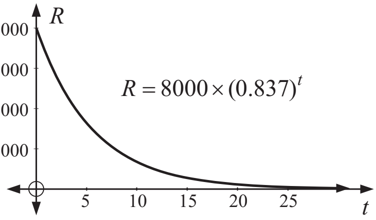
2 'to the power 3.5' means that
$(0.837)^{3.5} = (0.837)^3 \times \sqrt{0.837}$

3 When $R = 80$,
$8000 \times (0.837)^t = 80$
$\therefore \quad (0.837)^t = 0.01$

and by trial-and-error methods, $t \doteq 26$
i.e., it would take 26 weeks (approx)

4 Yes, after a very long time.
Notice that when $R = 1$
$8000(0.837)^t = 1$
$\therefore \quad (0.837)^t = 0.000\,125$
and by trial and error $t \doteq 51$
i.e., after 51 weeks

5



3 $T_t = 100 \times 2^{-0.02t}$

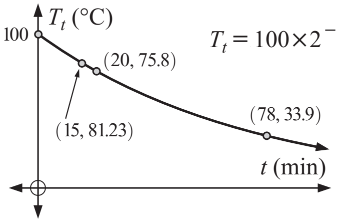
a $T_0 = 100 \times 2^0$
$= 100 \times 1$
$= 100^o\text{C}$

b **i** $T_{15} = 100 \times 2^{-0.02\times15}$
$= 100 \times 2^{-0.3}$
$\doteq 81.2^o\text{C}$

ii $T_{20} = 100 \times 2^{-0.02\times20}$
$= 100 \times 2^{-0.4}$
$\doteq 75.8^o\text{C}$

iii $T_{78} = 100 \times 2^{-0.02\times78}$
$= 100 \times 2^{-1.56}$
$\doteq 33.9^o\text{C}$

c



4 $W_t = 1000 \times 2^{-0.03t}$

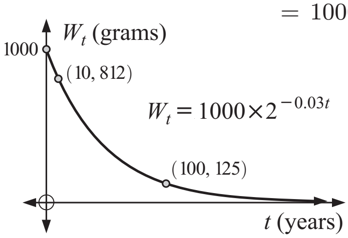
a $W_0 = 1000 \times 2^0$
$= 1000 \times 1$
$= 1000$ grams

b **i** W_{10}
$= 1000 \times 2^{-0.3}$
$\doteq 812$ g

ii W_{100}
$= 1000 \times 2^{-3}$
$= 125$ g

iii W_{1000}
$= 1000 \times 2^{-30}$
$\doteq 9.31 \times 10^{-7}$ g

c



5 **a** When $t = 0$, $W_0 = W_0 2^0$
$= W_0$

$\therefore$ the original weight was W_0

b % loss $= \left(\dfrac{W_{1000} - W_0}{W_0}\right) \times 100\%$
$= \left(\dfrac{W_0 \times 2^{-0.2} - W_0}{W_0}\right) \times 100\%$
$= (2^{-0.2} - 1) \times 100\%$
$\doteq -12.9\%$
i.e., a 12.9% weight loss

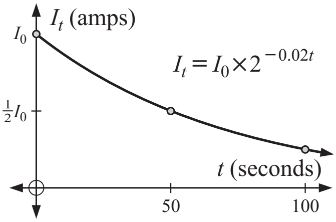
6 **a** When $t = 0$, $I_0 = I_0 \times 2^0$
$= I_0 \times 1$
$= I_0$

$\therefore$ the original current is I_0 amps

b When $t = 1$, $I_1 = I_0 \times 2^{-0.02}$
$\doteq 0.9862 \times I_0$

c $$\begin{aligned}\%\text{ change} &= \left(\frac{I_1 - I_0}{I_0}\right) \times 100\% \\ &\doteqdot \left(\frac{0.986I_0 - I_0}{I_0}\right) \times 100\% \\ &\doteqdot (0.9862 - 1) \times 100\% \\ &\doteqdot -0.0138 \times 100\% \\ &\doteqdot 1.38\%\text{ loss}\end{aligned}$$

d $$\begin{aligned}I_{50} &= I_0 \times 2^{-1} = \tfrac{1}{2}I_0 \\ I_{100} &= I_0 \times 2^{-2} = \tfrac{1}{4}I_0\end{aligned}$$



REVIEW SET 3A

1 **a** $$\begin{aligned}&-(-1)^{10} \\ &= -1 \quad \{(-1)^{10} = 1\}\end{aligned}$$

b $$\begin{aligned}&-(-3)^3 \\ &= -[-27] \\ &= 27\end{aligned}$$

c $$\begin{aligned}&3^0 - 3^{-1} \\ &= 1 - \tfrac{1}{3} \\ &= \tfrac{2}{3}\end{aligned}$$

2 **a** $$\begin{aligned}&a^4b^5 \times a^2b^2 \\ &= a^{4+2} \times b^{5+2} \\ &= a^6b^7\end{aligned}$$

b $$\begin{aligned}&6xy^5 \div 9x^2y^5 \\ &= \tfrac{6}{9}x^{1-2}y^{5-5} \\ &= \tfrac{2}{3}x^{-1}y^0 \\ &= \frac{2}{3x}\end{aligned}$$

c $$\begin{aligned}&\frac{5(x^2y)^2}{(5x^2)^2} \\ &= \frac{5 \times x^4y^2}{25x^4} \\ &= \tfrac{1}{5}x^0y^2 \quad \text{or} \quad \frac{y^2}{5}\end{aligned}$$

3 **a** $$\begin{aligned}&2 \times 2^{-4} \\ &= 2^1 \times 2^{-4} \\ &= 2^{1+(-4)} \\ &= 2^{-3}\end{aligned}$$

b $$\begin{aligned}&16 \div 2^{-3} \\ &= 2^4 \div 2^{-3} \\ &= 2^{4-(-3)} \\ &= 2^7\end{aligned}$$

c $$\begin{aligned}&8^4 \\ &= (2^3)^4 \\ &= 2^{12}\end{aligned}$$

4 **a** $$\begin{aligned}&b^{-3} \\ &= \frac{1}{b^3}\end{aligned}$$

b $$\begin{aligned}&(ab)^{-1} \\ &= \left(\frac{1}{ab}\right) \\ &= \frac{1}{ab}\end{aligned}$$

c $$\begin{aligned}&ab^{-1} \\ &= \frac{a}{1} \times \frac{1}{b} \\ &= \frac{a}{b}\end{aligned}$$

5 **a** $$\begin{aligned}2^{x-3} &= \tfrac{1}{32} \\ \therefore\ 2^{x-3} &= 2^{-5} \\ \therefore\ x - 3 &= -5 \\ \therefore\ x &= -2\end{aligned}$$

b $$\begin{aligned}9^x &= 27^{2-2x} \\ \therefore\ (3^2)^x &= (3^3)^{2-2x} \\ \therefore\ 2x &= 6 - 6x \\ \therefore\ 8x &= 6 \\ \therefore\ x &= \tfrac{6}{8} = \tfrac{3}{4}\end{aligned}$$

6 **a** $8^{\frac{2}{3}} = \left(2^3\right)^{\frac{2}{3}} = 2^2 = 4$ **b** $27^{-\frac{2}{3}} = \left(3^3\right)^{-\frac{2}{3}} = 3^{-2} = \dfrac{1}{3^2} = \tfrac{1}{9}$

7 **a** 2.28 **b** 0.517 **c** 3.16

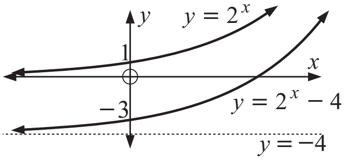
8 $f(x) = 3 \times 2^x$

a $$\begin{aligned}f(0) &= 3 \times 2^0 \\ &= 3 \times 1 \\ &= 3\end{aligned}$$

b $$\begin{aligned}f(3) &= 3 \times 2^3 \\ &= 3 \times 8 \\ &= 24\end{aligned}$$

c $$\begin{aligned}f(-2) &= 3 \times 2^{-2} \\ &= 3 \times \frac{1}{2^2} \\ &= \tfrac{3}{4}\end{aligned}$$

9



a $y = 2^x$ has y-intercept 1 and horizontal asymptote $y = 0$

b $y = 2^x - 4$ has y-intercept -3 and horizontal asymptote $y = -4$

10 $T = 80 \times (0.913)^t \ {}^o\text{C}$

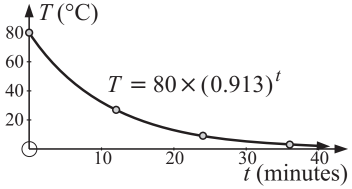
a When $t = 0$, $T = 80 \times (0.913)^0$
$= 80 \times 1$
$= 80 \quad \therefore$ initial temperature is 80^oC

b **i** When $t = 12$,
$T = 80 \times (0.913)^{12}$
$\doteqdot 26.8^o\text{C}$

ii When $t = 24$,
$T = 80 \times (0.913)^{24}$
$\doteqdot 9.00^o\text{C}$

iii When $t = 36$,
$T = 80 \times (0.913)^{36}$
$\doteqdot 3.02^o\text{C}$

c



d When $T = 25$
$80 \times (0.913)^t = 25$
$\therefore \ 0.913^t = 0.3125$
$\therefore \ t \doteqdot 12.8$ sec {technology}

REVIEW SET 3B

1 **a** $-(-2)^3$
$= -[-8]$
$= 8$

b $5^{-1} - 5^0$
$= \frac{1}{5} - 1$
$= -\frac{4}{5}$

2 **a** $(a^7)^3$
$= a^{7\times 3}$
$= a^{21}$

b $pq^2 \times p^3q^4$
$= p^{1+3}q^{2+4}$
$= p^4q^6$

c $\dfrac{8ab^5}{2a^4b^4}$
$= \frac{8}{2}a^{1-4}b^{5-4}$
$= 4a^{-3}b^1$
$= \dfrac{4}{1} \times \dfrac{1}{a^3} \times b$
$= \dfrac{4b}{a^3}$

3 **a** $\frac{1}{16}$
$= \dfrac{1}{2^4}$
$= 2^{-4}$

b $2^x \times 4$
$= 2^x \times 2^2$
$= 2^{x+2}$

c $4^x \div 8$
$= (2^2)^x \div 2^3$
$= 2^{2x} \div 2^3$
$= 2^{2x-3}$

4 **a** $x^{-2} \times x^{-3}$
$= x^{-2+(-3)}$
$= x^{-5}$
$= \dfrac{1}{x^5}$

b $2(ab)^{-2}$
$= 2 \times \dfrac{1}{(ab)^2}$
$= \dfrac{2}{a^2b^2}$

c $2ab^{-2}$
$= 2a \times \left(\dfrac{1}{b^2}\right)$
$= \dfrac{2a}{b^2}$

5 **a** $2^{x+1} = 32$
$\therefore \ 2^{x+1} = 2^5$
$\therefore \ x + 1 = 5$
$\therefore \ x = 4$

b $4^{x+1} = \left(\frac{1}{8}\right)^x$
$\therefore \ (2^2)^{x+1} = (2^{-3})^x$
$\therefore \ 2x + 2 = -3x$
$\therefore \ 5x = -2$
$\therefore \ x = -\frac{2}{5}$

6 **a** $81 = 3^4$ **b** $1 = 3^0$ **c** $\dfrac{1}{27} = \dfrac{1}{3^3} = 3^{-3}$ **d** $\dfrac{1}{243} = \dfrac{1}{3^5} = 3^{-5}$

7 **a** $$\begin{aligned}\frac{27}{9^a} &= \frac{3^3}{(3^2)^a}\\ &= 3^{3-2a}\end{aligned}$$

b $$\begin{aligned}\left(\sqrt{3}\right)^{1-x} \times 9^{1-2x} &= \left(3^{\frac{1}{2}}\right)^{1-x} \times (3^2)^{1-2x}\\ &= 3^{\frac{1}{2}-\frac{1}{2}x+2-4x}\\ &= 3^{\frac{5}{2}-\frac{9}{2}x}\end{aligned}$$

8 **a** When $x = 0$, $y = 3^0 - 5 = 1 - 5 = -4$

When $x = 1$, $y = 3^1 - 5 = 3 - 5 = -2$

When $x = 2$, $y = 3^2 - 5 = 9 - 5 = 4$

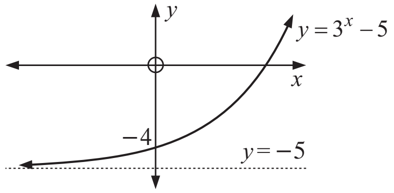
When $x = -1$, $y = 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3}$

When $x = -2$, $y = 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9}$

b as $x \to \infty$,
$3^x \to \infty$, $\therefore$ $y \to \infty$

as $x \to -\infty$,
$3^x \to 0$, $\therefore$ $y \to -5$
(from above)

c



d $y = -5$ is the horizontal asymptote

9 **a** $$\begin{aligned} 27^x &= 3\\ \therefore\ (3^3)^x &= 3^1\\ \therefore\ 3x &= 1\\ \therefore\ x &= \tfrac{1}{3}\end{aligned}$$

b $$\begin{aligned} 9^{1-x} &= 27^{x+2}\\ \therefore\ (3^2)^{1-x} &= (3^3)^{x+2}\\ \therefore\ 2-2x &= 3x+6\\ \therefore\ -5x &= 4 \quad \text{and so } x = -\tfrac{4}{5}\end{aligned}$$

10 $$\begin{aligned} 4^x \times 2^y &= 16\\ \therefore\ (2^2)^x \times 2^y &= 2^4\\ \therefore\ 2x+y &= 4 \quad \text{..... (1)}\end{aligned}$$

$$\begin{aligned} 8^x &= 2^{\frac{y}{2}}\\ \therefore\ (2^3)^x &= 2^{\frac{y}{2}}\\ \therefore\ 3x &= \frac{y}{2} \quad \text{..... (2)}\end{aligned}$$

$$\begin{aligned} \text{From (2),}\quad y &= 6x\\ \therefore\ \text{in (1),}\quad 2x+6x &= 4\\ \therefore\ 8x &= 4\\ \therefore\ x &= \tfrac{1}{2}\\ \text{and so}\quad y &= 6 \times \tfrac{1}{2} = 3\end{aligned}$$

REVIEW SET 3C

1 **a** $$\begin{aligned} &4 \times 2^n\\ &= 2^2 \times 2^n\\ &= 2^{2+n}\end{aligned}$$

b $$\begin{aligned} &7^{-1} - 7^0\\ &= \tfrac{1}{7} - 1\\ &= -\tfrac{6}{7}\end{aligned}$$

c $$\begin{aligned} &\left(\tfrac{2}{3}\right)^{-3}\\ &= \left(\tfrac{3}{2}\right)^3\\ &= \tfrac{27}{8}\\ &= 3\tfrac{3}{8}\end{aligned}$$

d $$\begin{aligned} &\left(\frac{2a^{-1}}{b^2}\right)^2\\ &= \frac{2^2a^{-2}}{b^4}\\ &= \frac{4}{a^2b^4}\end{aligned}$$

2 **a** $$\begin{array}{r|r} 2 & 288\\ \hline 2 & 144\\ \hline 2 & 72\\ \hline 2 & 36\\ \hline 2 & 18\\ \hline 3 & 9\\ \hline & 3 \end{array}$$

$\therefore$ $288 = 2^5 \times 3^2$

b $$\begin{aligned}\frac{2^{x+1}}{2^{1-x}} &= 2^{x+1-(1-x)}\\ &= 2^{x+1-1+x}\\ &= 2^{2x}\end{aligned}$$

3 **a** $1 = 5^0$

b $$\begin{aligned} &5\sqrt{5}\\ &= 5^1 \times 5^{\frac{1}{2}}\\ &= 5^{1\frac{1}{2}}\\ &= 5^{\frac{3}{2}}\end{aligned}$$

c $$\begin{aligned} &\frac{1}{\sqrt[4]{5}}\\ &= \frac{1}{5^{\frac{1}{4}}}\\ &= 5^{-\frac{1}{4}}\end{aligned}$$

d $$\begin{aligned} &25^{a+3}\\ &= (5^2)^{a+3}\\ &= 5^{2a+6}\end{aligned}$$

4 **a**
$$\begin{aligned}&-(-2)^2\\&=-[4]\\&=-4\end{aligned}$$

b
$$\begin{aligned}&\left(-\tfrac{1}{2}a^{-3}\right)^2\\&=\left(-\tfrac{1}{2}\right)^2a^{-6}\\&=\tfrac{1}{4}a^{-6}\\&=\tfrac{1}{4}\times\frac{1}{a^6}\\&=\frac{1}{4a^6}\end{aligned}$$

c
$$\begin{aligned}&(-3b^{-1})^{-3}\\&=(-3)^{-3}b^3\\&=\frac{1}{(-3)^3}\,b^3\\&=\frac{1}{-27}\,b^3\\&=-\frac{b^3}{27}\end{aligned}$$

5 **a**
$$\begin{aligned}&e^x(e^{-x}+e^x)\\&=e^0+e^{2x}\\&=1+e^{2x}\end{aligned}$$

b
$$\begin{aligned}&(2^x+5)^2\\&=(2^x)^2+2\times2^x\times5+5^2\\&=2^{2x}+5\times2^{x+1}+25\\&=4^x+5\times2^{x+1}+25\\&(\text{or}\quad 2^{2x}+10\times2^x+25)\end{aligned}$$

c
$$\begin{aligned}&(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+7)\\&=(x^{\frac{1}{2}})^2-7^2\\&=x^1-49\\&=x-49\end{aligned}$$

6 **a**
$$\begin{aligned}&(3-2^a)^2\\&=3^2-2\times3\times2^a+(2^a)^2\\&=9-3\times2^{a+1}+2^{2a}\\&(\text{or}\quad 9-6\times2^a+2^{2a})\end{aligned}$$

b
$$\begin{aligned}&(\sqrt{x}+2)(\sqrt{x}-2)\\&=(\sqrt{x})^2-2^2\\&=x-4\end{aligned}$$

c
$$\begin{aligned}&2^{-x}(2^{2x}+2^x)\\&=2^{-x+2x}+2^{-x+x}\\&=2^x+2^0\\&=2^x+1\end{aligned}$$

7 **a**
$$\begin{aligned}6\times2^x&=192\\\therefore\ 2^x&=32\\\therefore\ 2^x&=2^5\\\therefore\ x&=5\end{aligned}$$

b
$$\begin{aligned}4\times(\tfrac{1}{3})^x&=324\\\therefore\ (\tfrac{1}{3})^x&=81\\\therefore\ (3^{-1})^x&=3^4\\\therefore\ 3^{-x}&=3^4\\\therefore\ x&=-4\end{aligned}$$

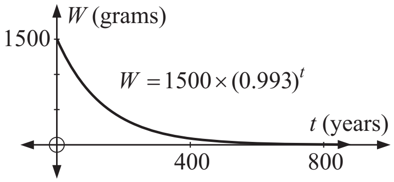
8 $W=1500\times(0.993)^t$ grams

a When $t=0$,
$$\begin{aligned}W&=1500\times(0.993)^0\\&=1500\times1\\&=1500\text{ grams}\end{aligned}$$

b **i** When $t=400$,
$$\begin{aligned}W&=1500\times(0.993)^{400}\\&\doteqdot 90.3\text{ grams}\end{aligned}$$

ii When $t=800$,
$$\begin{aligned}W&=1500\times(0.993)^{800}\\&\doteqdot 5.4\text{ grams}\end{aligned}$$

c



d When $W=100$,
$$\begin{aligned}1500\times(0.993)^t&=100\\\therefore\ (0.993)^t&\doteqdot 0.0667\end{aligned}$$

By trial and error
$$\begin{aligned}(0.993)^{300}&\doteqdot 0.1216\\(0.993)^{350}&\doteqdot 0.085\,56\\(0.993)^{380}&\doteqdot 0.069\,30\\(0.993)^{385}&\doteqdot 0.066\,91\\(0.993)^{386}&\doteqdot 0.066\,44\end{aligned}$$

i.e., about 386 years

Chapter 4

LOGARITHMS

EXERCISE 4A

1 **a** $10^4 = 10\,000$ **b** $10^{-1} = 0.1$ **c** $10^{\frac{1}{2}} = \sqrt{10}$ **d** $2^3 = 8$ **e** $2^{-2} = \frac{1}{4}$
f $3^{1.5} = \sqrt{27}$

2 **a** $\log_2 4 = 2$ **b** $\log_2(\frac{1}{8}) = -3$ **c** $\log_{10}(0.01) = -2$ **d** $\log_7 49 = 2$ **e** $\log_2 64 = 6$
f $\log_3(\frac{1}{27}) = -3$

3 **a** As $10^5 = 100\,000$
then $\log_{10}(100\,000) = 5$

b As $10^{-2} = 0.01$
then $\log_{10}(0.01) = -2$

c As $3^{\frac{1}{2}} = \sqrt{3}$
then $\log_3(\sqrt{3}) = \frac{1}{2}$

d As $2^3 = 8$
then $\log_2 8 = 3$

e As $2^6 = 64$
then $\log_2 64 = 6$

f As $2^7 = 128$
then $\log_2 128 = 7$

g As $5^2 = 25$
then $\log_5 25 = 2$

h As $5^3 = 125$
then $\log_5 125 = 3$

i As $2^{-3} = \frac{1}{8} = 0.125$
then $\log_2(0.125) = -3$

j As $9^{\frac{1}{2}} = 3$
then $\log_9 3 = \frac{1}{2}$

k As $4^2 = 16$
then $\log_4 16 = 2$

l As $36^{\frac{1}{2}} = \sqrt{36} = 6$
then $\log_{36} 6 = \frac{1}{2}$

m As $243 = 3^5$
then $\log_3 243 = 5$

n As $\sqrt[3]{2} = 2^{\frac{1}{3}}$
then $\log_2 \sqrt[3]{2} = \frac{1}{3}$

o As $a^n = a^n$
then $\log_n a^n = n$

p As $2 = 8^{\frac{1}{3}}$
then $\log_8 2 = \frac{1}{3}$

q As $\dfrac{1}{t} = t^{-1}$
then $\log_t \left(\dfrac{1}{t}\right) = -1$

r As $6\sqrt{6} = 6^1 \times 6^{\frac{1}{2}} = 6^{1\frac{1}{2}}$
then $\log_6(6\sqrt{6}) = 1\frac{1}{2}$

s As $1 = 4^0$
then $\log_4 1 = 0$

t As $9 = 9^1$
then $\log_9 9 = 1$

4 **a** $\doteqdot 2.18$ **b** $\doteqdot 1.40$ **c** $\doteqdot 1.87$ **d** $\doteqdot -0.0969$

5 **a**
$$\begin{aligned} \log_2 x &= 3 \\ \therefore\ x &= 2^3 \\ \therefore\ x &= 8 \end{aligned}$$

b
$$\begin{aligned} \log_4 x &= \tfrac{1}{2} \\ \therefore\ x &= 4^{\frac{1}{2}} \\ \therefore\ x &= 2 \end{aligned}$$

c
$$\begin{aligned} \log_x 81 &= 4 \\ \therefore\ 81 &= x^4 \\ \therefore\ x &= \pm\sqrt[4]{81} \\ \therefore\ x &= \pm 3 \\ \text{but}\ x &> 0 \\ \therefore\ x &= 3 \end{aligned}$$

d
$$\begin{aligned} \log_2(x-6) &= 3 \\ \therefore\ x-6 &= 2^3 \\ \therefore\ x-6 &= 8 \\ \therefore\ x &= 14 \end{aligned}$$

6 **a**
$$\begin{aligned} &\log_2 4 \\ &= \log_2 2^2 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} &\log_3(\tfrac{1}{3}) \\ &= \log_3 3^{-1} \\ &= -1 \end{aligned}$$

c
$$\begin{aligned} &\log_5 \left(25\sqrt{5}\right) \\ &= \log_5 \left(5^2 \times 5^{\frac{1}{2}}\right) \\ &= \log_5 5^{\frac{5}{2}} \\ &= \tfrac{5}{2} \end{aligned}$$

d
$$\begin{aligned} &\log_3 \left(\tfrac{1}{\sqrt{3}}\right) \\ &= \log_3 \left(\frac{1}{3^{\frac{1}{2}}}\right) \\ &= \log_3 3^{-\frac{1}{2}} \\ &= -\tfrac{1}{2} \end{aligned}$$

EXERCISE 4B

1 **a** $\log 10\,000 = \log_{10} 10^4 = 4$

b $\log 0.001 = \log_{10} 10^{-3} = -3$

c $\log 10 = \log_{10} 10^1 = 1$

d $\log 1 = \log_{10} 10^0 = 0$

e $\log \sqrt{10} = \log_{10} 10^{\frac{1}{2}} = \frac{1}{2}$

f $\log \sqrt[3]{10} = \log_{10} 10^{\frac{1}{3}} = \frac{1}{3}$

g $\log\left(\dfrac{1}{\sqrt[4]{10}}\right) = \log_{10} 10^{-\frac{1}{4}} = -\frac{1}{4}$

h $\log 10\sqrt{10} = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2}$

i $\log \sqrt[3]{100} = \log_{10}\left(10^2\right)^{\frac{1}{3}} = \log_{10} 10^{\frac{2}{3}} = \frac{2}{3}$

j $\log\left(\frac{100}{\sqrt{10}}\right) = \log_{10}\left(\dfrac{10^2}{10^{\frac{1}{2}}}\right) = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2}$

k $\log\left(10 \times \sqrt[3]{10}\right) = \log_{10}\left(10^1 \times 10^{\frac{1}{3}}\right) = \log_{10} 10^{\frac{4}{3}} = \frac{4}{3}$

l $\log 1000\sqrt{10} = \log_{10}\left(10^3 \times 10^{\frac{1}{2}}\right) = \log_{10} 10^{\frac{7}{2}} = \frac{7}{2}$

m $\log 10^n = \log_{10} 10^n = n$

n $\log\left(10^a \times 100\right) = \log_{10}\left(10^a \times 10^2\right) = \log_{10}\left(10^{a+2}\right) = a+2$

o $\log\left(\dfrac{10}{10^m}\right) = \log_{10}\left(10^{1-m}\right) = 1-m$

p $\log\left(\dfrac{10^a}{10^b}\right) = \log_{10}\left(10^{a-b}\right) = a-b$

2 Use:

a $\boxed{\textbf{log}}$ 10 000 $\boxed{\textbf{ENTER}}$ **b** $\boxed{\textbf{log}}$ 0.001 $\boxed{\textbf{ENTER}}$

c $\boxed{\textbf{log}}$ $\boxed{\textbf{2nd}}$ $\boxed{\sqrt{}}$ 10 $\boxed{)}$ $\boxed{)}$ $\boxed{\textbf{ENTER}}$

d $\boxed{\textbf{log}}$ 10 $\boxed{\wedge}$ $\boxed{(}$ 1 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{)}$ $\boxed{\textbf{ENTER}}$

e $\boxed{\textbf{log}}$ 100 $\boxed{\wedge}$ $\boxed{(}$ 1 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{)}$ $\boxed{\textbf{ENTER}}$

f $\boxed{\textbf{log}}$ 10 $\boxed{\times}$ $\boxed{\textbf{2nd}}$ $\boxed{\sqrt{}}$ 10 $\boxed{)}$ $\boxed{)}$ $\boxed{\textbf{ENTER}}$

g $\boxed{\textbf{log}}$ 1 $\boxed{\div}$ $\boxed{\textbf{2nd}}$ $\boxed{\sqrt{}}$ 10 $\boxed{)}$ $\boxed{)}$ $\boxed{\textbf{ENTER}}$

h $\boxed{\textbf{log}}$ 1 $\boxed{\div}$ 10 $\boxed{\wedge}$ 0.25 $\boxed{)}$ $\boxed{\textbf{ENTER}}$

3 **a** $6 = 10^{\log 6} \div 10^{0.7782}$

b $60 = 10^{\log 60} \div 10^{1.7782}$

c $6000 = 10^{\log 6000} \div 10^{3.7782}$

d $0.6 = 10^{\log(0.6)} = 10^{-0.2218}$

e $0.006 = 10^{\log(0.006)} = 10^{-2.2218}$

f $15 = 10^{\log 15} \div 10^{1.1761}$

g $1500 = 10^{\log 1500} = 10^{3.1761}$

h $1.5 = 10^{\log 1.5} = 10^{0.1761}$

i $0.15 = 10^{\log(0.15)} = 10^{-0.8239}$

j $0.000\,15 = 10^{\log(0.000\,15)} = 10^{-3.8239}$

4 **a** **i** $\log 3 \div 0.477$ **ii** $\log 300 \div 2.477$

b $300 = 3 \times 10^2 = 10^{\log 3} \times 10^2 = 10^{\log 3 + 2}$ $\therefore$ $\log 300 = \log 3 + 2$

5 **a** **i** $\log 5 \div 0.699$ **ii** $\log 0.05 = -1.301$

b $0.05 = 5 \times 10^{-2} = 10^{\log 5} \times 10^{-2} = 10^{\log 5 - 2}$

$\therefore$ $\log(0.05) = \log 5 - 2$

6 **a**
$$\begin{aligned}\log x &= 2\\ \therefore\quad x &= 10^2\\ \therefore\quad x &= 100\end{aligned}$$

b
$$\begin{aligned}\log x &= 1\\ \therefore\quad x &= 10^1\\ \therefore\quad x &= 10\end{aligned}$$

c
$$\begin{aligned}\log x &= 0\\ \therefore\quad x &= 10^o\\ \therefore\quad x &= 1\end{aligned}$$

d
$$\begin{aligned}\log x &= -1\\ \therefore\quad x &= 10^{-1}\\ \therefore\quad x &= \tfrac{1}{10}\end{aligned}$$

e
$$\begin{aligned}\log x &= \tfrac{1}{2}\\ \therefore\quad x &= 10^{\frac{1}{2}}\\ \therefore\quad x &= \sqrt{10}\end{aligned}$$

f
$$\begin{aligned}\log x &= -\tfrac{1}{2}\\ \therefore\quad x &= 10^{-\frac{1}{2}}\\ \therefore\quad x &= \frac{1}{10^{\frac{1}{2}}}\\ \therefore\quad x &= \tfrac{1}{\sqrt{10}}\end{aligned}$$

g
$$\begin{aligned}\log x &\doteq 0.8351\\ \therefore\quad x &\doteq 10^{0.8351}\\ \therefore\quad x &\doteq 6.84\end{aligned}$$

h
$$\begin{aligned}\log x &\doteq -3.1997\\ \therefore\quad x &\doteq 10^{-3.1997}\\ \therefore\quad x &\doteq 0.000\,631\end{aligned}$$

EXERCISE 4C

1 **a**
$$\begin{aligned}&\log 8+\log 2\\ &=\log(8\times 2)\\ &=\log 16\end{aligned}$$

b
$$\begin{aligned}&\log 8-\log 2\\ &=\log(\tfrac{8}{2})\\ &=\log 4\end{aligned}$$

c
$$\begin{aligned}&\log 40-\log 5\\ &=\log(\tfrac{40}{5})\\ &=\log 8\end{aligned}$$

d
$$\begin{aligned}&\log 4+\log 5\\ &=\log(4\times 5)\\ &=\log 20\end{aligned}$$

e
$$\begin{aligned}&\log 5+\log(0.4)\\ &=\log(5\times 0.4)\\ &=\log 2\end{aligned}$$

f
$$\begin{aligned}&\log 2+\log 3+\log 4\\ &=\log(2\times 3\times 4)\\ &=\log 24\end{aligned}$$

g
$$\begin{aligned}&1+\log 3\\ &=\log 10^1+\log 3\\ &=\log(10\times 3)\\ &=\log 30\end{aligned}$$

h
$$\begin{aligned}&\log 4-1\\ &=\log 4-\log 10^1\\ &=\log(\tfrac{4}{10})\\ &=\log(0.4)\end{aligned}$$

i
$$\begin{aligned}&\log 5+\log 4-\log 2\\ &=\log\left(\tfrac{5\times 4}{2}\right)\\ &=\log 10\end{aligned}$$

j
$$\begin{aligned}&2+\log 2\\ &=\log 10^2+\log 2\\ &=\log(100\times 2)\\ &=\log 200\end{aligned}$$

k
$$\begin{aligned}&\log 40-2\\ &=\log 40-\log 10^2\\ &=\log\left(\tfrac{40}{100}\right)\\ &=\log(0.4)\end{aligned}$$

l
$$\begin{aligned}&\log 6-\log 2-\log 3\\ &=\log(6\div 2\div 3)\\ &=\log 1\end{aligned}$$

m
$$\begin{aligned}&\log 50-4\\ &=\log 50-\log 10^4\\ &=\log\left(\tfrac{50}{10^4}\right)\\ &=\log(0.005)\end{aligned}$$

n
$$\begin{aligned}&3-\log 50\\ &=\log 10^3-\log 50\\ &=\log\left(\tfrac{1000}{50}\right)\\ &=\log 20\end{aligned}$$

o
$$\begin{aligned}&\log(\tfrac{4}{3})+\log 3+\log 7\\ &=\log\left(\tfrac{4}{3}\times 3\times 7\right)\\ &=\log 28\end{aligned}$$

2 **a**
$$\begin{aligned}&5\log 2+\log 3\\ &=\log 2^5+\log 3\\ &=\log(2^5\times 3)\\ &=\log 96\end{aligned}$$

b
$$\begin{aligned}&2\log 3+3\log 2\\ &=\log 3^2+\log 2^3\\ &=\log(9\times 8)\\ &=\log 72\end{aligned}$$

c
$$\begin{aligned}&3\log 4-\log 8\\ &=\log 4^3-\log 8\\ &=\log(\tfrac{64}{8})\\ &=\log 8\end{aligned}$$

d
$$\begin{aligned}&2\log 5-3\log 2\\ &=\log 5^2-\log 2^3\\ &=\log\left(\tfrac{25}{8}\right)\end{aligned}$$

e
$$\begin{aligned}&\tfrac{1}{2}\log 4+\log 3\\ &=\log 4^{\frac{1}{2}}+\log 3\\ &=\log(2\times 3)\\ &=\log 6\end{aligned}$$

f
$$\begin{aligned}&\tfrac{1}{3}\log\left(\tfrac{1}{8}\right)\\ &=\log\left(\tfrac{1}{8}\right)^{\frac{1}{3}}\\ &=\log\left(2^{-3}\right)^{\frac{1}{3}}\\ &=\log 2^{-1}\\ &=\log(\tfrac{1}{2})\quad\text{or}\quad -\log 2\end{aligned}$$

g
$$\begin{aligned}&3-\log 2-2\log 5\\ &=\log 10^3-\log 2-\log 5^2\\ &=\log(1000\div 2\div 25)\\ &=\log 20\end{aligned}$$

h
$$\begin{aligned}&1-3\log 2+\log 20\\ &=\log 10^1-\log 2^3+\log 20\\ &=\log(10\div 8\times 20)\\ &=\log 25\end{aligned}$$

i
$$\begin{aligned}&2-\tfrac{1}{2}\log 4-\log 5\\ &=\log 10^2-\log 4^{\frac{1}{2}}-\log 5\\ &=\log(100\div 2\div 5)\\ &=\log 10\\ &=1\end{aligned}$$

3 **a**
$$\begin{aligned}&\frac{\log 4}{\log 2}\\ &=\frac{\log 2^2}{\log 2}\\ &=\frac{2\log 2}{\log 2}\\ &=2\end{aligned}$$

b
$$\begin{aligned}&\frac{\log 27}{\log 9}\\ &=\frac{\log 3^3}{\log 3^2}\\ &=\frac{3\log 3}{2\log 3}\\ &=\tfrac{3}{2}\end{aligned}$$

c
$$\begin{aligned}&\frac{\log 8}{\log 2}\\ &=\frac{\log 2^3}{\log 2}\\ &=\frac{3\log 2}{\log 2}\\ &=3\end{aligned}$$

d
$$\begin{aligned}&\frac{\log 3}{\log 9}\\ &=\frac{\log 3}{\log 3^2}\\ &=\frac{\log 3}{2\log 3}\\ &=\tfrac{1}{2}\end{aligned}$$

e
$$\begin{aligned}&\frac{\log 25}{\log(0.2)}\\ &=\frac{\log 5^2}{\log 5^{-1}}\\ &=\frac{2\log 5}{-1\log 5}\\ &=-2\end{aligned}$$

f
$$\begin{aligned}&\frac{\log 8}{\log(0.25)}\\ &=\frac{\log 2^3}{\log 2^{-2}} \quad \{0.25=\tfrac{1}{4}=\tfrac{1}{2^2}\}\\ &=\frac{3\log 2}{-2\log 2}\\ &=-\tfrac{3}{2}\end{aligned}$$

4 **a**
$$\begin{aligned}\log 9 &= \log 3^2\\ &= 2\log 3\end{aligned}$$

b
$$\begin{aligned}\log\sqrt{2} &= \log 2^{\frac{1}{2}}\\ &= \tfrac{1}{2}\log 2\end{aligned}$$

c
$$\begin{aligned}\log\left(\tfrac{1}{8}\right) &= \log\left(\frac{1}{2^3}\right)\\ &= \log 2^{-3}\\ &= -3\log 2\end{aligned}$$

d
$$\begin{aligned}\log\left(\tfrac{1}{5}\right) &= \log 5^{-1}\\ &= -1\log 5\\ &= -\log 5\end{aligned}$$

e
$$\begin{aligned}\log 5 &= \log\left(\tfrac{10}{2}\right)\\ &= \log 10^1-\log 2\\ &= 1-\log 2\end{aligned}$$

f
$$\begin{aligned}\log 5000 &= \log\left(\tfrac{10\,000}{2}\right)\\ &= \log 10^4-\log 2\\ &= 4-\log 2\end{aligned}$$

5 **a**
$$\begin{aligned}y &= 2^x\\ \therefore\ \log y &= \log 2^x\\ \therefore\ \log y &= x\log 2\end{aligned}$$

b
$$\begin{aligned}y &= 20b^3\\ \therefore\ \log y &= \log(20b^3)\\ \therefore\ \log y &= \log 20+\log b^3\\ \therefore\ \log y &= \log 20+3\log b\end{aligned}$$

c
$$\begin{aligned}M &= ad^4\\ \therefore\ \log M &= \log(ad^4)\\ \therefore\ \log M &= \log a+\log d^4\\ \therefore\ \log M &= \log a+4\log d\end{aligned}$$

d
$$\begin{aligned}T &= 5\sqrt{d}=5d^{\frac{1}{2}}\\ \therefore\ \log T &= \log(5d^{\frac{1}{2}})\\ \therefore\ \log T &= \log 5+\log d^{\frac{1}{2}}\\ \therefore\ \log T &= \log 5+\tfrac{1}{2}\log d\end{aligned}$$

e
$$\begin{aligned}R &= b\sqrt{l}=bl^{\frac{1}{2}}\\ \therefore\ \log R &= \log(bl^{\frac{1}{2}})\\ \therefore\ \log R &= \log b+\log l^{\frac{1}{2}}\\ \therefore\ \log R &= \log b+\tfrac{1}{2}\log l\end{aligned}$$

f
$$\begin{aligned}Q &= \frac{a}{b^n}\\ \therefore\ \log Q &= \log\left(\frac{a}{b^n}\right)\\ \therefore\ \log Q &= \log a-\log b^n\\ \therefore\ \log Q &= \log a-n\log b\end{aligned}$$

g
$$\begin{aligned}y &= ab^x\\ \therefore\ \log y &= \log(ab^x)\\ \therefore\ \log y &= \log a+\log b^x\\ \therefore\ \log y &= \log a+x\log b\end{aligned}$$

h
$$\begin{aligned}F &= \frac{20}{\sqrt{n}}=\frac{20}{n^{\frac{1}{2}}}\\ \therefore\ \log F &= \log\left(\frac{20}{n^{\frac{1}{2}}}\right)\\ \therefore\ \log F &= \log 20-\log n^{\frac{1}{2}}\\ \therefore\ \log F &= \log 20-\tfrac{1}{2}\log n\end{aligned}$$

i
$$\begin{aligned} L &= \frac{ab}{c} \\ \therefore\ \log L &= \log\left(\frac{ab}{c}\right) \\ \therefore\ \log L &= \log ab - \log c \\ \therefore\ \log L &= \log a + \log b - \log c \end{aligned}$$

j
$$\begin{aligned} N &= \sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore\ \log N &= \log\left(\frac{a}{b}\right)^{\frac{1}{2}} \\ \therefore\ \log N &= \tfrac{1}{2}\log\left(\frac{a}{b}\right) \\ \therefore\ \log N &= \tfrac{1}{2}\{\log a - \log b\} \end{aligned}$$

k
$$\begin{aligned} S &= 200 \times 2^t \\ \therefore\ \log S &= \log(200 \times 2^t) \\ \therefore\ \log S &= \log 200 + \log 2^t \\ \therefore\ \log S &= \log 200 + t\log 2 \end{aligned}$$

l
$$\begin{aligned} y &= \frac{a^m}{b^n} \\ \therefore\ \log y &= \log\left(\frac{a^m}{b^n}\right) \\ \therefore\ \log y &= \log a^m - \log b^n \\ \therefore\ \log y &= m\log a - n\log b \end{aligned}$$

6 **a**
$$\begin{aligned} \log D &= \log e + \log 2 \\ &= \log(e \times 2) \\ \therefore\ D &= 2e \end{aligned}$$

b
$$\begin{aligned} \log F &= \log 5 - \log t \\ &= \log\left(\frac{5}{t}\right) \\ \therefore\ F &= \frac{5}{t} \end{aligned}$$

c
$$\begin{aligned} \log P &= \tfrac{1}{2}\log x \\ &= \log x^{\frac{1}{2}} \\ \therefore\ P &= \sqrt{x} \end{aligned}$$

d
$$\begin{aligned} \log M &= 2\log b + \log c \\ &= \log b^2 + \log c \\ &= \log(b^2c) \\ \therefore\ M &= b^2c \end{aligned}$$

e
$$\begin{aligned} \log B &= 3\log m - 2\log n \\ &= \log m^3 - \log n^2 \\ &= \log\left(\frac{m^3}{n^2}\right) \\ \therefore\ B &= \frac{m^3}{n^2} \end{aligned}$$

f
$$\begin{aligned} \log N &= -\tfrac{1}{3}\log p \\ &= \log p^{-\frac{1}{3}} \\ &= \log\left(\frac{1}{\sqrt[3]{p}}\right) \\ \therefore\ N &= \frac{1}{\sqrt[3]{p}} \end{aligned}$$

g
$$\begin{aligned} \log P &= 3\log x + 1 \\ &= \log x^3 + \log 10^1 \\ &= \log(10x^3) \\ \therefore\ P &= 10x^3 \end{aligned}$$

h
$$\begin{aligned} \log Q &= 2 - \log x \\ &= \log 10^2 - \log x \\ &= \log\left(\frac{100}{x}\right) \quad \therefore\ Q = \frac{100}{x} \end{aligned}$$

7 **a**
$$\begin{aligned} &\log_b 6 \\ &= \log_b(2 \times 3) \\ &= \log_b 2 + \log_b 3 \\ &= p + q \end{aligned}$$

b
$$\begin{aligned} &\log_b 108 \\ &= \log_b(2^2 3^3) \\ &= 2\log_b 2 + 3\log_b 3 \\ &= 2p + 3q \end{aligned}$$

c
$$\begin{aligned} &\log_b 45 \\ &= \log_b(3^2 5) \\ &= 2\log_b 3 + \log_b 5 \\ &= 2q + r \end{aligned}$$

d
$$\begin{aligned} &\log_b\left(\tfrac{5\sqrt{3}}{2}\right) \\ &= \log_b(5 \times 3^{\frac{1}{2}}) - \log_b 2 \\ &= \log_b 5 + \tfrac{1}{2}\log_b 3 - \log_b 2 \\ &= r + \tfrac{1}{2}q - p \end{aligned}$$

e
$$\begin{aligned} &\log_b(\tfrac{5}{32}) \\ &= \log_b 5 - \log_b 2^5 \\ &= \log_b 5 - 5\log_b 2 \\ &= r - 5p \end{aligned}$$

f
$$\begin{aligned} &\log_b(0.\bar{2}) \\ &= \log_b(\tfrac{2}{9}) \\ &= \log_b 2 - \log_b 3^2 \\ &= p - 2q \end{aligned}$$

8 **a**
$$\begin{aligned} &\log_2(PR) \\ &= \log_2 P + \log_2 R \\ &= x + z \end{aligned}$$

b
$$\begin{aligned} &\log_2(RQ^2) \\ &= \log_2 R + \log_2 Q^2 \\ &= \log_2 R + 2\log_2 Q \\ &= z + 2y \end{aligned}$$

c
$$\begin{aligned} &\log_2\left(\frac{PR}{Q}\right) \\ &= \log_2(PR) - \log_2 Q \\ &= \log_2 P + \log_2 R - \log_2 Q \\ &= x + z - y \end{aligned}$$

d
$$\begin{aligned}&\log_2\left(P^2\sqrt{Q}\right)\\&=\log_2 P^2+\log_2 Q^{\frac{1}{2}}\\&=2\log_2 P+\tfrac{1}{2}\log_2 Q\\&=2x+\tfrac{1}{2}y\end{aligned}$$

e
$$\begin{aligned}&\log_2\left(\frac{Q^3}{\sqrt{R}}\right)\\&=\log_2 Q^3-\log_2 R^{\frac{1}{2}}\\&=3\log_2 Q-\tfrac{1}{2}\log_2 R\\&=3y-\tfrac{1}{2}z\end{aligned}$$

f
$$\begin{aligned}&\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)\\&=\log_2 R^2+\log_2 Q^{\frac{1}{2}}-\log_2 P^3\\&=2\log_2 R+\tfrac{1}{2}\log_2 Q-3\log_2 P\\&=2z+\tfrac{1}{2}y-3x\end{aligned}$$

9 **a**
$$\begin{aligned}\log_t N^2&=1.72\\\therefore\quad 2\log_t N&=1.72\\\therefore\quad \log_t N&=1.72\div 2\\&=0.86\end{aligned}$$

b
$$\begin{aligned}&\log_t(MN)\\&=\log_t M+\log_t N\\&=1.29+0.86\\&=2.15\end{aligned}$$

c
$$\begin{aligned}&\log_t\left(\frac{N^2}{\sqrt{M}}\right)\\&=\log_t N^2-\log_t M^{\frac{1}{2}}\\&=2\log_t N-\tfrac{1}{2}\log_t M\\&=2(0.86)-\tfrac{1}{2}(1.29)\\&=1.075\end{aligned}$$

10 **a**
$$\begin{aligned}\log_3 27+\log_3(\tfrac{1}{3})&=\log_3 x\\\therefore\quad \log_3(27\times\tfrac{1}{3})&=\log_3 x\\\therefore\quad \log_3 9&=\log_3 x\\\therefore\quad x&=9\end{aligned}$$

b
$$\begin{aligned}\log_5 x&=\log_5 8-\log_5(6-x)\\\therefore\quad \log_5 x&=\log_5\left(\frac{8}{6-x}\right)\\\therefore\quad x&=\frac{8}{6-x}\\\therefore\quad 6x-x^2&=8\\\therefore\quad x^2-6x+8&=0\\\therefore\quad (x-2)(x-4)&=0\\\therefore\quad x&=2\text{ or }4\end{aligned}$$

Note: $x>0$
and $6-x>0$
i.e., $0<x<6$

c
$$\begin{aligned}\log_5 125-\log_5\sqrt{5}&=\log_5 x\\\therefore\quad \log_5\left(\tfrac{125}{\sqrt{5}}\right)&=\log_5 x\\\therefore\quad x&=\tfrac{125}{\sqrt{5}}\text{ or }25\sqrt{5}\end{aligned}$$

d
$$\begin{aligned}\log_{20} x&=1+\log_{20}10\\\therefore\quad \log_{20}x&=\log_{20}20^1+\log_{20}10\\&=\log_{20}200\\\therefore\quad x&=200\end{aligned}$$

e
$$\begin{aligned}\log x+\log(x+1)&=\log 30\\\therefore\quad \log[x(x+1)]&=\log 30\\\therefore\quad x^2+x&=30\\\therefore\quad x^2+x-30&=0\\\therefore\quad (x+6)(x-5)&=0\\\therefore\quad x&=-6\text{ or }5\end{aligned}$$

but $x>0$ for $\log x$ to exist

$$\therefore\quad x=5$$

f
$$\begin{aligned}\log(x+2)-\log(x-2)&=\log 5\\\therefore\quad \log\left(\frac{x+2}{x-2}\right)&=\log 5\\\therefore\quad \frac{x+2}{x-2}&=5\\\therefore\quad x+2&=5x-10\\\therefore\quad -4x&=-12\\\therefore\quad x&=3\end{aligned}$$

Note: $x+2>0$ and $x-2>0$ $\therefore\ x>2$ ✓

EXERCISE 4D

1 **a**
$$\begin{aligned}2^x&=10\\\therefore\quad \log 2^x&=\log 10\\\therefore\quad x\log 2&=\log 10\\\therefore\quad x&=\frac{\log 10}{\log 2}\doteq 3.32\end{aligned}$$

b
$$\begin{aligned}3^x&=20\\\therefore\quad \log 3^x&=\log 20\\\therefore\quad x\log 3&=\log 20\\\therefore\quad x&=\frac{\log 20}{\log 3}\doteq 2.73\end{aligned}$$

c
$$\begin{aligned}4^x&=100\\\therefore\quad \log 4^x&=\log 100\\\therefore\quad x\log 4&=\log 100\\\therefore\quad x&=\frac{\log 100}{\log 4}\doteq 3.32\end{aligned}$$

d
$$\begin{aligned}(1.2)^x&=1000\\\therefore\quad x\log(1.2)&=\log 1000\\\therefore\quad x&=\frac{\log 1000}{\log(1.2)}\\\therefore\quad x&\doteq 37.9\end{aligned}$$

e
$$\begin{aligned}2^x&=0.08\\\therefore\quad \log 2^x&=\log(0.08)\\\therefore\quad x\log 2&=\log(0.08)\\\therefore\quad x&=\frac{\log(0.08)}{\log 2}\\\therefore\quad x&\doteq\ -3.64\end{aligned}$$

f
$$\begin{aligned} 3^x &= 0.000\,25 \\ \therefore\ \log 3^x &= \log(0.000\,25) \\ \therefore\ x\log 3 &= \log(0.000\,25) \\ \therefore\ x &= \frac{\log(0.000\,25)}{\log 3} \doteq -7.55 \end{aligned}$$

g
$$\begin{aligned} \left(\tfrac{1}{2}\right)^x &= 0.005 \\ \therefore\ \log(0.5)^x &= \log(0.005) \\ \therefore\ x\log(0.5) &= \log(0.005) \\ \therefore\ x &= \frac{\log(0.005)}{\log(0.5)} \doteq 7.64 \end{aligned}$$

h
$$\begin{aligned} \left(\tfrac{3}{4}\right)^x &= 10^{-4} \\ \therefore\ \log(0.75)^x &= -4 \\ \therefore\ x\log(0.75) &= -4 \\ \therefore\ x &= \frac{-4}{\log(0.75)} \doteq 32.0 \end{aligned}$$

i
$$\begin{aligned} (0.99)^x &= 0.000\,01 \\ \therefore\ \log(0.99)^x &= \log(0.000\,01) \\ \therefore\ x\log(0.99) &= \log(0.000\,01) \\ \therefore\ x &= \frac{\log(0.000\,01)}{\log(0.99)} \\ \therefore\ x &\doteq 1146 \text{ or } 1150 \text{ (3 s.f.)} \end{aligned}$$

2 **a**
$$\begin{aligned} 200\times 2^{0.25t} &= 600 \\ \therefore\ 2^{0.25t} &= 3 \\ \therefore\ \log(2^{0.25t}) &= \log 3 \\ \therefore\ 0.25t\log 2 &= \log 3 \\ \therefore\ t &= \frac{\log 3}{0.25\times\log 2} \\ \therefore\ t &\doteq 6.34 \end{aligned}$$

b
$$\begin{aligned} 20\times 2^{0.06t} &= 450 \\ \therefore\ 2^{0.06t} &= 22.5 \\ \therefore\ \log(2^{0.06t}) &= \log(22.5) \\ \therefore\ 0.06t\log 2 &= \log(22.5) \\ \therefore\ t &= \frac{\log(22.5)}{0.06\times\log 2} \\ \therefore\ t &\doteq 74.9 \end{aligned}$$

c
$$\begin{aligned} 30\times 3^{-0.25t} &= 3 \\ \therefore\ 3^{-0.25t} &= \tfrac{1}{10} \\ \therefore\ \log 3^{-0.25t} &= \log(0.1) \\ \therefore\ -0.25t\log 3 &= \log(0.1) \\ \therefore\ t &= \frac{\log(0.1)}{-0.25\times\log 3} \doteq 8.38 \end{aligned}$$

d
$$\begin{aligned} 12\times 2^{-0.05t} &= 0.12 \\ \therefore\ 2^{-0.05t} &= \tfrac{1}{100} \\ \therefore\ \log 2^{-0.05t} &= \log(0.01) \\ \therefore\ -0.05t\log 2 &= \log(0.01) \\ \therefore\ t &= \frac{\log(0.01)}{-0.05\times\log 2} \doteq 133 \end{aligned}$$

e
$$\begin{aligned} 50\times 5^{-0.02t} &= 1 \\ \therefore\ 5^{-0.02t} &= \tfrac{1}{50} = 0.02 \\ \therefore\ \log 5^{-0.02t} &= \log(0.02) \\ \therefore\ -0.02t\log 5 &= \log(0.02) \\ \therefore\ t &= \frac{\log(0.02)}{-0.02\times\log 5} \\ \therefore\ t &\doteq 122 \end{aligned}$$

f
$$\begin{aligned} 300\times 2^{0.005t} &= 1000 \\ \therefore\ 2^{0.005t} &= \tfrac{10}{3} \\ \therefore\ \log 2^{0.005t} &= \log(\tfrac{10}{3}) \\ \therefore\ 0.005t\log 2 &= \log(\tfrac{10}{3}) \\ \therefore\ t &= \frac{\log(\frac{10}{3})}{0.005\times\log 2} \\ \therefore\ t &\doteq 347 \end{aligned}$$

EXERCISE 4E

1 $W_t = 20\times 2^{0.15t}$ grams

a
$$\begin{aligned} \text{When}\quad W_t &= 30, \\ 20\times 2^{0.15t} &= 30 \\ \therefore\ 2^{0.15t} &= 1.5 \\ \therefore\ \log 2^{0.15t} &= \log(1.5) \\ \therefore\ 0.15t\log 2 &= \log(1.5) \\ \therefore\ t &= \frac{\log(1.5)}{0.15\times\log 2} \\ \therefore\ t &\doteq 3.90 \text{ hours} \end{aligned}$$

b
$$\begin{aligned} \text{When}\quad W_t &= 100, \\ 20\times 2^{0.15t} &= 100 \\ \therefore\ 2^{0.15t} &= 5 \\ \therefore\ \log 2^{0.15t} &= \log 5 \\ \therefore\ 0.15t\log 2 &= \log 5 \\ \therefore\ t &= \frac{\log 5}{0.15\times\log 2} \\ \therefore\ t &\doteq 15.5 \text{ hours} \end{aligned}$$

2 $T = 100 \times 2^{-0.03t}$ ^{o}C

a

$$\begin{aligned} \text{When} \quad T &= 25, \\ 100 \times 2^{-0.03t} &= 25 \\ \therefore \quad 2^{-0.03t} &= \tfrac{1}{4} = 2^{-2} \\ \therefore \quad -0.03t &= -2 \\ \therefore \quad t &= \frac{2}{0.03} \\ \therefore \quad t &\doteqdot 66.7 \text{ min} \end{aligned}$$

b

$$\begin{aligned} \text{When} \quad T &= 1, \\ 100 \times 2^{-0.03t} &= 1 \\ \therefore \quad 2^{-0.03t} &= 0.01 \\ \therefore \quad \log 2^{-0.03t} &= \log(0.01) \\ \therefore \quad -0.03t \log 2 &= \log(0.01) \\ \therefore \quad t &= \frac{\log(0.01)}{-0.03 \times \log 2} \doteqdot 221 \text{ min} \end{aligned}$$

3 $W_t = 1000 \times 2^{-0.04t}$ has $W_0 = 1000 \times 2^0 = 1000$ grams

a For the weight to halve,

$$\begin{aligned} W_t &= 500 \\ \therefore \quad 1000 \times 2^{-0.04t} &= 500 \\ \therefore \quad 2^{-0.04t} &= \tfrac{1}{2} = 2^{-1} \\ \therefore \quad -0.04t &= -1 \\ \therefore \quad t &= \frac{1}{0.04} \\ \therefore \quad t &= 25 \text{ years} \end{aligned}$$

b

$$\begin{aligned} \text{For} \quad W_t &= 20, \\ 1000 \times 2^{-0.04t} &= 20 \\ \therefore \quad 2^{-0.04t} &= 0.02 \\ \therefore \quad \log 2^{-0.04t} &= \log(0.02) \\ \therefore \quad -0.04t \log 2 &= \log(0.02) \\ \therefore \quad t &= \frac{\log(0.02)}{-0.04 \times \log 2} \\ \therefore \quad t &\doteqdot 141 \text{ years} \end{aligned}$$

c

$$\begin{aligned} \text{When} \quad W_t &= 1\% \text{ of } 1000 \text{ grams} = 10 \text{ g}, \\ 1000 \times 2^{-0.04t} &= 10 \\ \therefore \quad 2^{-0.04t} &= 0.01 \\ \therefore \quad \log 2^{-0.04t} &= \log(0.01) \end{aligned}$$

$$\begin{aligned} \text{and so} \quad -0.04 \log 2 &= \log(0.01) \\ \therefore \quad t &= \frac{\log(0.01)}{-0.04 \times \log 2} \\ \therefore \quad t &\doteqdot 166 \text{ years} \end{aligned}$$

4 $W = W_0 \times 2^{-0.0002t}$ grams

a When W is 25% of original,

$$\begin{aligned} W &= \tfrac{1}{4} \text{ of } W_0 \\ \therefore \quad W_0 \times 2^{-0.0002t} &= \tfrac{1}{4} \times W_0 \\ \therefore \quad 2^{-0.0002t} &= 2^{-2} \\ \therefore \quad 0.0002t &= 2 \\ \therefore \quad t &= \frac{2}{0.0002} \\ \therefore \quad t &= 10\,000 \end{aligned}$$

$\therefore$ it would take 10 000 years

b When W is 0.1% of original,

$$\begin{aligned} W &= \tfrac{0.1}{100} \text{ of } W_0 \\ \therefore \quad W_0 \times 2^{-0.0002t} &= \tfrac{1}{1000} \times W_0 \\ \therefore \quad \log 2^{-0.0002t} &= \log(0.001) \\ \therefore \quad -0.0002t \log 2 &= \log(0.001) \\ \therefore \quad t &= \frac{\log(0.001)}{-0.0002 \times \log 2} \\ \therefore \quad t &\doteqdot 49\,829 \end{aligned}$$

i.e., it would take about 49 800 years

5 $V = V_0 \times 2^{0.1t}$

When $t = 0$, $V = V_0 \times 2^0 = V_0$

So, we need to find t when

$$\begin{aligned} V &= 3V_0 \\ \text{i.e.,} \quad V_0 \times 2^{0.1t} &= 3V_0 \\ \therefore \quad \log 2^{0.1t} &= \log 3 \\ \therefore \quad 0.1t \log 2 &= \log 3 \\ \therefore \quad t &= \frac{\log 3}{0.1 \times \log 2} \doteqdot 15.8 \end{aligned}$$

$\therefore$ the temperature would be 15.8°C.

6 $I = I_0 \times 2^{-0.02t}$ amps

When $t = 0$, $I = I_0 \times 2^0 = I_0$ amps

So, we need to find t when

$$\begin{aligned} I &= 10\% \text{ of } I_0 \\ \therefore \quad I_0 \times 2^{-0.02t} &= 0.1 \times I_0 \\ \therefore \quad \log 2^{-0.02t} &= \log(0.1) \\ \therefore \quad -0.02t \log 2 &= \log(0.1) \\ \therefore \quad t &= \frac{\log(0.1)}{-0.02 \times \log 2} \doteqdot 166 \end{aligned}$$

i.e., it would take 166 seconds.

7 $V = 50(1 - 2^{-0.2t})$

So when $V = 40$,
$$\begin{aligned} 50(1-2^{-0.2t}) &= 40 \\ \therefore\ 1-2^{-0.2t} &= 0.8 \\ \therefore\ 2^{-0.2t} &= 0.2 \\ \therefore\ \log 2^{-0.2t} &= \log(0.2) \\ \therefore\ -0.2t\log 2 &= \log(0.2) \\ \therefore\ t &= \frac{\log(0.2)}{-0.2\times\log 2} \doteqdot 11.6 \end{aligned}$$
i.e., it would take 11.6 sec.

EXERCISE 4F

1
$$\begin{aligned} r &= 107.5\%, \\ &= 1.075 \\ u_1 &= 160\,000 \\ u_{n+1} &= 250\,000 \end{aligned}$$
$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 250\,000 &= 160\,000 \times (1.075)^n \\ \therefore\ (1.075)^n &= \tfrac{25}{16} \\ \therefore\ \log(1.075)^n &= \log(\tfrac{25}{16}) \\ \therefore\ n\log(1.075) &= \log(\tfrac{25}{16}) \\ \therefore\ n &= \frac{\log(\frac{25}{16})}{\log(1.075)} \doteqdot 6.1709\,.... \end{aligned}$$
i.e., it would take 6.17 years
($\doteqdot$ 6 years, 62 days)

2
$$\begin{aligned} u_1 &= 10\,000 \\ u_{n+1} &= 15\,000 \\ r &= 104.8\% \\ &= 1.048 \end{aligned}$$
$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 15\,000 &= 10\,000 \times (1.048)^n \\ \therefore\ (1.048)^n &= 1.5 \\ \therefore\ \log(1.048)^n &= \log(1.5) \\ \therefore\ n\log(1.048) &= \log(1.5) \\ \therefore\ n &= \frac{\log(1.5)}{\log(1.048)} \\ \therefore\ n &\doteqdot 8.648\,.... \end{aligned}$$
i.e., it would take 8.65 years
($\doteqdot$ 8 years, 237 days)

3 **a** 8.4% p.a. compounded monthly

is $\dfrac{8.4\%}{12} = 0.7\%$ a month

So
$$\begin{aligned} T &= 100\% + 0.7\% \\ &= 100.7\% \\ &= 1.007 \end{aligned}$$

b $u_1 = 15\,000$ and $u_{n+1} = 25\,000$
$$\begin{aligned} u_{n+1} &= u_1 \times r^n \\ \therefore\ 25\,000 &= 15\,000 \times (1.007)^n \\ \therefore\ (1.007)^n &= \tfrac{25}{15} = \tfrac{5}{3} \\ \therefore\ \log(1.007)^n &= \log(\tfrac{5}{3}) \\ \therefore\ n\log(1.007) &= \log(\tfrac{5}{3}) \\ \therefore\ n &= \frac{\log(\frac{5}{3})}{\log(1.007)} \doteqdot 73.23\,.... \end{aligned}$$
i.e., it would take 74 months.

EXERCISE 4G

1 **a**
$$\begin{aligned} &\log_3 12 \\ &= \frac{\log 12}{\log 3} \\ &\doteqdot 2.26 \end{aligned}$$

b
$$\begin{aligned} &\log_{\frac{1}{2}} 1250 \\ &= \frac{\log 1250}{\log(0.5)} \\ &\doteqdot -10.3 \end{aligned}$$

c
$$\begin{aligned} &\log_3(0.067) \\ &= \frac{\log(0.067)}{\log 3} \\ &\doteqdot -2.46 \end{aligned}$$

d
$$\begin{aligned} &\log_{0.4}(0.006\,984) \\ &= \frac{\log(0.006\,984)}{\log(0.4)} \\ &\doteqdot 5.42 \end{aligned}$$

2 **a**
$$\begin{aligned} 2^x &= 0.051 \\ \therefore\ x &= \log_2(0.051) \\ \therefore\ x &= \frac{\log(0.051)}{\log 2} \\ \therefore\ x &\doteqdot -4.29 \end{aligned}$$

b
$$\begin{aligned} 4^x &= 213.8 \\ \therefore\ x &= \log_4 213.8 \\ \therefore\ x &= \frac{\log(213.8)}{\log 4} \\ \therefore\ x &\doteqdot 3.87 \end{aligned}$$

c
$$\begin{aligned} 3^{2x+1} &= 4.069 \\ \therefore\ 2x+1 &= \log_3(4.069) \\ \therefore\ 2x+1 &= \frac{\log(4.069)}{\log 3} \\ \therefore\ 2x+1 &\doteqdot 1.2774\,.... \\ \therefore\ 2x &\doteqdot 0.2774\,.... \\ \therefore\ x &\doteqdot 0.139 \end{aligned}$$

3 **a**
$$\begin{aligned} 25^x - 3\times 5^x &= 0\\ \therefore\ 5^{2x} - 3\times 5^x &= 0\\ \therefore\ 5^x(5^x-3) &= 0\\ \therefore\ 5^x &= 3\\ &\{\text{as } 5^x>0 \text{ for all } x\}\\ \therefore\ x &= \log_5 3\\ \therefore\ x &= \frac{\log 3}{\log 5}\\ \therefore\ x &\doteqdot 0.683 \end{aligned}$$

b
$$\begin{aligned} 8\times 9^x - 3^x &= 0\\ \therefore\ 8\times 3^{2x} - 3^x &= 0\\ \therefore\ 3^x(8\times 3^x - 1) &= 0\\ \therefore\ 8\times 3^x - 1 &= 0\\ &\{\text{as } 3^x>0 \text{ for all } x\}\\ \therefore\ 3^x &= \tfrac{1}{8}\\ \therefore\ x &= \log_3(\tfrac{1}{8})\\ \therefore\ x &= \frac{\log(\frac{1}{8})}{\log 3} \doteqdot -1.89 \end{aligned}$$

4 **a**
$$\begin{aligned} \log_4 x^3 + \log_2\sqrt{x} &= 8\\ \therefore\ \frac{\log x^3}{\log 4} + \frac{\log x^{\frac{1}{2}}}{\log 2} &= 8\\ \therefore\ \frac{3\log x}{2\log 2} + \frac{\frac{1}{2}\log x}{\log 2} &= 8\\ \therefore\ \frac{3\log x}{2\log 2} + \frac{\log x}{2\log 2} &= 8\\ \therefore\ \frac{4\log x}{2\log 2} &= 8\\ \therefore\ \log x &= 4\log 2\\ \therefore\ \log x &= \log 2^4\\ \therefore\ x &= 16 \end{aligned}$$

b
$$\begin{aligned} \log_{16} x^5 &= \log_{64} 125 - \log_4\sqrt{x}\\ \therefore\ \frac{\log x^5}{\log 16} &= \frac{\log 125}{\log 64} - \frac{\log x^{\frac{1}{2}}}{\log 4}\\ \therefore\ \frac{5\log x}{4\log 2} &= \frac{\log 125}{6\log 2} - \frac{\frac{1}{2}\log x}{2\log 2}\\ \therefore\ \frac{15\log x}{12\log 2} &= \frac{2\log 125}{12\log 2} - \frac{3\log x}{12\log 2}\\ \therefore\ 15\log x &= 2\log 125 - 3\log x\\ \therefore\ 18\log x &= 2\log 125\\ \therefore\ \log x &= \tfrac{1}{9}\log 5^3\\ \therefore\ \log x &= \log(5^3)^{\frac{1}{9}}\\ \therefore\ x &= 5^{\frac{1}{3}} \quad \textit{or} \quad x \doteqdot 1.71 \end{aligned}$$

5
$$\begin{aligned} 4^x\times 5^{4x+3} &= 10^{2x+3}\\ \therefore\ \log(4^x\times 5^{4x+3}) &= \log 10^{2x+3}\\ \therefore\ x\log 4 + (4x+3)\log 5 &= 2x+3\\ \therefore\ x\log 4 + 4x\log 5 + 3\log 5 &= 2x+3\\ \therefore\ x[\log 4 + 4\log 5 - 2] &= 3 - 3\log 5 \end{aligned}$$

$$\begin{aligned} \therefore\ x &= \frac{3-3\log 5}{\log 4 + 4\log 5 - 2}\\ \therefore\ x &= \frac{\log 10^3 - \log 5^3}{\log 4 + \log 5^4 - \log 10^2}\\ \therefore\ x &= \frac{\log(\frac{1000}{125})}{\log\left(\frac{4\times 5^4}{10^2}\right)}\\ \therefore\ x &= \frac{\log 8}{\log 25} \quad \textit{or} \quad \log_{25} 8 \end{aligned}$$

EXERCISE 4H

1 **a** $f(x) = \log_3(x+1)$

i We require $x+1>0$ $\therefore$ $x>-1$ So, the domain is $x\in\,]-1,\infty[$
the range is $y\in\mathcal{R}$

ii As $x\to -1$ (from the right), $y\to-\infty$, so $x=-1$ is a vertical asymptote.
As $x\to\infty$, $y\to\infty$.

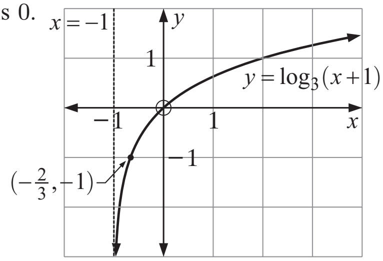
When $x=0$, $y=\log_3 1 = 0$ $\therefore$ y-intercept is 0.

When $y=0$, $\log_3(x+1)=0$

$\therefore$ $x+1=3^0$, so $x+1=1$, $x=0$

$\therefore$ x-intercept is 0.

iii We graph, using $f(x) = \dfrac{\log(x+1)}{\log 3}$



iv
$$\begin{aligned} \text{If}\quad f(x) &= -1 \\ \text{then}\quad \log_3(x+1) &= -1 \\ \therefore\quad x+1 &= 3^{-1} \\ \therefore\quad x &= \tfrac{1}{3}-1 \\ \text{i.e.,}\quad x &= -\tfrac{2}{3} \end{aligned}$$

which checks with the graph

v
$$\begin{aligned} f \text{ is defined by}\quad y &= \log_3(x+1) \\ \therefore\quad f^{-1} \text{ is defined by}\quad x &= \log_3(y+1) \\ \therefore\quad y+1 &= 3^x \\ \therefore\quad y &= 3^x-1 \\ \therefore\quad f^{-1}(x) &= 3^x-1 \end{aligned}$$

and has HA $y = -1$

Its domain is $x \in \mathcal{R}$

range is $y \in \,]-1, \infty[$

b $f(x) = 1 - \log_3(x+1)$

i We require $x+1>0$ $\therefore$ $x>-1$ So, domain is $x \in \,]-1, \infty[$
range is $y \in \mathcal{R}$

ii As $x \to -1$ (right), $y \to \infty$, so $x=-1$ is a vertical asymptote.

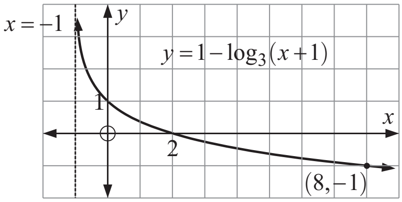
As $x \to \infty$, $y \to -\infty$.

When $x = 0$, $y = 1 - \log_3 1 = 1 - 0 = 1$ $\therefore$ y-intercept is 1.

When $y=0$, $1-\log_3(x+1)=0$ $\therefore$ $\log_3(x+1)=1$ $\therefore$ $x+1 = 3^1 = 3$
$\therefore$ $x = 2$

So, the x-intercept is 2.

iii We graph using $y = 1 - \dfrac{\log(x+1)}{\log 3}$



iv If $f(x) = -1$,
$$\begin{aligned} 1-\log_3(x+1) &= -1 \\ \therefore\quad \log_3(x+1) &= 2 \\ \therefore\quad x+1 &= 3^2 \\ \therefore\quad x &= 8 \end{aligned}$$

v
$$\begin{aligned} f \text{ is defined by}\quad y &= 1-\log_3(x+1) \\ \therefore\quad f^{-1} \text{ is defined by}\quad x &= 1-\log_3(y+1) \\ \therefore\quad \log_3(y+1) &= 1-x \\ \therefore\quad y+1 &= 3^{1-x} \\ \therefore\quad y &= 3^{1-x}-1 \\ \therefore\quad f^{-1}(x) &= 3^{1-x}-1 \end{aligned}$$

It has HA, $y = -1$

domain is $x \in \mathcal{R}$

range is $y \in \,]-1, \infty[$

c $y = \log_5(x-2) - 2$

i We require $x-2>0$ $\therefore$ $x>2$. So, domain is $x \in \,]2, \infty[$
range is $y \in \mathcal{R}$

ii As $x \to 2$ (right), $y \to -\infty$. So, $x = 2$ is a vertical asymptote.

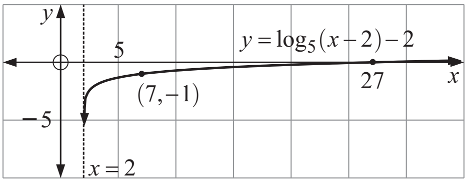
As $x \to \infty$, $y \to \infty$.

When $x=0$, y is undefined $\therefore$ no y-intercept.

When $y=0$, $\log_5(x-2)=2$ $\therefore$ $x-2 = 5^2 = 25$
$\therefore$ $x = 27$

$\therefore$ x-intercept is 27

iii We graph using $y = \dfrac{\log(x-2)}{\log 5} - 2$



iv
$$\begin{aligned} \text{If} \quad f(x) &= -1 \\ \log_5(x-2) - 2 &= -1 \\ \therefore \quad \log_5(x-2) &= 1 \\ \therefore \quad x - 2 &= 5^1 \\ \therefore \quad x &= 5 + 2 \\ \therefore \quad x &= 7 \end{aligned}$$

v
$$\begin{aligned} f \text{ is defined by} \quad y &= \log_5(x-2) - 2 \\ \therefore \quad f^{-1} \text{ is defined by} \quad x &= \log_5(y-2) - 2 \\ \therefore \quad x + 2 &= \log_5(y-2) \\ \therefore \quad y - 2 &= 5^{x+2} \\ \therefore \quad y &= 5^{x+2} + 2 \\ \text{i.e.,} \quad f^{-1}(x) &= 5^{x+2} + 2 \end{aligned}$$

It has a HA of $y = 2$
domain is $x \in \mathcal{R}$
range is $y \in \,]\, 2, \infty \,[$

d $y = 1 - \log_5(x-2)$

i We require $x - 2 > 0$ $\therefore$ $x > 2$, So domain is $x \in \,]\, 2, \infty \,[$
range is $y \in \mathcal{R}$

ii As $x \to 2$ (right), $y \to \infty$ $\therefore$ $x = 2$ is a VA.

As $x \to \infty$, $y \to -\infty$.

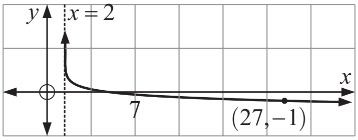
When $x = 0$, y is undefined $\therefore$ no y-intercept.

When $y = 0$, $1 - \log_5(x-2) = 0$
$$\begin{aligned} \therefore \quad \log_5(x-2) &= 1 \\ \therefore \quad x - 2 &= 5^1 \\ \therefore \quad x &= 7 \end{aligned}$$

So, x-intercept is 7.

iii We graph using $y = 1 - \dfrac{\log(x-2)}{\log 5}$

iv If $f(x) = -1$, then
$$\begin{aligned} 1 - \log_5(x-2) &= -1 \\ \therefore \quad \log_5(x-2) &= 2 \\ \therefore \quad x - 2 &= 5^2 \\ \therefore \quad x &= 27 \end{aligned}$$



v
$$\begin{aligned} f \text{ is defined by} \quad y &= 1 - \log_5(x-2) \\ \therefore \quad f^{-1} \text{ is defined by} \quad x &= 1 - \log_5(y-2) \\ \therefore \quad \log_5(y-2) &= 1 - x \\ \therefore \quad y - 2 &= 5^{1-x} \\ \therefore \quad y &= 5^{1-x} + 2 \\ \text{i.e.,} \quad f^{-1}(x) &= 5^{1-x} + 2 \end{aligned}$$

It has a HA of $y = 2$
domain is $x \in \mathcal{R}$
range is $y \in \,]\, 2, \infty \,[$

e $y = 1 - \log_2 x^2$

i We require $x^2 > 0$ which is true for all x except $x = 0$

$\therefore$ domain is $x \in \mathcal{R}$, but $x \neq 0$, range is $y \in \mathcal{R}$.

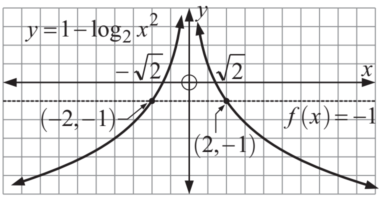
ii As $x \to 0$ (right or left), $y \to \infty$
$\therefore$ $x = 0$ is a VA.

As $x \to \infty$, $y \to -\infty$,
as $x \to -\infty$, $y \to -\infty$.

When $x = 0$, $y = 1 - \log_2 0$
which is undefined $\therefore$ no y-intercept.
When $y = 0$, $\log_2 x^2 = 1$
$$\begin{aligned} \therefore \quad x^2 &= 2^1 = 2 \\ \therefore \quad x &= \pm\sqrt{2} \end{aligned}$$

$\therefore$ x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$

iii We graph using $y = 1 - \dfrac{\log x^2}{\log 2}$



iv When $f(x) = -1$

$1 - \log_2 x^2 = -1$

$\therefore\ \log_2 x^2 = 2$

$\therefore\ x^2 = 2^2$

$\therefore\ x^2 = 4$

$\therefore\ x = \pm 2$

v If $f(x) = 1 - \log_2 x^2,\quad x > 0$

then $f^{-1}(x)$ exists and is defined

by $x = 1 - \log_2 y^2,\quad y > 0$

$\therefore\ \log_2 y^2 = 1 - x$

$\therefore\ y^2 = 2^{1-x}$

$\therefore\ y = 2^{\frac{1-x}{2}}$ as $y > 0$

i.e., $f^{-1}(x) = 2^{\frac{1-x}{2}}$

If $f(x) = 1 - \log_2 x^2,\quad x < 0$

then $f^{-1}(x)$ also exists and is defined

by $x = 1 - \log_2 y^2,\quad y < 0$

$\therefore\ \log_2 y^2 = 1 - x$

$\therefore\ y^2 = 2^{1-x}$

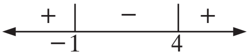
$\therefore\ y = -2^{\frac{1-x}{2}}$ as $y < 0$

$\therefore\ f^{-1}(x) = -2^{\frac{1-x}{2}}$

f $f(x) = \log_2(x^2 - 3x - 4)$

i We require $x^2 - 3x - 4 > 0$

i.e., $(x-4)(x+1) > 0$



$\therefore$ domain is $x \in\ ]-\infty, -1[$ or $]4, \infty[$

range is $y \in \mathcal{R}$

ii As $x \to -1$ (left), $y \to -\infty$

As $x \to 4$ (right), $y \to -\infty$ $\quad\therefore\ x = -1,\ x = 4$ are VAs.

As $x \to \infty,\ y \to \infty$ and as $x \to -\infty,\ y \to \infty$.

When $x = 0,\ y = \log_2(-4)$ which is undefined $\therefore$ no y-intercept.

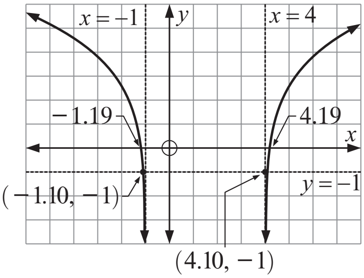
When $y = 0$, $\quad x^2 - 3x - 4 = 2^0 = 1$

$\therefore\ x^2 - 3x - 5 = 0$

$\therefore\ x \doteqdot -1.19$ or 4.19 {from technology}

i.e., x-intercepts are $\doteqdot -1.19,\ 4.19$

iii We graph using $y = \dfrac{\log(x^2 - 3x - 4)}{\log 2}$



iv If $f(x) = -1$ then

$\log_2(x^2 - 3x - 4) = -1$

$\therefore\ x^2 - 3x - 4 = 2^{-1}$

i.e., $x^2 - 3x - 4.5 = 0$

$\therefore\ x \doteqdot -1.10$ or 4.10

{using technology}

v If $f(x) = \log_2(x^2 - 3x - 4),\quad x > 4$

then f^{-1} is defined by

$x = \log_2(y^2 - 3y - 4),\ y > 4$

$\therefore\ y^2 - 3y - 4 = 2^x,\quad y > 0$

$\therefore\ y^2 - 3y - [4 + 2^x] = 0,\quad y > 0$

$$\therefore\ y = \frac{3 \pm \sqrt{9 + 4[4 + 2^x]}}{2},\quad y > 0$$

$$\therefore\ y = \frac{3 + \sqrt{25 + 2^{x+2}}}{2} \quad \text{as} \quad y > 0$$

$$\therefore\ f^{-1}(x) = \frac{3 + \sqrt{25 + 2^{x+2}}}{2}$$

If $f(x) = \log_2(x^2 - 3x - 4),\quad x < -1$

by the same working

$$y = \frac{3 - \sqrt{25 + 2^{x+2}}}{2} \quad \text{as} \quad y < 0$$

i.e., $f^{-1}(x) = \dfrac{3 - \sqrt{25 + 2^{x+2}}}{2}$

REVIEW SET 4A

1 **a**
$$\begin{aligned}&\log_4 64\\ &=\log_4 4^3\\ &=3\end{aligned}$$

b
$$\begin{aligned}&\log_2 256\\ &=\log_2 2^8\\ &=8\end{aligned}$$

c
$$\begin{aligned}&\log_2(0.25)\\ &=\log_2(\tfrac{1}{4})\\ &=\log_2 2^{-2}\\ &=-2\end{aligned}$$

d
$$\begin{aligned}&\log_{25} 5\\ &=\log_{25} 25^{\frac{1}{2}}\\ &=\tfrac{1}{2}\end{aligned}$$

e
$$\begin{aligned}&\log_8 1\\ &=\log_8 8^0\\ &=0\end{aligned}$$

f
$$\begin{aligned}&\log_6 6\\ &=\log_6 6^1\\ &=1\end{aligned}$$

g
$$\begin{aligned}&\log_{81} 3\\ &=\log_{81} 81^{\frac{1}{4}}\\ &=\tfrac{1}{4}\end{aligned}$$

h
$$\begin{aligned}&\log_9(0.\bar{1})\\ &=\log_9(\tfrac{1}{9})\\ &=\log_9 9^{-1}\\ &=-1\end{aligned}$$

i
$$\begin{aligned}&\log_{27} 3\\ &=\log_{27} 27^{\frac{1}{3}}\\ &=\tfrac{1}{3}\end{aligned}$$

j
$$\begin{aligned}&\log_k \sqrt{k}\\ &=\log_k k^{\frac{1}{2}}\\ &=\tfrac{1}{2}\\ &(k>0,\ \ k\neq 1)\end{aligned}$$

2 **a**
$$\begin{aligned}&\log\sqrt{10}\\ &=\log 10^{\frac{1}{2}}\\ &=\tfrac{1}{2}\end{aligned}$$

b
$$\begin{aligned}&\log\left(\frac{1}{\sqrt[3]{10}}\right)\\ &=\log 10^{-\frac{1}{3}}\\ &=-\tfrac{1}{3}\end{aligned}$$

c
$$\begin{aligned}&\log(10^a\times 10^{b+1})\\ &=\log 10^{a+b+1}\\ &=a+b+1\end{aligned}$$

3 **a**
$$\begin{aligned}\log_2 x &= -3\\ \therefore\quad x &= 2^{-3}\\ \therefore\quad x &= \tfrac{1}{8}\end{aligned}$$

b
$$\begin{aligned}\log_5 x &\doteqdot 2.743\\ \therefore\quad x &\doteqdot 5^{2.743}\\ \therefore\quad x &\doteqdot 82.7\end{aligned}$$

c
$$\begin{aligned}\log_3 x &\doteqdot -3.145\\ \therefore\quad x &\doteqdot 3^{-3.145}\\ \therefore\quad x &\doteqdot 0.0316\end{aligned}$$

4 **a**
$$\begin{aligned}P &= 3\times b^x\\ \therefore\quad \log P &= \log(3\times b^x)\\ \therefore\quad \log P &= \log 3+\log b^x\\ \therefore\quad \log P &= \log 3+x\log b\end{aligned}$$

b
$$\begin{aligned}m = \frac{n^3}{p^2}\qquad \therefore\quad \log m &= \log\left(\frac{n^3}{p^2}\right)\\ \therefore\quad \log m &= \log n^3-\log p^2\\ \therefore\quad \log m &= 3\log n-2\log p\end{aligned}$$

5 **a**
$$\begin{aligned}\log_2 k &\doteqdot 1.699+x\\ \therefore\quad k &\doteqdot 2^{1.699+x}\\ \therefore\quad k &\doteqdot 2^{1.699}\times 2^x\\ \therefore\quad k &\doteqdot 3.25\times 2^x\end{aligned}$$

b
$$\begin{aligned}\log_a Q &= 3\log_a P+\log_a R\\ &= \log_a(P^3\times R)\\ \therefore\quad Q &= P^3R\end{aligned}$$

c
$$\begin{aligned}&\log A \doteqdot 5\log B-2.602\\ \therefore\quad &\log A-\log B^5 \doteqdot -2.602\\ \therefore\quad &\log\left(\frac{A}{B^5}\right) \doteqdot -2.602\\ \therefore\quad &\frac{A}{B^5} \doteqdot 10^{-2.602} \doteqdot 0.0025\\ \therefore\quad &A \doteqdot \frac{B^5}{400}\end{aligned}$$

6 **a**
$$\begin{aligned}5^x &= 7\\ \therefore\quad \log 5^x &= \log 7\\ \therefore\quad x\log 5 &= \log 7\\ \therefore\quad x &= \frac{\log 7}{\log 5}\\ \therefore\quad x &\doteqdot 1.21\end{aligned}$$

b
$$\begin{aligned}20\times 2^{2x+1} &= 500\\ \therefore\quad 2^{2x+1} &= 25\\ \therefore\quad \log 2^{2x+1} &= \log 25\\ \therefore\quad (2x+1)\log 2 &= \log 25\\ \therefore\quad 2x+1 &= \frac{\log 25}{\log 2} \doteqdot 4.6438\ldots.\\ \therefore\quad 2x &\doteqdot 3.6438\ldots.\\ \therefore\quad x &\doteqdot 1.82\end{aligned}$$

7 $W_t = 2500 \times 3^{-\frac{t}{3000}}$ grams

a
$$\begin{aligned} W_0 &= 2500 \times 3^0 \\ &= 2500 \times 1 \\ &= 2500 \text{ grams} \end{aligned}$$

b We need t when $W_t = 30\%$ of 2500 g

$$\begin{aligned} \text{i.e.,}\quad 2500 \times 3^{-\frac{t}{3000}} &= 0.3 \times 2500 \\ \therefore\ \log 3^{-\frac{t}{3000}} &= \log(0.3) \\ \therefore\ -\frac{t}{3000} \times \log 3 &= \log(0.3) \\ \therefore\ t &= \frac{-\log(0.3) \times 3000}{\log 3} \\ \therefore\ t &\doteqdot 3287.7\,.... \end{aligned}$$

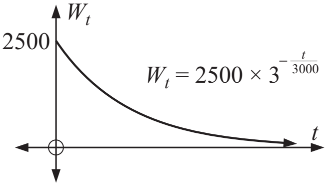
i.e., about 3290 years

c % loss

$$\begin{aligned} &= \left(\frac{W_{1500} - W_0}{W_0}\right) \times 100\% \\ &= \left(\frac{2500 \times 3^{-\frac{1}{2}} - 2500}{2500}\right) \times 100\% \\ &= (3^{-\frac{1}{2}} - 1) \times 100\% \\ &\doteqdot -42.3\% \end{aligned}$$

i.e., a loss of 42.3%

d



8
$$\begin{aligned} 16^x - 5 \times 8^x &= 0 \\ \therefore\ 2^x \times 8^x - 5 \times 8^x &= 0 \\ \therefore\ 8^x(2^x - 5) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore\ 2^x &= 5 \quad \text{as} \quad 8^x > 0 \quad \text{for all } x \\ \therefore\ x &= \log_2 5 = \frac{\log 5}{\log 2} \doteqdot 2.32 \end{aligned}$$

REVIEW SET 4B

1 **a**
$$\begin{aligned} &\log\sqrt{1000} \\ &= \log\left(10^3\right)^{\frac{1}{2}} \\ &= \log 10^{\frac{3}{2}} \\ &= \tfrac{3}{2} \end{aligned}$$

b
$$\begin{aligned} &\log\left(\frac{10}{\sqrt[3]{10}}\right) \\ &= \log\left(\frac{10^1}{10^{\frac{1}{3}}}\right) \\ &= \log 10^{\frac{2}{3}} = \tfrac{2}{3} \end{aligned}$$

c
$$\begin{aligned} &\log\left(\frac{10^a}{10^{-b}}\right) \\ &= \log\left(10^{a-(-b)}\right) \\ &= \log 10^{a+b} \\ &= a + b \end{aligned}$$

2 **a**
$$\begin{aligned} \log x &= 3 \\ \therefore\ x &= 10^3 \\ \therefore\ x &= 1000 \end{aligned}$$

b
$$\begin{aligned} \log_3(x+2) &= 1.732 \\ \therefore\ x + 2 &= 3^{1.732} \\ \therefore\ x + 2 &\doteqdot 6.7046 \\ \therefore\ x &\doteqdot 4.7046 \\ \therefore\ x &\doteqdot 4.70 \end{aligned}$$

c
$$\begin{aligned} \log_2\left(\frac{x}{10}\right) &= -0.671 \\ \therefore\ \frac{x}{10} &= 2^{-0.671} \\ \therefore\ \frac{x}{10} &\doteqdot 0.628\,07 \\ \therefore\ x &\doteqdot 6.28 \end{aligned}$$

3 **a**
$$\begin{aligned} &\log 16 + 2\log 3 \\ &= \log 16 + \log 3^2 \\ &= \log(16 \times 9) \\ &= \log 144 \end{aligned}$$

b
$$\begin{aligned} &\log_2 16 - 2\log_2 3 \\ &= \log_2 16 - \log_2 3^2 \\ &= \log_2(\tfrac{16}{9}) \end{aligned}$$

c
$$\begin{aligned} &2 + \log_4 5 \\ &= \log 4^2 + \log 5 \\ &= \log_4(16 \times 5) \\ &= \log_4 80 \end{aligned}$$

4 **a**
$$\begin{aligned} \log T &= 2\log x - \log y \\ \therefore\ \log T &= \log x^2 - \log y \\ \therefore\ \log T &= \log\left(\frac{x^2}{y}\right) \\ \therefore\ T &= \frac{x^2}{y} \end{aligned}$$

b
$$\begin{aligned} \log_2 K &= \log_2 n + \tfrac{1}{2}\log_2 t \\ \therefore\ \log_2 K &= \log_2 n + \log_2 t^{\frac{1}{2}} \\ \therefore\ \log_2 K &= \log_2(n \times \sqrt{t}) \\ \therefore\ K &= n\sqrt{t} \end{aligned}$$

5 **a**

$$\begin{aligned} 3^x &= 300 \\ \therefore\ \log 3^x &= \log 300 \\ \therefore\ x\log 3 &= \log 300 \\ \therefore\ x &= \frac{\log 300}{\log 3} \\ \therefore\ x &\doteqdot 5.19 \end{aligned}$$

b

$$\begin{aligned} 30\times 5^{1-x} &= 0.15 \\ \therefore\ 5^{1-x} &= 0.005 \\ \therefore\ \log 5^{1-x} &= \log(0.005) \\ \therefore\ (1-x)\log 5 &= \log(0.005) \\ \therefore\ 1-x &= \frac{\log(0.005)}{\log 5} \\ \therefore\ 1-x &\doteqdot -3.292 \\ \therefore\ x &\doteqdot 4.29 \end{aligned}$$

c

$$\begin{aligned} 3^{x+2} &= 2^{1-x} \\ \therefore\ \log 3^{x+2} &= \log 2^{1-x} \\ \therefore\ (x+2)\log 3 &= (1-x)\log 2 \\ \therefore\ x\log 3+2\log 3 &= \log 2 - x\log 2 \\ \therefore\ x(\log 3+\log 2) &= \log 2 - 2\log 3 \\ \therefore\ x\log 6 &= \log(\tfrac{2}{9}) \\ \therefore\ x = \frac{\log(\frac{2}{9})}{\log 6} &\doteqdot -0.839 \end{aligned}$$

6 **a**

$$\begin{aligned} &\log_2 36 \\ &= \log_2(2^2\times 3^2) \\ &= \log_2 2^2 + \log_2 3^2 \\ &= 2\log_2 2 + 2\log_2 3 \\ &= 2A+2B \end{aligned}$$

b

$$\begin{aligned} &\log 54 \\ &= \log_2(2\times 3^3) \\ &= \log_2 2 + \log_2 3^3 \\ &= \log_2 2 + 3\log_2 3 \\ &= A+3B \end{aligned}$$

c

$$\begin{aligned} &\log_2(8\sqrt{3}) \\ &= \log_2(2^3\times 3^{\frac{1}{2}}) \\ &= \log_2 2^3 + \log_2 3^{\frac{1}{2}} \\ &= 3\log_2 2 + \tfrac{1}{2}\log_2 3 \\ &= 3A+\tfrac{1}{2}B \end{aligned}$$

d

$$\begin{aligned} &\log_2(20.25) \\ &= \log_2(20\tfrac{1}{4}) \\ &= \log_2(\tfrac{81}{4}) \\ &= \log_2\left(\frac{3^4}{2^2}\right) \\ &= \log_2 3^4 - \log_2 2^2 \\ &= 4\log_2 3 - 2\log_2 2 \\ &= 4B-2A \end{aligned}$$

e

$$\begin{aligned} &\log_2(0.\bar{8}) \\ &= \log_2(\tfrac{8}{9}) \\ &= \log_2\left(\frac{2^3}{3^2}\right) \\ &= \log_2 2^3 - \log_2 3^2 \\ &= 3\log_2 2 - 2\log_2 3 \\ &= 3A-2B \end{aligned}$$

7 $g(x) = \log_3(x+2) - 2$

a We require $x+2>0$, i.e., $x>-2$ $\therefore$ domain is $x\in\ ]-2,\infty[$ range is $y\in R$

b If $x\to -2$ (right), $y\to-\infty$ $\therefore$ VA is $x=-2$.

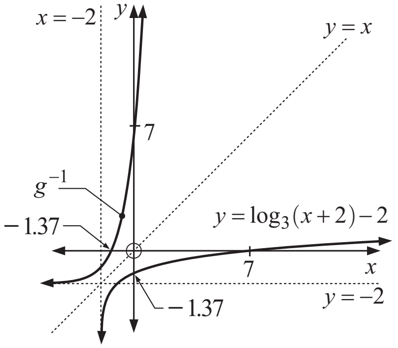
As $x\to\infty$, $y\to\infty$.

When $x=0$, $g(0)=\log_3 2 - 2 \doteqdot -1.37$ $\therefore$ y-intercept $\doteqdot -1.37$

When $y=0$, $\log_3(x+2)=2$ $\therefore$ $x+2 = 3^2$

$\therefore$ $x = 7$ So, the x-intercept is 7.

c, e



d g^{-1} is defined by $x = \log_3(y+2)-2$

$$\begin{aligned} \therefore\ \log_3(y+2) &= x+2 \\ \therefore\ y+2 &= 3^{x+2} \\ \therefore\ y &= 3^{x+2}-2 \\ \therefore\ g^{-1}(x) &= 3^{x+2}-2 \end{aligned}$$

8

$$\begin{aligned} \log_4 a^5 + \log_2 a^{\frac{3}{2}} &= \log_8 625 \\ \therefore\ \frac{\log a^5}{\log 4} + \frac{\log a^{\frac{3}{2}}}{\log 2} &= \frac{\log 5^4}{\log 8} \\ \therefore\ \frac{5\log a}{2\log 2} + \frac{\frac{3}{2}\log a}{\log 2} &= \frac{4\log 5}{3\log 2} \\ \therefore\ \tfrac{5}{2}\log a + \tfrac{3}{2}\log a &= \tfrac{4}{3}\log 5 \\ \therefore\ 4\log a &= \tfrac{4}{3}\log 5 \\ \therefore\ \log a &= \tfrac{1}{3}\log 5 \\ \therefore\ \log a &= \log\sqrt[3]{5} \end{aligned}$$

and so $a = \sqrt[3]{5}$

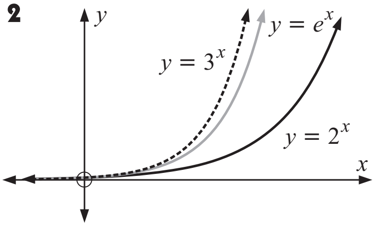
Chapter 5

NATURAL LOGARITHMS

EXERCISE 5A

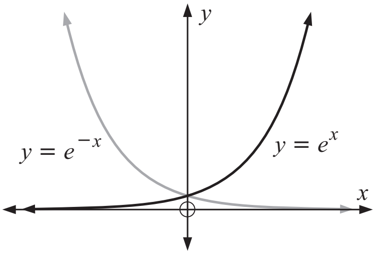
1 $e^1 \doteqdot 2.718\,281\,828\ldots.$

2



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

3



One is the other reflected in the y-axis.

4 When $x = 0$, $y = ae^0 = a \times 1 = a$ $\quad\therefore$ the y-intercept is a.

5 **a** The graph of $y = e^x$ is entirely above the x-axis.

$$\begin{aligned} &\text{So} & y &> 0 \quad \text{for all } x \\ &\text{i.e.,} & e^x &> 0 \quad \text{for all } x \\ &\therefore & 2e^x &> 0 \quad \text{for all } x \end{aligned}$$

$\therefore$ y cannot be negative if $y = 2e^x$

b **i** When $x = -20$, $y = 2e^{-20} \doteqdot 4.1 \times 10^{-9}$

ii When $x = 20$, $y = 2e^{20} \doteqdot 9.7 \times 10^{8}$

6 **a** $\doteqdot 7.39$ **b** $\doteqdot 20.1$ **c** $\doteqdot 2.01$ **d** $\doteqdot 1.65$ **e** $\doteqdot 0.368$

7 **a** $\sqrt{e} = e^{\frac{1}{2}}$

b $$\begin{aligned} &e\sqrt{e} \\ &= e^1 e^{\frac{1}{2}} \\ &= e^{\frac{3}{2}} \end{aligned}$$

c $$\begin{aligned} &\frac{1}{\sqrt{e}} \\ &= \frac{1}{e^{\frac{1}{2}}} \\ &= e^{-\frac{1}{2}} \end{aligned}$$

d $\dfrac{1}{e^2} = e^{-2}$

8 **a** $$\begin{aligned} &\left(e^{0.36}\right)^{\frac{t}{2}} \\ &= e^{0.36\times\frac{t}{2}} \\ &= e^{0.18t} \end{aligned}$$

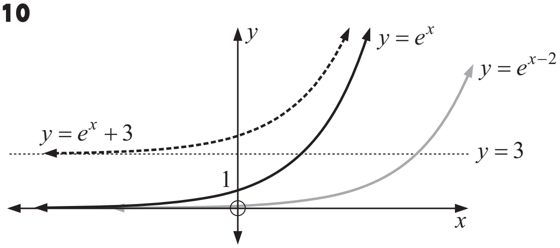
b $$\begin{aligned} &\left(e^{0.064}\right)^{\frac{t}{16}} \\ &= e^{0.064\times\frac{t}{16}} \\ &= e^{0.004t} \end{aligned}$$

c $$\begin{aligned} &\left(e^{-0.04}\right)^{\frac{t}{8}} \\ &= e^{-0.04\times\frac{t}{8}} \\ &= e^{-0.005t} \end{aligned}$$

d $$\begin{aligned} &\left(e^{-0.836}\right)^{\frac{t}{5}} \\ &= e^{-0.836\times\frac{t}{5}} \\ &= e^{-0.167t} \end{aligned}$$

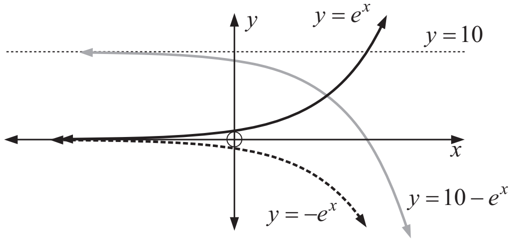
9 **a** 10.074 **b** 0.099 261 **c** 125.09 **d** 0.007 994 5 **e** 41.914 **f** 42.429 **g** 3540.3
h 0.006 342 4

10



Domain of f, g and h is $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$
Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > 3\}$

11



Domain of f, g and h is $\{x : x \in \mathcal{R}\}$

Range of f is $\{y : y > 0\}$

Range of g is $\{y : y < 0\}$

Range of h is $\{y : y < 10\}$

12 $W(t) = 2e^{\frac{t}{2}}$ grams

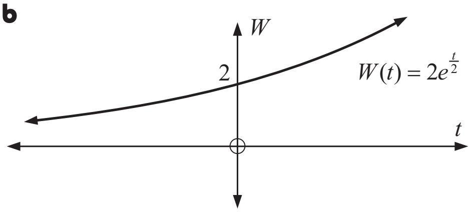
a **i** $W(0) = 2e^0 = 2 \times 1 = 2$ grams

ii $W(\frac{1}{2}) = 2e^{\frac{1}{4}} \doteqdot 2.57$ g

iii $W(1\frac{1}{2}) = 2e^{\frac{3}{4}} \doteqdot 4.23$ g

iv $W(6) = 2e^3 \doteqdot 40.2$ g

b



13 $I(t) = 75e^{-0.15t}$

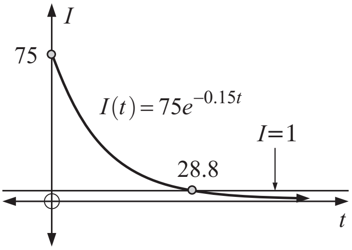
a **i** $I(1) = 75e^{-0.15} = 64.6$ amps

ii $I(10) = 75e^{-1.5} \doteqdot 16.7$ amps

c We need to solve $75e^{-0.15t} = 1$.

Using technology, $t \doteqdot 28.8$ sec.

b



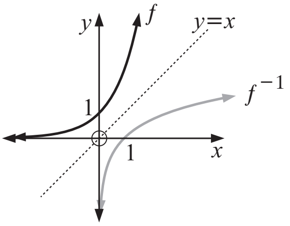
14 **a** $f(x) = e^x$

i.e., $y = e^x$ and has inverse $x = e^y$

$\therefore \ y = \log_e x$

i.e., $f^{-1}(x) = \log_e x$

b



EXERCISE 5B

1 **a** $\ln e^3 = 3$ $\{\ln e^a = a\}$

b $\ln 1 = \ln e^0 = 0$

c $\ln \sqrt[3]{e} = \ln e^{\frac{1}{3}} = \frac{1}{3}$

d $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2$

3 $\ln x$ exists only when $x > 0$. {See the graph of $y = \ln x$.}

So, $\ln(-2)$ and $\ln(0)$ do not exist.

Note: If $\ln(-2) = a$

then $-2 = e^a$

and $e^a = -2$ has no solutions as $e^a > 0$ for all a.

4 **a** $\ln e^a$
$= a$

b $\ln(e \times e^a)$
$= \ln e^{1+a}$
$= 1 + a$

c $\ln(e^a \times e^b)$
$= \ln(e^{a+b})$
$= a + b$

d $\ln(e^a)^b$
$= \ln e^{ab}$
$= ab$

f $\ln\left(\dfrac{e^a}{e^b}\right)$
$= \ln(e^{a-b})$
$= a - b$

5 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$ **e** $e^{-5.1160}$
f $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$ **i** $e^{-1.8971}$ **j** $e^{-8.8049}$

6 **a** $\ln x = 3$
$\therefore\ x = e^3$
$\therefore\ x \doteqdot 20.1$

b $\ln x = 1$
$\therefore\ x = e^1$
$\therefore\ x = e \doteqdot 2.72$

c $\ln x = 0$
$\therefore\ x = e^0$
$\therefore\ x = 1$

d $\ln x = -1$
$\therefore\ x = e^{-1}$
$\therefore\ x \doteqdot 0.368$

e $\ln x = -5$
$\therefore\ x = e^{-5}$
$\therefore\ x \doteqdot 0.006\,74$

f $\ln x \doteqdot 0.835$
$\therefore\ x \doteqdot e^{0.835}$
$\therefore\ x \doteqdot 2.30$

g $\ln x \doteqdot 2.145$
$\therefore\ x \doteqdot e^{2.145}$
$\therefore\ x \doteqdot 8.54$

h $\ln x \doteqdot -3.2971$
$\therefore\ x \doteqdot e^{-3.2971}$
$\therefore\ x \doteqdot 0.0370$

EXERCISE 5C

1 **a** $\ln 8 + \ln 2$
$= \ln(8 \times 2)$
$= \ln 16$

b $\ln 8 - \ln 2$
$= \ln(\frac{8}{2})$
$= \ln 4$

c $\ln 40 - \ln 5$
$= \ln(\frac{40}{5})$
$= \ln 8$

d $\ln 4 + \ln 5$
$= \ln(4 \times 5)$
$= \ln 20$

e $\ln 5 + \ln(0.4)$
$= \ln(5 \times 0.4)$
$= \ln 2$

f $\ln 2 + \ln 3 + \ln 4$
$= \ln(2 \times 3 \times 4)$
$= \ln 24$

g $1 + \ln 3$
$= \ln e^1 + \ln 3$
$= \ln(e \times 3)$
$= \ln 3e$

h $\ln 4 - 1$
$= \ln 4 - \ln e^1$
$= \ln\left(\frac{4}{e}\right)$

i $\ln 5 + \ln 4 - \ln 2$
$= \ln(5 \times 4) - \ln 2$
$= \ln(\frac{20}{2})$
$= \ln 10$

j $2 + \ln 2$
$= \ln e^2 + \ln 2$
$= \ln(e^2 \times 2)$
$= \ln(2e^2)$

k $\ln 40 - 2$
$= \ln 40 - \ln e^2$
$= \ln\left(\frac{40}{e^2}\right)$

l $\ln 6 - \ln 2 - \ln 3$
$= \ln(\frac{6}{2}) - \ln 3$
$= \ln 3 - \ln 3$
$= \ln(\frac{3}{3}) = \ln 1$

2 **a** $5\ln 2 + \ln 3$
$= \ln 2^5 + \ln 3$
$= \ln(2^5 \times 3)$
$= \ln 96$

b $2\ln 3 + 3\ln 2$
$= \ln 3^2 + \ln 2^3$
$= \ln(9 \times 8)$
$= \ln 72$

c $3\ln 4 - \ln 8$
$= \ln 4^3 - \ln 8$
$= \ln(\frac{64}{8})$
$= \ln 8$

d $2\ln 5 - 3\ln 2$
$= \ln 5^2 - \ln 2^3$
$= \ln(\frac{25}{8})$

e $\frac{1}{2}\ln 4 + \ln 3$
$= \ln 4^{\frac{1}{2}} + \ln 3$
$= \ln 2 + \ln 3$
$= \ln(2 \times 3)$
$= \ln 6$

f $\frac{1}{3}\ln(\frac{1}{8})$
$= \frac{1}{3}\ln\left(2^{-3}\right)$
$= \frac{1}{3} \times -3\ln 2$
$= -\ln 2 \quad \left(\text{or } \ln(\frac{1}{2})\right)$

g $-\ln 2$
$= -1\ln 2$
$= \ln 2^{-1}$
$= \ln(\frac{1}{2})$

h $-\ln(\frac{1}{3})$
$= -1\ln(3^{-1})$
$= \ln(3^{-1})^{-1}$
$= \ln 3$

i $-2\ln(\frac{1}{4})$
$= \ln(2^{-2})^{-2}$
$= \ln 2^4$
$= \ln 16 \text{ or } 4\ln 2$

3 **a** $\ln 9$
$= \ln 3^2$
$= 2\ln 3$

b $\ln\sqrt{2}$
$= \ln 2^{\frac{1}{2}}$
$= \frac{1}{2}\ln 2$

c $\ln(\frac{1}{8})$
$= \ln\left(\dfrac{1}{2^3}\right)$
$= \ln(2^{-3})$
$= -3\ln 2$

d $\ln(\frac{1}{5})$
$= \ln 5^{-1}$
$= -1\ln 5$
$= -\ln 5$

e $\ln\left(\frac{1}{\sqrt{2}}\right)$
$= \ln 2^{-\frac{1}{2}}$
$= -\frac{1}{2}\ln 2$

f $\ln\left(\frac{e}{5}\right)$

$= \ln e^1 - \ln 5$

$= 1 - \ln 5$

g $\ln\sqrt[3]{5}$

$= \ln 5^{\frac{1}{3}}$

$= \frac{1}{3}\ln 5$

h $\ln(\frac{1}{32})$

$= \ln 2^{-5}$

$= -5\ln 2$

i $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = \ln\left(\frac{1}{2^{\frac{1}{5}}}\right)$

$= \ln 2^{-\frac{1}{5}} = -\frac{1}{5}\ln 2$

4 **a**
$$\begin{aligned} \ln D &= \ln x + 1 \\ \therefore\ \ln D - \ln x &= 1 \\ \therefore\ \ln\left(\frac{D}{x}\right) &= 1 \\ \therefore\ \frac{D}{x} &= e^1 \\ \therefore\ D &= ex \end{aligned}$$

b
$$\begin{aligned} \ln F &= -\ln p + 2 \\ \therefore\ \ln F + \ln p &= 2 \\ \therefore\ \ln(Fp) &= 2 \\ \therefore\ Fp &= e^2 \\ \therefore\ F &= \frac{e^2}{p} \end{aligned}$$

c
$$\begin{aligned} \ln P &= \tfrac{1}{2}\ln x \\ \therefore\ \ln P &= \ln x^{\frac{1}{2}} \\ \therefore\ P &= \sqrt{x} \end{aligned}$$

d
$$\begin{aligned} \ln M &= 2\ln y + 3 \\ \therefore\ \ln M - 2\ln y &= 3 \\ \therefore\ \ln\left(\frac{M}{y^2}\right) &= 3 \\ \therefore\ \frac{M}{y^2} &= e^3 \\ \therefore\ M &= e^3y^2 \end{aligned}$$

e
$$\begin{aligned} \ln B &= 3\ln t - 1 \\ \therefore\ \ln B - \ln t^3 &= -1 \\ \therefore\ \ln\left(\frac{B}{t^3}\right) &= -1 \\ \therefore\ \frac{B}{t^3} &= e^{-1} \\ \therefore\ B &= \frac{t^3}{e} \end{aligned}$$

f
$$\begin{aligned} \ln N &= -\tfrac{1}{3}\ln g \\ \therefore\ \ln N &= \ln g^{-\frac{1}{3}} \\ \therefore\ N &= g^{-\frac{1}{3}} \\ \therefore\ N &= \frac{1}{\sqrt[3]{g}} \end{aligned}$$

g
$$\begin{aligned} \ln Q &\doteq 3\ln x + 2.159 \\ \therefore\ \ln Q - 3\ln x &\doteq 2.159 \\ \therefore\ \ln\left(\frac{Q}{x^3}\right) &\doteq 2.159 \\ \therefore\ \frac{Q}{x^3} &\doteq e^{2.159} \\ \therefore\ \frac{Q}{x^3} &\doteq 8.66 \\ \therefore\ Q &\doteq 8.66x^3 \end{aligned}$$

h
$$\begin{aligned} \ln D &\doteq 0.4\ln n - 0.6582 \\ \therefore\ \ln D - \ln n^{0.4} &\doteq -0.6582 \\ \therefore\ \ln\left(\frac{D}{n^{0.4}}\right) &\doteq -0.6582 \\ \therefore\ \frac{D}{n^{0.4}} &\doteq e^{-0.6582} \\ \therefore\ \frac{D}{n^{0.4}} &\doteq 0.518 \\ \therefore\ D &\doteq 0.518n^{0.4} \end{aligned}$$

EXERCISE 5D

1 **a**
$$\begin{aligned} e^x &= 10 \\ \therefore\ x &= \ln 10 \\ \therefore\ x &\doteq 2.303 \end{aligned}$$

b
$$\begin{aligned} e^x &= 1000 \\ \therefore\ x &= \ln 1000 \\ \therefore\ x &\doteq 6.908 \end{aligned}$$

c
$$\begin{aligned} e^x &= 0.008\,62 \\ \therefore\ x &= \ln(0.008\,62) \\ \therefore\ x &\doteq -4.754 \end{aligned}$$

d
$$\begin{aligned} e^{\frac{x}{2}} &= 5 \\ \therefore\ \frac{x}{2} &= \ln 5 \\ \therefore\ x &= 2\ln 5 \\ \therefore\ x &\doteq 3.219 \end{aligned}$$

e
$$\begin{aligned} e^{\frac{x}{3}} &= 157.8 \\ \therefore\ \frac{x}{3} &= \ln(157.8) \\ \therefore\ x &= 3\ln(157.8) \\ \therefore\ x &\doteq 15.18 \end{aligned}$$

f
$$\begin{aligned} e^{\frac{x}{10}} &= 0.016\,82 \\ \therefore\ \frac{x}{10} &= \ln(0.016\,82) \\ \therefore\ x &= 10\ln(0.016\,82) \\ \therefore\ x &\doteq -40.85 \end{aligned}$$

g
$$\begin{aligned} 20 \times e^{0.06x} &= 8.312 \\ \therefore\ e^{0.06x} &= 0.4156 \\ \therefore\ 0.06x &= \ln(0.4156) \\ \therefore\ x &= \frac{\ln(0.4156)}{0.06} \\ \therefore\ x &\doteq -14.63 \end{aligned}$$

h
$$\begin{aligned} 50e^{-0.03x} &= 0.816 \\ \therefore\ e^{-0.03x} &= 0.016\,32 \\ \therefore\ -0.03x &= \ln(0.016\,32) \\ \therefore\ x &= \frac{\ln(0.016\,32)}{-0.03} \\ \therefore\ x &\doteq 137.2 \end{aligned}$$

i
$$\begin{aligned} 41.83e^{0.652x} &= 1000 \\ \therefore\ e^{0.652x} &\doteq 23.91 \\ \therefore\ 0.652x &\doteq \ln(23.91) \\ \therefore\ x &\doteq \frac{\ln(23.91)}{0.652} \\ \therefore\ x &\doteq 4.868 \end{aligned}$$

EXERCISE 5E

1 $M_t = 20e^{0.15t}$

a When $M_t = 25$,

$$\begin{aligned} 20e^{0.15t} &= 25 \\ \therefore\ e^{0.15t} &= \tfrac{25}{20} \\ \therefore\ 0.15t &= \ln(\tfrac{25}{20}) \\ \therefore\ t &= \frac{\ln(1.25)}{0.15} \\ \therefore\ t &\doteqdot 1.488 \end{aligned}$$

i.e., after 1 hour 29 min

b When $M_t = 100$,

$$\begin{aligned} 20e^{0.15t} &= 100 \\ \therefore\ e^{0.15t} &= 5 \\ \therefore\ 0.15t &= \ln 5 \\ \therefore\ t &= \frac{\ln 5}{0.15} \\ \therefore\ t &\doteqdot 10.73 \end{aligned}$$

i.e., after 10 hours 44 min

2 $M_t = 1000e^{-0.04t}$ $\quad\therefore\ M_0 = 1000e^0 = 1000$ g

a For M_t to halve,

$$\begin{aligned} M_t &= 500 \\ \therefore\ 1000e^{-0.04t} &= 500 \\ \therefore\ e^{-0.04t} &= 0.5 \\ \therefore\ -0.04t &= \ln(0.5) \\ \therefore\ t &= \frac{\ln(0.5)}{-0.04} \\ \therefore\ t &\doteqdot 17.3 \text{ years} \end{aligned}$$

b For $M_t = 25$ g,

$$\begin{aligned} \therefore\ 1000e^{-0.04t} &= 25 \\ \therefore\ e^{-0.04t} &= 0.025 \\ \therefore\ -0.04t &= \ln(0.025) \\ \therefore\ t &= \frac{\ln(0.025)}{-0.04} \\ \therefore\ t &\doteqdot 92.2 \text{ years} \end{aligned}$$

c For $M_t = 1\%$ of M_0

$$\begin{aligned} \therefore\ 1000e^{-0.04t} &= 0.01 \times 1000 \\ \therefore\ e^{-0.04t} &= 0.01 \\ \therefore\ -0.04t &= \ln(0.01) \\ \therefore\ t &= \frac{\ln(0.01)}{-0.04} \\ \therefore\ t &\doteqdot 115 \text{ years} \end{aligned}$$

3 $V = 50\left(1 - e^{-0.2t}\right)$ m/s

So, when $V = 40$,

$$\begin{aligned} 50\left(1 - e^{-0.2t}\right) &= 40 \\ \therefore\ 1 - e^{-0.2t} &= 0.8 \\ \therefore\ e^{-0.2t} &= 0.2 \\ \therefore\ -0.2t &= \ln(0.2) \\ \therefore\ t &= \frac{\ln(0.2)}{-0.2} \\ \therefore\ t &\doteqdot 8.05 \text{ sec} \end{aligned}$$

i.e., it would take 8.05 sec

4 $T_m = \left(225 \times e^{-0.17m} - 6\right)$ ^{o}C

So, when $T_m = 0$,

$$\begin{aligned} 225e^{-0.17m} - 6 &= 0 \\ \therefore\ 225e^{-0.17m} &= 6 \\ \therefore\ e^{-0.17m} &= \tfrac{6}{225} \\ \therefore\ -0.17m &= \ln(\tfrac{6}{225}) \\ \therefore\ m &= \frac{\ln(\frac{6}{225})}{-0.17} \\ \therefore\ m &\doteqdot 21.3 \end{aligned}$$

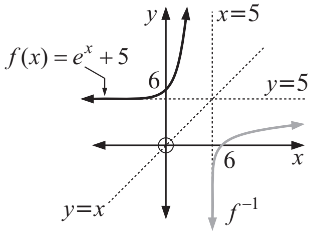
i.e., it would take 21.3 minutes

EXERCISE 5F

1 **a** **i**

$$\begin{aligned} f(x) &= e^x + 5 \\ \text{i.e.,}\quad y &= e^x + 5 \\ &\text{has inverse function} \\ x &= e^y + 5 \\ \therefore\quad x - 5 &= e^y \\ \therefore\quad y &= \ln(x-5) \\ \therefore\quad f^{-1}(x) &= \ln(x-5) \end{aligned}$$

ii

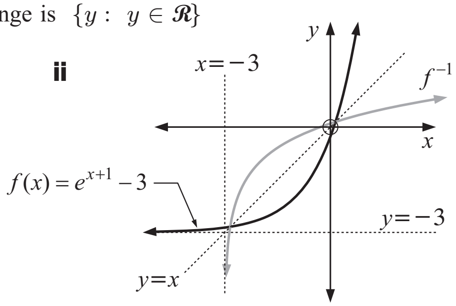


iii domain of f is $\{x :\ x \in \mathcal{R}\}$, range is $\{y :\ y > 5\}$
domain of f^{-1} is $\{x :\ x > 5\}$, range is $\{y :\ y \in \mathcal{R}\}$

b **i**

$$\begin{aligned} f(x) &= e^{x+1} - 3 \\ \text{i.e.,}\quad y &= e^{x+1} - 3 \\ &\text{has inverse function} \\ x &= e^{y+1} - 3 \\ \text{i.e.,}\quad x + 3 &= e^{y+1} \\ \text{i.e.,}\quad y + 1 &= \ln(x+3) \\ \therefore\quad f^{-1}(x) &= \ln(x+3) - 1 \end{aligned}$$

ii

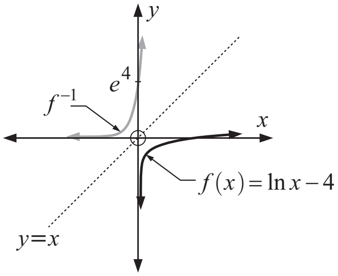


iii domain of f is $\{x :\ x \in \mathcal{R}\}$, range is $\{y :\ y > -3\}$
domain of f^{-1} is $\{x :\ x > -3\}$, range is $\{y :\ y \in \mathcal{R}\}$

c **i**

$$\begin{aligned} f(x) &= \ln x - 4 \\ \text{i.e.,}\quad y &= \ln x - 4 \quad \text{and} \\ &\text{has inverse function} \\ x &= \ln y - 4 \\ \text{i.e.,}\quad x + 4 &= \ln y \\ \therefore\quad y &= e^{x+4} \\ \text{i.e.,}\quad f^{-1}(x) &= e^{x+4} \end{aligned}$$

ii

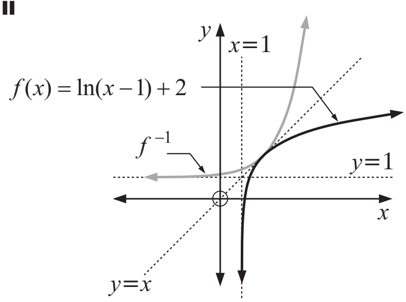


iii domain of f is $\{x :\ x > 0\}$, range is $\{y :\ y \in \mathcal{R}\}$
domain of f^{-1} is $\{x :\ x \in \mathcal{R}\}$, range is $\{y :\ y > 0\}$

d **i**

$$\begin{aligned} f(x) &= \ln(x-1) + 2,\ x > 1 \\ \text{i.e.,}\quad y &= \ln(x-1) + 2 \quad \text{and} \\ &\text{has inverse function} \\ x &= \ln(y-1) + 2 \\ \text{i.e.,}\quad \ln(y-1) &= x - 2 \\ \therefore\quad y - 1 &= e^{x-2} \\ \therefore\quad y &= e^{x-2} + 1 \\ \textit{or}\quad f^{-1}(x) &= e^{x-2} + 1 \end{aligned}$$

ii



iii domain of f is $\{x :\ x > 1\}$, range is $\{y :\ y \in \mathcal{R}\}$
domain of f^{-1} is $\{x :\ x \in \mathcal{R}\}$, range is $\{y :\ y > 1\}$

2 $f(x) = e^{2x}$ and $g(x) = 2x - 1$

a Now for $f(x) = e^{2x}$, i.e., $y = e^{2x}$ the inverse function is

$$\begin{aligned} x &= e^{2y} \\ \therefore\ 2y &= \ln x \\ \therefore\ y &= \tfrac{1}{2}\ln x \\ \text{i.e., } f^{-1}(x) &= \tfrac{1}{2}\ln x \\ \therefore\ (f^{-1}\circ g)(x) &= f^{-1}(g(x)) \\ &= \tfrac{1}{2}\ln(g(x)) \\ &= \tfrac{1}{2}\ln(2x-1) \end{aligned}$$

b

$$\begin{aligned} &(g\circ f)(x) \\ &= g(f(x)) \\ &= 2f(x) - 1 \\ &= 2e^{2x} - 1 \quad \text{i.e., } y = 2e^{2x} - 1 \end{aligned}$$

which has inverse function

$$\begin{aligned} x &= 2e^{2y} - 1 \\ \text{i.e., } x + 1 &= 2e^{2y} \\ \therefore\ \frac{x+1}{2} &= e^{2y} \\ \therefore\ 2y &= \ln\left(\frac{x+1}{2}\right) \\ \therefore\ y &= \tfrac{1}{2}\ln\left(\frac{x+1}{2}\right) \\ \therefore\ (g\circ f)^{-1}(x) &= \tfrac{1}{2}\ln\left(\frac{x+1}{2}\right) \end{aligned}$$

3 **a** $y = \ln x$ cuts the x-axis when $y = 0$

$$\begin{aligned} \therefore\ \ln x &= 0 \\ \therefore\ x &= e^0 = 1 \end{aligned}$$

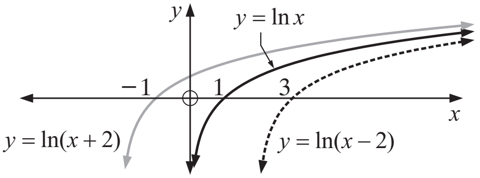
So, graph A is that of $y = \ln x$.

Note: x-intercept of $y = \ln(x-2)$ is when $x - 2 = e^0 = 1$

i.e., $x = 3$

b The x-intercept of $y = \ln(x+2)$ occurs when $x + 2 = e^0 = 1$

$\therefore\ x = -1$



c $y = \ln x$ has a VA of $x = 0$

$y = \ln(x+2)$ has a VA of $x = -2$

$y = \ln(x-2)$ has a VA of $x = 2$

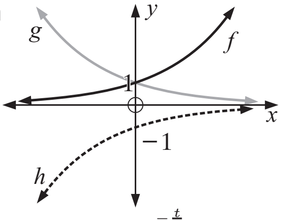
4 Since $y = \ln x^2$ then $y = 2\ln x$ {log law}

i.e., new y-values are 2 × old y-values. So, she is correct.

REVIEW SET 5A

1 **a** $\doteqdot 54.6$ **b** $\doteqdot 22.2$ **c** $\doteqdot 0.0613$ **d** $\doteqdot 6.07$

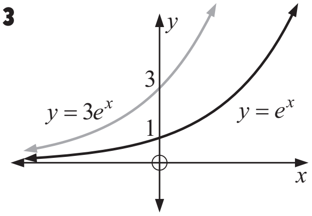
2 **a**



b **i** g is the reflection of f in the y-axis

ii h is the reflection of g in the x-axis

3



4 $s(t) = 120t - 40e^{-\frac{t}{5}}$ metres

a **i** $s(0) = 0 - 40e^0$

$= -40$ m

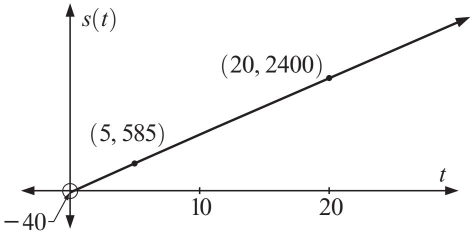
ii $s(5) = 600 - 40e^{-1}$

$\doteqdot 585$ m

iii $s(20) = 2400 - 40e^{-4}$

$\doteqdot 2399.3$ m

b



5 **a** $\ln(e^5) = 5$
$\{\text{as } \ln e^a = a\}$

b $\ln\left(\sqrt{e}\right) = \ln e^{\frac{1}{2}}$
$= \frac{1}{2}$

c $\ln\left(\frac{1}{e}\right) = \ln e^{-1}$
$= -1$

6 **a** $\ln(e^{2x}) = 2x$

b $\ln(e^2 e^x) = \ln(e^{2+x})$
$= 2 + x$

c $\ln\left(\dfrac{e}{e^x}\right) = \ln(e^{1-x})$
$= 1 - x$

7 **a** $\ln x = 5$
$\therefore\ x = e^5$
$\therefore\ x \doteqdot 148$

b $3\ln x + 2 = 0$
$\therefore\ 3\ln x = -2$
$\therefore\ \ln x = -\frac{2}{3}$
$\therefore\ x = e^{-\frac{2}{3}}$
$\therefore\ x \doteqdot 0.513$

8 **a** $\ln 6 + \ln 4$
$= \ln(6 \times 4)$
$= \ln 24$

b $\ln 60 - \ln 20$
$= \ln(\frac{60}{20})$
$= \ln 3$

c $\ln 4 + \ln 1$
$= \ln 4 + 0$
$= \ln 4$

d $\ln 200 - \ln 8 + \ln 5$
$= \ln(\frac{200}{8}) + \ln 5$
$= \ln\left(\frac{200}{8} \times 5\right)$
$= \ln 125$

9 **a** $\ln 32 = \ln 2^5$
$= 5\ln 2$

b $\ln 125 = \ln 5^3$
$= 3\ln 5$

c $\ln 729 = \ln 3^6$
$= 6\ln 3$

10 **a** $e^x = 400$
$\therefore\ x = \ln 400$
$\therefore\ x \doteqdot 5.99$

b $e^{2x+1} = 11$
$\therefore\ 2x + 1 = \ln 11$
$\therefore\ 2x = \ln 11 - 1$
$\therefore\ x = \dfrac{\ln 11 - 1}{2}$
$\therefore\ x \doteqdot 0.699$

c $25e^{\frac{x}{2}} = 750$
$\therefore\ e^{\frac{x}{2}} = 30$
$\therefore\ \dfrac{x}{2} = \ln 30$
$\therefore\ x = 2\ln 30$
$\therefore\ x \doteqdot 6.80$

d $e^{2x} = 7e^x - 12$
$\therefore\ e^{2x} - 7e^x + 12 = 0$
$\therefore\ (e^x - 3)(e^x - 4) = 0$
$\therefore\ e^x = 3 \text{ or } e^x = 4$
$\therefore\ x = \ln 3 \text{ or } \ln 4$
$\therefore\ x \doteqdot 1.10 \text{ or } 1.39$

REVIEW SET 5B

1 **a** $\dfrac{1}{e^3}$
$= e^{-3}$

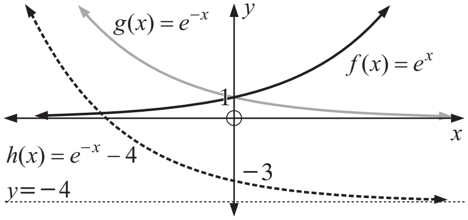
b $\dfrac{\sqrt{e}}{e^2}$
$= \dfrac{e^{\frac{1}{2}}}{e^2}$
$= e^{\frac{1}{2}-2}$
$= e^{-\frac{3}{2}}$

c $e^3\sqrt{e^3}$
$= e^3 \times \left(e^3\right)^{\frac{1}{2}}$
$= e^{3+\frac{3}{2}}$
$= e^{\frac{9}{2}}$

d $\sqrt{10e}$
$= \left(e^{\ln 10}e\right)^{\frac{1}{2}}$
$= e^{\frac{1}{2}(\ln 10+1)}$
$(\doteqdot e^{1.65})$

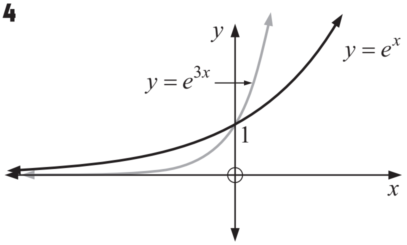
2 **a** $\doteqdot 26.9401$ **b** $\doteqdot 0.109\,447$

3



Each function has domain $\{x : x \in \mathcal{R}\}$
Range of f is $\{y : y > 0\}$
Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > -4\}$

4



5 **a**
$$\begin{aligned}&\ln\left(e\sqrt{e}\right)\\ &= \ln\left(e^1 e^{\frac{1}{2}}\right)\\ &= \ln e^{1\frac{1}{2}}\\ &= 1\tfrac{1}{2}\end{aligned}$$

b
$$\begin{aligned}&\ln\left(\frac{1}{e^3}\right)\\ &= \ln e^{-3}\\ &= -3\end{aligned}$$

c
$$\ln\left(\frac{e}{\sqrt{e^5}}\right) = \ln\left(\frac{e^1}{e^{\frac{5}{2}}}\right) = \ln\left(e^{1-\frac{5}{2}}\right) = \ln e^{-\frac{3}{2}} = -\tfrac{3}{2}$$

6 **a**
$$\begin{aligned}20 &= e^{\ln 20}\\ &\doteqdot e^{3.00}\end{aligned}$$

b
$$\begin{aligned}3000 &= e^{\ln 3000}\\ &\doteqdot e^{8.01}\end{aligned}$$

c
$$\begin{aligned}0.075 &= e^{\ln(0.075)}\\ &\doteqdot e^{-2.59}\end{aligned}$$

7 **a**
$$\begin{aligned}&4\ln 2 + 2\ln 3\\ &= \ln 2^4 + \ln 3^2\\ &= \ln(16\times 9)\\ &= \ln 144\end{aligned}$$

b
$$\begin{aligned}&\tfrac{1}{2}\ln 9 - \ln 2\\ &= \ln 9^{\frac{1}{2}} - \ln 2\\ &= \ln 3 - \ln 2\\ &= \ln(\tfrac{3}{2})\end{aligned}$$

c
$$\begin{aligned}&2\ln 5 - 1\\ &= \ln 5^2 - \ln e^1\\ &= \ln\left(\frac{25}{e}\right)\end{aligned}$$

d
$$\begin{aligned}&\tfrac{1}{4}\ln 81\\ &= \ln\left(3^4\right)^{\frac{1}{4}}\\ &= \ln 3^1\\ &= \ln 3\end{aligned}$$

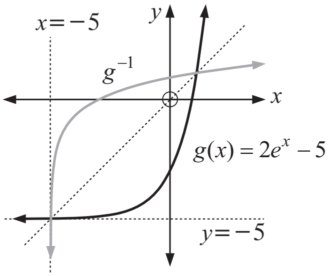
8 **a**
$$\begin{aligned}\ln P &= 1.5\ln Q + \ln T\\ \therefore\ \ln P &= \ln Q^{\frac{3}{2}} + \ln T\\ &= \ln\left(Q^{\frac{3}{2}}T\right)\\ \therefore\ P &= Q^{\frac{3}{2}}T\end{aligned}$$

b
$$\begin{aligned}\ln M &= 1.2 - 0.5\ln N\\ \therefore\ \ln M + \ln N^{\frac{1}{2}} &= 1.2\\ \therefore\ \ln\left(M\sqrt{N}\right) &= 1.2\\ \therefore\ M\sqrt{N} &= e^{1.2}\\ \therefore\ M &\doteqdot \frac{3.32}{\sqrt{N}}\end{aligned}$$

9 **a** $g(x) = 2e^x - 5$ has inverse function $x = 2e^y - 5$

$$\begin{aligned}\therefore\ 2e^y &= x + 5\\ \therefore\ e^y &= \frac{x+5}{2}\\ \text{i.e., }\ y &= \ln\left(\frac{x+5}{2}\right)\\ \therefore\ g^{-1}(x) &= \ln\left(\frac{x+5}{2}\right)\end{aligned}$$

b



c domain of g is $\{x:\ x\in\mathcal{R}\}$, range is $\{y:\ y > -5\}$
domain of g^{-1} is $\{x:\ x > -5\}$, range is $\{y:\ y\in\mathcal{R}\}$

10
$$\begin{aligned}W_t &= 8000\times e^{-\frac{t}{20}}\ \text{grams}\\ W_0 &= 8000e^0\\ &= 8000\times 1\\ &= 8000\ \text{grams}\end{aligned}$$

a When $W_t = \frac{1}{2}\times 8000$ grams,
$$\begin{aligned}8000e^{-\frac{t}{20}} &= 4000\\ \therefore\ e^{-\frac{t}{20}} &= 0.5\\ \therefore\ -\frac{t}{20} &= \ln(0.5)\\ \therefore\ t &= -20\ln(0.5) \doteqdot 13.9\ \text{weeks}\end{aligned}$$

b When $W_t = 1000$,
$$\begin{aligned}8000e^{-\frac{t}{20}} &= 1000\\ \therefore\ e^{-\frac{t}{20}} &= \tfrac{1}{8}\\ \therefore\ -\frac{t}{20} &= \ln(\tfrac{1}{8})\\ \therefore\ t &= -20\ln(\tfrac{1}{8})\\ \therefore\ t &\doteqdot 41.6\ \text{weeks}\end{aligned}$$

c When $W_t = 0.1\%$ of W_0
$$\begin{aligned}&= \tfrac{1}{1000}\ \text{of 8000 g}\\ &= 8\ \text{g},\\ 8000e^{-\frac{t}{20}} &= 8\\ \therefore\ e^{-\frac{t}{20}} &= 0.001\\ \therefore\ -\frac{t}{20} &= \ln(0.001)\\ \therefore\ t &= -20\ln(0.001) \doteqdot 138\ \text{weeks}\end{aligned}$$

Chapter 6

GRAPHING AND TRANSFORMING FUNCTIONS

EXERCISE 6A

1 $f(x) = x$

a $f(2x) = 2x$

b $f(x) + 2$
$= x + 2$

c $\frac{1}{2}\,f(x) = \frac{1}{2}\,x$

d $2\,f(x) + 3$
$= 2x + 3$

2 $f(x) = x^3$

a $f(4x)$
$= (4x)^3$
$= 64x^3$

b $\frac{1}{2}\,f(2x)$
$= \frac{1}{2}(2x)^3$
$= \frac{1}{2} \times 8x^3$
$= 4x^3$

c $f(x+1)$
$= (x+1)^3$
$= x^3 + 3x^2 + 3x + 1$

d $2\,f(x+1) - 3$
$= 2(x+1)^3 - 3$
$= 2(x^3 + 3x^2 + 3x + 1) - 3$
$= 2x^3 + 6x^2 + 6x - 1$

3 $f(x) = 2^x$

a $f(2x) = 2^{2x}$
$= 4^x$

b $f(-x) + 1$
$= 2^{-x} + 1$

c $f(x-2) + 3$
$= 2^{x-2} + 3$

d $2\,f(x) + 3$
$= 2 \times 2^x + 3$
$= 2^{1+x} + 3$

4 $f(x) = \dfrac{1}{x}$

a $f(-x)$
$= \dfrac{1}{(-x)}$
$= -\dfrac{1}{x}$

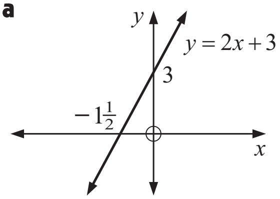
b $f(\frac{1}{2}x)$
$= \dfrac{1}{\frac{1}{2}x}$
$= \dfrac{2}{x}$

c $2\,f(x) + 3$
$= 2\left(\dfrac{1}{x}\right) + 3$
$= \dfrac{2}{x} + 3$

d $3\,f(x-1) + 2$
$= 3\left(\dfrac{1}{x-1}\right) + 2$
$= \dfrac{3}{x-1} + 2$

EXERCISE 6B

1 **a**

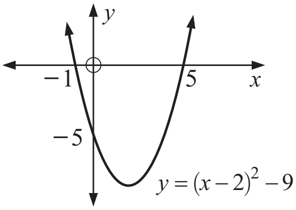


b **i** When $y = 0$, $2x + 3 = 0$ $\therefore$ $x = -\frac{3}{2}$
$\therefore$ x-intercept is $-1\frac{1}{2}$

ii When $x = 0$, $y = 0 + 3 = 3$
$\therefore$ y-intercept is 3

iii As $y = 2x + 3$, the slope is 2 {the coefficient of x}

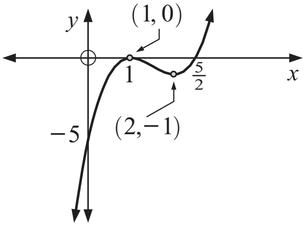
2 **a**



b When $y = 0$,
$(x-2)^2 - 9 = 0$
$\therefore$ $(x-2)^2 = 9$
$\therefore$ $x - 2 = \pm 3$
$\therefore$ $x = 2 + 3$ or $2 - 3$
$\therefore$ $x = 5$ or -1
$\therefore$ x-intercepts are 5 and -1

When $x = 0$,
$y = (-2)^2 - 9$
$= 4 - 9$
$= -5$
$\therefore$ y-intercept is -5

3 **a**, **b**



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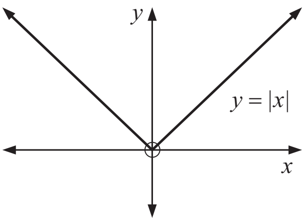

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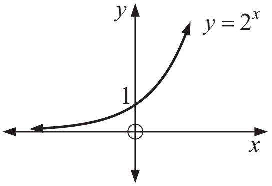
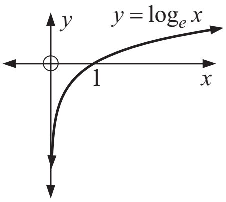


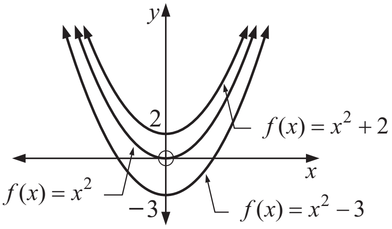
When $x = 0$,
$y = 2^0 = 1$ ✓
$2^x > 0$ for all x as the graph is always above the y-axis. ✓

6



EXERCISE 6C.1

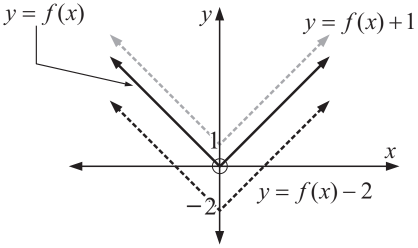
1 **a**, **b**



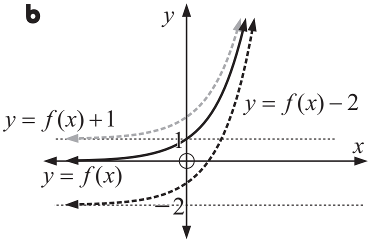
c **i** If $b > 0$, the function is translated vertically upwards through b units.

ii If $b < 0$, the function is translated vertically downwards $|b|$ units.

2 **a**



b



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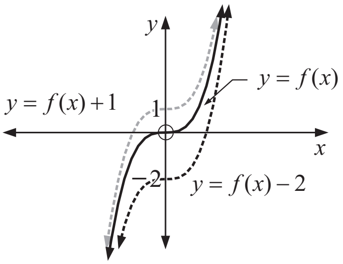

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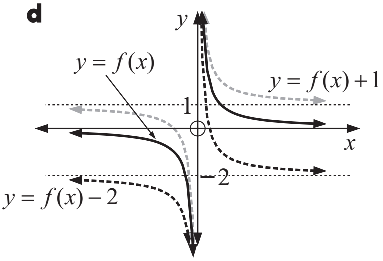
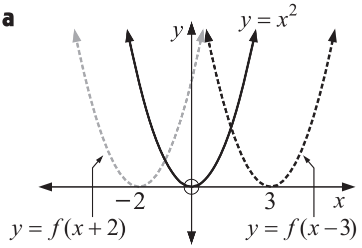


Summary: For $y = f(x) + b$, $y = f(x)$ is translated upwards b units.
If $b > 0$ movement is vertically upwards.
If $b < 0$ movement is vertically downwards.

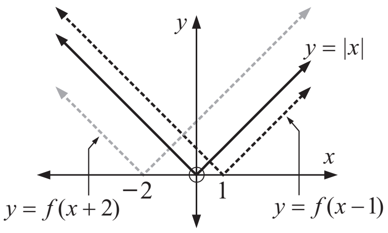
3 **a**



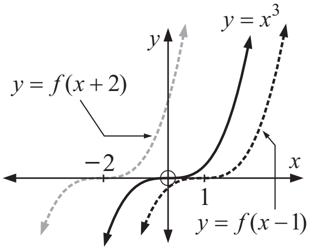
b **i** If $a > 0$, the graph is translated a units right.

ii If $a < 0$, the graph is translated $|a|$ units left.

4 **a**



b



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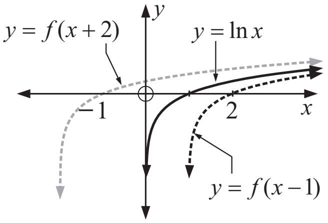
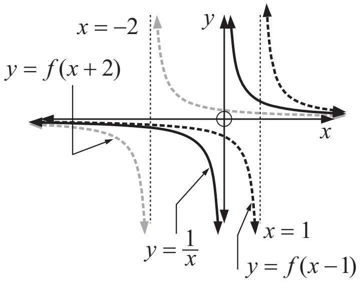
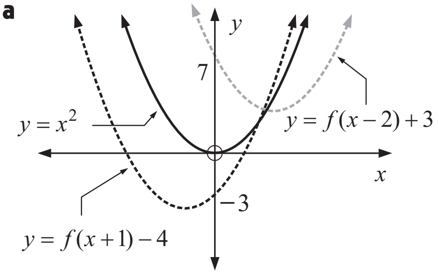


d



$y = f(x-a)$ is a horizontal translation of $y = f(x)$ through $\begin{bmatrix} a \\ 0 \end{bmatrix}$.

5 **a**



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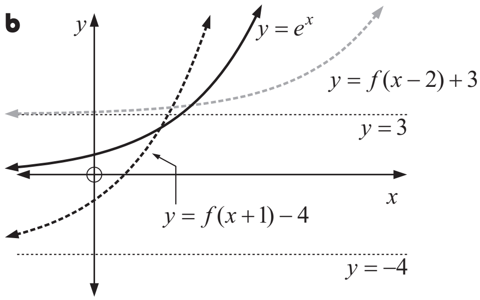

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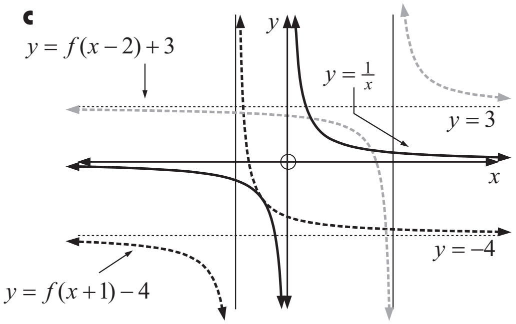
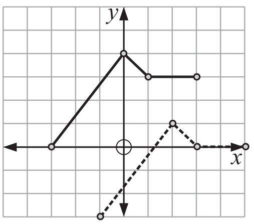
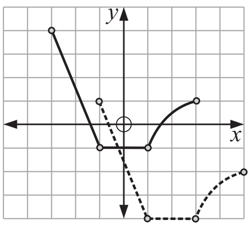


6 A translation of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

a

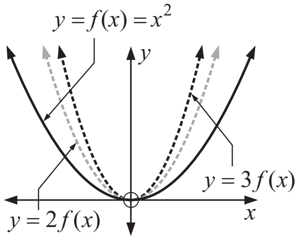


b

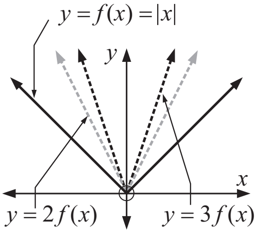


EXERCISE 6C.2

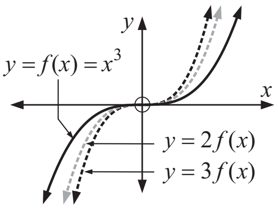
1 **a**



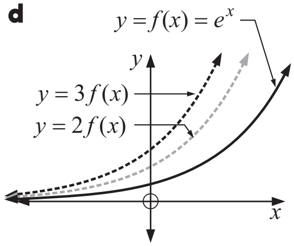
b



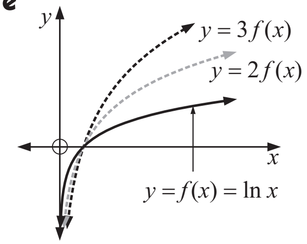
c



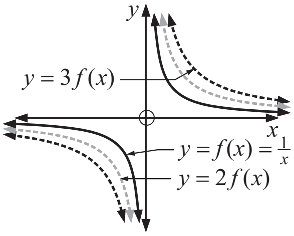
d



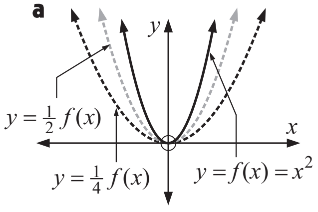
e



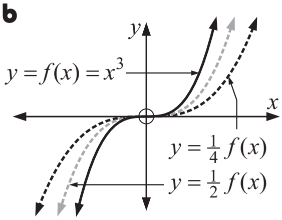
f



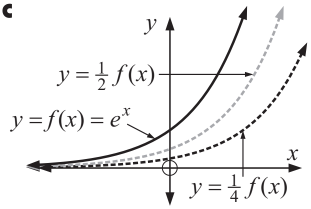
2 **a**



b

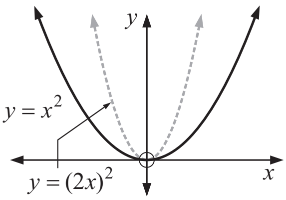


c

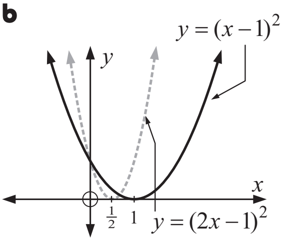


3 p affects the vertical stretching or compression of the graph of $y=f(x)$ by a factor of p. If $p>1$, stretching occurs. If $0<p<1$, compression occurs.

4 **a**



b



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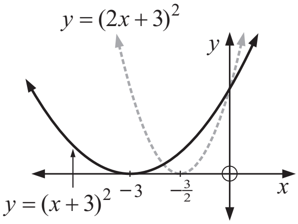
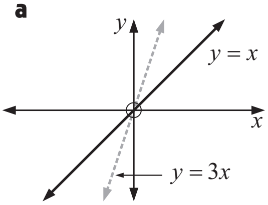
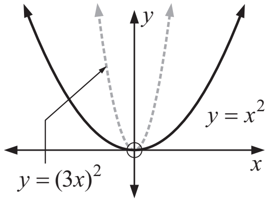


5 **a**



b



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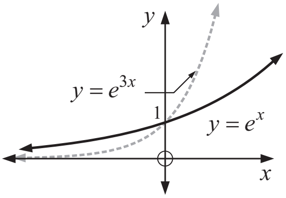
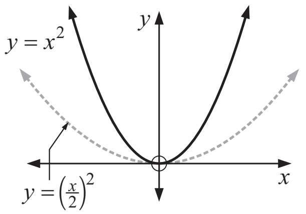


6 **a**



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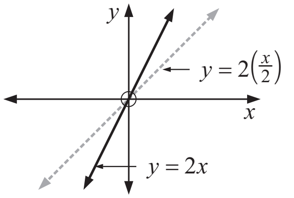

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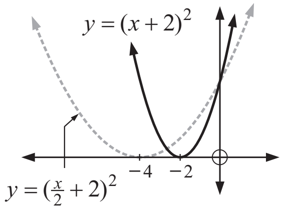
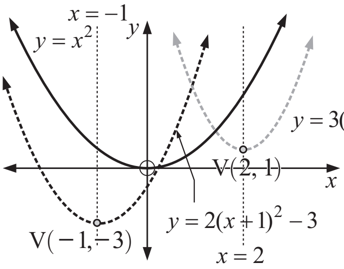


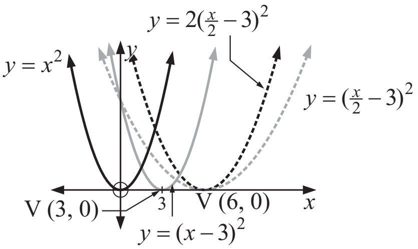
7 k affects the horizontal stretching or compression of $y = f(x)$ by a factor of $\dfrac{1}{k}$.

If $k > 1$, it moves closer to the y-axis. If $0 < k < 1$, it moves further from the y-axis.

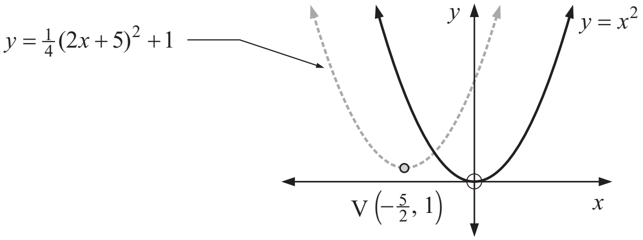
8 **a**



b

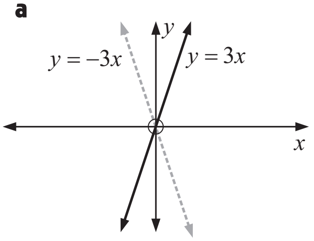


c

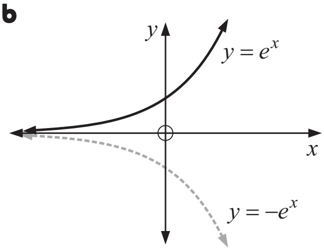


EXERCISE 6C.3

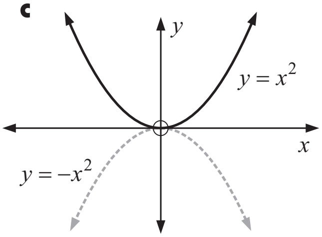
1 **a**



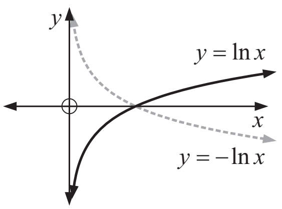
b



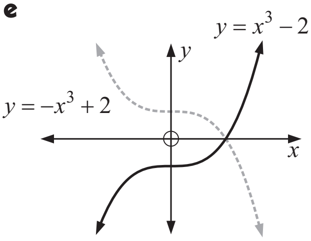
c



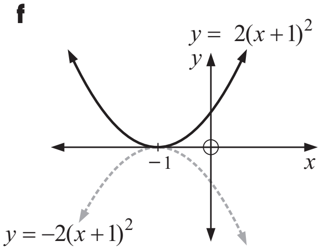
d



e



f



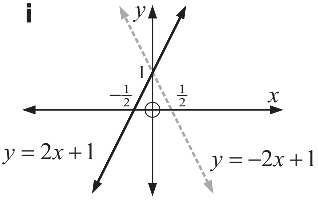
2 $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis.

3 **a** **i** $f(x) = 2x + 1$
$\therefore \quad f(-x)$
$= 2(-x) + 1$
$= -2x + 1$

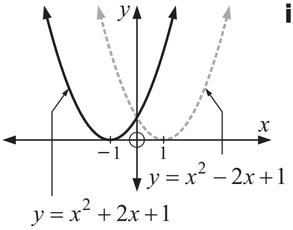
ii $f(x) = x^2 + 2x + 1$
$\therefore \quad f(-x)$
$= (-x)^2 + 2(-x) + 1$
$= x^2 - 2x + 1$

iii $f(x) = |x - 3|$
$\therefore \quad f(-x) = |-x - 3|$
$= |-(x + 3)|$
$= |x + 3|$

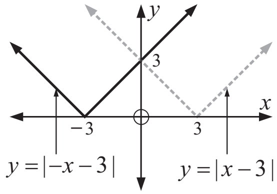
b **i**



ii



iii

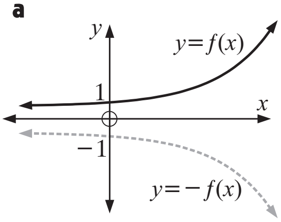


4 $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

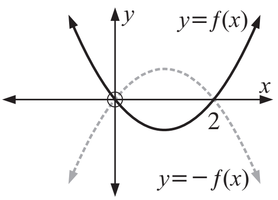
EXERCISE 6D

1 $y = -f(x)$ is obtained from $y = f(x)$ by reflecting it in the x-axis.

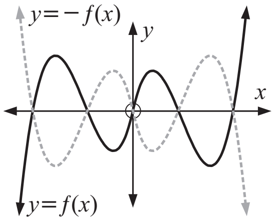
a



b

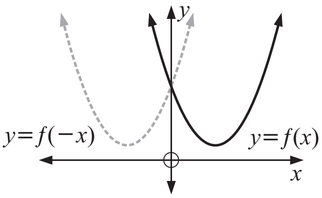


c



2 $y = f(-x)$ is obtained from $y = f(x)$ by reflecting it in the y-axis.

a



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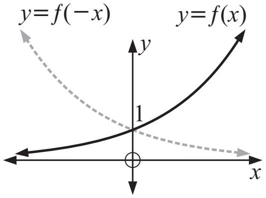
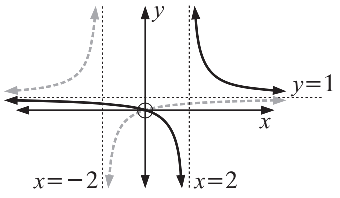
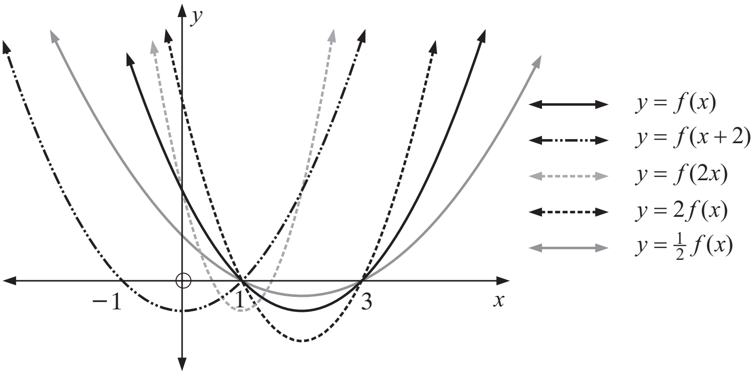


c

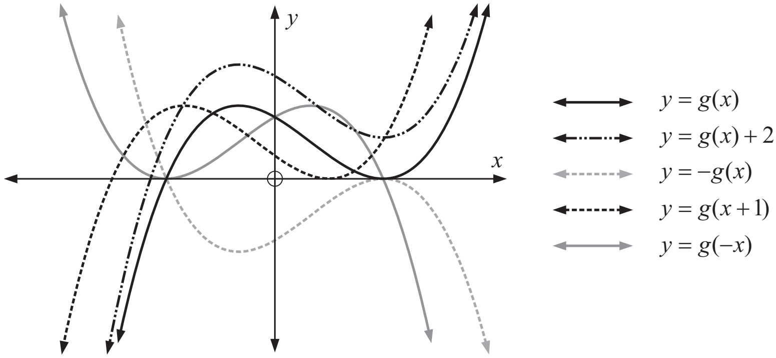


3 $y = 2x^4$ and $y = 6x^4$ are 'thinner' than $y = x^4$ and $y = \frac{1}{2}x^4$ is 'fatter'
$\therefore$ **a** is **A**, **b** is **B**, **c** is **D** and **d** is **C**

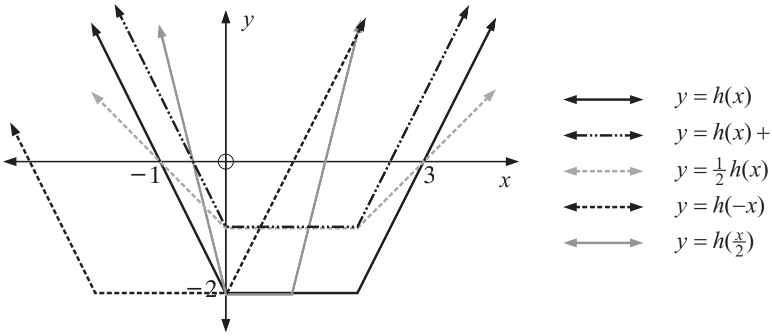
4



5



6



EXERCISE 6E

1 **a** **i** Under a vertical stretch of factor $\frac{1}{2}$, $y = \dfrac{1}{x}$ becomes $y = \frac{1}{2}\left(\dfrac{1}{x}\right)$ i.e., $y = \dfrac{1}{2x}$.

ii Under a horizontal stretch of factor 3, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{\left(\frac{x}{3}\right)}$, i.e., $y = \dfrac{3}{x}$.

iii Under a horizontal translation of -3, $y = \dfrac{1}{x}$ becomes $y = \dfrac{1}{x+3}$.

iv Under a vertical translation of 4, $y = \dfrac{1}{x}$ becomes $y - 4 = \dfrac{1}{x}$ i.e., $y = \dfrac{1}{x} + 4$

i.e., $y = \dfrac{4x+1}{x}$

v Under all four transformations

$$\frac{1}{x} \text{ becomes } \frac{1}{2x} \text{ becomes } \frac{1}{2\left(\frac{x}{3}\right)} \text{ or } \frac{3}{2x} \text{ becomes } \frac{3}{2(x+3)}$$

So, finally $y - 4 = \dfrac{3}{2(x+3)}$ i.e., $y = \dfrac{3}{2(x+3)} + 4$

or $y = \dfrac{3 + 8(x+3)}{2x+6}$

i.e., $y = \dfrac{8x+27}{2x+6}$

b From $y = \dfrac{3}{2(x+3)} + 4$ we see that y is undefined when $x + 3 = 0$ i.e., $x = -3$

$\therefore$ VA is $x = -3$. Domain is $\{x : x \in R, \ x \neq -3\}$

Also, if $y = 4$, $\dfrac{3}{2(x+3)} = 0$ which is not possible as $3 \neq 0$

$\therefore$ $y = 4$ is the HA. Range is $\{y : y \in R, \ y \neq 4\}$

2 **a**

$$\begin{aligned} y &= \frac{2x+4}{x-1} \\ &= \frac{2(x-1)+6}{x-1} \\ &= 2+\frac{6}{x-1} \end{aligned}$$

$\therefore\ y-2 = \dfrac{6}{x-1}$

$\therefore\ (x-1)(y-2) = 6$

$\therefore\ x-1 \neq 0,\ \ y-2 \neq 0$

$\{\text{otherwise}\ \ 0=6\}$

$\therefore$ VA is $x=1$, HA is $y=2$

b To get $y = \dfrac{2x+4}{x-1} = 2+\dfrac{6}{x-1}$ from $y=\dfrac{1}{x}$ we

- vertically stretch by factor 6

  $\dfrac{1}{x}$ becomes $6\left(\dfrac{1}{x}\right)=\dfrac{6}{x}$ then

- translate $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ so $y=\dfrac{6}{x}$ becomes

  $y-2 = \dfrac{6}{x-1}$ i.e., $y = 2+\dfrac{6}{x-1}$

3 **a** **i**

$y = \dfrac{2x+3}{x+1}$

$\therefore\ y = \dfrac{2(x+1)+1}{x+1}$

$\therefore\ y = 2+\dfrac{1}{x+1}$

$\therefore\ (y-2)(x+1)=1$

$\therefore\ y \neq 2,\ \ x \neq -1$

VA is $x=-1$

HA is $y=2$

ii Cuts x-axis when

$y=0\ \therefore\ 2x+3=0$

$\therefore\ x = -\frac{3}{2}$

$\therefore$ x-intercept is $-\frac{3}{2}$

Cuts y-axis when $x=0$

$\therefore\ y = \frac{3}{1} = 3$

$\therefore$ y-intercept is 3

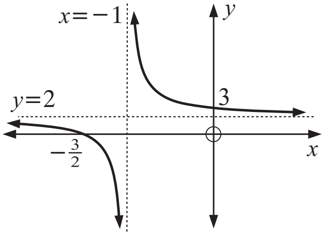
iii As $x \to -1$ (from left), $y \to -\infty$

As $x \to -1$ (from right), $y \to \infty$

As $x \to -\infty$, $y \to 2$ (from below)

As $x \to \infty$, $y \to 2$ (from above)

iv



v Translate it $\left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]$

b **i**

$y = \dfrac{3}{x-2}$

$\therefore\ (x-2)y=3$

$\therefore\ x-2 \neq 0,\ \ y \neq 0$

$\therefore$ VA is $x=2$

HA is $y=0$

ii Cuts x-axis when

$y=0\ \ \therefore\ \dfrac{3}{x-2}=0$

which is not possible.

$\therefore$ no x-intercept

Cuts y-axis when $x=0$

$\therefore\ y = \frac{3}{-2} = -1\frac{1}{2}$

$\therefore$ y-intercept is $-1\frac{1}{2}$

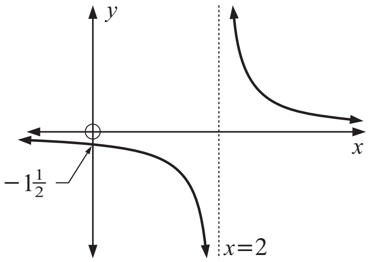
iii As $x \to 2$ (from left), $y \to -\infty$

As $x \to 2$ (from right), $y \to \infty$

As $x \to \infty$, $y \to 0$ (from above)

As $x \to -\infty$, $y \to 0$ (from below)

iv



v Vertically stretch with factor 3, then translate $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$.

c **i**

$$\begin{aligned} y &= \frac{2x-1}{3-x} \\ &= \frac{-2x+1}{x-3} \\ &= \frac{-2(x-3)-5}{x-3} \\ &= -2-\frac{5}{x-3} \end{aligned}$$

$\therefore\ (y+2)(x-3)=-5$

$\therefore\ x \neq 3,\ \ y \neq -2$

$\therefore$ VA is $x=3$

HA is $y=-2$

ii Cuts x-axis when

$y=0\ \therefore\ 2x-1=0$

$\therefore\ x=\frac{1}{2}$

$\therefore$ x-intercept is $\frac{1}{2}$

Cuts y-axis when $x=0$

$\therefore\ y = \frac{-1}{3} = -\frac{1}{3}$

$\therefore$ y-intercept is $-\frac{1}{3}$

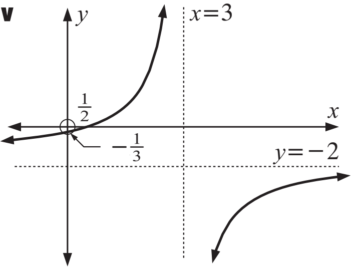
iii As $x \to 3$ (from left), $y \to \infty$

As $x \to 3$ (from right), $y \to -\infty$

As $x \to -\infty$, $y \to -2$ (from above)

As $x \to \infty$, $y \to -2$ (from below)

iv



v Vertically stretch with factor -5, then translate $\left[\begin{smallmatrix}3\\-2\end{smallmatrix}\right]$.

d **i**
$$\begin{aligned} y &= \frac{5x-1}{2x+1} \\ &= \frac{\frac{5}{2}x-\frac{1}{2}}{x+\frac{1}{2}} \\ &= \frac{\frac{5}{2}\left(x+\frac{1}{2}\right)-\frac{7}{4}}{x+\frac{1}{2}} \\ &= \frac{5}{2}-\frac{\frac{7}{4}}{x+\frac{1}{2}} \end{aligned}$$

$\therefore \quad \left(y-\frac{5}{2}\right)\left(x+\frac{1}{2}\right)=-\frac{7}{4}$

$\therefore \quad y \neq \frac{5}{2}, \quad x \neq -\frac{1}{2}$

$\therefore$ VA is $x=-\frac{1}{2}$, HA is $y=\frac{5}{2}$

ii Cuts x-axis when

$y=0 \quad \therefore \quad 5x-1=0$

$\therefore \quad x=\frac{1}{5}$

$\therefore$ x-intercept is $\frac{1}{5}$

Cuts y-axis when $x=0$

$\therefore \quad y=\frac{-1}{1}=-1$

$\therefore$ y-intercept is -1

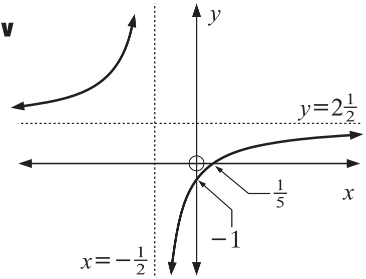
iii As $x \to -\frac{1}{2}$ (from left), $y \to \infty$

As $x \to -\frac{1}{2}$ (from right), $y \to -\infty$

As $x \to \infty$, $y \to 2\frac{1}{2}$ (from below)

As $x \to -\infty$, $y \to 2\frac{1}{2}$ (from above)

iv



v Vertically stretch with factor $-\frac{7}{4}$, then translate $\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$.

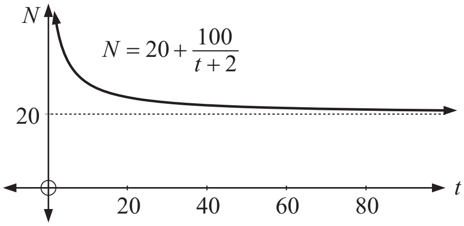
4 $N=20+\dfrac{100}{t+2}$ weeds/ha

a When $t=0$
$$\begin{aligned} N &= 20+\tfrac{100}{2} \\ &= 20+50 \\ &= 70 \text{ weeds/ha} \end{aligned}$$

b When $t=8$,
$$\begin{aligned} N &= 20+\tfrac{100}{10} \\ &= 20+10 \\ &= 30 \text{ weeds/ha} \end{aligned}$$

c When $N=40$,
$$\begin{aligned} 20+\tfrac{100}{t+2} &= 40 \\ \therefore \quad \frac{100}{t+2} &= 20 \\ \therefore \quad t+2 &= 5 \\ \therefore \quad t &= 3 \text{ days} \end{aligned}$$

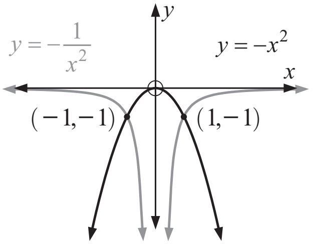
d



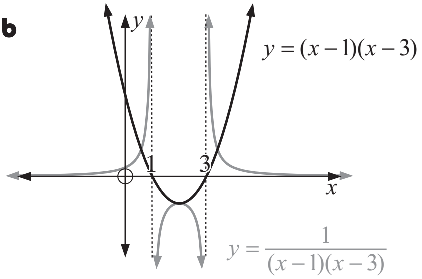
e No, the number of weeds/ha will approach 20 (from above).

EXERCISE 6F

1 **a**



b



2 Notice that $f(x)=\dfrac{1}{f(x)}$ i.e., $y=\dfrac{1}{y} \quad \therefore \quad y^2=1 \quad \therefore \quad y=\pm 1$.

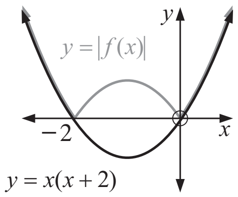
For **1a** When $y=1$, $-x^2=1$ which has no real solutions
When $y=-1$, $-x^2=-1 \quad \therefore \quad x^2=1 \quad \therefore \quad x=\pm 1$
So invariant points are (1, −1) and (−1, −1).

For **1b** When $y=1$,
$$\begin{aligned} (x-1)(x-3) &= 1 \\ \therefore \quad x^2-4x+3 &= 1 \\ \therefore \quad x^2-4x+2 &= 0 \\ \therefore \quad x = \frac{4\pm\sqrt{8}}{2} &= 2\pm\sqrt{2} \end{aligned}$$

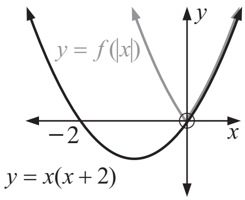
when $y=-1$,
$$\begin{aligned} (x-1)(x-3) &= -1 \\ \therefore \quad x^2-4x+3 &= -1 \\ \therefore \quad x^2-4x+4 &= 0 \\ \therefore \quad (x-2)^2 &= 0 \\ \therefore \quad x &= 2 \end{aligned}$$

$\therefore$ invariant points are $(2-\sqrt{2}, 1)$, $(2, -1)$, $(2+\sqrt{2}, 1)$

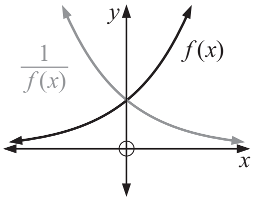
3 **a**



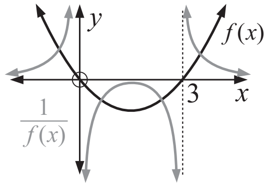
b



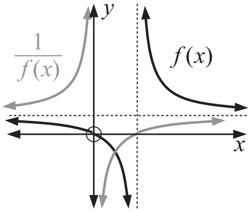
4 **a**



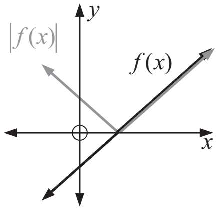
b



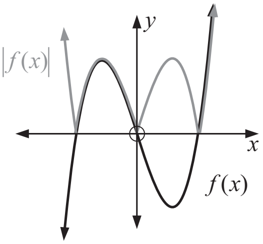
c



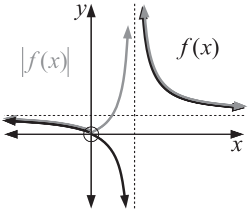
5 **a**



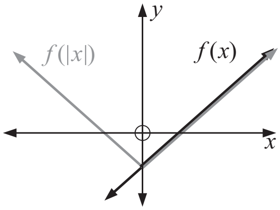
b



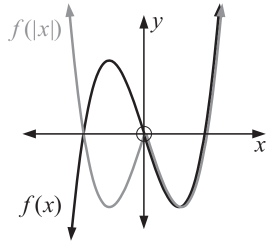
c



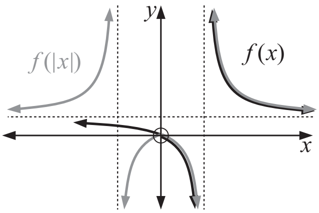
6 **a**



b



c



REVIEW SET 6A

1 $f(x) = x^2 - 2x$

a $f(3)$
$= 3^2 - 2(3)$
$= 9 - 6$
$= 3$

b $f(-2)$
$= (-2)^2 - 2(-2)$
$= 4 + 4$
$= 8$

c $f(2x)$
$= (2x)^2 - 2(2x)$
$= 4x^2 - 4x$

d $f(-x)$
$= (-x)^2 - 2(-x)$
$= x^2 + 2x$

e $3f(x) - 2$
$= 3(x^2 - 2x) - 2$
$= 3x^2 - 6x - 2$

2 $f(x) = 5 - x - x^2$

a $f(4)$
$= 5 - 4 - 4^2$
$= 1 - 16$
$= -15$

b $f(-1)$
$= 5 - (-1) - (-1)^2$
$= 5 + 1 - 1$
$= 5$

c $f(x - 1)$
$= 5 - (x - 1) - (x - 1)^2$
$= 5 - x + 1 - [x^2 - 2x + 1]$
$= 6 - x - x^2 + 2x - 1$
$= -x^2 + x + 5$

d $f\left(\frac{x}{2}\right)$

$= 5 - \left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2$

$= 5 - \dfrac{x}{2} - \dfrac{x^2}{4}$

e $2\,f(x) - f(-x)$

$= 2(5 - x - x^2) - \left[5 - (-x) - (-x)^2\right]$

$= 10 - 2x - 2x^2 - [5 + x - x^2]$

$= 10 - 2x - 2x^2 - 5 - x + x^2$

$= -x^2 - 3x + 5$

3 $f(x) = \dfrac{4}{x}$

a $f(-4)$

$= \dfrac{4}{-4}$

$= -1$

b $f(2x)$

$= \dfrac{4}{2x}$

$= \dfrac{2}{x}$

c $f\left(\frac{x}{2}\right)$

$= \dfrac{4}{\frac{x}{2}}$

$= 4 \times \frac{2}{x}$

$= \dfrac{8}{x}$

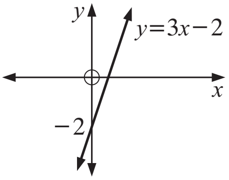
d $4\,f(x+2) - 3$

$= 4\left(\dfrac{4}{x+2}\right) - 3$

$= \dfrac{16}{x+2} - 3$

$\left(or \quad \dfrac{16 - 3(x+2)}{x+2} = \dfrac{10 - 3x}{x+2}\right)$

4 **a**



b **i** When $y = 0$,

$3x - 2 = 0$

$\therefore \quad x = \frac{2}{3}$

$\therefore \quad x$-intercept is $\frac{2}{3}$

ii When $x = 0$,

$y = 0 - 2 = -2$

$\therefore \quad y$-intercept is -2

iii As $y = 3x - 2$, the slope is 3 {coefficient of x}

c **i** When $x = 0.3$,

$y = 3(0.3) - 2$

$= 0.9 - 2$

$= -1.1$

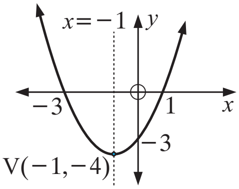
ii When $y = 0.7$,

$3x - 2 = 0.7$

$\therefore \quad 3x = 2.7$

$\therefore \quad x = 0.9$

5 **a**



b **i** When $y = 0$,

$(x+1)^2 - 4 = 0$

$\therefore \quad (x+1)^2 = 4$

$\therefore \quad x + 1 = \pm 2$

$\therefore \quad x = 2 - 1$ or $-2 - 1$

i.e., $x = 1$ or -3

$\therefore \quad x$-intercepts are 1, -3

ii When $x = 0$,

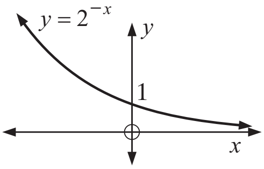
$y = 1^2 - 4$

$= -3$

$\therefore \quad y$-intercept is -3

c Since $y = (x+1)^2 - 4$ then $y + 4 = (x+1)^2$ which is obtained from $y = x^2$ under a translation of $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So, the vertex must be $(-1, -4)$.

6 **a**



b **i** $x \to \infty$ means x is very large and positive.

We see the graph approaching the x-axis

i.e., $y \to 0 \quad \therefore$ **true**.

ii $x \to -\infty$ means x is very large and negative.

We see the graph heading for ∞ and positive

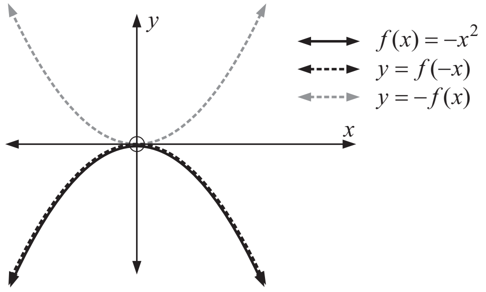
$\therefore$ statement is **false**.

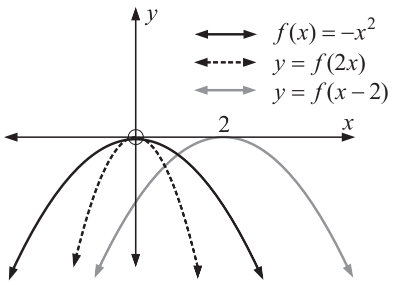
iii The y-intercept is 1, when $x = 0$, $y = 2^0 = 1$ $\therefore$ **false**.

iv The graph is above the x-axis for all x $\therefore$ $2^{-x} > 0$ for all x $\therefore$ **true**.

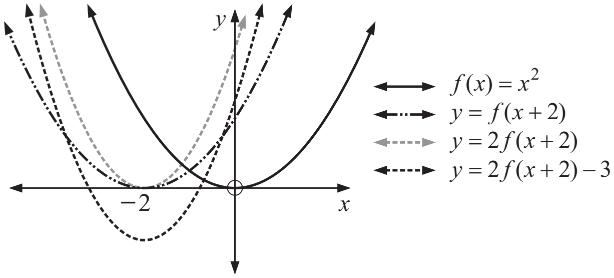
So that you can see the answers more easily in question **7**, they have been drawn on two graphs.

7



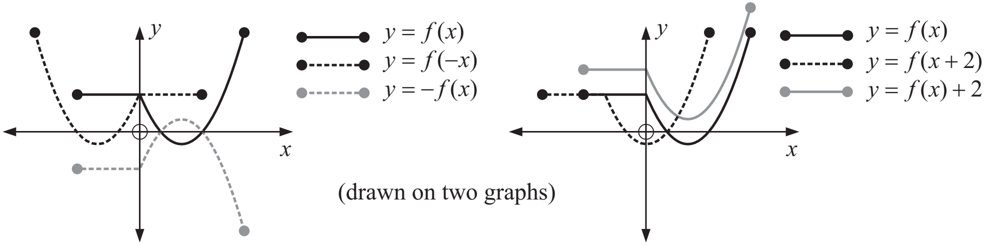


8



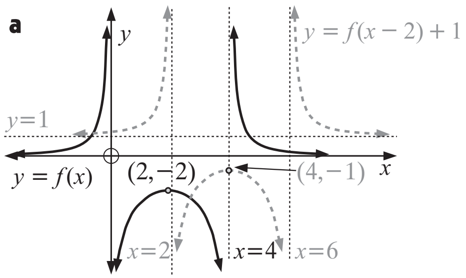
REVIEW SET 6B

1

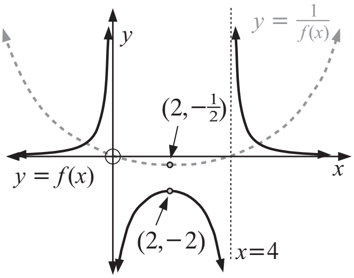


(drawn on two graphs)

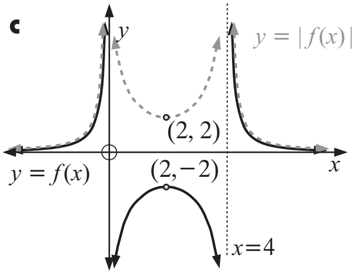
2 **a**



b



c



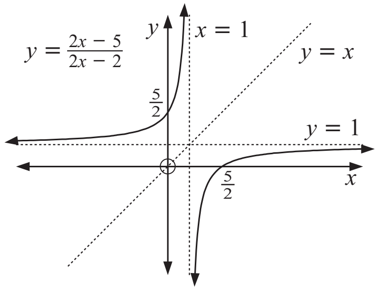
3 **a** $y = \dfrac{1}{x}$ under a reflection in the y-axis becomes $y = \dfrac{1}{(-x)} = -\dfrac{1}{x}$

$y = -\dfrac{1}{x}$ under a vertical stretch of factor 3 becomes $y = 3\left(\dfrac{-1}{x}\right) = -\dfrac{3}{x}$

$y = -\dfrac{3}{x}$ under a horizontal stretch of factor $\frac{1}{2}$ becomes $y = \dfrac{-3}{(2x)} = \dfrac{-3}{2x}$

$y = -\dfrac{3}{2x}$ under a translation of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ becomes $y - 1 = \dfrac{-3}{2(x-1)}$

b



$$\begin{aligned} \text{i.e.,}\quad y &= 1 - \frac{3}{2x-2} \\ \text{i.e.,}\quad y &= \frac{2x-2}{2x-2} - \frac{3}{2x-2} \\ \text{i.e.,}\quad y &= \frac{2x-5}{2x-2} \end{aligned}$$

domain of $f(x)$ is $\{x\colon\ x \neq 1\}$
range of $f(x)$ is $\{y\colon\ y \neq 1\}$

c Yes, since it is a one-to-one function (passes both the vertical and horizontal line tests).

d

$$\begin{aligned} f(x) = y &= \frac{2x-5}{2x-2} \\ \therefore\ \text{inverse function is}\quad x &= \frac{2y-5}{2y-2} \\ \therefore\ (2y-2)x &= 2y-5 \\ \therefore\ 2xy - 2x &= 2y-5 \\ \therefore\ y(2x-2) &= 2x-5 \\ \therefore\ y &= \frac{2x-5}{2x-2} \\ \therefore\ f^{-1}(x) = f(x) &= \frac{2x-5}{2x-2} \end{aligned}$$

i.e., it is a self-inverse function.

Also, the graph of $f(x)$ is symmetrical about the line $y = x$.

4

$$\begin{aligned} y &= \frac{2x-3}{3x+5} \\ &= \frac{\frac{2}{3}x - 1}{x + \frac{5}{3}} \\ &= \frac{\frac{2}{3}\left(x+\frac{5}{3}\right) - 1 - \frac{10}{9}}{x+\frac{5}{3}} \\ &= \frac{\frac{2}{3}(x+\frac{5}{3}) - \frac{19}{9}}{x+\frac{5}{3}} \\ &= \frac{2}{3} - \frac{\frac{19}{9}}{\left(x+\frac{5}{3}\right)} \end{aligned}$$

$\therefore\ (y - \frac{2}{3})(x + \frac{5}{3}) = -\frac{19}{9}$

$\therefore\ x \neq -\frac{5}{3},\ \ y \neq \frac{2}{3}$

$\therefore$ VA is $x = -\frac{5}{3}$, HA is $y = \frac{2}{3}$

When $x = 0$, $y = -\frac{3}{5}$

$\therefore$ y-intercept is $-\frac{3}{5}$

When $y = 0$, $2x - 3 = 0$

$\therefore$ $x = \frac{3}{2}$

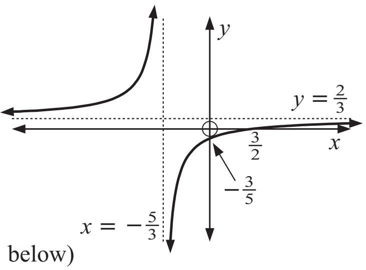
$\therefore$ x-intercept is $\frac{3}{2}$

As $x \to +\infty$, $y \to \frac{2}{3}$ (from below)

As $x \to -\infty$, $y \to \frac{2}{3}$ (from above)

As $x \to -\frac{5}{3}$ (from the left), $y \to +\infty$

As $x \to -\frac{5}{3}$ (from the right), $y \to -\infty$



5 VA is $x = 3$, HA is $y = -2$

$\therefore$ function has form $(x-3)(y+2) = k$ where k is a constant.

But (0, 0) lies on the curve $\quad \therefore \quad (0-3)(0+2) = k$

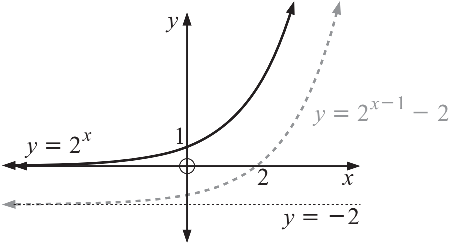
$$\therefore \quad k = -6$$

$$\begin{aligned} \therefore \text{ curve is } \quad (x-3)(y+2) &= -6 \\ \text{i.e., } \quad y+2 &= \frac{-6}{x-3} \\ \therefore \quad y &= \frac{-6}{x-3} - \frac{2}{1} \\ \therefore \quad y &= \frac{-6-2x+6}{x-3} \\ \text{i.e., } \quad y &= \frac{-2x}{x-3} \end{aligned}$$

6 **a** **i** Under translation $\begin{bmatrix}1\\-2\end{bmatrix}$,

$$\begin{aligned} y &= 2^x \\ \text{becomes} \quad y+2 &= 2^{x-1} \\ \text{i.e., } \quad y &= 2^{x-1} - 2 \end{aligned}$$

ii



iii For $y = 2^x$, HA is $y = 0$, no VA

For $y = 2^{x-1} - 2$, HA is $y = -2$, no VA

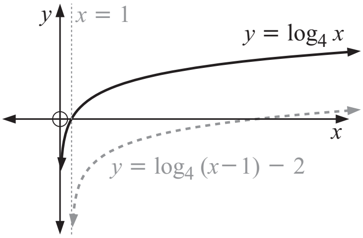
iv For $y = 2^x$, domain is $\{x\colon\ x \in \mathcal{R}\}$, range is $\{y\colon\ y > 0\}$

For $y = 2^{x-1} - 2$, domain is $\{x\colon\ x \in \mathcal{R}\}$, range is $\{y\colon\ y > -2\}$

b **i** Under translation $\begin{bmatrix}1\\-2\end{bmatrix}$,

$$\begin{aligned} y &= \log_4 x \\ \text{becomes} \quad y+2 &= \log_4(x-1) \\ \text{i.e., } \quad y &= \log_4(x-1) - 2 \end{aligned}$$

ii



iii For $y = \log_4 x$, VA is $x = 0$, no HA

For $y = \log_4(x-1) - 2$, VA is $x = 1$, no HA

iv For $y = \log_4 x$, domain is $\{x\colon\ x > 0\}$, range is $\{y\colon\ y \in \mathcal{R}\}$

For $y = \log_4(x-1) - 2$, domain is $\{x\colon\ x > 1\}$, range is $\{y \in \mathcal{R}\}$

Chapter 7

QUADRATIC EQUATIONS AND FUNCTIONS

EXERCISE 7A

1 **a** $y = 3x^2 - 4x + 1$ is of the form $y = ax^2 + bx + c$, where $a = 3$, $b = -4$ and $c = 1$.
Hence it is a quadratic function.

b $y = 5x - 7$ is of the form $y = ax^2 + bx + c$, but requires $a = 0$.
Hence it is not a quadratic function.

c $y = -x^2$ is of the form $y = ax^2 + bx + c$, where $a = -1$, $b = 0$ and $c = 0$.
Hence it is a quadratic function.

d $y = \frac{2}{3}x^2 + 4$ is of the form $y = ax^2 + bx + c$, where $a = \frac{2}{3}$, $b = 0$ and $c = 4$.
Hence it is a quadratic function.

e If $2y + 3x^2 - 5 = 0$, then $2y = -3x^2 + 5$ $\therefore$ $y = -\frac{3}{2}x^2 + \frac{5}{2}$

This is of the form $f(x) = ax^2 + bx + c$, where $a = -\frac{3}{2}$, $b = 0$ and $c = \frac{5}{2}$.
Hence it is a quadratic function.

f $y = 5x^3 + x - 6$ contains the term x^3, so is not of the form $y = ax^2 + bx + c$.
Hence it is not a quadratic function.

2 **a** When $x = 3$,
$$\begin{aligned} y &= 3^2 + 5 \times 3 - 4 \\ &= 9 + 15 - 4 \\ &= 20 \end{aligned}$$

b When $x = -3$,
$$\begin{aligned} y &= 2 \times (-3)^2 + 9 \\ &= 2 \times 9 + 9 \\ &= 27 \end{aligned}$$

c When $x = 1$,
$$\begin{aligned} y &= -2 \times 1^2 + 3 \times 1 - 5 \\ &= -2 + 3 - 5 \\ &= -4 \end{aligned}$$

d When $x = 4$,
$$\begin{aligned} y &= 4 \times 4^2 - 7 \times 4 + 1 \\ &= 64 - 28 + 1 \\ &= 37 \end{aligned}$$

3 **a** $f(2)$
$$\begin{aligned} &= 2^2 - 2 \times 2 + 3 \\ &= 4 - 4 + 3 \\ &= 3 \end{aligned}$$

b $f(-3)$
$$\begin{aligned} &= 4 - (-3)^2 \\ &= 4 - 9 \\ &= -5 \end{aligned}$$

c $f(0)$
$$\begin{aligned} &= -\tfrac{1}{4} \times 0^2 + 3 \times 0 - 4 \\ &= -4 \end{aligned}$$

d $f(2)$
$$\begin{aligned} &= \tfrac{1}{2} \times 2^2 + 3 \times 2 \\ &= 2 + 6 \\ &= 8 \end{aligned}$$

4 **a** $f(x) = 5x^2 - 10$
$$\begin{aligned} \therefore\ f(0) &= 5 \times 0 - 10 \\ &= -10 \end{aligned}$$
$\therefore$ $f(0) \neq 5$

$\therefore$ the function is not satisfied by the ordered pair.

b $y = 2x^2 + 5x - 3$

when $x = 4$,
$$\begin{aligned} y &= 2 \times 4^2 + 5 \times 4 - 3 \\ &= 32 + 20 - 3 \\ &= 49 \end{aligned}$$

$\therefore$ the function is not satisfied by the ordered pair.

c $y = -2x^2 + 3x$

when $x = -\frac{1}{2}$,
$$\begin{aligned} y &= -2 \times (-\tfrac{1}{2})^2 + 3(-\tfrac{1}{2}) \\ &= -2 \times \tfrac{1}{4} - \tfrac{3}{2} \\ &= -\tfrac{1}{2} - \tfrac{3}{2} \\ &= -2 \end{aligned}$$

$\therefore$ the function is not satisfied by the ordered pair.

d $y = -7x^2 + 8x + 15$

when $x = -1$,
$$\begin{aligned} y &= -7 \times (-1)^2 + 8 \times (-1) + 15 \\ &= -7 - 8 + 15 \\ &= 0 \end{aligned}$$

$\therefore$ the function is not satisfied by the ordered pair.

EXERCISE 7B.1

1 **a** **i** $y = (x+2)(x-2)$ has x-intercepts -2 and 2.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 0$.

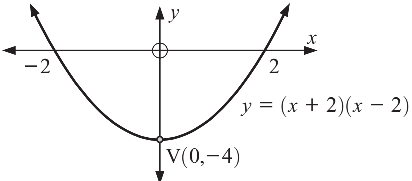
iii When $x = 0$, $y = 2 \times (-2) = -4$

$\therefore$ the vertex is at $(0, -4)$

iv From **iii**, when $x = 0$, $y = -4$

$\therefore$ the y-intercept is -4.

v



b **i** $y = 2(x-1)(x-3)$ has x-intercepts 1 and 3.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 2$.

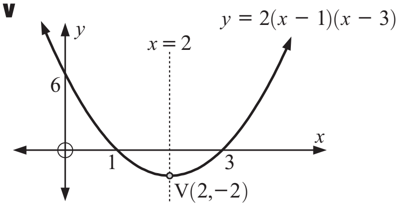
iii When $x = 2$, $y = 2 \times 1 \times (-1) = -2$

$\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 2(-1)(-3) = 6$

$\therefore$ the y-intercept is 6.

v



c **i** $y = 3(x-1)(x-2)$ has x-intercepts 1 and 2.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = \frac{3}{2}$.

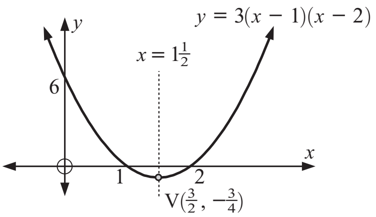
iii When $x = \frac{3}{2}$, $y = 3 \times \frac{1}{2} \times (-\frac{1}{2}) = -\frac{3}{4}$

$\therefore$ the vertex is at $(\frac{3}{2}, -\frac{3}{4})$

iv When $x = 0$, $y = 3(-1)(-2) = 6$

$\therefore$ the y-intercept is 6.

v



d **i** $y = \frac{1}{2}x(x-4)$ has x-intercepts 0 and 4.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 2$.

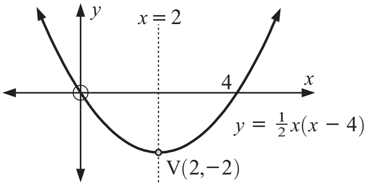
iii When $x = 2$, $y = \frac{1}{2} \times 2 \times (-2) = -2$

$\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 0$ also.

$\therefore$ the y-intercept is 0.

v



e **i** $y = -2x(x+3)$ has x-intercepts 0 and -3.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -\frac{3}{2}$.

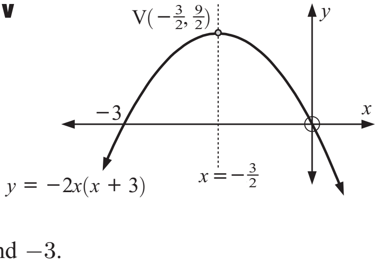
iii When $x = -\frac{3}{2}$, $y = -2 \times (-\frac{3}{2}) \times (\frac{3}{2}) = \frac{9}{2}$

$\therefore$ the vertex is at $(-\frac{3}{2}, \frac{9}{2})$

iv When $x = 0$, $y = 0$ also.

$\therefore$ the y-intercept is 0.

v



f **i** $y = -\frac{1}{2}(x+2)(x+3)$ has x-intercepts -2 and -3.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -\frac{5}{2}$.

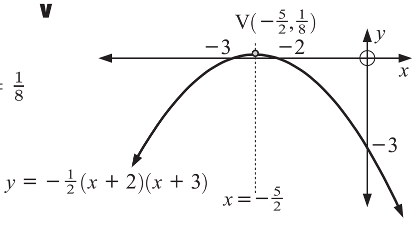
iii When $x = -\frac{5}{2}$, $y = -\frac{1}{2} \times (-\frac{1}{2}) \times (\frac{1}{2}) = \frac{1}{8}$

$\therefore$ the vertex is at $(-\frac{5}{2}, \frac{1}{8})$

iv When $x = 0$, $y = -\frac{1}{2} \times 2 \times 3 = -3$

$\therefore$ the y-intercept is -3.

v



2 **a** $y = x(x-2)$ has x-intercepts 0 and 2.
Its axis of symmetry is $x = 1$, so its vertex has $y = 1 \times (-1) = -1$.
Hence the graph is **B**.

b $y = 3x(x-2)$ also has x-intercepts 0 and 2.
Its axis of symmetry is $x = 1$, so its vertex has $y = 3 \times 1 \times (-1) = -3$.
Hence the graph is **A**.

c $y = -x(x-2)$ also has x-intercepts 0 and 2.
$\therefore$ its graph must be the remaining one like this, i.e., **F**.
Check: axis of symmetry is $x = 1$,
so its vertex has $y = -1 \times (-1) = 1$, which is true.

d $y = (x+2)(x-1)$ has x-intercepts -2 and 1.
Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{4}$.
Hence the graph is **D**.

e $y = 2(x+2)(x-1)$ also has x-intercepts -2 and 1.
Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = 2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{2}$.
Hence the graph is **E**.

f $y = -2(x+2)(x-1)$ also has x-intercepts -2 and 1.
Its graph must be the only one remaining, i.e., **C**.
Check: axis of symmetry is $x = -\frac{1}{2}$,
so its vertex has $y = -2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = \frac{9}{2}$, which is true.

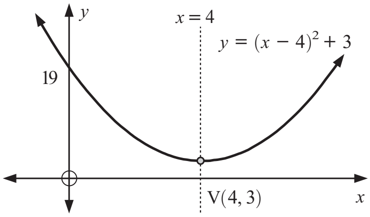
EXERCISE 7B.2

1 **a** **i** $y = (x-4)^2 + 3$ has axis of symmetry $x = 4$.

ii Its vertex is (4, 3).

iii When $x = 0$, $y = (-4)^2 + 3 = 19$
$\therefore$ the y-intercept is 19.

iv

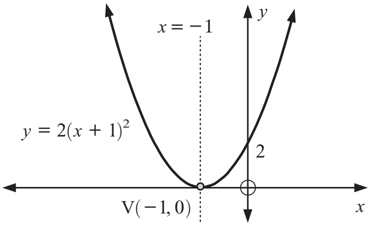


b **i** $y = 2(x+1)^2$ has axis of symmetry $x = -1$.

ii Its vertex is $(-1, 0)$.

iii When $x = 0$, $y = 2 \times 1^2 = 2$
$\therefore$ the y-intercept is 2.

iv

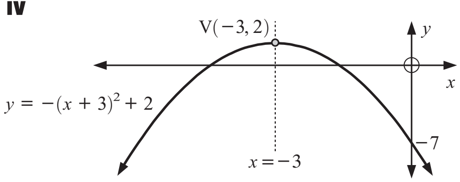


c **i** $y = -(x+3)^2 + 2$ has axis of symmetry $x = -3$.

ii Its vertex is $(-3, 2)$.

iii When $x = 0$, $y = -3^2 + 2 = -7$
$\therefore$ the y-intercept is -7.

iv

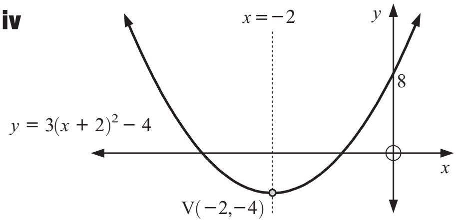


d **i** $y = 3(x+2)^2 - 4$ has axis of symmetry $x = -2$.

ii Its vertex is $(-2, -4)$.

iii When $x = 0$, $y = 3 \times 2^2 - 4 = 8$
$\therefore$ the y-intercept is 8.

iv



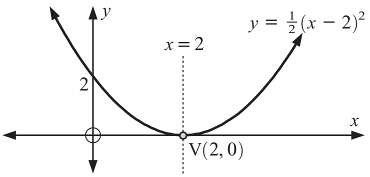
e **i** $y = \frac{1}{2}(x-2)^2$ has axis of symmetry $x = 2$.

ii Its vertex is $(2,\ 0)$.

iii When $x = 0$, $y = \frac{1}{2} \times (-2)^2 = 2$

$\therefore$ the y-intercept is 2.

iv



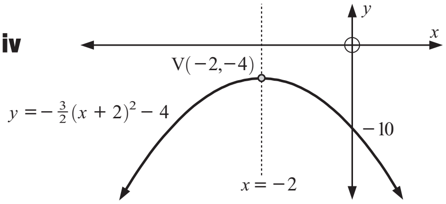
f **i** $y = -\frac{3}{2}(x+2)^2 - 4$ has axis of symmetry $x = -2$.

ii Its vertex is $(-2,\ -4)$.

iii When $x = 0$, $y = -\frac{3}{2} \times 2^2 - 4 = -10$

$\therefore$ the y-intercept is -10.

iv



2 **a** $y = -(x+1)^2 + 3$ has vertex $(-1,\ 3)$, so its graph must be **G**.

b $y = -2(x-3)^2 + 2$ has vertex $(3,\ 2)$, so its graph must be **A**.

c $y = x^2 + 2$ has vertex $(0,\ 2)$, so its graph must be **E**.

d $y = -(x-1)^2 + 1$ has vertex $(1,\ 1)$, so its graph must be **B**.

e $y = (x-2)^2 - 2$ has vertex $(2,\ -2)$, so its graph must be **I**.

f $y = \frac{1}{3}(x+3)^2 - 3$ has vertex $(-3,\ -3)$, so its graph must be **C**.

g $y = -x^2$ has vertex $(0,\ 0)$, so its graph must be **D**.

h $y = -\frac{1}{2}(x-1)^2 + 1$ has vertex $(1,\ 1)$, so its graph must be **F**.

i $y = 2(x+2)^2 - 1$ has vertex $(-2,\ -1)$, so its graph must be **H**.

3 **a** The graph has x-intercepts 0 and 4,
and its axis of symmetry is midway between them, i.e., $x = 2$.

b The graph has x-intercepts -5 and 0,
and its axis of symmetry is midway between them, i.e., $x = -\frac{5}{2}$.

c The graph has x-intercepts -1 and 3,
and its axis of symmetry is midway between them, i.e., $x = 1$.

d The graph touches the x-axis at its vertex $(3,\ 0)$
$\therefore$ its axis of symmetry is $x = 3$.

e The graph touches the x-axis at its vertex $(-4,\ 0)$
$\therefore$ its axis of symmetry is $x = -4$.

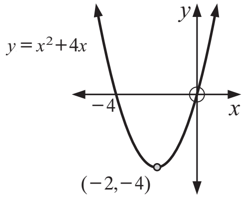
f Both $(-8,\ -5)$ and $(0,\ -5)$ lie on the graph. Since they have the same y-coordinate, the axis of symmetry must lie midway between them, i.e., $x = -4$.

4 **a** **i** $y = x^2 + 4x$ has x-intercepts -4 and 0, and y-intercept 0.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = -2$.

iii When $x = -2$, $y = (-2)^2 + 4(-2) = -4$

$\therefore$ the vertex is $(-2,\ -4)$.

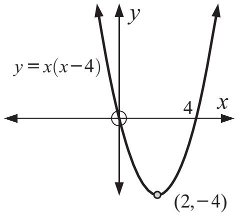


b **i** $y = x(x-4)$ has x-intercepts 0 and 4, and y-intercept 0.

ii The axis of symmetry is midway between the x-intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = 2(-2) = -4$

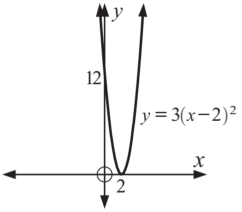
$\therefore$ the vertex is $(2,\ -4)$.



c **i** $y = 3(x-2)^2$ has x-intercept 2
and y-intercept $= 3 \times (-2)^2 = 12$.

ii Since the vertex is when $x = 2$,
$x = 2$ is the axis of symmetry.

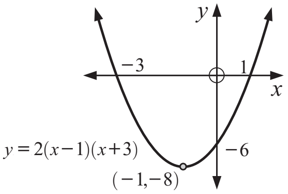
iii The vertex is (2, 0).



d **i** $y = 2(x-1)(x+3)$ has x-intercepts -3 and 1,
and y-intercept -6.

ii The axis of symmetry is midway
between the x-intercepts, i.e., $x = -1$.

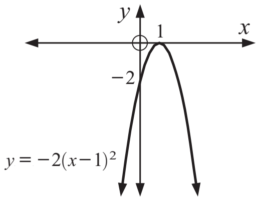
iii When $x = -1$, $y = 2(-2)(2) = -8$
$\therefore$ the vertex is $(-1, -8)$.



e **i** $y = -2(x-1)^2$ has x-intercept 1
and y-intercept $= -2 \times (-1)^2 = -2$.

ii Since the vertex is when $x = 1$,
$x = 1$ is the axis of symmetry.

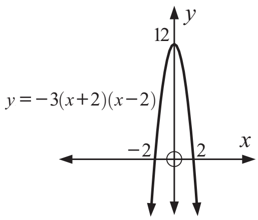
iii The vertex is (1, 0).



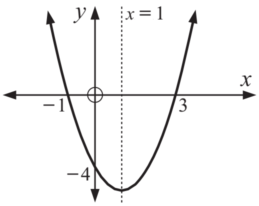
f **i** $y = -3(x+2)(x-2)$ has x-intercepts -2 and 2,
and y-intercept $= -3(2)(-2) = 12$.

ii The axis of symmetry is midway
between the x-intercepts, i.e., $x = 0$.

iii Since the y-intercept $= 12$,
the vertex is (0, 12).

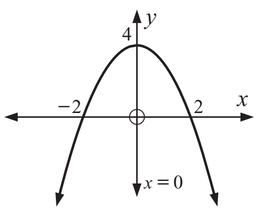


5 **a** **i**



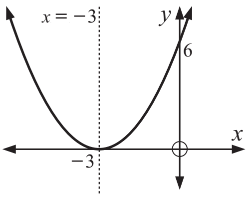
ii The axis of symmetry lies halfway between the x-intercepts,
i.e., the axis of symmetry is $x = 1$.

b **i**



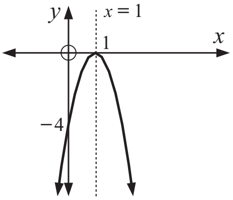
ii The axis of symmetry lies halfway between the x-intercepts,
i.e., the axis of symmetry is $x = 0$.

c **i**



ii Since the vertex is at $x = -3$,
the axis of symmetry is $x = -3$.

d **i**



ii Since the vertex is at $x = 1$, the axis of symmetry is $x = 1$.

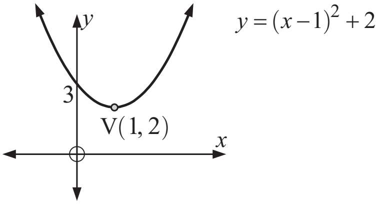
6 **a** The axis of symmetry, $x = -3$, lies midway between the x-intercepts.
$\therefore$ since one x-intercept is -1, the other is -5
$\therefore$ the x-intercepts are -1 and -5.

b Since the graph touches the x-axis at 3 and it is a quadratic function, the only x-intercept is 3 (touching).

EXERCISE 7C

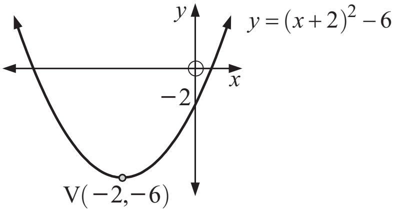
1 **a**

$$y = x^2 - 2x + 3$$
$$\therefore\ y = x^2 - 2x + 1^2 + 3 - 1^2$$
$$\therefore\ y = (x-1)^2 + 2$$
$\therefore$ vertex is (1, 2),
y-intercept is 3



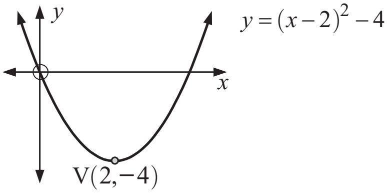
b

$$y = x^2 + 4x - 2$$
$$\therefore\ y = x^2 + 4x + 2^2 - 2 - 2^2$$
$$\therefore\ y = (x+2)^2 - 6$$
$\therefore$ vertex is (−2, −6),
y-intercept is −2



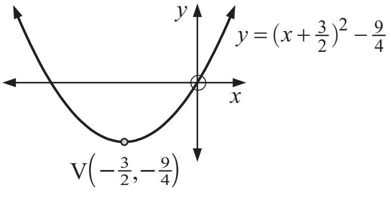
c

$$y = x^2 - 4x$$
$$\therefore\ y = x^2 - 4x + 2^2 - 2^2$$
$$\therefore\ y = (x-2)^2 - 4$$
$\therefore$ vertex is (2, −4),
y-intercept is 0



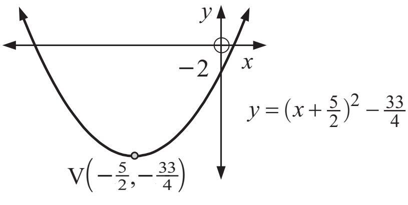
d

$$y = x^2 + 3x$$
$$\therefore\ y = x^2 + 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2$$
$$\therefore\ y = (x + \tfrac{3}{2})^2 - \tfrac{9}{4}$$
$\therefore$ vertex is $(-\frac{3}{2}, -\frac{9}{4})$,
y-intercept is 0



e

$$y = x^2 + 5x - 2$$
$$\therefore\ y = x^2 + 5x + (\tfrac{5}{2})^2 - 2 - (\tfrac{5}{2})^2$$
$$\therefore\ y = (x + \tfrac{5}{2})^2 - \tfrac{33}{4}$$
$\therefore$ vertex is $(-\frac{5}{2}, -\frac{33}{4})$,
y-intercept is −2



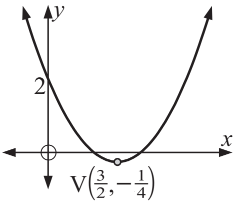
f $y = x^2 - 3x + 2$

$\therefore\ y = x^2 - 3x + (\frac{3}{2})^2 + 2 - (\frac{3}{2})^2$

$\therefore\ y = (x - \frac{3}{2})^2 - \frac{1}{4}$

$\therefore$ vertex is $(\frac{3}{2}, -\frac{1}{4})$,

y-intercept is 2



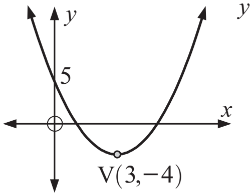
g $y = x^2 - 6x + 5$

$\therefore\ y = x^2 - 6x + 3^2 + 5 - 3^2$

$\therefore\ y = (x - 3)^2 - 4$

$\therefore$ vertex is $(3, -4)$,

y-intercept is 5



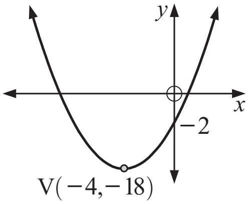
h $y = x^2 + 8x - 2$

$\therefore\ y = x^2 + 8x + 4^2 - 2 - 4^2$

$\therefore\ y = (x + 4)^2 - 18$

$\therefore$ vertex is $(-4, -18)$,

y-intercept is -2



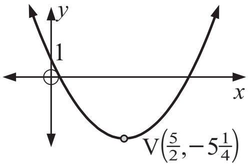
i $y = x^2 - 5x + 1$

$\therefore\ y = x^2 - 5x + (\frac{5}{2})^2 + 1 - (\frac{5}{2})^2$

$\therefore\ y = (x - \frac{5}{2})^2 - \frac{21}{4}$

$\therefore$ vertex is $(\frac{5}{2}, -5\frac{1}{4})$,

y-intercept is 1



2 **a** **i**

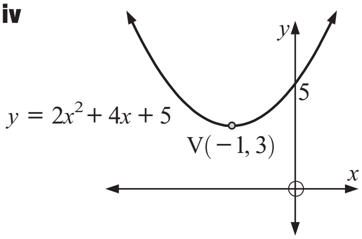
$$\begin{aligned} y &= 2x^2 + 4x + 5 \\ &= 2\left[x^2 + 2x + \tfrac{5}{2}\right] \\ &= 2\left[x^2 + 2x + 1^2 - 1^2 + \tfrac{5}{2}\right] \\ &= 2\left[(x+1)^2 + \tfrac{3}{2}\right] \\ &= 2(x+1)^2 + 3 \end{aligned}$$

ii The vertex is $(-1, 3)$.

iii When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5

iv



b **i**

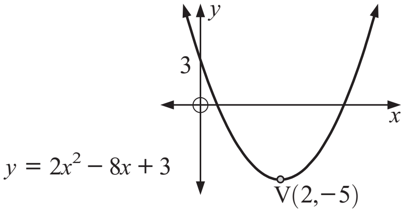
$$\begin{aligned} y &= 2x^2 - 8x + 3 \\ &= 2\left[x^2 - 4x + \tfrac{3}{2}\right] \\ &= 2\left[x^2 - 4x + 2^2 - 2^2 + \tfrac{3}{2}\right] \\ &= 2\left[(x-2)^2 - \tfrac{5}{2}\right] \\ &= 2(x-2)^2 - 5 \end{aligned}$$

ii The vertex is $(2, -5)$.

iii When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3

iv



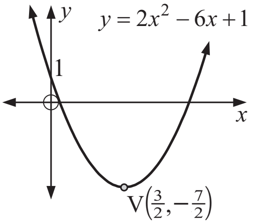
c **i**
$$\begin{aligned} y &= 2x^2 - 6x + 1 \\ &= 2\left[x^2 - 3x + \tfrac{1}{2}\right] \\ &= 2\left[x^2 - 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + \tfrac{1}{2}\right] \\ &= 2\left[(x - \tfrac{3}{2})^2 - \tfrac{7}{4}\right] \\ &= 2(x - \tfrac{3}{2})^2 - \tfrac{7}{2} \end{aligned}$$

ii The vertex is $(\frac{3}{2}, -\frac{7}{2})$.

iii When $x = 0$, $y = 1$

$\therefore$ the y-intercept is 1

iv



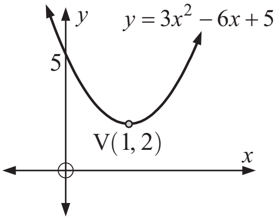
d **i**
$$\begin{aligned} y &= 3x^2 - 6x + 5 \\ &= 3\left[x^2 - 2x + \tfrac{5}{3}\right] \\ &= 3\left[x^2 - 2x + 1^2 - 1^2 + \tfrac{5}{3}\right] \\ &= 3\left[(x - 1)^2 + \tfrac{2}{3}\right] \\ &= 3(x - 1)^2 + 2 \end{aligned}$$

ii The vertex is $(1, 2)$.

iii When $x = 0$, $y = 5$

$\therefore$ the y-intercept is 5

iv



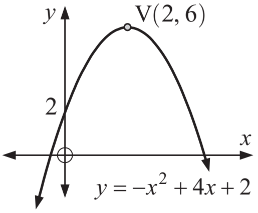
e **i**
$$\begin{aligned} y &= -x^2 + 4x + 2 \\ &= -\left[x^2 - 4x - 2\right] \\ &= -\left[x^2 - 4x + 2^2 - 2^2 - 2\right] \\ &= -\left[(x - 2)^2 - 6\right] \\ &= -(x - 2)^2 + 6 \end{aligned}$$

ii The vertex is $(2, 6)$.

iii When $x = 0$, $y = 2$

$\therefore$ the y-intercept is 2

iv



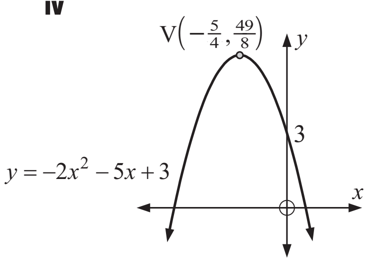
f **i**
$$\begin{aligned} y &= -2x^2 - 5x + 3 \\ &= -2\left[x^2 + \tfrac{5}{2}x - \tfrac{3}{2}\right] \\ &= -2\left[x^2 + \tfrac{5}{2}x + (\tfrac{5}{4})^2 - (\tfrac{5}{4})^2 - \tfrac{3}{2}\right] \\ &= -2\left[(x + \tfrac{5}{4})^2 - \tfrac{25}{16} - \tfrac{24}{16}\right] \\ &= -2\left[(x + \tfrac{5}{4})^2 - \tfrac{49}{16}\right] \\ &= -2(x + \tfrac{5}{4})^2 + \tfrac{49}{8} \end{aligned}$$

ii The vertex is $(-\frac{5}{4}, \frac{49}{8})$.

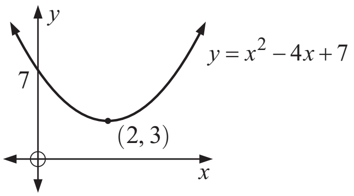
iii When $x = 0$, $y = 3$

$\therefore$ the y-intercept is 3

iv



3 **a** Using technology, the graph is



Since the vertex is at (2, 3), the function must be of the form

$y = a(x-2)^2 + 3$ for some a.

$\therefore$ when $x = 0$,

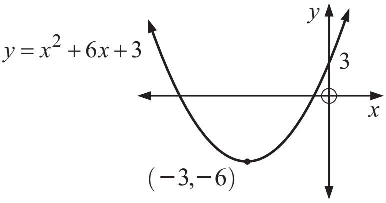
$y = a(-2)^2 + 3 = 4a + 3$

but the y-intercept is 7

$$\therefore \quad 4a + 3 = 7$$
$$\therefore \quad a = 1$$

$\therefore$ the equation is $y = (x-2)^2 + 3$

b Using technology, the graph is



Since the vertex is at $(-3, -6)$, the function must be of the form

$y = a(x+3)^2 - 6$ for some a.

$\therefore$ when $x = 0$,

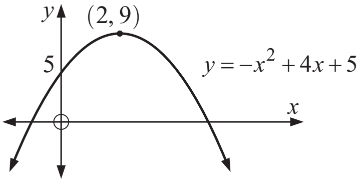
$y = a \times 3^2 - 6 = 9a - 6$

but the y-intercept is 3

$$\therefore \quad 9a - 6 = 3$$
$$\therefore \quad a = 1$$

$\therefore$ the equation is $y = (x+3)^2 - 6$

c Using technology, the graph is



Since the vertex is at (2, 9), the function must be of the form

$y = a(x-2)^2 + 9$ for some a.

$\therefore$ when $x = 0$,

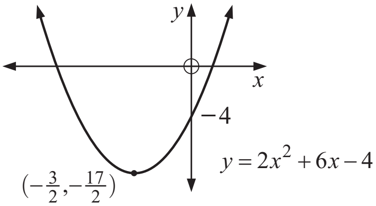
$y = a(-2)^2 + 9 = 4a + 9$

but the y-intercept is 5

$$\therefore \quad 4a + 9 = 5$$
$$\therefore \quad a = -1$$

$\therefore$ the equation is $y = -(x-2)^2 + 9$

d Using technology, the graph is



Since the vertex is at $(-\frac{3}{2}, -\frac{17}{2})$,

the function must be of the form

$y = a(x + \frac{3}{2})^2 - \frac{17}{2}$ for some a.

$\therefore$ when $x = 0$,

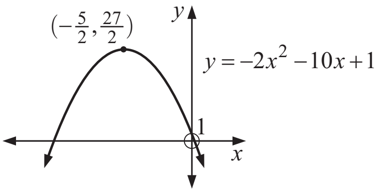
$y = a(\frac{3}{2})^2 - \frac{17}{2} = \frac{9}{4}a - \frac{17}{2}$

but the y-intercept is -4

$$\therefore \quad \tfrac{9}{4}a - \tfrac{17}{2} = -4$$
$$\therefore \quad \tfrac{9}{4}a = \tfrac{9}{2}$$
$$\therefore \quad a = \tfrac{4}{9} \times \tfrac{9}{2} = 2$$

$\therefore$ the equation is $y = 2(x + \frac{3}{2})^2 - \frac{17}{2}$

e Using technology, the graph is



Since the vertex is at $(-\frac{5}{2}, \frac{27}{2})$,

the function must be of the form

$y = a(x + \frac{5}{2})^2 + \frac{27}{2}$ for some a.

$\therefore$ when $x = 0$,

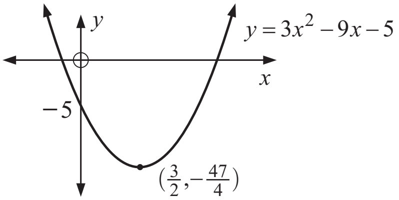
$y = a(\frac{5}{2})^2 + \frac{27}{2} = \frac{25}{4}a + \frac{27}{2}$

but the y-intercept is 1

$$\therefore \ \frac{25}{4}a + \frac{27}{2} = 1$$
$$\therefore \ \frac{25}{4}a = -\frac{25}{2}$$
$$\therefore \ a = -\frac{25}{2} \times \frac{4}{25} = -2$$

$\therefore$ the equation is $y = -2(x + \frac{5}{2})^2 + \frac{27}{2}$

f Using technology, the graph is



Since the vertex is at $(\frac{3}{2}, -\frac{47}{4})$,

the function must be of the form

$y = a(x - \frac{3}{2})^2 - \frac{47}{4}$ for some a.

$\therefore$ when $x = 0$,

$y = a(-\frac{3}{2})^2 - \frac{47}{4} = \frac{9}{4}a - \frac{47}{4}$

but the y-intercept is -5

$$\therefore \ \frac{9}{4}a - \frac{47}{4} = -5$$
$$\therefore \ \frac{9}{4}a = \frac{27}{4}$$
$$\therefore \ a = 3$$

$\therefore$ the equation is $y = 3(x - \frac{3}{2})^2 - \frac{47}{4}$

EXERCISE 7D.1

1 **a**
$$4x^2 + 7x = 0$$
$$\therefore \ x(4x + 7) = 0$$
$$\therefore \ x = 0 \text{ or } 4x + 7 = 0$$
{Null Factor law}
$$\therefore \ x = 0 \text{ or } -\tfrac{7}{4}$$

b
$$6x^2 + 2x = 0$$
$$\therefore \ 2x(3x + 1) = 0$$
$$\therefore \ x = 0 \text{ or } 3x + 1 = 0$$
{Null Factor law}
$$\therefore \ x = 0 \text{ or } -\tfrac{1}{3}$$

c
$$3x^2 - 7x = 0$$
$$\therefore \ x(3x - 7) = 0$$
$$\therefore \ x = 0 \text{ or } 3x - 7 = 0$$
{Null Factor law}
$$\therefore \ x = 0 \text{ or } \tfrac{7}{3}$$

d
$$2x^2 - 11x = 0$$
$$\therefore \ x(2x - 11) = 0$$
$$\therefore \ x = 0 \text{ or } 2x - 11 = 0$$
{Null Factor law}
$$\therefore \ x = 0 \text{ or } \tfrac{11}{2}$$

e
$$3x^2 = 8x$$
$$\therefore \ 3x^2 - 8x = 0$$
$$\therefore \ x(3x - 8) = 0$$
$$\therefore \ x = 0 \text{ or } 3x - 8 = 0$$
{Null Factor law}
$$\therefore \ x = 0 \text{ or } \tfrac{8}{3}$$

f
$$9x = 6x^2$$
$$\therefore \ 6x^2 - 9x = 0$$
$$\therefore \ 3x(2x - 3) = 0$$
$$\therefore \ x = 0 \text{ or } 2x - 3 = 0$$
{Null Factor law}
$$\therefore \ x = 0 \text{ or } \tfrac{3}{2}$$

g
$$x^2 - 5x + 6 = 0$$
$$\therefore \ (x - 2)(x - 3) = 0$$
$$\therefore \ x - 2 = 0 \text{ or } x - 3 = 0$$
{Null Factor law}
$$\therefore \ x = 2 \text{ or } 3$$

h
$$x^2 = 2x + 8$$
$$\therefore \ x^2 - 2x - 8 = 0$$
$$\therefore \ (x - 4)(x + 2) = 0$$
$$\therefore \ x - 4 = 0 \text{ or } x + 2 = 0$$
{Null Factor law}
$$\therefore \ x = -2 \text{ or } 4$$

i
$$x^2 + 21 = 10x$$
$$\therefore \ x^2 - 10x + 21 = 0$$
$$\therefore \ (x - 3)(x - 7) = 0$$
$$\therefore \ x - 3 = 0 \text{ or } x - 7 = 0$$
{Null Factor law}
$$\therefore \ x = 3 \text{ or } 7$$

j
$$9 + x^2 = 6x$$
$$\therefore \ x^2 - 6x + 9 = 0$$
$$\therefore \ (x - 3)^2 = 0$$
$$\therefore \ x - 3 = 0$$
$$\therefore \ x = 3$$

k
$$x^2 + x = 12$$
$$\therefore \ x^2 + x - 12 = 0$$
$$\therefore \ (x + 4)(x - 3) = 0$$
$$\therefore \ x + 4 = 0 \text{ or } x - 3 = 0$$
{Null Factor law}
$$\therefore \ x = -4 \text{ or } 3$$

l
$$x^2 + 8x = 33$$
$$\therefore \ x^2 + 8x - 33 = 0$$
$$\therefore \ (x + 11)(x - 3) = 0$$
$$\therefore \ x + 11 = 0 \text{ or } x - 3 = 0$$
{Null Factor law}
$$\therefore \ x = -11 \text{ or } 3$$

2 **a** $9x^2 - 12x + 4 = 0$
$\therefore\ (3x-2)^2 = 0$
$\therefore\ x = \frac{2}{3}$

b $2x^2 - 13x - 7 = 0$
$\therefore\ (2x+1)(x-7) = 0$
$\therefore\ x = -\frac{1}{2}$ or 7

c $3x^2 = 16x + 12$
$\therefore\ 3x^2 - 16x - 12 = 0$
$\therefore\ (3x+2)(x-6) = 0$
$\therefore\ x = -\frac{2}{3}$ or 6

d $3x^2 + 5x = 2$
$\therefore\ 3x^2 + 5x - 2 = 0$
$\therefore\ (3x-1)(x+2) = 0$
$\therefore\ x = \frac{1}{3}$ or -2

e $2x^2 + 3 = 5x$
$\therefore\ 2x^2 - 5x + 3 = 0$
$\therefore\ (2x-3)(x-1) = 0$
$\therefore\ x = \frac{3}{2}$ or 1

f $3x^2 = 4x + 4$
$\therefore\ 3x^2 - 4x - 4 = 0$
$\therefore\ (3x+2)(x-2) = 0$
$\therefore\ x = -\frac{2}{3}$ or 2

g $3x^2 = 10x + 8$
$\therefore\ 3x^2 - 10x - 8 = 0$
$\therefore\ (3x+2)(x-4) = 0$
$\therefore\ x = -\frac{2}{3}$ or 4

h $4x^2 + 4x = 3$
$\therefore\ 4x^2 + 4x - 3 = 0$
$\therefore\ (2x+3)(2x-1) = 0$
$\therefore\ x = -\frac{3}{2}$ or $\frac{1}{2}$

i $4x^2 = 11x + 3$
$\therefore\ 4x^2 - 11x - 3 = 0$
$\therefore\ (4x+1)(x-3) = 0$
$\therefore\ x = -\frac{1}{4}$ or 3

j $12x^2 = 11x + 15$
$\therefore\ 12x^2 - 11x - 15 = 0$
$\therefore\ (4x+3)(3x-5) = 0$
$\therefore\ x = -\frac{3}{4}$ or $\frac{5}{3}$

k $7x^2 + 6x = 1$
$\therefore\ 7x^2 + 6x - 1 = 0$
$\therefore\ (7x-1)(x+1) = 0$
$\therefore\ x = \frac{1}{7}$ or -1

l $15x^2 + 2x = 56$
$\therefore\ 15x^2 + 2x - 56 = 0$
$\therefore\ (15x-28)(x+2) = 0$
$\therefore\ x = \frac{28}{15}$ or -2

3 **a** $(x+1)^2 = 2x^2 - 5x + 11$
$\therefore\ x^2 + 2x + 1 = 2x^2 - 5x + 11$
$\therefore\ x^2 - 7x + 10 = 0$
$\therefore\ (x-2)(x-5) = 0$
$\therefore\ x = 2$ or 5

b $(x+2)(1-x) = -4$
$\therefore\ x - x^2 + 2 - 2x = -4$
$\therefore\ x^2 + x - 6 = 0$
$\therefore\ (x+3)(x-2) = 0$
$\therefore\ x = -3$ or 2

c $5 - 4x^2 = 3(2x+1) + 2$
$\therefore\ 5 - 4x^2 = 6x + 3 + 2$
$\therefore\ 4x^2 + 6x = 0$
$\therefore\ 2x(2x+3) = 0$
$\therefore\ x = 0$ or $-\frac{3}{2}$

d $x + \dfrac{2}{x} = 3$
$\therefore\ x^2 + 2 = 3x$
$\therefore\ x^2 - 3x + 2 = 0$
$\therefore\ (x-1)(x-2) = 0$
$\therefore\ x = 1$ or 2

e $2x - \dfrac{1}{x} = -1$
$\therefore\ 2x^2 - 1 = -x$
$\therefore\ 2x^2 + x - 1 = 0$
$\therefore\ (2x-1)(x+1) = 0$
$\therefore\ x = \frac{1}{2}$ or -1

f $\dfrac{x+3}{1-x} = -\dfrac{9}{x}$
$\therefore\ x(x+3) = -9(1-x)$
$\therefore\ x^2 + 3x = -9 + 9x$
$\therefore\ x^2 - 6x + 9 = 0$
$\therefore\ (x-3)^2 = 0$
$\therefore\ x = 3$

4 **a** If $y = 1$, then $x^2 + 6x + 10 = 1$
$\therefore\ x^2 + 6x + 9 = 0$
$\therefore\ (x+3)^2 = 0$
$\therefore\ x = -3$

b If $y = 2$, then $x^2 + 5x + 8 = 2$
$\therefore\ x^2 + 5x + 6 = 0$
$\therefore\ (x+3)(x+2) = 0$
$\therefore\ x = -3$ or -2

c If $y = -3$, then $x^2 - 5x + 1 = -3$
$\therefore\ x^2 - 5x + 4 = 0$
$\therefore\ (x-4)(x-1) = 0$
$\therefore\ x = 4$ or 1

d If $y = -3$, then $3x^2 = -3$
$\therefore\ x^2 = -1$
$\therefore$ there is no solution

5 **a**
$$\begin{aligned} f(x) &= 5 \\ \therefore\ 3x^2-2x+5 &= 5 \\ \therefore\ 3x^2-2x &= 0 \\ \therefore\ x(3x-2) &= 0 \\ \therefore\ x &= 0 \text{ or } \tfrac{2}{3} \end{aligned}$$

b
$$\begin{aligned} f(x) &= 1 \\ \therefore\ x^2-x-5 &= 1 \\ \therefore\ x^2-x-6 &= 0 \\ \therefore\ (x+2)(x-3) &= 0 \\ \therefore\ x &= -2 \text{ or } 3 \end{aligned}$$

c
$$\begin{aligned} f(x) &= -4 \\ \therefore\ -2x^2-13x+3 &= -4 \\ \therefore\ 2x^2+13x-7 &= 0 \\ \therefore\ (2x-1)(x+7) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } -7 \end{aligned}$$

d
$$\begin{aligned} f(x) &= -17 \\ \therefore\ 2x^2-12x+1 &= -17 \\ \therefore\ 2x^2-12x+18 &= 0 \\ \therefore\ 2(x^2-6x+9) &= 0 \\ \therefore\ 2(x-3)^2 &= 0 \\ \therefore\ x &= 3 \end{aligned}$$

6 **a** **i**
$$\begin{aligned} h(1) &= 30-5 \\ &= 25 \text{ m} \end{aligned}$$

ii
$$\begin{aligned} h(5) &= 30\times 5-5\times 5^2 \\ &= 150-125 \\ &= 25 \text{ m} \end{aligned}$$

iii
$$\begin{aligned} h(3) &= 30\times 3-5\times 3^2 \\ &= 90-45 \\ &= 45 \text{ m} \end{aligned}$$

b **i**
$$\begin{aligned} \text{When } h &= 40 \text{ m}, \\ 30t-5t^2 &= 40 \\ \therefore\ 5t^2-30t+40 &= 0 \\ \therefore\ 5(t^2-6t+8) &= 0 \\ \therefore\ 5(t-4)(t-2) &= 0 \\ \therefore\ t &= 4 \text{ or } 2 \text{ seconds} \end{aligned}$$

ii
$$\begin{aligned} \text{When } h &= 0 \text{ m}, \\ 30t-5t^2 &= 0 \\ \therefore\ 5t(6-t) &= 0 \\ \therefore\ t &= 0 \text{ or } 6 \text{ seconds} \end{aligned}$$

c We get two answers in each case because the object is projected up, and gravity brings it back down.

7 **a** **i**
$$\begin{aligned} P(0) &= -\tfrac{1}{4}\times 0+16\times 0-30 \\ &= -\$30 \end{aligned}$$

ii
$$\begin{aligned} P(10) &= -\tfrac{1}{4}\times 10^2+16\times 10-30 \\ &= -25+160-30 \\ &= \$105 \end{aligned}$$

b If $P(x)=57$, then
$$\begin{aligned} -\tfrac{1}{4}x^2+16x-30 &= 57 \\ \therefore\ \tfrac{1}{4}x^2-16x+87 &= 0 \\ \therefore\ x^2-64x+348 &= 0 \\ \therefore\ (x-6)(x-58) &= 0 \\ \therefore\ x &= 6 \text{ or } 58 \end{aligned}$$

i.e., either 6 or 58 cakes are made.

EXERCISE 7D.2

1 **a**
$$\begin{aligned} (x+5)^2 &= 2 \\ \therefore\ x+5 &= \pm\sqrt{2} \\ \therefore\ x &= -5\pm\sqrt{2} \end{aligned}$$

b
$$\begin{aligned} (x+6)^2 &= 11 \\ \therefore\ x+6 &= \pm\sqrt{11} \\ \therefore\ x &= -6\pm\sqrt{11} \end{aligned}$$

c
$$\begin{aligned} (x-4)^2 &= 8 \\ \therefore\ x-4 &= \pm\sqrt{8} \\ \therefore\ x &= 4\pm 2\sqrt{2} \end{aligned}$$

d
$$\begin{aligned} (x-8)^2 &= 7 \\ \therefore\ x-8 &= \pm\sqrt{7} \\ \therefore\ x &= 8\pm\sqrt{7} \end{aligned}$$

e
$$\begin{aligned} 2(x+3)^2 &= 10 \\ \therefore\ (x+3)^2 &= 5 \\ \therefore\ x+3 &= \pm\sqrt{5} \\ \therefore\ x &= -3\pm\sqrt{5} \end{aligned}$$

f
$$\begin{aligned} 3(x-2)^2 &= 18 \\ \therefore\ (x-2)^2 &= 6 \\ \therefore\ x-2 &= \pm\sqrt{6} \\ \therefore\ x &= 2\pm\sqrt{6} \end{aligned}$$

g
$$\begin{aligned} (x+1)^2+1 &= 11 \\ \therefore\ (x+1)^2 &= 10 \\ \therefore\ x+1 &= \pm\sqrt{10} \\ \therefore\ x &= -1\pm\sqrt{10} \end{aligned}$$

h
$$\begin{aligned} (2x+1)^2 &= 3 \\ \therefore\ 2x+1 &= \pm\sqrt{3} \\ \therefore\ 2x &= -1\pm\sqrt{3} \\ \therefore\ x &= -\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{3} \end{aligned}$$

2 **a**

$$\begin{aligned} x^2-4x+1 &= 0 \\ \therefore\ x^2-4x &= -1 \\ \therefore\ x^2-4x+(-2)^2 &= -1+(-2)^2 \\ \therefore\ (x-2)^2 &= 3 \\ \therefore\ x-2 &= \pm\sqrt{3} \\ \therefore\ x &= 2\pm\sqrt{3} \end{aligned}$$

b

$$\begin{aligned} x^2+6x+2 &= 0 \\ \therefore\ x^2+6x &= -2 \\ \therefore\ x^2+6x+3^2 &= -2+3^2 \\ \therefore\ (x+3)^2 &= 7 \\ \therefore\ x+3 &= \pm\sqrt{7} \\ \therefore\ x &= -3\pm\sqrt{7} \end{aligned}$$

c

$$\begin{aligned} x^2-14x+46 &= 0 \\ \therefore\ x^2-14x &= -46 \\ \therefore\ x^2-14x+(-7)^2 &= -46+(-7)^2 \\ \therefore\ (x-7)^2 &= 3 \\ \therefore\ x-7 &= \pm\sqrt{3} \\ \therefore\ x &= 7\pm\sqrt{3} \end{aligned}$$

d

$$\begin{aligned} x^2 &= 4x+3 \\ \therefore\ x^2-4x &= 3 \\ \therefore\ x^2-4x+(-2)^2 &= 3+(-2)^2 \\ \therefore\ (x-2)^2 &= 7 \\ \therefore\ x-2 &= \pm\sqrt{7} \\ \therefore\ x &= 2\pm\sqrt{7} \end{aligned}$$

e

$$\begin{aligned} x^2+6x+7 &= 0 \\ \therefore\ x^2+6x &= -7 \\ \therefore\ x^2+6x+3^2 &= -7+3^2 \\ \therefore\ (x+3)^2 &= 2 \\ \therefore\ x+3 &= \pm\sqrt{2} \\ \therefore\ x &= -3\pm\sqrt{2} \end{aligned}$$

f

$$\begin{aligned} x^2 &= 2x+6 \\ \therefore\ x^2-2x &= 6 \\ \therefore\ x^2-2x+(-1)^2 &= 6+(-1)^2 \\ \therefore\ (x-1)^2 &= 7 \\ \therefore\ x-1 &= \pm\sqrt{7} \\ \therefore\ x &= 1\pm\sqrt{7} \end{aligned}$$

g

$$\begin{aligned} x^2+6x &= 2 \\ \therefore\ x^2+6x+3^2 &= 2+3^2 \\ \therefore\ (x+3)^2 &= 11 \\ \therefore\ x+3 &= \pm\sqrt{11} \\ \therefore\ x &= -3\pm\sqrt{11} \end{aligned}$$

h

$$\begin{aligned} x^2+10 &= 8x \\ \therefore\ x^2-8x &= -10 \\ \therefore\ x^2-8x+(-4)^2 &= -10+(-4)^2 \\ \therefore\ (x-4)^2 &= 6 \\ \therefore\ x-4 &= \pm\sqrt{6} \\ \therefore\ x &= 4\pm\sqrt{6} \end{aligned}$$

i

$$\begin{aligned} x^2+6x &= -11 \\ \therefore\ x^2+6x+3^2 &= -11+3^2 \\ \therefore\ (x+3)^2 &= -2 \end{aligned}$$

$\therefore$ x has no real solutions, since the perfect square cannot be negative.

3 **a**

$$\begin{aligned} 2x^2+4x+1 &= 0 \\ \therefore\ x^2+2x+\tfrac{1}{2} &= 0 \\ \therefore\ x^2+2x &= -\tfrac{1}{2} \\ \therefore\ x^2+2x+1^2 &= -\tfrac{1}{2}+1^2 \\ \therefore\ (x+1)^2 &= \tfrac{1}{2} \\ \therefore\ x+1 &= \pm\tfrac{1}{\sqrt{2}} \\ \therefore\ x &= -1\pm\tfrac{1}{\sqrt{2}} \end{aligned}$$

b

$$\begin{aligned} 2x^2-10x+3 &= 0 \\ \therefore\ x^2-5x+\tfrac{3}{2} &= 0 \\ \therefore\ x^2-5x &= -\tfrac{3}{2} \\ \therefore\ x^2-5x+(-\tfrac{5}{2})^2 &= -\tfrac{3}{2}+(-\tfrac{5}{2})^2 \\ \therefore\ (x-\tfrac{5}{2})^2 &= -\tfrac{3}{2}+\tfrac{25}{4} \\ \therefore\ (x-\tfrac{5}{2})^2 &= \tfrac{19}{4} \\ \therefore\ x-\tfrac{5}{2} &= \pm\tfrac{\sqrt{19}}{2} \\ \therefore\ x &= \tfrac{5}{2}\pm\tfrac{\sqrt{19}}{2} \end{aligned}$$

c

$$\begin{aligned} 3x^2+12x+5 &= 0 \\ \therefore\ x^2+4x+\tfrac{5}{3} &= 0 \\ \therefore\ x^2+4x &= -\tfrac{5}{3} \\ \therefore\ x^2+4x+2^2 &= -\tfrac{5}{3}+2^2 \\ \therefore\ (x+2)^2 &= \tfrac{7}{3} \\ \therefore\ x+2 &= \pm\sqrt{\tfrac{7}{3}} \\ \therefore\ x &= -2\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

d

$$\begin{aligned} 3x^2 &= 6x+4 \\ \therefore\ x^2 &= 2x+\tfrac{4}{3} \\ \therefore\ x^2-2x &= \tfrac{4}{3} \\ \therefore\ x^2-2x+(-1)^2 &= \tfrac{4}{3}+(-1)^2 \\ \therefore\ (x-1)^2 &= \tfrac{7}{3} \\ \therefore\ x-1 &= \pm\sqrt{\tfrac{7}{3}} \\ \therefore\ x &= 1\pm\sqrt{\tfrac{7}{3}} \end{aligned}$$

e
$$\begin{aligned} 5x^2-15x+2 &= 0 \\ \therefore\ x^2-3x+\tfrac{2}{5} &= 0 \\ \therefore\ x^2-3x &= -\tfrac{2}{5} \\ \therefore\ x^2-3x+(-\tfrac{3}{2})^2 &= -\tfrac{2}{5}+(-\tfrac{3}{2})^2 \\ \therefore\ (x-\tfrac{3}{2})^2 &= -\tfrac{2}{5}+\tfrac{9}{4}=\tfrac{37}{20} \\ \therefore\ x-\tfrac{3}{2} &= \pm\sqrt{\tfrac{37}{20}} \\ \therefore\ x &= \tfrac{3}{2}\pm\sqrt{\tfrac{37}{20}} \end{aligned}$$

f
$$\begin{aligned} 4x^2+4x &= 5 \\ \therefore\ x^2+x &= \tfrac{5}{4} \\ \therefore\ x^2+x+(\tfrac{1}{2})^2 &= \tfrac{5}{4}+(\tfrac{1}{2})^2 \\ \therefore\ (x+\tfrac{1}{2})^2 &= \tfrac{6}{4} \\ \therefore\ x+\tfrac{1}{2} &= \pm\tfrac{\sqrt{6}}{2} \\ \therefore\ x &= -\tfrac{1}{2}\pm\tfrac{\sqrt{6}}{2} \end{aligned}$$

EXERCISE 7E

1 **a** $x^2-4x-3=0$

has $a=1,\quad b=-4,\quad c=-3$

$$\begin{aligned} \therefore\ x &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)} \\ &= \frac{4\pm\sqrt{28}}{2} \\ &= \frac{4\pm2\sqrt{7}}{2} \\ &= 2\pm\sqrt{7} \end{aligned}$$

b $x^2+6x+7=0$

has $a=1,\quad b=6,\quad c=7$

$$\begin{aligned} \therefore\ x &= \frac{-6\pm\sqrt{6^2-4(1)(7)}}{2(1)} \\ &= \frac{-6\pm\sqrt{8}}{2} \\ &= \frac{-6\pm2\sqrt{2}}{2} \\ &= -3\pm\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^2+1 &= 4x \\ \therefore\ x^2-4x+1 &= 0 \end{aligned}$$

which has $a=1,\quad b=-4,\quad c=1$

$$\begin{aligned} \therefore\ x &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)} \\ &= \frac{4\pm\sqrt{12}}{2} \\ &= \frac{4\pm2\sqrt{3}}{2} \\ &= 2\pm\sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2+4x &= 1 \\ \therefore\ x^2+4x-1 &= 0 \end{aligned}$$

which has $a=1,\quad b=4,\quad c=-1$

$$\begin{aligned} \therefore\ x &= \frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)} \\ &= \frac{-4\pm\sqrt{20}}{2} \\ &= \frac{-4\pm2\sqrt{5}}{2} \\ &= -2\pm\sqrt{5} \end{aligned}$$

e $x^2-4x+2=0$

has $a=1,\quad b=-4,\quad c=2$

$$\begin{aligned} \therefore\ x &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)} \\ &= \frac{4\pm\sqrt{8}}{2} \\ &= \frac{4\pm2\sqrt{2}}{2} \\ &= 2\pm\sqrt{2} \end{aligned}$$

f $2x^2-2x-3=0$

has $a=2,\quad b=-2,\quad c=-3$

$$\begin{aligned} \therefore\ x &= \frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)} \\ &= \frac{2\pm\sqrt{28}}{4} \\ &= \frac{2\pm2\sqrt{7}}{4} \\ &= \tfrac{1}{2}\pm\tfrac{\sqrt{7}}{2} \end{aligned}$$

g $x^2-2\sqrt{2}x+2=0$ has $a=1,\quad b=-2\sqrt{2},\quad c=2$

$$\begin{aligned} \therefore\ x &= \frac{-(-2\sqrt{2})\pm\sqrt{(-2\sqrt{2})^2-4(1)(2)}}{2(1)} \\ &= \frac{2\sqrt{2}\pm\sqrt{8-8}}{2}=\sqrt{2} \end{aligned}$$

h

$$\begin{aligned} (3x+1)^2 &= -2x \\ \therefore\ 9x^2+6x+1 &= -2x \\ \therefore\ 9x^2+8x+1 &= 0 \end{aligned}$$

which has $a=9,\quad b=8,\quad c=1$

$$\begin{aligned} \therefore\ x &= \frac{-8\pm\sqrt{8^2-4(9)(1)}}{2(9)} \\ &= \frac{-8\pm\sqrt{28}}{18} \\ &= \frac{-8\pm2\sqrt{7}}{18} \quad\text{or}\quad -\tfrac{4}{9}\pm\tfrac{\sqrt{7}}{9} \end{aligned}$$

i

$$\begin{aligned} (x+3)(2x+1) &= 9 \\ \therefore\ 2x^2+x+6x+3 &= 9 \\ \therefore\ 2x^2+7x-6 &= 0 \end{aligned}$$

which has $a=2,\quad b=7,\quad c=-6$

$$\begin{aligned} \therefore\ x &= \frac{-7\pm\sqrt{7^2-4(2)(-6)}}{2(2)} \\ &= \frac{-7\pm\sqrt{49+48}}{4} \\ &= -\tfrac{7}{4}\pm\tfrac{\sqrt{97}}{4} \end{aligned}$$

2 **a**

$$\begin{aligned} (x+2)(x-1) &= 2-3x \\ \therefore\ x^2-x+2x-2 &= 2-3x \\ \therefore\ x^2+4x-4 &= 0 \end{aligned}$$

which has $a=1,\quad b=4,\quad c=-4$

$$\begin{aligned} \therefore\ x &= \frac{-4\pm\sqrt{4^2-4(1)(-4)}}{2(1)} \\ &= \frac{-4\pm\sqrt{32}}{2} \\ &= \frac{-4\pm4\sqrt{2}}{2} \\ &= -2\pm2\sqrt{2} \end{aligned}$$

b

$$\begin{aligned} (2x+1)^2 &= 3-x \\ \therefore\ 4x^2+4x+1 &= 3-x \\ \therefore\ 4x^2+5x-2 &= 0 \end{aligned}$$

which has $a=4,\quad b=5,\quad c=-2$

$$\begin{aligned} \therefore\ x &= \frac{-5\pm\sqrt{5^2-4(4)(-2)}}{2(4)} \\ &= \frac{-5\pm\sqrt{25+32}}{8} \\ &= -\tfrac{5}{8}\pm\tfrac{\sqrt{57}}{8} \end{aligned}$$

c

$$\begin{aligned} (x-2)^2 &= 1+x \\ \therefore\ x^2-4x+4 &= 1+x \\ \therefore\ x^2-5x+3 &= 0 \end{aligned}$$

which has $a=1,\quad b=-5,\quad c=3$

$$\begin{aligned} \therefore\ x &= \frac{-(-5)\pm\sqrt{(-5)^2-4(1)(3)}}{2(1)} \\ &= \frac{5\pm\sqrt{25-12}}{2} \\ &= \tfrac{5}{2}\pm\tfrac{\sqrt{13}}{2} \end{aligned}$$

d

$$\begin{aligned} \frac{x-1}{2-x} &= 2x+1 \\ \therefore\ x-1 &= (2x+1)(2-x) \\ \therefore\ x-1 &= 4x-2x^2+2-x \\ \therefore\ 2x^2-2x-3 &= 0 \end{aligned}$$

which has $a=2,\quad b=-2,\quad c=-3$

$$\begin{aligned} \therefore\ x &= \frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)} \\ &= \frac{2\pm\sqrt{28}}{4} \\ &= \frac{2\pm2\sqrt{7}}{4} \quad\text{or}\quad \tfrac{1}{2}\pm\tfrac{\sqrt{7}}{2} \end{aligned}$$

e

$$\begin{aligned} x-\frac{1}{x} &= 1 \\ \therefore\ x^2-1 &= x \\ \therefore\ x^2-x-1 &= 0 \end{aligned}$$

which has $a=1,\quad b=-1,\quad c=-1$

$$\begin{aligned} \therefore\ x &= \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)} \\ &= \frac{1\pm\sqrt{1+4}}{2} \\ &= \tfrac{1}{2}\pm\tfrac{\sqrt{5}}{2} \end{aligned}$$

f

$$\begin{aligned} 2x-\frac{1}{x} &= 3 \\ \therefore\ 2x^2-1 &= 3x \\ \therefore\ 2x^2-3x-1 &= 0 \end{aligned}$$

which has $a=2,\quad b=-3,\quad c=-1$

$$\begin{aligned} \therefore\ x &= \frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-1)}}{2(2)} \\ &= \frac{3\pm\sqrt{9+8}}{4} \\ &= \tfrac{3}{4}\pm\tfrac{\sqrt{17}}{4} \end{aligned}$$

EXERCISE 7F

1 **a** $x = -3.414$ or -0.586 **b** $x = 0.317$ or -6.317 **c** $x = 2.77$ or -1.27
d $x = -1.08$ or 3.41 **e** $x = -0.892$ or 3.64 **f** $x = 1.34$ or -2.54

2 **a** $x = -4.83$ or 0.828 **b** $x = -1.57$ or 0.319 **c** $x = 0.697$ or 4.30
d $x = -0.823$ or 1.82 **e** $x = -0.618$ or 1.62 **f** $x = -0.281$ or 1.78

EXERCISE 7G

1 Let the smaller of the integers be x.
Since they differ by 12,
the other integer is $(x + 12)$.

$\therefore$ the sum of their squares is

$$\begin{aligned} x^2 + (x+12)^2 &= 74 \\ \therefore\ x^2 + x^2 + 24x + 144 &= 74 \\ \therefore\ 2x^2 + 24x + 70 &= 0 \\ \therefore\ x^2 + 12x + 35 &= 0 \\ \therefore\ (x+7)(x+5) &= 0 \\ \therefore\ x &= -7 \text{ or } -5 \end{aligned}$$

$\therefore$ the larger integer is 5 or 7

i.e., the integers are -7 and 5, or -5 and 7

2 Let the number be x, so its reciprocal is $\dfrac{1}{x}$.

They have sum

$$\begin{aligned} x + \frac{1}{x} &= 5\tfrac{1}{5} \\ \therefore\ x^2 + 1 &= \tfrac{26}{5}x \\ \therefore\ x^2 - \tfrac{26}{5}x + 1 &= 0 \\ \therefore\ 5x^2 - 26x + 5 &= 0 \\ \therefore\ (5x-1)(x-5) &= 0 \\ \therefore\ x &= \tfrac{1}{5} \text{ or } 5 \end{aligned}$$

i.e., the number is either $\frac{1}{5}$ or 5

3 Let the number be x so its square is x^2.

$$\begin{aligned} \therefore\ \text{the sum is}\quad x + x^2 &= 210 \\ \therefore\ x^2 + x - 210 &= 0 \\ \therefore\ (x+15)(x-14) &= 0 \\ \therefore\ x &= -15 \text{ or } 14 \end{aligned}$$

But x is a natural number, so $x > 0$,

$\therefore$ the number is 14.

4 Suppose the numbers are x and $(x+2)$.

$$\begin{aligned} \text{Then}\quad x(x+2) &= 360 \\ \therefore\ x^2 + 2x - 360 &= 0 \\ \therefore\ (x+20)(x-18) &= 0 \\ \therefore\ x &= -20 \text{ or } 18 \end{aligned}$$

$\therefore$ the numbers are -20 and -18,
or 18 and 20.

5 Suppose the numbers are x and $(x+2)$.

$$\begin{aligned} \text{Then}\quad x(x+2) &= 255 \\ \therefore\ x^2 + 2x - 255 &= 0 \\ \therefore\ (x+17)(x-15) &= 0 \\ \therefore\ x &= -17 \text{ or } 15 \end{aligned}$$

$\therefore$ the numbers are -17 and -15,
or 15 and 17.

6 If the polygon has n sides, then

$$\begin{aligned} \frac{n}{2}(n-3) &= 90 \\ \therefore\ \tfrac{1}{2}n^2 - \tfrac{3}{2}n &= 90 \\ \therefore\ n^2 - 3n - 180 &= 0 \\ \therefore\ (n-15)(n+12) &= 0 \\ \therefore\ n &= -12 \text{ or } 15 \end{aligned}$$

$\therefore$ the polygon has 15 sides. $\{$as $n > 0\}$

7 If the width of the rectangle is w cm,
then its length is $(w+4)$ cm.

$$\begin{aligned} \therefore\ \text{the area is}\quad w(w+4) &= 26 \\ \therefore\ w^2 + 4w - 26 &= 0 \end{aligned}$$

which has $a = 1$, $b = 4$, $c = -26$

$$\begin{aligned} \therefore\ w &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{120}}{2} = -2 \pm \sqrt{30} \end{aligned}$$

But $w > 0$, so

$$\begin{aligned} w &= -2 + \sqrt{30} \\ &\doteqdot 3.477 \text{ cm} \end{aligned}$$

i.e., the width is approximately 3.48 cm.

8 **a** The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm^2.

The box has height $(x+1)$ cm, so the area of each of the side faces is $x(x+1)$ cm.

$\therefore$ the total surface area is

$$\begin{aligned} A &= 2x^2 + 4x(x+1) \\ &= 2x^2 + 4x^2 + 4x \\ &= 6x^2 + 4x \text{ cm}^2 \end{aligned}$$

b

$$\begin{aligned} 6x^2 + 4x &= 240 \\ \therefore\ 3x^2 + 2x - 120 &= 0 \\ \therefore\ (3x+20)(x-6) &= 0 \\ \therefore\ x &= -\tfrac{20}{3} \text{ or } 6 \end{aligned}$$

but $x > 0$, so $x = 6$ cm

$\therefore$ the box is 6 cm × 6 cm × 7 cm

9 Suppose the tin plate was x cm × x cm. When 3 cm × 3 cm squares are cut from the corners, the base of the open box formed is $(x-6)$ cm × $(x-6)$ cm.

The open box has height 3 cm, so its volume is $3 \times (x-6) \times (x-6) = 80$

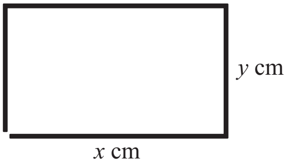
$$\begin{aligned} \therefore\ 3(x^2 - 12x + 36) &= 80 \\ \therefore\ 3x^2 - 36x + 108 &= 80 \\ \therefore\ 3x^2 - 36x + 28 &= 0 \end{aligned}$$

which has $a = 3$, $b = -36$, $c = 28$

$$\begin{aligned} \therefore\ x &= \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)} \\ &= \frac{36 \pm \sqrt{960}}{6} \quad \text{and since } x > 0, \\ x &= 6 + \frac{\sqrt{960}}{6} \doteqdot 11.16 \text{ cm} \end{aligned}$$

$\therefore$ the original piece of tinplate was about 11.2 cm square.

10



Suppose one side of the rectangle has length x cm and the other has length y cm.

The perimeter is $(2x + 2y)$ cm,

$$\begin{aligned} \text{so}\quad 2x + 2y &= 20 \\ \therefore\ 2y &= 20 - 2x \\ \therefore\ y &= 10 - x \end{aligned}$$

The area of the rectangle is therefore $x(10-x)$ cm^2.

$\therefore$ if the area is 30 cm^2, then

$$\begin{aligned} x(10-x) &= 30 \\ \therefore\ 10x - x^2 &= 30 \\ \therefore\ x^2 - 10x + 30 &= 0 \end{aligned}$$

which has $a = 1$, $b = -10$, $c = 30$

$$\begin{aligned} \therefore\ x &= \frac{-(-10)\sqrt{(-10)^2 - 4(1)(30)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 120}}{2} \\ &= \frac{10 \pm \sqrt{-120}}{2} \end{aligned}$$

$\therefore$ x has no real solutions, so it is not possible.

11 The smaller rectangle is similar to the original rectangle.

$$\therefore\ \frac{\text{AB}}{\text{AD}} = \frac{\text{BC}}{\text{BY}}$$

But AD = BC and

$$\begin{aligned} \text{BY} &= \text{AB} - \text{AY} \\ &= \text{AB} - \text{AD} \end{aligned}$$

$$\therefore\ \frac{\text{AB}}{\text{AD}} = \frac{\text{AD}}{\text{AB} - \text{AD}}$$

Suppose AB = x units, and AD = BC = 1 unit

$$\begin{aligned} \therefore\ \frac{x}{1} &= \frac{1}{x-1} \\ \therefore\ x(x-1) &= 1 \\ \therefore\ x^2 - x - 1 &= 0 \end{aligned}$$

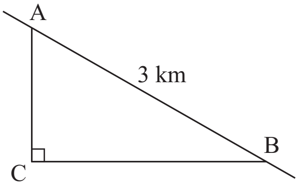
which has $a = 1$, $b = -1$, $c = -1$

$$\begin{aligned} \therefore\ x &= \frac{-(-1)\sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1+4}}{2} \\ &= \frac{1 \pm \sqrt{5}}{2} \\ \therefore\ x &= \frac{1 + \sqrt{5}}{2}, \quad \text{since } x > 0 \end{aligned}$$

But $\dfrac{\text{AB}}{\text{AD}} = x$, which is the golden ratio

$\therefore$ the golden ratio is $\dfrac{1+\sqrt{5}}{2}$

12



Suppose AC is x hundred metres,
so BC is $(x+4)$ hundred metres.

$$\begin{aligned} \text{Now}\quad \text{AC}^2+\text{BC}^2 &= \text{AB}^2 \quad \{\text{Pythagoras}\}\\ \therefore\quad x^2+(x+4)^2 &= 30^2\\ \therefore\quad x^2+x^2+8x+16 &= 900\\ \therefore\quad 2x^2+8x-884 &= 0\\ \therefore\quad x^2+4x-442 &= 0 \end{aligned}$$

which has $a=1,\quad b=4,\quad c=-442$

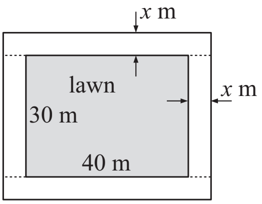
$$\begin{aligned} \therefore\quad x &= \frac{-4\pm\sqrt{4^2-4(1)(-442)}}{2(1)}\\ &= \frac{-4\pm\sqrt{1784}}{2} \end{aligned}$$

$$\therefore\quad \text{since}\ x>0,\quad x=\frac{-4\pm\sqrt{1784}}{2}\doteqdot 19.12$$

$\therefore$ AC $\doteqdot$ 19.12 hundred metres and BC $\doteqdot$ 23.12 hundred metres

$\therefore$ since the paddock is triangular, its area is $\frac{1}{2}\times 19.12\times 23.12\doteqdot 221$ hectares.

13



Suppose the concrete has width x m around the lawn. We divide the concrete up into four regions as shown.

The smaller regions have area $30x$ m^2, whilst the larger regions have area $x(40+2x)$ m^2.

Now the total area of concrete is one quarter the area of the lawn.

$$\begin{aligned} \therefore\quad 2\times 30x+2\times x(40+2x) &= \tfrac{1}{4}\times 30\times 40\\ \therefore\quad 60x+80x+4x^2 &= 300\\ \therefore\quad 4x^2+140x &= 300\\ \therefore\quad x^2+35x-75 &= 0 \end{aligned}$$

which has $a=1,\quad b=35,\quad c=-75$

$$\begin{aligned} \therefore\quad x &= \frac{-35\pm\sqrt{35^2-4(1)(-75)}}{2(1)}\\ &= \frac{-35\pm\sqrt{1525}}{2} \end{aligned}$$

But $x>0$,

$$\text{so}\quad x=\frac{-35+\sqrt{1525}}{2}\doteqdot 2.026\ \text{m}$$

$\therefore$ the path is about 2.03 m wide.

14 Suppose Hassan's speed is h kmph.

We know that $\text{speed}=\dfrac{\text{distance}}{\text{time}}$,

so $\text{time}=\dfrac{\text{distance}}{\text{speed}}$

$\therefore$ if it takes Hassan t hours, $t=\dfrac{40}{h}$ .... (1)

Now Chuong says he will drive home at speed $(h+40)$ kmph and arrive in time $(t-\frac{1}{3})$ hrs.

$$\therefore\quad t-\tfrac{1}{3}=\frac{40}{h+40}$$

$$\text{i.e.,}\quad t=\frac{40}{h+40}+\tfrac{1}{3}\quad \text{..... (2)}$$

$$\therefore\ \text{using (1) and (2),}\quad \frac{40}{h}=\frac{40}{h+40}+\tfrac{1}{3}$$

$$\begin{aligned} \therefore\quad 40(h+40) &= 40h+\tfrac{1}{3}h(h+40)\\ \therefore\quad 40h+1600 &= 40h+\tfrac{1}{3}h^2+\tfrac{40}{3}h\\ \therefore\quad h^2+40h-4800 &= 0 \end{aligned}$$

which has $a=1,\quad b=40,\quad c=-4800$

$$\begin{aligned} \therefore\quad h &= \frac{-40\pm\sqrt{40^2-4(1)(-4800)}}{2(1)}\\ &= \frac{-40\pm\sqrt{1600+19\,200}}{2}\\ &= \frac{-40\pm\sqrt{20\,800}}{2} \end{aligned}$$

$$\text{But}\ h>0,\ \text{so}\ h=\frac{-40+\sqrt{20\,800}}{2}\doteqdot 52.1\ \text{kmph}$$

i.e., Hassan's speed is approximately 52.1 kmph.

15 Suppose the speed of the plane is x kmph.

We know $\text{speed} = \dfrac{\text{distance}}{\text{time}}$, so $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

Using the information given, $\dfrac{1000}{x} = \dfrac{1000}{x-120} - \frac{1}{2}$

$$\therefore\ 1000(x-120) = 1000x - \tfrac{1}{2}x(x-120)$$
$$\therefore\ 1000x - 120\,000 = 1000x - \tfrac{1}{2}x^2 + 60x$$
$$\therefore\ x^2 - 120x - 240\,000 = 0$$

which has $a = 1$, $b = -120$, $c = -240\,000$

$$\therefore\ x = \frac{-(-120) \pm \sqrt{(-120)^2 - 4(1)(-240\,000)}}{2(1)}$$
$$= \frac{120 \pm \sqrt{974\,400}}{2}$$

But $x > 0$, so $x = \dfrac{120 + \sqrt{974\,400}}{2} \doteqdot 553.6$ kmph, $\therefore$ the plane has speed approximately 554 kmph.

16 Suppose the express train travels at x kmph (on average).

We know $\text{speed} = \dfrac{\text{distance}}{\text{time}}$, so $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$\therefore$ it takes the express train $\dfrac{105}{x}$ hours and the normal train $\dfrac{105}{x-10}$ hours.

$$\therefore\ \frac{105}{x} + \tfrac{1}{2} = \frac{105}{x-10}$$
$$\therefore\ 105(x-10) + \tfrac{1}{2}x(x-10) = 105x$$
$$\therefore\ 105x - 1050 + \tfrac{1}{2}x^2 - 5x = 105x$$
$$\therefore\ x^2 - 10x - 2100 = 0$$

which has $a = 1$, $b = -10$, $c = -2100$

$$\therefore\ x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-2100)}}{2(1)} = \frac{10 \pm \sqrt{8500}}{2}$$

But $x > 0$, so $x = \dfrac{10 + \sqrt{8500}}{2} \doteqdot 51.1$ kmph

$\therefore$ the express train travels on average at about 51.1 kmph.

17 Suppose n elderly citizens ended up going on the trip, so the cost per person was $\$\dfrac{160}{n}$.

If the original number of elderly citizens had gone, there would have been $(n+8)$,

and the cost per person would have been $\$\dfrac{160}{n+8}$.

$$\text{Hence}\ \frac{160}{n} = \frac{160}{n+8} + 1$$
$$\therefore\ 160(n+8) = 160n + n(n+8)$$
$$\therefore\ 160n + 1280 = 160n + n^2 + 8n$$
$$\therefore\ n^2 + 8n - 1280 = 0$$
$$\therefore\ (n-32)(n+40) = 0$$

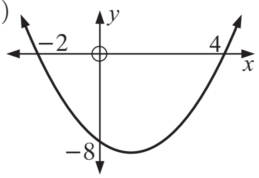
$\therefore$ since $n > 0$, $n = 32$, i.e., 32 elderly citizens went on the trip.

EXERCISE 7H

1 **a** $y = (x-4)(x+2)$

has x-intercepts -2 and 4

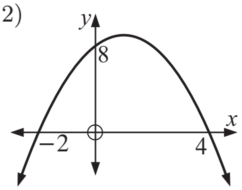
and y-intercept -8



b $y = -(x-4)(x+2)$

has x-intercepts -2 and 4

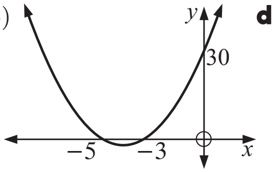
and y-intercept 8



c $y = 2(x+3)(x+5)$

has x-intercepts -5 and -3

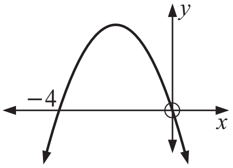
and y-intercept 30



d $y = -3x(x+4)$

has x-intercepts 0 and -4

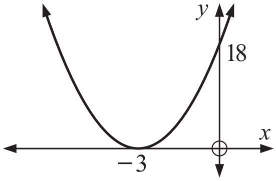
and y-intercept 0



e $y = 2(x+3)^2$

has x-intercept -3

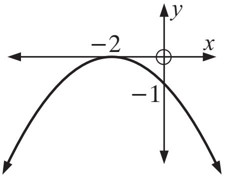
and y-intercept 18



f $y = -\frac{1}{4}(x+2)^2$

has x-intercept -2

and y-intercept -1



2 **a** The average of the x-intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.

b The average of the x-intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.

c The average of the x-intercepts is -4 $\therefore$ the axis of symmetry is $x = -4$.

d The average of the x-intercepts is -2 $\therefore$ the axis of symmetry is $x = -2$.

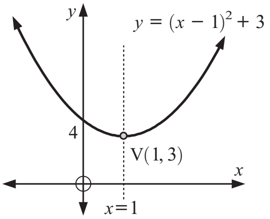
e The only x-intercept is -3, so the axis of symmetry is $x = -3$.

f The only x-intercept is -2, so the axis of symmetry is $x = -2$.

3 **a** The vertex is (1, 3).

The axis of symmetry is $x = 1$.

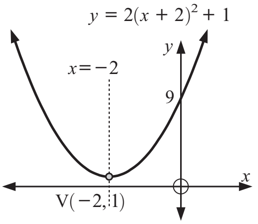
The y-intercept is 4.



b The vertex is (-2, 1).

The axis of symmetry is $x = -2$.

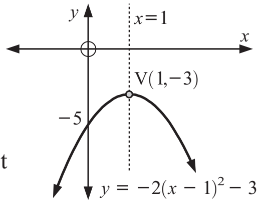
The y-intercept is 9.



c The vertex is (1, -3).

The axis of symmetry is $x = 1$.

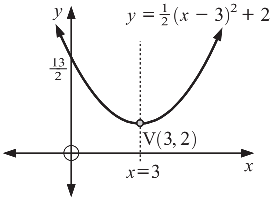
The y-intercept is -5.



d The vertex is (3, 2).

The axis of symmetry is $x = 3$.

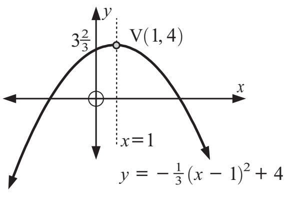
The y-intercept is $\frac{13}{2}$.



e The vertex is (1, 4).

The axis of symmetry is $x = 1$.

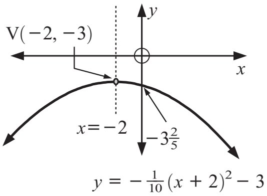
The y-intercept is $3\frac{2}{3}$.



f The vertex is (-2, -3).

The axis of symmetry is $x = -2$.

The y-intercept is $-3\frac{2}{5}$.



4 **a** $y = x^2 - 4x + 2$

has $a = 1,\quad b = -4,\quad c = 2$

$$\therefore\quad -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$$

$\therefore$ the axis of symmetry is $x = 2$

When $x = 2,$

$$y = 2^2 - 4\times 2 + 2 = -2$$

$\therefore$ the vertex is at $(2, -2)$.

b $y = x^2 + 2x - 3$

has $a = 1,\quad b = 2,\quad c = -3$

$$\therefore\quad -\frac{b}{2a} = -\frac{2}{2(1)} = -1$$

$\therefore$ the axis of symmetry is $x = -1$

When $x = -1,$

$$y = (-1)^2 + 2(-1) - 3 = -4$$

$\therefore$ the vertex is at $(-1, -4)$.

c $y = 2x^2 + 4$

has $a = 2,\quad b = 0,\quad c = 4$

$$\therefore\quad -\frac{b}{2a} = -\frac{0}{2(2)} = 0$$

$\therefore$ the axis of symmetry is $x = 0$

When $x = 0,\quad y = 4$

$\therefore$ the vertex is at $(0, 4)$.

d $y = -3x^2 + 1$

has $a = -3,\quad b = 0,\quad c = 1$

$$\therefore\quad -\frac{b}{2a} = -\frac{0}{2(-3)} = 0$$

$\therefore$ the axis of symmetry is $x = 0$

When $x = 0,\quad y = 1$

$\therefore$ the vertex is at $(0, 1)$.

e $y = 2x^2 + 8x - 7$

has $a = 2,\quad b = 8,\quad c = -7$

$$\therefore\quad -\frac{b}{2a} = -\frac{8}{2(2)} = -2$$

$\therefore$ the axis of symmetry is $x = -2$

When $x = -2,$

$$y = 2(-2)^2 + 8(-2) - 7 = -15$$

$\therefore$ the vertex is at $(-2, -15)$.

f $y = -x^2 - 4x - 9$

has $a = -1,\quad b = -4,\quad c = -9$

$$\therefore\quad -\frac{b}{2a} = -\frac{(-4)}{2(-1)} = -2$$

$\therefore$ the axis of symmetry is $x = -2$

When $x = -2,$
$$\begin{aligned} y &= -(-2)^2 - 4(-2) - 9 \\ &= -4 + 8 - 9 \\ &= -5 \end{aligned}$$

$\therefore$ the vertex is at $(-2, -5)$.

g $y = 2x^2 + 6x - 1$

has $a = 2,\quad b = 6,\quad c = -1$

$$\therefore\quad -\frac{b}{2a} = -\frac{6}{2(2)} = -\frac{3}{2}$$

$\therefore$ the axis of symmetry is $x = -\frac{3}{2}$

When $x = -\frac{3}{2},$

$$\begin{aligned} y &= 2(-\tfrac{3}{2})^2 + 6(-\tfrac{3}{2}) - 1 \\ &= \tfrac{9}{2} - 9 - 1 \\ &= -\tfrac{11}{2} \end{aligned}$$

$\therefore$ the vertex is at $(-\frac{3}{2}, -\frac{11}{2})$.

h $y = 2x^2 - 10x + 3$

has $a = 2,\quad b = -10,\quad c = 3$

$$\therefore\quad -\frac{b}{2a} = -\frac{(-10)}{2(2)} = \frac{5}{2}$$

$\therefore$ the axis of symmetry is $x = \frac{5}{2}$

When $x = \frac{5}{2},$
$$\begin{aligned} y &= 2(\tfrac{5}{2})^2 - 10(\tfrac{5}{2}) + 3 \\ &= \tfrac{25}{2} - \tfrac{50}{2} + 3 \\ &= -\tfrac{19}{2} \end{aligned}$$

$\therefore$ the vertex is at $(\frac{5}{2}, -\frac{19}{2})$.

i $y = -\frac{1}{2}x^2 + x - 5$

has $a = -\frac{1}{2},\quad b = 1,\quad c = -5$

$$\therefore\quad -\frac{b}{2a} = -\frac{1}{2(-\frac{1}{2})} = 1$$

$\therefore$ the axis of symmetry is $x = 1$

When $x = 1,$

$$y = -\tfrac{1}{2}(1)^2 + 1 - 5 = -\tfrac{9}{2}$$

$\therefore$ the vertex is at $(1, -\frac{9}{2})$.

j $y = -2x^2 + 8x - 2$

has $a = -2,\quad b = 8,\quad c = -2$

$$\therefore\quad -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$$

$\therefore$ the axis of symmetry is $x = 2$

When $x = 2,$
$$\begin{aligned} y &= -2(2)^2 + 8(2) - 2 \\ &= -8 + 16 - 2 \\ &= 6 \end{aligned}$$

$\therefore$ the vertex is at $(2, 6)$.

5 **a** When $y=0$, $x^2-9=0$

$\therefore\ (x+3)(x-3)=0$

$\therefore\ x=\pm 3$

$\therefore$ the x-intercepts are ± 3

b When $y=0$, $2x^2-6=0$

$\therefore\ x^2-3=0$

$\therefore\ (x+\sqrt{3})(x-\sqrt{3})=0$

$\therefore\ x=\pm\sqrt{3}$

$\therefore$ the x-intercepts are $\pm\sqrt{3}$

c When $y=0$, $x^2+7x+10=0$

$\therefore\ (x+5)(x+2)=0$

$\therefore\ x=-5$ or -2

$\therefore$ the x-intercepts are -5 and -2

d When $y=0$, $x^2+x-12=0$

$\therefore\ (x+4)(x-3)=0$

$\therefore\ x=-4$ or 3

$\therefore$ the x-intercepts are -4 and 3

e When $y=0$, $4x-x^2=0$

$\therefore\ x(4-x)=0$

$\therefore\ x=0$ or 4

$\therefore$ the x-intercepts are 0 and 4

f When $y=0$, $-x^2-6x-8=0$

$\therefore\ x^2+6x+8=0$

$\therefore\ (x+4)(x+2)=0$

$\therefore\ x=-4$ or -2

$\therefore$ the x-intercepts are -4 and -2

g When $y=0$, $-2x^2-4x-2=0$

$\therefore\ x^2+2x+1=0$

$\therefore\ (x+1)^2=0$

$\therefore\ x=-1$

$\therefore$ the x-intercept is -1 (touching)

h When $y=0$, $4x^2-24x+36=0$

$\therefore\ x^2-6x+9=0$

$\therefore\ (x-3)^2=0$

$\therefore\ x=3$

$\therefore$ the x-intercept is 3 (touching)

i When $y=0$, $x^2-4x+1=0$

$a=1$, $b=-4$ and $c=1$

$$\therefore\ x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)}$$

$$=\frac{4\pm\sqrt{12}}{2}$$

$$=\frac{4\pm 2\sqrt{3}}{2}$$

$$=2\pm\sqrt{3}$$

$\therefore$ the x-intercepts are $2\pm\sqrt{3}$

j When $y=0$, $x^2+4x-3=0$

$a=1$, $b=4$ and $c=-3$

$$\therefore\ x=\frac{-4\pm\sqrt{4^2-4(1)(-3)}}{2(1)}$$

$$=\frac{-4\pm\sqrt{28}}{2}$$

$$=\frac{-4\pm 2\sqrt{7}}{2}$$

$$=-2\pm\sqrt{7}$$

$\therefore$ the x-intercepts are $-2\pm\sqrt{7}$

k When $y=0$, $x^2-6x-2=0$

$a=1$, $b=-6$ and $c=-2$

$$\therefore\ x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-2)}}{2(1)}$$

$$=\frac{6\pm\sqrt{44}}{2}$$

$$=\frac{6\pm 2\sqrt{11}}{2}$$

$$=3\pm\sqrt{11}$$

$\therefore$ the x-intercepts are $3\pm\sqrt{11}$

l When $y=0$, $x^2+8x+11=0$

$a=1$, $b=8$ and $c=11$

$$\therefore\ x=\frac{-8\pm\sqrt{8^2-4(1)(11)}}{2(1)}$$

$$=\frac{-8\pm\sqrt{20}}{2}$$

$$=\frac{-8\pm 2\sqrt{5}}{2}$$

$$=-4\pm\sqrt{5}$$

$\therefore$ the x-intercepts are $-4\pm\sqrt{5}$

6 **a** **i** $y = x^2 - 2x + 5$

has $a = 1, \quad b = -2, \quad c = 5$

$$\therefore \quad -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$$

$\therefore$ the axis of symmetry is $x = 1$

ii When $x = 1$,

$$\begin{aligned} y &= 1^2 - 2(1) + 5 \\ &= 1 - 2 + 5 \\ &= 4 \end{aligned}$$

$\therefore$ the vertex is at $(1, 4)$

iii When $x = 0, \quad y = 5$,

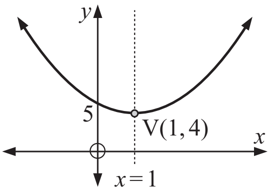
so the y-intercept is 5

When $y = 0, \quad x^2 - 2x + 5 = 0$

$$\begin{aligned} \therefore \quad x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{2 \pm \sqrt{4 - 20}}{2} \end{aligned}$$

This has no real solutions,
so there are no x-intercepts.

iv



b **i** $y = x^2 + 4x - 1$

has $a = 1, \quad b = 4, \quad c = -1$

$$\therefore \quad -\frac{b}{2a} = -\frac{4}{2(1)} = -2$$

$\therefore$ the axis of symmetry is $x = -2$

ii When $x = -2$,

$$\begin{aligned} y &= (-2)^2 + 4(-2) - 1 \\ &= 4 - 8 - 1 \\ &= -5 \end{aligned}$$

$\therefore$ the vertex is at $(-2, -5)$

iii When $x = 0, \quad y = -1$,

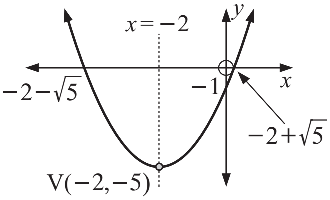
so the y-intercept is -1.

When $y = 0, \quad x^2 + 4x - 1 = 0$

$$\begin{aligned} \therefore \quad x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{20}}{2} \\ &= \frac{-4 \pm 2\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5} \end{aligned}$$

$\therefore$ the x-intercepts are $-2 \pm \sqrt{5}$

iv



c **i** $y = 2x^2 - 5x + 2$

has $a = 2, \quad b = -5, \quad c = 2$

$$\therefore \quad -\frac{b}{2a} = -\frac{(-5)}{2(2)} = \frac{5}{4}$$

$\therefore$ the axis of symmetry is $x = \frac{5}{4}$

ii When $x = \frac{5}{4}$,

$$\begin{aligned} y &= 2(\tfrac{5}{4})^2 - 5(\tfrac{5}{4}) + 2 \\ &= \tfrac{50}{16} - \tfrac{25}{4} + 2 \\ &= -\tfrac{9}{8} \end{aligned}$$

$\therefore$ the vertex is at $(\frac{5}{4}, -\frac{9}{8})$

iii When $x = 0, \quad y = 2$,

so the y-intercept is 2.

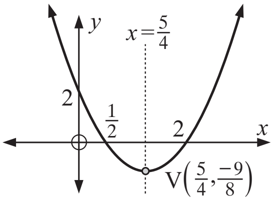
When $y = 0, \quad 2x^2 - 5x + 2 = 0$

$$\therefore \quad (2x - 1)(x - 2) = 0$$

$$\therefore \quad x = \tfrac{1}{2} \text{ or } 2$$

$\therefore$ the x-intercepts are $\frac{1}{2}$ and 2

iv



d **i** $y = -x^2 + 3x - 2$

has $a = -1$, $b = 3$, $c = -2$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{3}{2(-1)} = \dfrac{3}{2}$

$\therefore$ the axis of symmetry is $x = \frac{3}{2}$

ii When $x = \frac{3}{2}$,

$$\begin{aligned} y &= -(\tfrac{3}{2})^2 + 3(\tfrac{3}{2}) - 2 \\ &= -\tfrac{9}{4} + \tfrac{9}{2} - 2 \\ &= \tfrac{1}{4} \end{aligned}$$

$\therefore$ the vertex is at $(\frac{3}{2}, \frac{1}{4})$

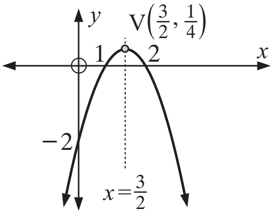
iii When $x = 0$, $y = -2$,

so the y-intercept is -2.

$$\begin{aligned} \text{When } y = 0, \quad -x^2 + 3x - 2 &= 0 \\ \therefore \quad x^2 - 3x + 2 &= 0 \\ \therefore \quad (x-1)(x-2) &= 0 \\ \therefore \quad x &= 1 \text{ or } 2 \end{aligned}$$

$\therefore$ the x-intercepts are 1 and 2

iv



e **i** $y = -3x^2 + 4x - 1$

has $a = -3$, $b = 4$, $c = -1$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{4}{2(-3)} = \dfrac{2}{3}$

$\therefore$ the axis of symmetry is $x = \frac{2}{3}$

ii When $x = \frac{2}{3}$,

$$\begin{aligned} y &= -3(\tfrac{2}{3})^2 + 4(\tfrac{2}{3}) - 1 \\ &= -\tfrac{4}{3} + \tfrac{8}{3} - 1 \\ &= \tfrac{1}{3} \end{aligned}$$

$\therefore$ the vertex is at $(\frac{2}{3}, \frac{1}{3})$

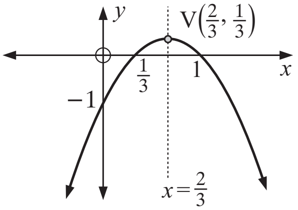
iii When $x = 0$, $y = -1$,

so the y-intercept is -1.

$$\begin{aligned} \text{When } y = 0, \quad -3x^2 + 4x - 1 &= 0 \\ \therefore \quad 3x^2 - 4x + 1 &= 0 \\ \therefore \quad (3x-1)(x-1) &= 0 \\ \therefore \quad x &= \tfrac{1}{3} \text{ or } 1 \end{aligned}$$

$\therefore$ the x-intercepts are $\frac{1}{3}$ and 1

iv



f **i** $y = -2x^2 + x + 1$

has $a = -2$, $b = 1$, $c = 1$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{1}{2(-2)} = \dfrac{1}{4}$

$\therefore$ the axis of symmetry is $x = \frac{1}{4}$

ii When $x = \frac{1}{4}$,

$$\begin{aligned} y &= -2(\tfrac{1}{4})^2 + \tfrac{1}{4} + 1 \\ &= -\tfrac{1}{8} + \tfrac{1}{4} + 1 \\ &= \tfrac{9}{8} \end{aligned}$$

$\therefore$ the vertex is at $(\frac{1}{4}, \frac{9}{8})$

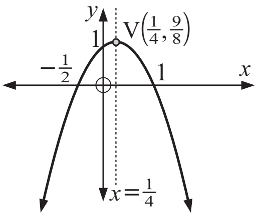
iii When $x = 0$, $y = 1$,

so the y-intercept is 1.

$$\begin{aligned} \text{When } y = 0, \quad -2x^2 + x + 1 &= 0 \\ \therefore \quad 2x^2 - x - 1 &= 0 \\ \therefore \quad (2x+1)(x-1) &= 0 \\ \therefore \quad x &= -\tfrac{1}{2} \text{ or } 1 \end{aligned}$$

$\therefore$ the x-intercepts are $-\frac{1}{2}$ and 1

iv



g **i** $y = 6x - x^2$

has $a = -1, \quad b = 6, \quad c = 0$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{6}{2(-1)} = 3$

$\therefore$ the axis of symmetry is $x = 3$

ii When $x = 3$,

$$\begin{aligned} y &= 6 \times 3 - 3^2 \\ &= 9 \end{aligned}$$

$\therefore$ the vertex is at (3, 9)

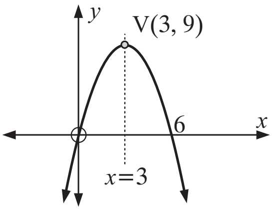
iii When $x = 0, \quad y = 0$,
so the y-intercept is 0.

When $y = 0$,

$$\begin{aligned} 6x - x^2 &= 0 \\ \therefore \quad x(6 - x) &= 0 \\ \therefore \quad x &= 0 \text{ or } 6 \end{aligned}$$

$\therefore$ the x-intercepts are 0 and 6

iv



h **i** $y = -x^2 - 6x - 8$

has $a = -1, \quad b = -6, \quad c = -8$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{(-6)}{2(-1)} = -3$

$\therefore$ the axis of symmetry is $x = -3$

ii When $x = -3$,

$$\begin{aligned} y &= -(-3)^2 - 6(-3) - 8 \\ &= -9 + 18 - 8 \\ &= 1 \end{aligned}$$

$\therefore$ the vertex is at (−3, 1)

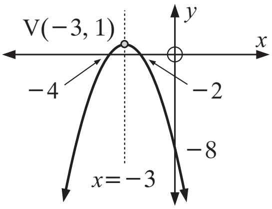
iii When $x = 0, \quad y = -8$,
so the y-intercept is −8.

When $y = 0$,

$$\begin{aligned} -x^2 - 6x - 8 &= 0 \\ \therefore \quad x^2 + 6x + 8 &= 0 \\ \therefore \quad (x + 4)(x + 2) &= 0 \\ \therefore \quad x &= -4 \text{ or } -2 \end{aligned}$$

$\therefore$ the x-intercepts are −4 and −2

iv



i **i** $y = -\frac{1}{4}x^2 + 2x + 1$

has $a = -\frac{1}{4}, \quad b = 2, \quad c = 1$

$\therefore \quad -\dfrac{b}{2a} = -\dfrac{2}{2(-\frac{1}{4})} = 4$

$\therefore$ the axis of symmetry is $x = 4$

ii When $x = 4$,

$$\begin{aligned} y &= -\tfrac{1}{4}(4)^2 + 2(4) + 1 \\ &= -4 + 8 + 1 \\ &= 5 \end{aligned}$$

$\therefore$ the vertex is at (4, 5)

iii When $x = 0, \quad y = 1$,
so the y-intercept is 1.

When $y = 0$,

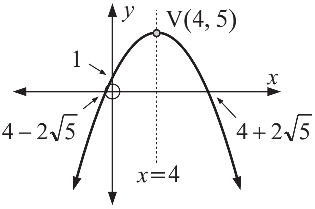
$$\begin{aligned} -\tfrac{1}{4}x^2 + 2x + 1 &= 0 \\ \therefore \quad x^2 - 8x - 4 &= 0 \end{aligned}$$

which has $a = 1, \quad b = -8, \quad c = -4$

$$\begin{aligned} \therefore \quad x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{8 \pm \sqrt{80}}{2} \\ &= \frac{8 \pm 4\sqrt{5}}{2} \\ &= 4 \pm 2\sqrt{5} \end{aligned}$$

$\therefore$ the x-intercepts are $4 \pm 2\sqrt{5}$

iv



EXERCISE 7I.1

1 **a** $x^2+7x-2=0$

has $a=1,\quad b=7,\quad c=-2$

$$\therefore\quad \Delta = b^2-4ac = 7^2-4(1)(-2) = 57$$

Since $\Delta>0$, there are 2 distinct real solutions.

b $x^2+4\sqrt{2}x+8=0$

has $a=1,\quad b=4\sqrt{2},\quad c=8$

$$\therefore\quad \Delta = b^2-4ac = (4\sqrt{2})^2-4(1)(8) = 32-32 = 0$$

$\therefore$ there is one repeated real root.

c $2x^2+3x-1=0$

has $a=2,\quad b=3,\quad c=-1$

$$\therefore\quad \Delta = b^2-4ac = 3^2-4(2)(-1) = 17$$

Since $\Delta>0$, there are 2 distinct real solutions.

d $6x^2+5x-4=0$

has $a=6,\quad b=5,\quad c=-4$

$$\therefore\quad \Delta = b^2-4ac = 5^2-4(6)(-4) = 121$$

Since $\Delta>0$, there are 2 distinct real solutions.

e $x^2+x+6=0$

has $a=1,\quad b=1,\quad c=6$

$$\therefore\quad \Delta = b^2-4ac = 1^2-4(1)(6) = -23$$

Since $\Delta<0$, there are no real roots.

f $9x^2+6x+1=0$

has $a=9,\quad b=6,\quad c=1$

$$\therefore\quad \Delta = b^2-4ac = 6^2-4(9)(1) = 0$$

$\therefore$ there is one repeated real root.

2 **a** $2x^2+7x-4=0$

has $a=2,\quad b=7,\quad c=-4$

$$\therefore\quad \Delta = b^2-4ac = 7^2-4(2)(-4) = 81$$

$\therefore$ $\sqrt{\Delta}=9$, so the equation has rational roots.

b $3x^2-7x-6=0$

has $a=3,\quad b=-7,\quad c=-6$

$$\therefore\quad \Delta = b^2-4ac = (-7)^2-4(3)(-6) = 121$$

$\therefore$ $\sqrt{\Delta}=11$, so the equation has rational roots.

c $2x^2+6x+1=0$

has $a=2,\quad b=6,\quad c=1$

$$\therefore\quad \Delta = b^2-4ac = 6^2-4(2)(1) = 28$$

$\therefore$ $\sqrt{\Delta}=\sqrt{28}=2\sqrt{7}$, so the equation does not have rational roots.

d $6x^2+19x+10=0$

has $a=6,\quad b=19,\quad c=10$

$$\therefore\quad \Delta = b^2-4ac = 19^2-4(6)(10) = 121$$

$\therefore$ $\sqrt{\Delta}=11$, so the equation has rational roots.

e $4x^2-3x+3=0$

has $a=4,\quad b=-3,\quad c=3$

$$\therefore\quad \Delta = b^2-4ac = (-3)^2-4(4)3 = -39$$

Since $\Delta<0$, the equation does not have rational roots.

f $8x^2-10x-3=0$

has $a=8,\quad b=-10,\quad c=-3$

$$\therefore\quad \Delta = b^2-4ac = (-10)^2-4(8)(-3) = 196$$

$\therefore$ $\sqrt{\Delta}=14$, so the equation has rational roots.

3 **a** $x^2+3x+m=0$ has $a=1,\quad b=3,\quad c=m$

$$\begin{aligned}\therefore\quad \Delta = b^2-4ac &= 3^2-4(1)m\\ &= 9-4m\end{aligned}$$

i For a repeated root,
$$\begin{aligned}\Delta &= 0\\ \therefore\quad 9-4m &= 0\\ \therefore\quad 4m &= 9\\ \therefore\quad m &= \tfrac{9}{4}\end{aligned}$$

ii For two distinct real roots,
$$\begin{aligned}\Delta &> 0\\ \therefore\quad 9-4m &> 0\\ \therefore\quad 4m &< 9\\ \therefore\quad m &< \tfrac{9}{4}\end{aligned}$$

iii For no real roots,
$$\begin{aligned}\Delta &< 0\\ \therefore\quad 9-4m &< 0\\ \therefore\quad 4m &> 9\\ \therefore\quad m &> \tfrac{9}{4}\end{aligned}$$

b $x^2-5x+m=0$ has $a=1,\quad b=-5,\quad c=m$

$$\begin{aligned}\therefore\quad \Delta = b^2-4ac &= (-5)^2-4(1)m\\ &= 25-4m\end{aligned}$$

i For a repeated root,
$$\begin{aligned}\Delta &= 0\\ \therefore\quad 25-4m &= 0\\ \therefore\quad 4m &= 25\\ \therefore\quad m &= \tfrac{25}{4}\end{aligned}$$

ii For two distinct real roots,
$$\begin{aligned}\Delta &> 0\\ \therefore\quad 25-4m &> 0\\ \therefore\quad 4m &< 25\\ \therefore\quad m &< \tfrac{25}{4}\end{aligned}$$

iii For no real roots,
$$\begin{aligned}\Delta &< 0\\ \therefore\quad 25-4m &< 0\\ \therefore\quad 4m &> 25\\ \therefore\quad m &> \tfrac{25}{4}\end{aligned}$$

c $mx^2-x+1=0$ has $a=m,\quad b=-1,\quad c=1$

$$\begin{aligned}\therefore\quad \Delta = b^2-4ac &= (-1)^2-4m(1)\\ &= 1-4m\end{aligned}$$

i For a repeated root,
$$\begin{aligned}\Delta &= 0\\ \therefore\quad 1-4m &= 0\\ \therefore\quad 4m &= 1\\ \therefore\quad m &= \tfrac{1}{4}\end{aligned}$$

ii For two distinct real roots,
$$\begin{aligned}\Delta &> 0\\ \therefore\quad 1-4m &> 0\\ \therefore\quad 4m &< 1\\ \therefore\quad m &< \tfrac{1}{4}\end{aligned}$$

iii For no real roots,
$$\begin{aligned}\Delta &< 0\\ \therefore\quad 1-4m &< 0\\ \therefore\quad 4m &> 1\\ \therefore\quad m &> \tfrac{1}{4}\end{aligned}$$

d $mx^2+2x+3=0$ has $a=m,\quad b=2,\quad c=3$

$$\begin{aligned}\therefore\quad \Delta = b^2-4ac &= 2^2-4m(3)\\ &= 4-12m\end{aligned}$$

i For a repeated root,
$$\begin{aligned}\Delta &= 0\\ \therefore\quad 4-12m &= 0\\ \therefore\quad 12m &= 4\\ \therefore\quad m &= \tfrac{1}{3}\end{aligned}$$

ii For two distinct real roots,
$$\begin{aligned}\Delta &> 0\\ \therefore\quad 4-12m &> 0\\ \therefore\quad 12m &< 4\\ \therefore\quad m &< \tfrac{1}{3}\end{aligned}$$

iii For no real roots,
$$\begin{aligned}\Delta &< 0\\ \therefore\quad 4-12m &< 0\\ \therefore\quad 12m &> 4\\ \therefore\quad m &> \tfrac{1}{3}\end{aligned}$$

e $2x^2+7x+m=0$ has $a=2,\quad b=7,\quad c=m$

$$\begin{aligned}\therefore\quad \Delta = b^2-4ac &= 7^2-4(2)m\\ &= 49-8m\end{aligned}$$

i For a repeated root,
$$\begin{aligned}\Delta &= 0\\ \therefore\quad 49-8m &= 0\\ \therefore\quad 8m &= 49\\ \therefore\quad m &= \tfrac{49}{8}\end{aligned}$$

ii For two distinct real roots,
$$\begin{aligned}\Delta &> 0\\ \therefore\quad 49-8m &> 0\\ \therefore\quad 8m &< 49\\ \therefore\quad m &< \tfrac{49}{8}\end{aligned}$$

iii For no real roots,
$$\begin{aligned}\Delta &< 0\\ \therefore\quad 49-8m &< 0\\ \therefore\quad 8m &> 49\\ \therefore\quad m &> \tfrac{49}{8}\end{aligned}$$

f $mx^2-5x+4=0$ has $a=m,\quad b=-5,\quad c=4$

$$\begin{aligned}\therefore\quad \Delta = b^2-4ac &= (-5)^2-4m(4)\\ &= 25-16m\end{aligned}$$

i For a repeated root,

$$\begin{aligned} \Delta &= 0 \\ \therefore\ 25 - 16m &= 0 \\ \therefore\ 16m &= 25 \\ \therefore\ m &= \tfrac{25}{16} \end{aligned}$$

ii For two distinct real roots,

$$\begin{aligned} \Delta &> 0 \\ \therefore\ 25 - 16m &> 0 \\ \therefore\ 16m &< 25 \\ \therefore\ m &< \tfrac{25}{16} \end{aligned}$$

iii For no real roots,

$$\begin{aligned} \Delta &< 0 \\ \therefore\ 25 - 16m &< 0 \\ \therefore\ 16m &> 25 \\ \therefore\ m &> \tfrac{25}{16} \end{aligned}$$

EXERCISE 7I.2

1 **a** $y = x^2 + 7x - 2$
has $a = 1,\quad b = 7,\quad c = -2$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 7^2 - 4(1)(-2) \\ &= 57 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

b $y = x^2 + 4\sqrt{2}x + 8$
has $a = 1,\quad b = 4\sqrt{2},\quad c = 8$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= (4\sqrt{2})^2 - 4(1)(8) \\ &= 0 \end{aligned}$$

$\therefore$ the graph touches the x-axis.

c $y = -2x^2 + 3x + 1$
has $a = -2,\quad b = 3,\quad c = 1$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(1) \\ &= 17 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

d $y = 6x^2 + 5x - 4$
has $a = 6,\quad b = 5,\quad c = -4$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 5^2 - 4(6)(-4) \\ &= 121 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

e $y = -x^2 + x + 6$
has $a = -1,\quad b = 1,\quad c = 6$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 1^2 - 4(-1)(6) \\ &= 25 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

f $y = 9x^2 + 6x + 1$
has $a = 9,\quad b = 6,\quad c = 1$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 6^2 - 4(9)(1) \\ &= 0 \end{aligned}$$

$\therefore$ the graph touches the x-axis.

2 **a** $x^2 - 3x + 6$
has $a = 1,\quad b = -3,\quad c = 6$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(1)(6) \\ &= -15 \end{aligned}$$

$\therefore$ since $a > 0$ and $\Delta < 0$,
$x^2 - 3x + 6 > 0$ for all x.

b $4x - x^2 - 6$
has $a = -1,\quad b = 4,\quad c = -6$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 4^2 - 4(-1)(-6) \\ &= -8 \end{aligned}$$

$\therefore$ since $a < 0$ and $\Delta < 0$,
$4x - x^2 - 6 < 0$ for all x.

c $2x^2 - 4x + 7$
has $a = 2,\quad b = -4,\quad c = 7$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(2)(7) \\ &= -40 \end{aligned}$$

$\therefore$ since $a > 0$ and $\Delta < 0$,
$2x^2 - 4x + 7 > 0$ for all x.
$\therefore$ it is positive definite.

d $-2x^2 + 3x - 4$
has $a = -2,\quad b = 3,\quad c = -4$

$$\begin{aligned} \therefore\ \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(-4) \\ &= -23 \end{aligned}$$

$\therefore$ since $a < 0$ and $\Delta < 0$,
$-2x^2 + 3x - 4 < 0$ for all x.
$\therefore$ it is negative definite.

3 $3x^2 + kx - 1$

has $a = 3,\quad b = k,\quad c = -1$

$$\begin{aligned} \therefore\quad \Delta &= b^2 - 4ac \\ &= k^2 - 4(3)(-1) \\ &= k^2 + 12 \end{aligned}$$

Now $k^2 \geqslant 0$ for all k

$\therefore\quad k^2 + 12 > 0$ for all k

$\therefore\quad \Delta > 0$ for all k

$\therefore\quad 3x^2 + kx - 1$ has two real distinct roots for all k.

$\therefore$ it can never be positive definite.

4 $2x^2 + kx + 2$

has $a = 2,\quad b = k,\quad c = 2$

$$\begin{aligned} \therefore\quad \Delta &= b^2 - 4ac \\ &= k^2 - 4(2)(2) \\ &= k^2 - 16 \end{aligned}$$

Now $2x^2 + kx + 2$ has $a > 0$.

$\therefore$ it is positive definite provided $k^2 - 16 < 0$

$\therefore\quad k^2 < 16$

$\therefore\quad -4 < k < 4$

EXERCISE 7J

1 **a** The x-intercepts are 1 and 2.

$\therefore\quad y = a(x-1)(x-2)$ for some $a \neq 0$.

But the y-intercept is 4.

$$\begin{aligned} \therefore\quad a(-1)(-2) &= 4 \\ \therefore\quad 2a &= 4 \\ \therefore\quad a &= 2 \end{aligned}$$

$\therefore\quad y = 2(x-1)(x-2)$

b The graph touches the x-axis when $x = 2$.

$\therefore\quad y = a(x-2)^2$ for some $a \neq 0$.

But the y-intercept is 8.

$$\begin{aligned} \therefore\quad a(-2)^2 &= 8 \\ \therefore\quad 4a &= 8 \\ \therefore\quad a &= 2 \\ \therefore\quad y &= 2(x-2)^2 \end{aligned}$$

c The x-intercepts are 1 and 3.

$\therefore\quad y = a(x-1)(x-3)$ for some $a \neq 0$.

But the y-intercept is 3.

$$\begin{aligned} \therefore\quad a(-1)(-3) &= 3 \\ \therefore\quad 3a &= 3 \\ \therefore\quad a &= 1 \end{aligned}$$

$\therefore\quad y = (x-1)(x-3)$

d The x-intercepts are -1 and 3.

$\therefore\quad y = a(x+1)(x-3)$ for some $a \neq 0$.

But the y-intercept is 3.

$$\begin{aligned} \therefore\quad a(1)(-3) &= 3 \\ \therefore\quad a &= -1 \end{aligned}$$

$\therefore\quad y = -(x+1)(x-3)$

e The graph touches the x-axis when $x = 1$.

$\therefore\quad y = a(x-1)^2$ for some $a \neq 0$.

But the y-intercept is -3.

$$\begin{aligned} \therefore\quad a(-1)^2 &= -3 \\ \therefore\quad a &= -3 \\ \therefore\quad y &= -3(x-1)^2 \end{aligned}$$

f The x-intercepts are -2 and 3.

$\therefore\quad y = a(x+2)(x-3)$ for some $a \neq 0$.

But the y-intercept is 12.

$$\begin{aligned} \therefore\quad a(2)(-3) &= 12 \\ \therefore\quad -6a &= 12 \\ \therefore\quad a &= -2 \end{aligned}$$

$\therefore\quad y = -2(x+2)(x-3)$

2 **a** $y = 2(x-1)(x-4)$ has x-intercepts 1 and 4, and y-intercept 8

$\therefore$ its graph is **C**

b $y = -(x+1)(x-4)$ has x-intercepts -1 and 4, and y-intercept 4

$\therefore$ its graph is **E**

c $y = (x-1)(x-4)$ has x-intercepts 1 and 4, and y-intercept 4

$\therefore$ its graph is **B**

d $y = (x+1)(x-4)$ has x-intercepts -1 and 4, and y-intercept -4

$\therefore$ its graph is **F**

e $y = 2(x+4)(x-1)$ has x-intercepts -4 and 1, and y-intercept -8

$\therefore$ its graph is **G**

f $y = -3(x+4)(x-1)$ has x-intercepts -4 and 1, and y-intercept 12

$\therefore$ its graph is **H**

g $y = -(x-1)(x-4)$
has x-intercepts 1 and 4,
and y-intercept -4
$\therefore$ its graph is **A**

h $y = -3(x-1)(x-4)$
has x-intercepts 1 and 4,
and y-intercept -12
$\therefore$ its graph is **D**

3 **a** As the axis of symmetry is $x = 3$, the other x-intercept is 4.

$\therefore \; y = a(x-2)(x-4)$ for some $a \neq 0$.

But the y-intercept $= 12$

$$\begin{aligned} \therefore \; a(-2)(-4) &= 12 \\ \therefore \; 8a &= 12 \\ \therefore \; a &= \tfrac{12}{8} = \tfrac{3}{2} \end{aligned}$$

$\therefore \; y = \frac{3}{2}(x-2)(x-4)$

b As the axis of symmetry is $x = -1$, the other x-intercept is 2.

$\therefore \; y = a(x+4)(x-2)$ for some $a \neq 0$.

But the y-intercept $= 4$

$$\begin{aligned} \therefore \; a(4)(-2) &= 4 \\ \therefore \; -8a &= 4 \\ \therefore \; a &= -\tfrac{1}{2} \end{aligned}$$

$\therefore \; y = -\frac{1}{2}(x+4)(x-2)$

c The graph touches the x-axis at $x = -3$,

$\therefore \; y = a(x+3)^2$ for some $a \neq 0$.

But the y-intercept $= -12$

$$\begin{aligned} \therefore \; a(3)^2 &= -12 \\ \therefore \; 9a &= -12 \\ \therefore \; a &= -\tfrac{12}{9} = -\tfrac{4}{3} \end{aligned}$$

$\therefore \; y = -\frac{4}{3}(x+3)^2$

4 **a** Since the x-intercepts are 5 and 1, the equation is $y = a(x-5)(x-1)$ for some $a \neq 0$.

But when $x = 2$, $y = -9$

$$\begin{aligned} \therefore \; -9 &= a(2-5)(2-1) \\ \therefore \; -9 &= a(-3)(1) \\ \therefore \; -3a &= -9 \\ \therefore \; a &= 3 \end{aligned}$$

$\therefore$ the equation is

$$\begin{aligned} y &= 3(x-5)(x-1) \\ \text{i.e., } y &= 3(x^2-6x+5) \\ \text{i.e., } y &= 3x^2-18x+15 \end{aligned}$$

b Since the x-intercepts are 2 and $-\frac{1}{2}$, the equation is $y = a(x-2)(x+\frac{1}{2})$ for some $a \neq 0$.

But when $x = 3$, $y = -14$

$$\begin{aligned} \therefore \; -14 &= a(3-2)(3+\tfrac{1}{2}) \\ \therefore \; -14 &= a(1)(\tfrac{7}{2}) \\ \therefore \; \tfrac{7}{2}a &= -14 \\ \therefore \; a &= -4 \end{aligned}$$

$\therefore$ the equation is

$$\begin{aligned} y &= -4(x-2)(x+\tfrac{1}{2}) \\ \text{i.e., } y &= -4(x^2-\tfrac{3}{2}x-1) \\ \text{i.e., } y &= -4x^2+6x+4 \end{aligned}$$

c Since the graph touches the x-axis at 3, its equation is $y = a(x-3)^2$, for some $a \neq 0$.

But when $x = -2$, $y = -25$

$$\begin{aligned} \therefore \; -25 &= a(-2-3)^2 \\ \therefore \; -25 &= 25a \\ \therefore \; a &= -1 \end{aligned}$$

$\therefore$ the equation is

$$\begin{aligned} y &= -(x-3)^2 \\ \text{i.e., } y &= -(x^2-6x+9) \\ \text{i.e., } y &= -x^2+6x-9 \end{aligned}$$

d Since the graph touches the x-axis at -2, its equation is $y = a(x+2)^2$, for some $a \neq 0$.

But when $x = -1$, $y = 4$

$$\begin{aligned} \therefore \; 4 &= a(-1+2)^2 \\ \therefore \; 4 &= a \end{aligned}$$

$\therefore$ the equation is

$$\begin{aligned} y &= 4(x+2)^2 \\ \text{i.e., } y &= 4(x^2+4x+4) \\ \text{i.e., } y &= 4x^2+16x+16 \end{aligned}$$

e Since the graph cuts the x-axis at 3 and has axis of symmetry $x = 2$, it must also cut the x-axis at 1.

$\therefore$ the x-intercepts are 3 and 1, and the equation is $y = a(x-3)(x-1)$ for some $a \neq 0$.

But when $x = 5$, $y = 12$

$$\begin{aligned}\therefore\ 12 &= a(5-3)(5-1)\\ \therefore\ 12 &= a(2)(4)\\ \therefore\ 8a &= 12\\ \therefore\ a &= \tfrac{3}{2}\end{aligned}$$

$\therefore$ the equation is

$$\begin{aligned} y &= \tfrac{3}{2}(x-3)(x-1)\\ \text{i.e., } y &= \tfrac{3}{2}(x^2-4x+3)\\ \text{i.e., } y &= \tfrac{3}{2}x^2 - 6x + \tfrac{9}{2}\end{aligned}$$

f Since the graph cuts the x-axis at 5 and has axis of symmetry $x = 1$, it must also cut the x-axis at -3.

$\therefore$ the x-intercepts are 5 and -3, and the equation is $y = a(x-5)(x+3)$ for some $a \neq 0$.

But when $x = 2$, $y = 5$

$$\begin{aligned}\therefore\ 5 &= a(2-5)(2+3)\\ \therefore\ 5 &= a(-3)(5)\\ \therefore\ -3a &= 1\\ \therefore\ a &= -\tfrac{1}{3}\end{aligned}$$

$\therefore$ the equation is

$$\begin{aligned} y &= -\tfrac{1}{3}(x-5)(x+3)\\ \text{i.e., } y &= -\tfrac{1}{3}(x^2-2x-15)\\ \text{i.e., } y &= -\tfrac{1}{3}x^2 + \tfrac{2}{3}x + 5\end{aligned}$$

5 **a** The vertex is (2, 4), so the quadratic has equation $y = a(x-2)^2 + 4$ for some $a \neq 0$.

But the graph passes through the origin

$$\begin{aligned}\therefore\ 0 &= a(0-2)^2 + 4\\ \therefore\ 4a + 4 &= 0\\ \therefore\ a &= -1\end{aligned}$$

$\therefore$ the equation is $y = -(x-2)^2 + 4$

b The vertex is (2, -1), so the quadratic has equation $y = a(x-2)^2 - 1$ for some $a \neq 0$.

But the graph passes through (0, 7)

$$\begin{aligned}\therefore\ 7 &= a(0-2)^2 - 1\\ \therefore\ 7 &= 4a - 1\\ \therefore\ 4a &= 8\\ \therefore\ a &= 2\end{aligned}$$

$\therefore$ the equation is $y = 2(x-2)^2 - 1$

c The vertex is (3, 8), so the quadratic has equation $y = a(x-3)^2 + 8$ for some $a \neq 0$.

But the graph passes through (1, 0)

$$\begin{aligned}\therefore\ 0 &= a(1-3)^2 + 8\\ \therefore\ 0 &= 4a + 8\\ \therefore\ a &= -2\end{aligned}$$

$\therefore$ the equation is $y = -2(x-3)^2 + 8$

d The vertex is (4, -6), so the quadratic has equation $y = a(x-4)^2 - 6$ for some $a \neq 0$.

But the graph passes through (7, 0)

$$\begin{aligned}\therefore\ 0 &= a(7-4)^2 - 6\\ \therefore\ 9a - 6 &= 0\\ \therefore\ a &= \tfrac{2}{3}\end{aligned}$$

$\therefore$ the equation is $y = \frac{2}{3}(x-4)^2 - 6$

e The vertex is (2, 3), so the quadratic has equation $y = a(x-2)^2 + 3$ for some $a \neq 0$.

But the graph passes through (3, 1)

$$\begin{aligned}\therefore\ 1 &= a(3-2)^2 + 3\\ \therefore\ 1 &= a + 3\\ \therefore\ a &= -2\end{aligned}$$

$\therefore$ the equation is $y = -2(x-2)^2 + 3$

f The vertex is ($\frac{1}{2}$, $-\frac{3}{2}$), so the quadratic has equation $y = a(x - \frac{1}{2})^2 - \frac{3}{2}$ for some $a \neq 0$.

But the graph passes through ($\frac{3}{2}$, $\frac{1}{2}$)

$$\begin{aligned}\therefore\ \tfrac{1}{2} &= a(\tfrac{3}{2} - \tfrac{1}{2})^2 - \tfrac{3}{2}\\ \therefore\ \tfrac{1}{2} &= a - \tfrac{3}{2}\\ \therefore\ a &= 2\end{aligned}$$

$\therefore$ the equation is $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 7K

1 **a** $y = x^2 - 2x + 8$ meets $y = x + 6$

when $x^2 - 2x + 8 = x + 6$

$\therefore\ x^2 - 3x + 2 = 0$

$\therefore\ (x-1)(x-2) = 0$

$\therefore\ x = 1$ or 2

Substituting into $y = x + 6$,

when $x = 1,\ y = 7$

and when $x = 2,\ y = 8$

$\therefore$ the graphs meet at (1, 7) and (2, 8)

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$

when $-x^2 + 3x + 9 = 2x - 3$

$\therefore\ x^2 - x - 12 = 0$

$\therefore\ (x-4)(x+3) = 0$

$\therefore\ x = 4$ or -3

Substituting into $y = 2x - 3$,

when $x = -3,\ y = 2(-3) - 3 = -9$

and when $x = 4,\ y = 2(4) - 3 = 5$

$\therefore$ the graphs meet at $(-3, -9)$ and (4, 5)

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$

when $x^2 - 4x + 3 = 2x - 6$

$\therefore\ x^2 - 6x + 9 = 0$

$\therefore\ (x-3)^2 = 0$

$\therefore\ x = 3$

Substituting into $y = 2x - 6$,

when $x = 3,\ y = 0$

$\therefore$ the graphs touch at (3, 0)

d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$

when $-x^2 + 4x - 7 = 5x - 4$

$\therefore\ x^2 + x + 3 = 0$

which has $a = 1,\ b = 1,\ c = 3$

$$\therefore\ x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2} = \frac{-1 \pm \sqrt{-11}}{2}$$

$\therefore$ there are no real solutions

$\therefore$ the graphs do not meet.

2 **a** (0.59, 5.59) and (3.41, 8.41) **b** $(3, -4)$ touching **c** graphs do not meet

d $(-2.56, -18.81)$ and (1.56, 1.81)

3 **a** $y = x^2$ meets $y = x + 2$

when $x^2 = x + 2$

$\therefore\ x^2 - x - 2 = 0$

$\therefore\ (x+1)(x-2) = 0$

$\therefore\ x = -1$ or 2

Substituting into $y = x + 2$,

when $x = -1,\ y = 1$

and when $x = 2,\ y = 4$

$\therefore$ the graphs meet at $(-1, 1)$ and (2, 4).

b $y = x^2 + 2x - 3$ meets $y = x - 1$

when $x^2 + 2x - 3 = x - 1$

$\therefore\ x^2 + x - 2 = 0$

$\therefore\ (x-1)(x+2) = 0$

$\therefore\ x = 1$ or -2

Substituting into $y = x - 1$,

when $x = 1,\ y = 0$

and when $x = -2,\ y = -3$

$\therefore$ the graphs meet at (1, 0) and $(-2, -3)$.

c $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$

when $2x^2 - x + 3 = 2 + x + x^2$

$\therefore\ x^2 - 2x + 1 = 0$

$\therefore\ (x-1)^2 = 0$

$\therefore\ x = 1$

Substituting into $y = 2 + x + x^2$,

when $x = 1,\ y = 2 + 1 + 1 = 4$

$\therefore$ the graphs meet at (1, 4)

d Now if $xy = 4$, then $y = \dfrac{4}{x}$

$xy = 4$ meets $y = x + 3$

when $\dfrac{4}{x} = x + 3$

$\therefore\ 4 = x^2 + 3x$

$\therefore\ x^2 + 3x - 4 = 0$

$\therefore\ (x+4)(x-1) = 0$

$\therefore\ x = -4$ or 1

Substituting into $y = x + 3$,

when $x = -4,\ y = -1$

and when $x = 1,\ y = 4$

$\therefore$ the graphs meet at $(-4, -1)$ and (1, 4)

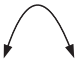
EXERCISE 7L

1 **a** $H(t) = 36t - 2t^2$

has $a = -2$ and $b = 36$

Since $a < 0$,

the graph is



The maximum height reached occurs

when $t = -\frac{b}{2a} = -\frac{36}{2(-2)} = 9$

i.e., the maximum height is reached after 9 seconds.

b $$\begin{aligned} H(9) &= 36 \times 9 - 2 \times 9^2 \\ &= 324 - 162 \\ &= 162 \end{aligned}$$

$\therefore$ the maximum height reached is 162 m.

c The ball hits the ground when $H(t) = 0$

$$\begin{aligned} \therefore \quad 36t - 2t^2 &= 0 \\ \therefore \quad 2t(18 - t) &= 0 \\ \therefore \quad t &= 0 \text{ or } 18 \end{aligned}$$

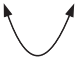
$\therefore$ the ball hits the ground after 18 seconds.

2 **a** $C(x) = x^2 - 24x + 244$

has $a = 1$, $b = -24$ and $c = 244$

Since $a > 0$,

the graph is



The minimum cost occurs

when $x = -\frac{b}{2a} = -\frac{(-24)}{2(1)} = 12$

i.e., the minimum cost is when twelve skateboards are produced.

b $$\begin{aligned} C(12) &= 12^2 - 24(12) + 244 \\ &= 144 - 288 + 244 \\ &= 100 \end{aligned}$$

$\therefore$ the minimum cost is $100.

c $$\begin{aligned} C(0) &= 0 - 24(0) + 244 \\ &= 244 \end{aligned}$$

$\therefore$ if no skateboards are made, there is a fixed cost of $244.

3 $v(t) = -4t^2 + 12t + 80$ m/s

a $v(0) = 80$ m/s was the speed when the brakes were applied.

b When $v(t) = 88$,

$$\begin{aligned} -4t^2 + 12t + 80 &= 88 \\ \therefore \quad -4t^2 + 12t - 8 &= 0 \\ \therefore \quad t^2 - 3t + 2 &= 0 \\ \therefore \quad (t-1)(t-2) &= 0 \\ \therefore \quad t &= 1 \text{ or } 2 \end{aligned}$$

Note: Actually the units for this question should have been kmph to be physically realistic.

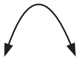
Since the car was travelling downhill, it was accelerating. Therefore, when the brake was applied, it took a while before the deceleration due to the brake cancelled the acceleration due to the hill. $\therefore$ the speed of the vehicle increased for a short time after the brake was applied. It passed 88 m/s after one second, then again after two seconds as the car slowed down.

c Axis of symmetry is $t = \frac{-b}{2a} = \frac{-12}{-8} = 1\frac{1}{2}$, $\therefore$ maximum velocity occurred at $1\frac{1}{2}$ sec.

d $\therefore$ maximum velocity was
$$\begin{aligned} &-4(\tfrac{3}{2})^2 + 12(\tfrac{3}{2}) + 80 \text{ m/s} \\ &= -9 + 18 + 80 \text{ m/s} \\ &= 89 \text{ m/s} \end{aligned}$$

4 **a** $P(n) = 84n - 45 - 2n^2$ has $a = -2$, $b = 84$ and $c = -45$

Since $a < 0$, the graph is



$\therefore$ the maximum value of P is when $n = -\frac{b}{2a} = -\frac{84}{2(-2)} = 21$

$\therefore$ the maximum profit occurs for a fleet of 21 taxis.

b $$\begin{aligned} P(21) &= 84 \times 21 - 45 - 2 \times 21^2 \\ &= 837 \end{aligned}$$

$\therefore$ the maximum profit is $837 per hour.

c $P(0) = -45$, so if no taxis are on the road, $45 is lost per hour.

5 **a** Dusk corresponds to $t = 0$.

Now $T(t) = \frac{1}{4}t^2 - 5t + 30$

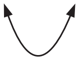
$\therefore$ $T(0) = 30^o$C

$\therefore$ the temperature at dusk was 30^oC.

b $T(t) = \frac{1}{4}t^2 - 5t + 30$

has $a = \frac{1}{4}$, $b = -5$ and $c = 30$

Since $a > 0$,
the graph is



$\therefore$ the temperature was a minimum

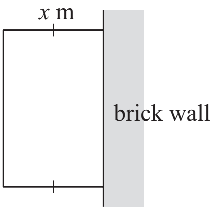
when $t = -\dfrac{b}{2a} = -\dfrac{(-5)}{2(\frac{1}{4})} = 10$

Now 10 hours after 7 pm is 5 am, so the temperature was a minimum at 5 am.

c $T(10) = \frac{1}{4} \times 10^2 - 5 \times 10 + 30$
$= 5^o$C

$\therefore$ the minimum temperature was 5^oC.

6 **a**



If the sides shown are x m long, and the total length of fencing is 40 m, then the other side must have length $(40 - 2x)$ m.

$\therefore$ the area is $A = x(40 - 2x)$ m^2
i.e., $A = -2x^2 + 40x$ m^2

b $A = -2x^2 + 40x$

has $a = -2$ and $b = 40$

Since $a < 0$,

the graph of A has shape

$\therefore$ the area A is maximised

when $x = -\dfrac{b}{2a} = -\dfrac{40}{2(-2)} = 10$

$\therefore$ the area is maximised when $x = 10$.

c $A(10) = -2 \times 10^2 + 40 \times 10$
$= -200 + 400$
$= 200$ m^2

$\therefore$ the maximum area is 200 m^2.

7 **a** With the axes as described, the parabola has vertex (0, 70).

$\therefore$ its equation is $y = ax^2 + 70$ for some $a \neq 0$.

The end of the bridge is 80 m from A,
so the arch meets the vertical end supports at the point (80, 6).

Since (80, 6) must lie on the curve, $6 = a(80)^2 + 70$

$\therefore$ $6400a = -64$ and so $a = -\frac{1}{100}$

$\therefore$ the arch has equation $y = -\frac{1}{100}x^2 + 70$

b The supports occur every 10 m.

When $x = 10$, $y = -\frac{1}{100} \times 10^2 + 70 = 69$ m

When $x = 20$, $y = -\frac{1}{100} \times 20^2 + 70 = 66$ m

When $x = 30$, $y = -\frac{1}{100} \times 30^2 + 70 = 61$ m

When $x = 40$, $y = -\frac{1}{100} \times 40^2 + 70 = 54$ m

When $x = 50$, $y = -\frac{1}{100} \times 50^2 + 70 = 45$ m

When $x = 60$, $y = -\frac{1}{100} \times 60^2 + 70 = 34$ m

When $x = 70$, $y = -\frac{1}{100} \times 70^2 + 70 = 21$ m

$\therefore$ the other supports have lengths 21 m, 34 m, 45 m, 54 m, 61 m, 66 m and 69 m.

REVIEW SET 7A

1 **a** The x-intercepts are -2 and 1.

b The axis of symmetry lies midway between the x-intercepts, so its equation is $x = -\frac{1}{2}$.

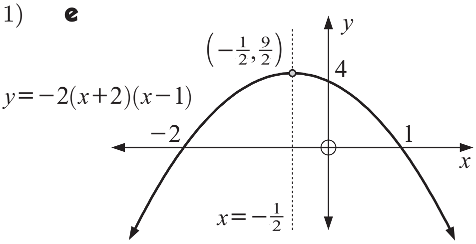
c When $x = -\frac{1}{2}$,
$$\begin{aligned} y &= -2(-\tfrac{1}{2}+2)(-\tfrac{1}{2}-1) \\ &= -2(\tfrac{3}{2})(-\tfrac{3}{2}) \\ &= \tfrac{9}{2} \end{aligned}$$

$\therefore$ the vertex is $(-\frac{1}{2}, \frac{9}{2})$

d When $x = 0$, $y = -2(2)(-1) = 4$

$\therefore$ the y-intercept is 4

e



2 **a** The axis of symmetry is $x = 2$.

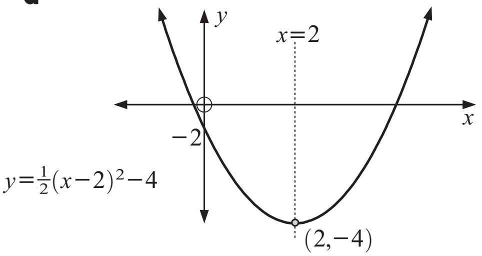
b When $x = 2$,
$$\begin{aligned} y &= \tfrac{1}{2}(2-2)^2 - 4 \\ &= -4 \end{aligned}$$

$\therefore$ the vertex is $(2, -4)$

c When $x = 0$,
$$\begin{aligned} y &= \tfrac{1}{2}(-2)^2 - 4 \\ &= -2 \end{aligned}$$

$\therefore$ the y-intercept is -2

d



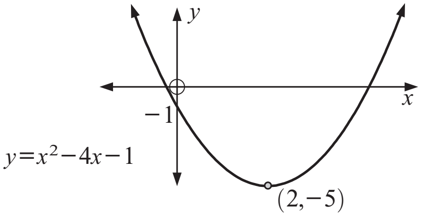
3 **a**
$$\begin{aligned} y &= x^2 - 4x - 1 \\ &= x^2 - 4x + 4 - 4 - 1 \\ &= (x-2)^2 - 5 \end{aligned}$$

b The vertex is $(2, -5)$

c When $x = 0$, $y = (-2)^2 - 5 = -1$

$\therefore$ the y-intercept is -1

d



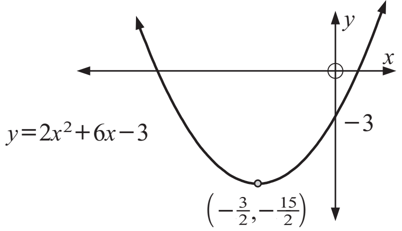
4 **a**
$$\begin{aligned} y &= 2x^2 + 6x - 3 \\ &= 2[x^2 + 3x - \tfrac{3}{2}] \\ &= 2\left[x^2 + 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 - \tfrac{3}{2}\right] \\ &= 2\left[x^2 + 3x + \tfrac{9}{4} - \tfrac{9}{4} - \tfrac{3}{2}\right] \\ &= 2\left(x^2 + 3x + \tfrac{9}{4}\right) - \tfrac{9}{2} - 3 \\ &= 2(x + \tfrac{3}{2})^2 - \tfrac{15}{2} \end{aligned}$$

b The vertex is $(-\frac{3}{2}, -\frac{15}{2})$.

c When $x = 0$, $y = -3$

$\therefore$ the y-intercept is -3.

d



5 **a**
$$\begin{aligned} x^2 - 11x &= 60 \\ \therefore\ x^2 - 11x - 60 &= 0 \\ \therefore\ (x+4)(x-15) &= 0 \\ \therefore\ x &= -4 \text{ or } 15 \end{aligned}$$

b
$$\begin{aligned} 3x^2 - x - 10 &= 0 \\ \therefore\ (3x+5)(x-2) &= 0 \\ \therefore\ x &= -\tfrac{5}{3} \text{ or } 2 \end{aligned}$$

c
$$\begin{aligned} 3x^2 - 12x &= 0 \\ \therefore\ 3x(x-4) &= 0 \\ \therefore\ x &= 0 \text{ or } 4 \end{aligned}$$

6 **a**
$$\begin{aligned} x^2 + 10 &= 7x \\ \therefore\ x^2 - 7x + 10 &= 0 \\ \therefore\ (x-2)(x-5) &= 0 \\ \therefore\ x &= 2 \text{ or } 5 \end{aligned}$$

b
$$\begin{aligned} x + \frac{12}{x} &= 7 \\ \therefore\ x^2 + 12 &= 7x \\ \therefore\ x^2 - 7x + 12 &= 0 \\ \therefore\ (x-3)(x-4) &= 0 \\ \therefore\ x &= 3 \text{ or } 4 \end{aligned}$$

c
$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ \therefore\ (2x-1)(x-3) &= 0 \\ \therefore\ x &= \tfrac{1}{2} \text{ or } 3 \end{aligned}$$

7

$$\begin{aligned} x^2+7x-4 &= 0 \\ \therefore\quad x^2+7x+(\tfrac{7}{2})^2-(\tfrac{7}{2})^2-4 &= 0 \\ \therefore\quad (x+\tfrac{7}{2})^2-\tfrac{49}{4}-4 &= 0 \\ \therefore\quad (x+\tfrac{7}{2})^2 &= \tfrac{65}{4} \\ \therefore\quad x+\tfrac{7}{2} &= \pm\tfrac{\sqrt{65}}{2} \\ \therefore\quad x &= -\tfrac{7}{2}\pm\tfrac{\sqrt{65}}{2} \end{aligned}$$

8

$$\begin{aligned} x^2+4x+1 &= 0 \\ \therefore\quad x^2+4x+4-4+1 &= 0 \\ \therefore\quad (x+2)^2 &= 3 \\ \therefore\quad x+2 &= \pm\sqrt{3} \\ \therefore\quad x &= -2\pm\sqrt{3} \end{aligned}$$

9 **a** $x^2-7x+3=0$

has $a=1$, $b=-7$ and $c=3$

$$\begin{aligned} \therefore\quad x &= \frac{-(-7)\pm\sqrt{(-7)^2-4(1)(3)}}{2(1)} \\ &= \frac{7\pm\sqrt{49-12}}{2} \\ &= \frac{7\pm\sqrt{37}}{2} \end{aligned}$$

i.e., $x = \frac{7}{2}\pm\frac{\sqrt{37}}{2}$

b $2x^2-5x+4=0$

has $a=2$, $b=-5$ and $c=4$

$$\begin{aligned} \therefore\quad x &= \frac{-(-5)\pm\sqrt{(-5)^2-4(2)(4)}}{2(2)} \\ &= \frac{5\pm\sqrt{25-32}}{4} \end{aligned}$$

i.e., $x = \dfrac{5\pm\sqrt{-7}}{4}$

$\therefore$ x has no real solutions.

REVIEW SET 7B

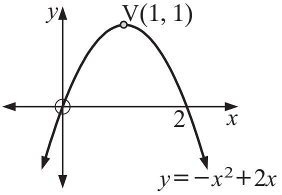
1 $y = -x^2+2x = x(2-x)$

$\therefore$ the graph has x-intercepts 0 and 2, and y-intercept 0

Its axis of symmetry is midway between the x-intercepts, i.e., at $x = 1$

and when $x = 1$, $y=-1^2+2=1$

$\therefore$ the vertex is $(1, 1)$



2 $y=-3x^2+8x+7$ has $a=-3$, $b=8$ and $c=7$

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{8}{2(-3)}$

i.e., $x = \frac{4}{3}$

$$\begin{aligned} \text{When } x=\tfrac{4}{3},\quad y &= -3(\tfrac{4}{3})^2+8(\tfrac{4}{3})+7 \\ &= -\tfrac{16}{3}+\tfrac{32}{3}+7 \\ &= \tfrac{37}{3} \end{aligned}$$

$\therefore$ the axis of symmetry is $x=\frac{4}{3}$ and the vertex is $(\frac{4}{3}, \frac{37}{3})$.

3 $y=2x^2+4x-3$

has $a=2$, $b=4$, $c=-3$

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{4}{2(2)}$ i.e., $x = -1$

$$\begin{aligned} \text{When } x=-1,\quad y &= 2(-1)^2+4(-1)-3 \\ &= 2-4-3 \\ &= -5 \end{aligned}$$

$\therefore$ the axis of symmetry is $x=-1$ and the vertex is $(-1, -5)$.

4 **a** $3x^2 - 5x + 7 = 0$

has $a = 3, \quad b = -5 \quad$ and $\quad c = 7$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(3)(7) \\ &= -59 \end{aligned}$$

Since $\Delta < 0$, there are no real solutions.

b $-2x^2 - 4x + 3 = 0$

has $a = -2, \quad b = -4 \quad$ and $\quad c = 3$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(-2)(3) \\ &= 40 \end{aligned}$$

Since $\Delta > 0$, there are two real solutions.

5 $5 + 7x + 3x^2$ has $a = 3, \quad b = 7 \quad$ and $\quad c = 5$

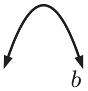
$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 7^2 - 4(3)(5) \\ &= -11 \end{aligned}$$

Since $\Delta < 0$, the graph of $y = 5 + 7x + 3x^2$ never cuts or touches the x-axis.

$\therefore$ since $a > 0$, the graph lies completely above the x-axis,

and $5 + 7x + 3x^2$ is positive definite.

6 $y = -2x^2 + 4x - 3$ has $a = -2, \quad b = 4 \quad$ and $\quad c = 3$

Since $a < 0$, the graph is  and will have a maximum.

The axis of symmetry is $x = -\dfrac{b}{2a} = -\dfrac{4}{2(-2)}$ i.e., $x = 1$

$$\begin{aligned} \text{When } x = 1, \quad y &= -2(1)^2 + 4(1) + 3 \\ &= 5 \end{aligned}$$

$\therefore$ the maximum is 5, and this occurs when $x = 1$.

7 $y = x^2 - 3x \quad$ meets $\quad y = 3x^2 - 5x - 24$

$$\begin{aligned} \text{when } \quad x^2 - 3x &= 3x^2 - 5x - 24 \\ \therefore \quad 2x^2 - 2x - 24 &= 0 \\ \therefore \quad x^2 - x - 12 &= 0 \\ \therefore \quad (x-4)(x+3) &= 0 \\ \therefore \quad x &= 4 \text{ or } -3 \end{aligned}$$

$$\begin{aligned} \text{Substituting into } \quad y &= x^2 - 3x, \\ \text{when } x = 4, \quad y &= 4^2 - 3 \times 4 = 4 \\ \text{and when } x = -3, \quad y &= (-3)^2 - 3(-3) \\ &= 9 + 9 = 18 \end{aligned}$$

$\therefore$ the graphs meet at (4, 4) and (−3, 18).

8 $y = -2x^2 + 5x + k$

has $a = -2, \quad b = 5 \quad$ and $\quad c = k$.

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 5^2 - 4(-2)k \\ &= 25 + 8k \end{aligned}$$

The graph does not cut the x-axis

$$\begin{aligned} \text{if } \Delta &< 0 \\ \therefore \quad 25 + 8k &< 0 \\ \therefore \quad 8k &< -25 \\ \therefore \quad k &< -\tfrac{25}{8} \\ \text{i.e., } \quad k &< -3\tfrac{1}{8} \end{aligned}$$

9 **a** The total length of wire for the fence is 60 m.

$\therefore$ AB + BC + CD = 60

Since the enclosure is rectangular,

$$\begin{aligned} \text{CD} &= \text{AB} \\ \therefore \quad 2\text{AB} + x &= 60 \\ \therefore \quad 2\text{AB} &= 60 - x \\ \therefore \quad \text{AB} &= 30 - \tfrac{1}{2}x \end{aligned}$$

$\therefore$ the area of the rectangle is

$$\begin{aligned} A &= x\left(30 - \tfrac{1}{2}x\right) \\ &= \left(30x - \tfrac{1}{2}x^2\right) \text{ m}^2 \end{aligned}$$

b $A = 30x - \frac{1}{2}x^2$

has $a = -\frac{1}{2} \quad$ and $\quad b = 30$.

Since $a < 0$, A has a maximum at the axis of symmetry, and this is at

$$x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{2})} = 30$$

$$\begin{aligned} \text{When } x = 30, \quad \text{AB} &= 30 - \tfrac{1}{2} \times 30 \\ &= 15 \text{ m} \end{aligned}$$

$\therefore$ the enclosure is 15 m by 30 m.

REVIEW SET 7C

1 **a** $x^2+5x+3=0$ has $a=1,\quad b=5,\quad c=3$

$$\therefore\ x=\frac{-5\pm\sqrt{5^2-4(1)(3)}}{2(1)}=-\tfrac{5}{2}\pm\tfrac{\sqrt{13}}{2}$$

b $3x^2+11x-2=0$ has $a=3,\quad b=11,\quad c=-2$

$$\therefore\ x=\frac{-11\pm\sqrt{11^2-4(3)(-2)}}{2(3)}=-\tfrac{11}{6}\pm\tfrac{\sqrt{145}}{6}$$

2 **a** $x^2-5x-3=0$ has $a=1,\quad b=-5,\quad c=-3$

$$\therefore\ x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-3)}}{2(1)}=\tfrac{5}{2}\pm\tfrac{\sqrt{37}}{2}$$

b $2x^2-7x-3=0$ has $a=2,\quad b=-7,\quad c=-3$

$$\therefore\ x=\frac{-(-7)\pm\sqrt{(-7)^2-4(2)(-3)}}{2(2)}=\tfrac{7}{4}\pm\tfrac{\sqrt{73}}{4}$$

3 **a** $x \doteqdot -5.828$ or -0.1716 **b** $x \doteqdot -1.135$ or 1.468

4 **a** $x \doteqdot 0.5858$ or 3.414 **b** $x \doteqdot -0.1861$ or 2.686

5 **a** $2x^2-5x-7=0$ has $a=2,\quad b=-5,\quad c=-7$

$$\begin{aligned}\therefore\ \Delta &= b^2-4ac\\ &= (-5)^2-4(2)(-7)\\ &= 25+56\\ &= 81\end{aligned}$$

$\therefore\ \Delta>0$ and $\sqrt{\Delta}=9$

$\therefore$ there are two distinct real rational roots

b $3x^2-24x+48=0$ has $a=3,\quad b=-24,\quad c=48$

$$\begin{aligned}\therefore\ \Delta &= b^2-4ac\\ &= (-24)^2-4(3)(48)\\ &= 576-576\\ &= 0\end{aligned}$$

$\therefore$ there is a repeated real root

6 $2x^2-3x+m=0$ has $a=2,\quad b=-3$ and $c=m$

$$\begin{aligned}\therefore\ \Delta &= b^2-4ac\\ &= (-3)^2-4(2)m\\ &= 9-8m\end{aligned}$$

a There is a repeated root if $\Delta=0$

i.e., $9-8m=0$

$\therefore\ m=\frac{9}{8}$

b There are two distinct real roots if $\Delta>0$

i.e., $9-8m>0$

$\therefore\ 8m<9$

$\therefore\ m<\frac{9}{8}$

c There are no real roots if $\Delta<0$

i.e., $9-8m<0$

$\therefore\ 8m>9$

$\therefore\ m>\frac{9}{8}$

7 $3x^2+4x+t=0$ has $a=3,\quad b=4$ and $c=t$

$$\begin{aligned}\therefore\ \Delta &= b^2-4ac\\ &= 4^2-4(3)t\\ &= 16-12t\end{aligned}$$

a There is a repeated root if $\Delta=0$

i.e., $16-12t=0$

$\therefore\ 12t=16$

$\therefore\ t=\frac{4}{3}$

b There are two distinct real roots if $\Delta>0$

i.e., $16-12t>0$

$\therefore\ 12t<16$

$\therefore\ t<\frac{4}{3}$

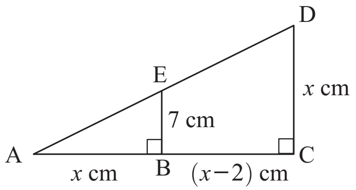
c There are no real roots if $\Delta<0$

i.e., $16-12t<0$

$\therefore\ 12t>16$

$\therefore\ t>\frac{4}{3}$

8 Suppose AB is x cm in length. Then, using the information given, we can label the diagram:



Now by similar triangles, $\dfrac{\text{BE}}{\text{AB}} = \dfrac{\text{CD}}{\text{AC}}$

$$\therefore \quad \frac{7}{x} = \frac{x}{x+(x-2)}$$

$$\therefore \quad \frac{7}{x} = \frac{x}{2x-2}$$

$$\therefore \quad 7(2x-2) = x^2$$

$$\therefore \quad 14x - 14 = x^2$$

$$\therefore \quad x^2 - 14x + 14 = 0$$

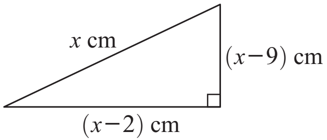
which has $a = 1$, $b = -14$ and $c = 14$

$$\therefore \quad x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(14)}}{2(1)} = \frac{14 \pm \sqrt{140}}{2}$$

Now $x > 0$, so $x = \dfrac{14 + \sqrt{140}}{2} \doteqdot 12.92$ cm

$\therefore$ AB is approximately 12.9 cm long.

9 Let the hypotenuse have length x cm.



$\therefore$ the longer of the remaining sides has length $(x-2)$ cm,

and the third side has length $(x-9)$ cm.

By Pythagoras' theorem,

$$(x-2)^2 + (x-9)^2 = x^2$$

$$\therefore \quad x^2 - 4x + 4 + x^2 - 18x + 81 = x^2$$

$$\therefore \quad x^2 - 22x + 85 = 0$$

$$\therefore \quad (x-5)(x-17) = 0$$

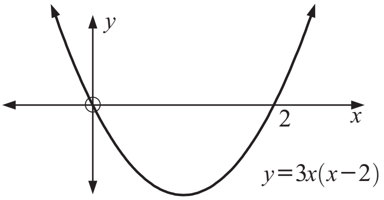
$$\therefore \quad x = 5 \text{ or } 17$$

But if x was 5, the shortest side would have negative length.

$\therefore$ the only solution is that the hypotenuse has length 17 cm.

REVIEW SET 7D

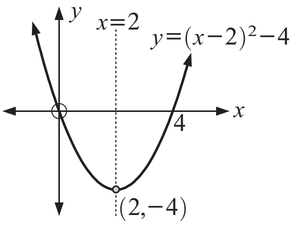
1 $y = 3x(x-2)$ has x-intercepts 0 and 2 and y-intercept 0



2 **a** $y = (x-2)^2 - 4$ has vertex $(2, -4)$ and axis of symmetry $x = 2$.

When $x = 0$, $y = (-2)^2 - 4 = 0$

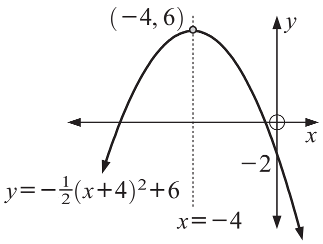
so the y-intercept is 0.



b $y = -\frac{1}{2}(x+4)^2 + 6$ has vertex $(-4, 6)$ and axis of symmetry $x = -4$.

When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$

so the y-intercept is -2.



3 **a** $y = 2x^2 + 4x - 1$

has $a = 2$, $b = 4$ and $c = -1$

$$\text{The axis of symmetry is} \quad x = -\frac{b}{2a}$$

$$\text{i.e.,} \quad x = -\frac{4}{2 \times 2}$$

$$\text{i.e.,} \quad x = -1$$

b When $x = -1$,

$$\begin{aligned} y &= 2(-1)^2 + 4(-1) - 1 \\ &= 2 - 4 - 1 \\ &= -3 \end{aligned}$$

$\therefore$ the vertex is $(-1, -3)$

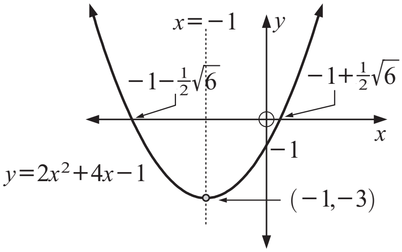
c When $x = 0$, $y = -1$,
so the y-intercept is -1

When $y = 0$, $2x^2 + 4x - 1 = 0$

$$\begin{aligned} \therefore \quad x &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{24}}{4} \\ &= \frac{-4 \pm 2\sqrt{6}}{4} = -1 \pm \tfrac{1}{2}\sqrt{6} \end{aligned}$$

$\therefore$ the x-intercepts are $-1 \pm \frac{1}{2}\sqrt{6}$

d



4 **a** $y = 2x^2 + 3x - 7$

has $a = 2$, $b = 3$ and $c = -7$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 3^2 - 4(2)(-7) \\ &= 65 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a > 0$, the graph is

b $y = -3x^2 - 7x + 4$

has $a = -3$, $b = -7$ and $c = 4$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(-3)4 \\ &= 97 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a < 0$, the graph is

5 **a** $y = -2x^2 + 3x + 2$

has $a = -2$, $b = 3$ and $c = 2$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(2) \\ &= 25 \end{aligned}$$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

b $y = 3x^2 + x + 11$

has $a = 3$, $b = 1$ and $c = 11$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 1^2 - 4(3)(11) \\ &= -131 \end{aligned}$$

$\therefore$ $\Delta < 0$, and since $a > 0$, the function is positive definite.

6 **a** The graph has x-intercepts ± 3, so its equation is

$y = a(x + 3)(x - 3)$ for some $a \neq 0$.

Its y-intercept is -27, so

$$\begin{aligned} a(3)(-3) &= -27 \\ \therefore \quad -9a &= -27 \\ \therefore \quad a &= 3 \end{aligned}$$

$\therefore$ the equation is $y = 3(x + 3)(x - 3)$

b The quadratic has vertex $(2, 25)$

$\therefore$ its equation is $y = a(x - 2)^2 + 25$

The y-intercept is 1, so

$$\begin{aligned} a(-2)^2 + 25 &= 1 \\ \therefore \quad 4a &= -24 \\ \therefore \quad a &= -6 \end{aligned}$$

$\therefore$ the equation is $y = -6(x - 2)^2 + 25$

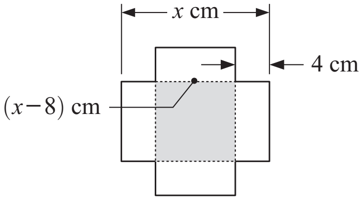
7 Let the number be x.

$\therefore$ its reciprocal is $\dfrac{1}{x}$.

$$\therefore\ x+\frac{1}{x} = 2\tfrac{1}{30} = \tfrac{61}{30}$$
$$\therefore\ x^2+1 = \tfrac{61}{30}x$$
$$\therefore\ 30x^2+30 = 61x$$
$$\therefore\ 30x^2-61x+30 = 0$$
$$\therefore\ (6x-5)(5x-6) = 0$$
$$\therefore\ x = \tfrac{5}{6} \text{ or } \tfrac{6}{5}$$

$\therefore$ the number is $\tfrac{5}{6}$ or $\tfrac{6}{5}$

8



Since the container has a square base, the original tinplate must have been square.

Suppose its side was x cm long, so the base of the container is $(x-8)$ cm by $(x-8)$ cm.

The height of the container is 4 cm, so its capacity is $4(x-8)(x-8)$ cm^3.

$$\therefore\ 4(x-8)^2 = 120$$
$$\therefore\ (x-8)^2 = 30$$
$$\therefore\ x-8 = \pm\sqrt{30}$$
$$\therefore\ x = 8\pm\sqrt{30}$$

Clearly, $x > 8$, so $x = 8+\sqrt{30} \doteqdot 13.48$

$\therefore$ the tinplate was about 13.5 cm by 13.5 cm

9 $y=-x^2-5x+3$ meets $y=x^2+3x+11$

when $-x^2-5x+3 = x^2+3x+11$
$$\therefore\ 2x^2+8x+8 = 0$$
$$\therefore\ x^2+4x+4 = 0$$
$$\therefore\ (x+2)^2 = 0$$
$$\therefore\ x = -2$$

Substituting into $y = x^2+3x+11$,

when $x=-2$, $y = (-2)^2+3(-2)+11$
$= 4-6+11$
$= 9$

$\therefore$ the graphs touch at $(-2, 9)$.

REVIEW SET 7E

1 **a** The graph has x-intercepts -3 and 1, so its equation is

$y=a(x+3)(x-1)$ for some $a\neq 0$.

Its y-intercept is 18, so
$$a(3)(-1) = 18$$
$$\therefore\ a = -6$$

So the equation is

$y=-6(x+3)(x-1)$.

b The graph has vertex $(2, -20)$, so its equation is

$y=a(x-2)^2-20$ for some $a\neq 0$.

Now an x-intercept is 5
$$\therefore\ a(5-2)^2-20 = 0$$
$$\therefore\ 9a = 20 \text{ and so } a=\tfrac{20}{9}$$

So the equation is $y=\tfrac{20}{9}(x-2)^2-20$.

2 **a** Since one x-intercept is 7 and the axis of symmetry is $x=4$, the other x-intercept is $x=1$.

$\therefore$ the graph has equation

$y=a(x-7)(x-1)$ for some $a\neq 0$.

The y-intercept is -2
$$\therefore\ a(-7)(-1) = -2$$
$$\therefore\ a = -\tfrac{2}{7}$$

$\therefore$ the equation is $y=-\tfrac{2}{7}(x-7)(x-1)$.

b The graph has vertex $(-3, 0)$, so its equation is

$y=a(x+3)^2$ for some $a\neq 0$.

The y-intercept is 2
$$\therefore\ a(3)^2 = 2$$
$$\therefore\ 9a = 2 \text{ and so } a=\tfrac{2}{9}$$

So the equation is $y=\tfrac{2}{9}(x+3)^2$.

3 The x-intercepts are 3 and -2, so the equation is $y = a(x-3)(x+2)$ for some $a \neq 0$.

But the y-intercept is 24 $\quad \therefore \; a(-3)(2) = 24$

$$\therefore \; -6a = 24$$
$$\therefore \; a = -4$$

$\therefore$ the equation is $y = -4(x-3)(x+2)$ i.e., $y = -4(x^2 - x - 6)$

i.e., $y = -4x^2 + 4x + 24$

4 The graph touches the x-axis at 4, so its vertex is (4, 0).

$\therefore$ its equation is $y = a(x-4)^2$ for some $a \neq 0$.

The graph also passes through (2, 12) $\quad \therefore \; a(2-4)^2 = 12$

$$\therefore \; 4a = 12$$
$$\therefore \; a = 3$$

$\therefore$ the equation is $y = 3(x-4)^2$ i.e., $y = 3(x^2 - 8x + 16)$

i.e., $y = 3x^2 - 24x + 48$

5 The quadratic has vertex $(-4, 1)$, so its equation is $y = a(x+4)^2 + 1$ for some $a \neq 0$.

The graph also passes through (1, 11)

$$\therefore \; 11 = a(1+4)^2 + 1$$
$$\therefore \; 25a = 10$$
$$\therefore \; a = \tfrac{2}{5}$$

$\therefore$ the equation is $y = \frac{2}{5}(x+4)^2 + 1$ i.e., $y = \frac{2}{5}(x^2 + 8x + 16) + 1$

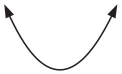
i.e., $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

6 **a** $y = 3x^2 + 4x + 7$

has $a = 3$, $b = 4$ and $c = 7$

Since $a > 0$,

the graph is



and so has a minimum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

i.e., $x = -\dfrac{4}{2(3)} = -\dfrac{2}{3}$

When $x = -\frac{2}{3}$,

$$\begin{aligned} y &= 3(-\tfrac{2}{3})^2 + 4(-\tfrac{2}{3}) + 7 \\ &= \tfrac{4}{3} - \tfrac{8}{3} + 7 \\ &= \tfrac{17}{3} \end{aligned}$$

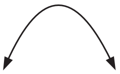
$\therefore$ the minimum is $\frac{17}{3}$ when $x = -\frac{2}{3}$

b $y = -2x^2 - 5x + 2$

has $a = -2$, $b = -5$ and $c = 2$

Since $a < 0$,

the graph is



and so has a maximum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

i.e., $x = -\dfrac{(-5)}{2(-2)} = -\dfrac{5}{4}$

When $x = -\frac{5}{4}$,

$$\begin{aligned} y &= -2(-\tfrac{5}{4})^2 - 5(-\tfrac{5}{4}) + 2 \\ &= -\tfrac{50}{16} + \tfrac{25}{4} + 2 \\ &= \frac{-25 + 50 + 16}{8} \\ &= \tfrac{41}{8} \end{aligned}$$

$\therefore$ the maximum is $\frac{41}{8}$ when $x = -\frac{5}{4}$

7 $y = x^2 - 2x + k$ has $a = 1$, $b = -2$ and $c = k$

$$\begin{aligned}\therefore\quad \Delta &= b^2 - 4ac\\ &= (-2)^2 - 4(1)k\\ &= 4 - 4k\end{aligned}$$

The graph cuts the x-axis twice if $\Delta > 0$

$$\begin{aligned}\therefore\quad 4 - 4k &> 0\\ \therefore\quad 4k &< 4\\ \therefore\quad k &< 1\end{aligned}$$

8 **a** The total length of fencing is

$(8x + 9y)$ m

$$\begin{aligned}\therefore\quad 8x + 9y &= 600\\ \therefore\quad 9y &= 600 - 8x\\ \therefore\quad y &= \frac{600 - 8x}{9}\end{aligned}$$

b The area of each pen is

$$\begin{aligned}A &= xy\\ &= x\left(\frac{600 - 8x}{9}\right)\ \text{m}^2\end{aligned}$$

c $$\begin{aligned}A &= x\left(\frac{600 - 8x}{9}\right)\\ &= \tfrac{600}{9}x - \tfrac{8}{9}x^2\end{aligned}$$

which has $a = -\frac{8}{9}$, $b = \frac{600}{9}$

Since $a < 0$, A is maximised at the axis of symmetry, which is $x = -\dfrac{b}{2a}$

i.e., $x = -\dfrac{\frac{600}{9}}{2(-\frac{8}{9})} = \frac{600}{16}$

i.e., $x = = \frac{75}{2}$

When $x = \frac{75}{2}$, $y = \dfrac{600 - 8(\frac{75}{2})}{9} = 33\frac{1}{3}$

$\therefore$ for maximum area, each pen should be $37\frac{1}{2}$ m × $33\frac{1}{3}$ m

d The maximum area of each pen is

$$\begin{aligned}&37\tfrac{1}{2} \times 33\tfrac{1}{3}\\ &= \tfrac{75}{2} \times \tfrac{100}{3}\\ &= 1250\ \text{m}^2\end{aligned}$$

Chapter 8

COMPLEX NUMBERS AND POLYNOMIALS

EXERCISE 8A

1 **a**
$$\begin{aligned}&\sqrt{-9}\\ &=\sqrt{9}\times\sqrt{-1}\\ &=3i\end{aligned}$$

b
$$\begin{aligned}&\sqrt{-64}\\ &=\sqrt{64}\times\sqrt{-1}\\ &=8i\end{aligned}$$

c
$$\begin{aligned}&\sqrt{-\tfrac{1}{4}}\\ &=\sqrt{\tfrac{1}{4}}\times\sqrt{-1}\\ &=\tfrac{1}{2}i\end{aligned}$$

d
$$\begin{aligned}&\sqrt{-5}\\ &=\sqrt{5}\times\sqrt{-1}\\ &=i\sqrt{5}\end{aligned}$$

e
$$\begin{aligned}&\sqrt{-8}\\ &=\sqrt{8}\times\sqrt{-1}\\ &=2i\sqrt{2}\end{aligned}$$

2 **a**
$$\begin{aligned}&x^2-9\\ &=(x+3)(x-3)\end{aligned}$$

b
$$\begin{aligned}&x^2+9\\ &=x^2-9i^2\\ &=(x+3i)(x-3i)\end{aligned}$$

c
$$\begin{aligned}&x^2-7\\ &=(x+\sqrt{7})(x-\sqrt{7})\end{aligned}$$

d
$$\begin{aligned}&x^2+7\\ &=x^2+(\sqrt{7}i)^2\\ &=(x+i\sqrt{7})(x-i\sqrt{7})\end{aligned}$$

e
$$\begin{aligned}&4x^2-1\\ &=(2x+1)(2x-1)\end{aligned}$$

f
$$\begin{aligned}&4x^2+1\\ &=4x^2-i^2\\ &=(2x+i)(2x-i)\end{aligned}$$

g
$$\begin{aligned}&2x^2-9\\ &=(\sqrt{2}x+3)(\sqrt{2}x-3)\end{aligned}$$

h
$$\begin{aligned}&2x^2+9\\ &=2x^2-9i^2\\ &=(\sqrt{2}x+3i)(\sqrt{2}x-3i)\end{aligned}$$

i
$$\begin{aligned}&x^3-x\\ &=x(x^2-1)\\ &=x(x+1)(x-1)\end{aligned}$$

j
$$\begin{aligned}&x^3+x\\ &=x(x^2+1)\\ &=x(x^2-i^2)\\ &=x(x+i)(x-i)\end{aligned}$$

k
$$\begin{aligned}&x^4-1\\ &=(x^2+1)(x^2-1)\\ &=(x^2-i^2)(x^2-1)\\ &=(x+i)(x-i)(x+1)(x-1)\end{aligned}$$

l
$$\begin{aligned}&x^4-16\\ &=(x^2+4)(x^2-4)\\ &=(x^2-4i^2)(x^2-4)\\ &=(x+2i)(x-2i)(x+2)(x-2)\end{aligned}$$

3 **a**
$$\begin{aligned}x^2-25&=0\\ (x+5)(x-5)&=0\\ \therefore\quad x&=\pm5\end{aligned}$$

b
$$\begin{aligned}x^2+25&=0\\ x^2-25i^2&=0\\ (x+5i)(x-5i)&=0\\ \therefore\quad x&=\pm5i\end{aligned}$$

c
$$\begin{aligned}x^2-5&=0\\ (x+\sqrt{5})(x-\sqrt{5})&=0\\ x&=\pm\sqrt{5}\end{aligned}$$

d
$$\begin{aligned}x^2+5&=0\\ x^2-5i^2&=0\\ (x+i\sqrt{5})(x-i\sqrt{5})&=0\\ \therefore\quad x&=\pm i\sqrt{5}\end{aligned}$$

e
$$\begin{aligned}4x^2-9&=0\\ (2x+3)(2x-3)&=0\\ \therefore\quad x&=\pm\tfrac{3}{2}\end{aligned}$$

f
$$\begin{aligned}4x^2+9&=0\\ 4x^2-9i^2&=0\\ (2x+3i)(2x-3i)&=0\\ \therefore\quad x&=\pm\tfrac{3}{2}i\end{aligned}$$

g
$$\begin{aligned}x^3-4x&=0\\ x(x^2-4)&=0\\ x(x+2)(x-2)&=0\\ \therefore\quad x&=0\ \text{ or }\ \pm2\end{aligned}$$

h
$$\begin{aligned}x^3+4x&=0\\ x(x^2+4)&=0\\ x(x^2-4i^2)&=0\\ x(x+2i)(x-2i)&=0\\ \therefore\quad x&=0\ \text{ or }\ \pm2i\end{aligned}$$

i
$$\begin{aligned} x^3 - 3x &= 0 \\ x(x^2 - 3) &= 0 \\ x(x + \sqrt{3})(x - \sqrt{3}) &= 0 \\ \therefore \quad x &= 0 \text{ or } \pm\sqrt{3} \end{aligned}$$

j
$$\begin{aligned} x^3 + 3x &= 0 \\ x(x^2 + 3) &= 0 \\ x(x^2 - 3i^2) &= 0 \\ x(x + i\sqrt{3})(x - i\sqrt{3}) &= 0 \\ \therefore \quad x &= 0 \text{ or } \pm i\sqrt{3} \end{aligned}$$

k
$$\begin{aligned} x^4 - 1 &= 0 \\ (x^2 + 1)(x^2 - 1) &= 0 \\ (x^2 - i^2)(x^2 - 1) &= 0 \\ (x + i)(x - i)(x + 1)(x - 1) &= 0 \\ \therefore \quad x &= \pm i \text{ or } \pm 1 \end{aligned}$$

l
$$\begin{aligned} x^4 &= 81 \\ x^4 - 81 &= 0 \\ (x^2 + 9)(x^2 - 9) &= 0 \\ (x^2 - 9i^2)(x^2 - 9) &= 0 \\ (x + 3i)(x - 3i)(x + 3)(x - 3) &= 0 \\ \therefore \quad x = \pm 3i &\text{ or } \pm 3 \end{aligned}$$

4 **a** If $x^2 - 10x + 29 = 0$

$$\begin{aligned} \text{then} \quad x &= \frac{10 \pm \sqrt{100 - 4 \times 1 \times 29}}{2} \\ &= \frac{10 \pm \sqrt{-16}}{2} \\ &= 5 \pm \sqrt{-4} \\ &= 5 \pm 2i \end{aligned}$$

b If $x^2 + 6x + 25 = 0$

$$\begin{aligned} \text{then} \quad x &= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 25}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= -3 \pm \sqrt{-16} \\ &= -3 \pm 4i \end{aligned}$$

c If $x^2 + 14x + 50 = 0,$

$$\begin{aligned} \text{then} \quad x &= \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 50}}{2} \\ &= \frac{-14 \pm \sqrt{-4}}{2} \\ &= -7 \pm \sqrt{-1} \\ &= -7 \pm i \end{aligned}$$

d
$$\begin{aligned} \text{If} \quad 2x^2 + 5 &= 6x, \\ \text{then} \quad 2x^2 - 6x + 5 &= 0 \end{aligned}$$
$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 4 \times 2 \times 5}}{4} \\ &= \frac{6 \pm \sqrt{-4}}{4} \\ &= \frac{3 \pm \sqrt{-1}}{2} \\ &= \tfrac{3}{2} \pm \tfrac{1}{2}i \end{aligned}$$

e If $x^2 - 2\sqrt{3}x + 4 = 0,$

$$\begin{aligned} \text{then} \quad x &= \frac{2\sqrt{3} \pm \sqrt{12 - 4 \times 1 \times 4}}{2} \\ &= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} \\ &= \sqrt{3} \pm \sqrt{-1} \\ &= \sqrt{3} \pm i \end{aligned}$$

f
$$\begin{aligned} \text{If} \quad 2x + \frac{1}{x} &= 1, \\ \text{then} \quad 2x^2 + 1 &= x \\ \therefore \quad 2x^2 - x + 1 &= 0 \end{aligned}$$
$$\begin{aligned} x &= \frac{1 \pm \sqrt{1 - 4 \times 2 \times 1}}{4} \\ &= \frac{1 \pm \sqrt{-7}}{4} \\ &= \tfrac{1}{4} \pm i\tfrac{\sqrt{7}}{4} \end{aligned}$$

5 **a**
$$\begin{aligned} x^4 + 2x^2 &= 3 \\ x^4 + 2x^2 - 3 &= 0 \\ (x^2 + 3)(x^2 - 1) &= 0 \\ (x^2 - 3i^2)(x^2 - 1) &= 0 \\ (x + i\sqrt{3})(x - i\sqrt{3})(x + 1)(x - 1) &= 0 \\ \therefore \quad x &= \pm i\sqrt{3} \text{ or } \pm 1 \end{aligned}$$

b
$$\begin{aligned} x^4 &= x^2+6 \\ x^4-x^2-6 &= 0 \\ (x^2-3)(x^2+2) &= 0 \\ (x^2-3)(x^2-2i^2) &= 0 \\ (x+\sqrt{3})(x-\sqrt{3})(x+i\sqrt{2})(x-i\sqrt{2}) &= 0 \quad \therefore \quad x = \pm\sqrt{3} \text{ or } \pm i\sqrt{2} \end{aligned}$$

c
$$\begin{aligned} x^4+5x^2 &= 36 \\ x^4+5x^2-36 &= 0 \\ (x^2+9)(x^2-4) &= 0 \\ (x^2-9i^2)(x^2-4) &= 0 \\ (x+3i)(x-3i)(x+2)(x-2) &= 0 \quad \therefore \quad x = \pm 3i \text{ or } \pm 2 \end{aligned}$$

d
$$\begin{aligned} x^4+9x^2+14 &= 0 \\ (x^2+7)(x^2+2) &= 0 \\ (x^2-7i^2)(x^2-2i^2) &= 0 \\ (x+i\sqrt{7})(x-i\sqrt{7})(x+i\sqrt{2})(x-i\sqrt{2}) &= 0 \quad \therefore \quad x = \pm i\sqrt{7} \text{ or } \pm i\sqrt{2} \end{aligned}$$

e
$$\begin{aligned} x^4+1 &= 2x^2 \\ x^4-2x^2+1 &= 0 \\ (x^2-1)^2 &= 0 \\ (x+1)^2(x-1)^2 &= 0 \\ \therefore \quad x &= \pm 1 \end{aligned}$$

f
$$\begin{aligned} x^4+2x^2+1 &= 0 \\ (x^2+1)^2 &= 0 \\ (x^2-i^2)^2 &= 0 \\ (x+i)^2(x-i)^2 &= 0 \\ \therefore \quad x &= \pm i \end{aligned}$$

EXERCISE 8B.1

1

z	$\text{Re}(z)$	$\text{Im}(z)$	z	$\text{Re}(z)$	$\text{Im}(z)$
$3+2i$	3	2	$-3+4i$	-3	4
$5-i$	5	-1	$-7-2i$	-7	-2
3	3	0	$-11i$	0	-11
0	0	0	$i\sqrt{3}$	0	$\sqrt{3}$

2 **a**
$$\begin{aligned} &z+w \\ &= (5-2i)+(2+i) \\ &= 7-i \end{aligned}$$

b
$$\begin{aligned} &2z \\ &= 2(5-2i) \\ &= 10-4i \end{aligned}$$

c
$$\begin{aligned} iw &= i(2+i) \\ &= 2i+i^2 \\ &= -1+2i \end{aligned}$$

d
$$\begin{aligned} &z-w \\ &= (5-2i)-(2+i) \\ &= 5-2i-2-i \\ &= 3-3i \end{aligned}$$

e
$$\begin{aligned} &2z-3w \\ &= 2(5-2i)-3(2+i) \\ &= 10-4i-6-3i \\ &= 4-7i \end{aligned}$$

f
$$\begin{aligned} &zw \\ &= (5-2i)(2+i) \\ &= 10-4i+5i-2i^2 \\ &= 12+i \end{aligned}$$

g
$$\begin{aligned} w^2 &= (2+i)^2 \\ &= 4+4i+i^2 \\ &= 3+4i \end{aligned}$$

h
$$\begin{aligned} z^2 &= (5-2i)^2 \\ &= 25-20i+4i^2 \\ &= 21-20i \end{aligned}$$

3 **a**
$$\begin{aligned} &z+2w \\ &= (1+i)+2(-2+3i) \\ &= 1+i-4+6i \\ &= -3+7i \end{aligned}$$

b
$$\begin{aligned} &z^2 \\ &= (1+i)^2 \\ &= 1+2i+i^2 \\ &= 2i \end{aligned}$$

c
$$\begin{aligned} z^3 &= z^2 \times z \\ &= 2i(1+i) \\ &= 2i+2i^2 \\ &= -2+2i \end{aligned}$$

d
$$\begin{aligned} iz &= i(1+i) \\ &= i+i^2 \\ &= -1+i \end{aligned}$$

e
$$\begin{aligned} w^2 &= (-2+3i)^2 \\ &= 4-12i+9i^2 \\ &= -5-12i \end{aligned}$$

f
$$\begin{aligned} &zw \\ &= (1+i)(-2+3i) \\ &= -2+3i-2i+3i^2 \\ &= -5+i \end{aligned}$$

g
$$\begin{aligned} z^2w &= (1+i)^2(-2+3i) \\ &= 2i(-2+3i) \\ &= -4i+6i^2 \\ &= -6-4i \end{aligned}$$

h
$$\begin{aligned} izw &= i(1+i)(-2+3i) \\ &= i(-5+i) \\ &= -5i+i^2 \\ &= -1-5i \end{aligned}$$

4
$$\begin{array}{llll} i^0 = 1 & i^4 = 1 & i^8 = 1 & i^{-1} = -i \\ i^1 = i & i^5 = i & i^9 = i & i^{-2} = -1 \\ i^2 = -1 & i^6 = -1 & & i^{-3} = i \\ i^3 = -i & i^7 = -i & & i^{-4} = 1 \\ & & & i^{-5} = -i \end{array}$$

$\therefore\ i^{4n+3} = -i$ where n is any integer

5 **a** If
$$\begin{aligned} z &= \cos\theta + i\sin\theta \quad \text{then} \\ z^2 &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta \\ &= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta) \\ &= \cos 2\theta + i\sin 2\theta \end{aligned}$$

b
$$\begin{aligned} \frac{1}{z} &= \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta} \\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\ &= \frac{\cos\theta - i\sin\theta}{1} \\ &= \cos\theta - i\sin\theta \end{aligned}$$

6
$$\begin{aligned} (1+i)^4 &= [(1+i)^2]^2 \\ &= (1+2i+i^2)^2 \\ &= (2i)^2 \\ &= -4 \end{aligned}$$

$$\begin{aligned} (1+i)^{101} &= (1+i)^{100} \times (1+i) \\ &= \left[(1+i)^4\right]^{25} \times (1+i) \\ &= [-4]^{25}\,(1+i) \\ &= -2^{50}(1+i) \end{aligned}$$

7
$$\begin{aligned} (a+bi)^2 &= -16-30i \\ a^2+2abi+b^2i^2 &= -16-30i \\ \therefore\quad a^2-b^2 &= -16 \quad \text{and} \quad 2ab = -30, \\ \text{i.e.,}\quad ab &= \quad -15 \quad \text{and} \quad \therefore \quad b = -\frac{15}{a} \\ \text{so}\quad a^2 - \left(-\frac{15}{a}\right)^2 &= -16 \\ \therefore\quad a^2 - \frac{225}{a^2} &= -16 \\ \therefore\quad a^4+16a^2-225 &= 0 \\ \therefore\quad (a^2+25)(a^2-9) &= 0 \\ \therefore\quad a &= \pm 3 \quad \text{or} \quad \pm 5i \end{aligned}$$

But a is real and > 0 $\quad\therefore\quad a = 3$ and $b = -\frac{15}{3} = -5$

8 **a**
$$\begin{aligned} \frac{z}{w} &= \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i} \\ &= \frac{2-6i-i+3i^2}{1-9i^2} \\ &= \frac{-1-7i}{10} \\ &= -\tfrac{1}{10} - \tfrac{7}{10}i \end{aligned}$$

b
$$\begin{aligned} \frac{i}{z} &= \frac{i}{2-i} \times \frac{2+i}{2+i} \\ &= \frac{2i+i^2}{4-i^2} \\ &= \frac{-1+2i}{5} \\ &= -\tfrac{1}{5} + \tfrac{2}{5}i \end{aligned}$$

c
$$\begin{aligned}\frac{w}{iz} &= \frac{1+3i}{i(2-i)}\\ &= \frac{1+3i}{2i-i^2}\\ &= \frac{1+3i}{1+2i}\times\frac{1-2i}{1-2i}\\ &= \frac{1-2i+3i-6i^2}{1-4i^2}\\ &= \frac{7+i}{5}\\ &= \tfrac{7}{5}+\tfrac{1}{5}i\end{aligned}$$

d
$$\begin{aligned}z^{-2} &= \frac{1}{(2-i)^2}\\ &= \frac{1}{4-4i+i^2}\\ &= \frac{1}{3-4i}\times\frac{3+4i}{3+4i}\\ &= \frac{3+4i}{9-16i^2}\\ &= \frac{3+4i}{25}\\ &= \tfrac{3}{25}+\tfrac{4}{25}i\end{aligned}$$

9 **a**
$$\begin{aligned}&\frac{i}{1-2i}\\ &= \frac{i}{1-2i}\times\frac{1+2i}{1+2i}\\ &= \frac{i+2i^2}{1-4i^2}\\ &= \frac{-2+i}{5}\\ &= -\tfrac{2}{5}+\tfrac{1}{5}i\end{aligned}$$

b
$$\begin{aligned}&\frac{i(2-i)}{3-2i}\\ &= \frac{2i-i^2}{3-2i}\\ &= \frac{1+2i}{3-2i}\times\frac{3+2i}{3+2i}\\ &= \frac{3+2i+6i+4i^2}{9-4i^2}\\ &= \frac{-1+8i}{13}\\ &= -\tfrac{1}{13}+\tfrac{8}{13}i\end{aligned}$$

c
$$\begin{aligned}&\frac{1}{2-i}-\frac{2}{2+i}\\ &= \frac{1}{2-i}\left(\frac{2+i}{2+i}\right)-\frac{2}{2+i}\left(\frac{2-i}{2-i}\right)\\ &= \frac{2+i-2(2-i)}{(2-i)(2+i)}\\ &= \frac{2+i-4+2i}{4-i^2}\\ &= \frac{-2+3i}{5}\\ &= -\tfrac{2}{5}+\tfrac{3}{5}i\end{aligned}$$

10 **a**
$$\begin{aligned}4z-3w &= 4(2+i)-3(-1+2i)\\ &= 8+4i+3-6i\\ &= 11-2i\\ \therefore\ \operatorname{Im}(4z-3w) &= -2\end{aligned}$$

b
$$\begin{aligned}zw &= (2+i)(-1+2i)\\ &= -2+4i-i+2i^2\\ &= -4+3i\\ \therefore\ \operatorname{Re}(zw) &= -4\end{aligned}$$

c
$$\begin{aligned}iz^2 &= i(2+i)^2\\ &= i(4+4i+i^2)\\ &= i(3+4i)\\ &= 3i+4i^2\\ &= -4+3i\\ \therefore\ \operatorname{Im}(iz^2) &= 3\end{aligned}$$

d
$$\begin{aligned}\frac{z}{w} &= \frac{2+i}{-1+2i}\times\frac{-1-2i}{-1-2i}\\ &= \frac{-2-4i-i-2i^2}{1-4i^2}\\ &= \frac{0-5i}{5}\\ &= -i\\ \therefore\ \operatorname{Re}\left(\frac{z}{w}\right) &= 0\end{aligned}$$

EXERCISE 8B.2

1 **a** $2x+3iy = -x-6i$

Equating real and imaginary parts,

$2x = -x$ and $3y=-6$

$\{x,\ y$ are real$\}$

$\therefore\ 3x = 0$ and $y=-2$

$\therefore\ x = 0$ and $y=-2$

b $x^2+ix = 4-2i$

Equating real and imaginary parts,

$x^2 = 4$ and $x=-2$ $\{x$ is real$\}$

$\therefore\ x = \pm 2$ and $x=-2$

$\therefore\ x = -2$

c
$$\begin{aligned}(x+iy)(2-i) &= 8+i\\ \therefore\quad x+iy &= \frac{8+i}{2-i}\times\frac{2+i}{2+i}\\ \therefore\quad x+iy &= \frac{16+8i+2i+i^2}{4-i^2}\\ \therefore\quad x+iy &= \frac{15+10i}{5}\\ \therefore\quad x+iy &= 3+2i\end{aligned}$$

Equating real and imag. parts, for real x, y

$x=3$ and $y=2$

d
$$\begin{aligned}(3+2i)(x+iy) &= -i\\ \therefore\quad x+iy &= \frac{-i}{3+2i}\times\frac{3-2i}{3-2i}\\ \therefore\quad x+iy &= \frac{-3i+2i^2}{9-4i^2}\\ \therefore\quad x+iy &= \frac{-2-3i}{13}\end{aligned}$$

Equating real and imag. parts, for real x, y

$x=-\frac{2}{13}$ and $y=-\frac{3}{13}$

2 **a**
$$\begin{aligned}2(x+iy) &= x-iy\\ \therefore\quad 2x+2iy &= x-iy\end{aligned}$$

Equating real and imaginary parts,

$$\begin{aligned}2x &= x \quad\text{and}\quad 2y=-y\\ \therefore\quad x &= 0 \quad\text{and}\quad 3y=0\\ \therefore\quad x &= 0 \quad\text{and}\quad y=0\end{aligned}$$

b
$$\begin{aligned}(x+2i)(y-i) &= -4-7i\\ \therefore\quad xy-ix+2iy-2i^2 &= -4-7i\\ \therefore\quad (xy+2)+i(2y-x) &= -4-7i\end{aligned}$$

Equating real and imaginary parts,

$xy+2=-4$ and $2y-x=-7$

$xy=-6$ and $x=2y+7$

$$\begin{aligned}(2y+7)y &= -6\\ 2y^2+7y &= -6\\ 2y^2+7y+6 &= 0\\ (2y+3)(y+2) &= 0\\ \therefore\quad y=-\tfrac{3}{2} \quad&\text{or}\quad y=-2\end{aligned}$$

When $y=-2$, $x=2(-2)+7=3$

When $y=-\frac{3}{2}$, $x=2(-\frac{3}{2})+7=4$

$\therefore\ x=3$ and $y=-2$

or $x=4$ and $y=-\frac{3}{2}$

c
$$\begin{aligned}(x+i)(3-iy) &= 1+13i\\ 3x-ixy+3i-i^2y &= 1+13i\\ (3x+y)+i(3-xy) &= 1+13i\end{aligned}$$

Equating real and imaginary parts,

$3x+y=1$ and $3-xy=13$

$y=1-3x$ and $xy=-10$

$$\begin{aligned}x(1-3x) &= -10\\ x-3x^2 &= -10\\ 0 &= 3x^2-x-10\\ 0 &= (3x+5)(x-2)\end{aligned}$$

$\therefore\ x=-\frac{5}{3}$ *or* $x=2$

When $x=-\frac{5}{3}$, $y=1-3(-\frac{5}{3})=6$

and when $x=2$, $y=1-3\times 2=-5$

$\therefore\ x=-\frac{5}{3}$ and $y=6$

or $x=2$ and $y=-5$

d
$$\begin{aligned}(x+iy)(2+i) &= 2x-i(y+1)\\ 2x+ix+2iy+yi^2 &= 2x+i(-y-1)\\ (2x-y)+i(x+2y) &= 2x+i(-y-1)\end{aligned}$$

Equating real and imaginary parts,

$2x-y=2x$ and $x+2y=-y-1$

$\therefore\ -y=2x-2x$

$\therefore\ y=0$ and consequently

$x+0=0-1=-1$

$\therefore\ x=-1$ and $y=0$

EXERCISE 8B.3

1 **a** roots α and β are $3\pm i$ $\quad\therefore\ \alpha+\beta=6$ and $\alpha\beta=9-i^2=10$

$\therefore$ quadratics have form $k(x^2-6x+10)=0$ $\quad k\neq 0$

b roots α and β are $1\pm 3i$ $\quad\therefore\ \alpha+\beta=2$ and $\alpha\beta=1-9i^2=10$

$\therefore$ quadratics have form $k(x^2-2x+10)=0$ $\quad k\neq 0$

c roots α and β are $-2\pm 5i$ $\quad\therefore\ \alpha+\beta=-4$ and $\alpha\beta=4-25i^2=29$

$\therefore$ quadratics have form $k(x^2+4x+29)=0$ $\quad k\neq 0$

d roots α and β are $\sqrt{2}\pm i$ $\quad\therefore\ \alpha+\beta=2\sqrt{2}$ and $\alpha\beta=2-i^2=3$

$\therefore$ quadratics have form $k(x^2-2\sqrt{2}x+3)=0$ $\quad k\neq 0$

e roots α and β are $2 \pm \sqrt{3}$ $\quad \therefore \quad \alpha + \beta = 4$ and $\alpha\beta = 4 - 3 = 1$

$\therefore$ quadratics have form $k(x^2 - 4x + 1) = 0 \quad k \neq 0$

f roots α and β are 0 and $-\frac{2}{3}$ $\therefore$ factors are $x,\ 3x + 2$

$\therefore$ quadratics have form $kx(3x + 2) = 0$

$\therefore \quad k(3x^2 + 2x) = 0, \quad k \neq 0$

g roots α and β are $\pm i\sqrt{2}$ $\therefore \quad \alpha + \beta = 0$ and $\alpha\beta = -2i^2 = +2$

$\therefore$ quadratics have form $k(x^2 + 2) = 0 \quad k \neq 0$

h roots α and β are $-6 \pm i$ $\quad \therefore \quad \alpha + \beta = -12$ and $\alpha\beta = 36 - i^2 = 37$

$\therefore$ quadratics have form $k(x^2 + 12x + 37) = 0 \quad k \neq 0$

2 **a** If $3 + i$ is a root then so is $3 - i$ (if a and b are real)

$\therefore \quad \alpha + \beta = 6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore \quad x^2 - 6x + 10 = 0$ and so $a = -6, \quad b = 10$

b If $1 - \sqrt{2}$ is a root then so is $1 + \sqrt{2}$ if a, b are rational.

$\therefore \quad \alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$\therefore \quad x^2 - 2x - 1 = 0$

$\therefore \quad a = -2$ and $b = -1$

c If $a + ai$ is a root then so is $a - ai$ (if a and b are real and $a \neq 0$.)

$$\therefore \quad \alpha + \beta = 2a \quad \text{and} \quad \alpha\beta = (a + ai)(a - ai)$$
$$= a^2 - (ai^2)$$
$$= 2a^2$$

$\therefore \quad x^2 - 2ax + 2a^2 = 0$

$\therefore \quad -2a = 4$ and $b = 2a^2$

$\therefore \quad a = -2$ and $b = 8$

However if $a = 0$, $a + i = 0$ (i.e., is *not* complex) so the other root could be any real number.

But $\alpha\beta = 0 \quad \therefore \quad b = 0$

$\therefore \quad a = 0, \quad b = 0$ is also a solution.

EXERCISE 8B.4

1 *To Prove:* $(z_1 - z_2)^* = {z_1}^* - {z_2}^*$

Let $z_1 = a + ib$, and $z_2 = c + id$.

$$\begin{aligned} \therefore \quad (z_1 - z_2)^* &= [(a + ib) - (c + id)]^* \\ &= [(a - c) + i(b - d)]^* \\ &= (a - c) - i(b - d) \\ &= a - c - bi + di \\ &= (a - bi) - (c - di) \\ &= {z_1}^* - {z_2}^* \end{aligned}$$

2 $(w^* - z)^* - (w - 2z^*)$

$$\begin{aligned} &= w^{**} - z^* - w + 2z^* \\ &= w - z^* - w + 2z^* \\ &= -z^* + 2z^* \\ &= z^* \end{aligned}$$

3 Let $z = a + bi \quad \therefore \quad z^* = a - bi$

If $z^* = -z$, then $a - bi = -a - bi$

$\therefore \quad a = -a$

$\therefore \quad 2a = 0$

i.e., $a = 0$ and b is any real number

i.e., z is purely imaginary

or $a = 0, \quad b = 0$ i.e., z is zero.

4 **a** Let $z_1 = a+bi \quad z_2 = c+di$

$$\therefore \quad \frac{z_1}{z_2} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$
$$= \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$= \frac{ac-adi+bci-bdi^2}{c^2-i^2d^2}$$
$$= \frac{(ac+bd)+i(-ad+bc)}{(c^2+d^2)}$$
$$= \left[\frac{ac+bd}{c^2+d^2}\right] + \left[\frac{bc-ad}{c^2+d^2}\right] i$$

b
$$\frac{{z_1}^*}{{z_2}^*} = \frac{a-bi}{c-di} \times \frac{c+di}{c+di}$$
$$= \frac{(a-bi)(c+di)}{(c-di)(c+di)}$$
$$= \frac{ac+adi-bci-bdi^2}{c^2-i^2d^2}$$
$$= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$
$$= \left[\frac{(ac+bd)}{c^2+d^2}\right] - \left[\frac{bc-ad}{c^2+d^2}\right] i$$
$$= \left(\frac{z_1}{z_2}\right)^* \quad \text{for all } z_2 \neq 0$$

5
$$\left(\frac{z_1}{z_2}\right)^* \times {z_2}^* = \left(\frac{z_1}{z_2} \times z_2\right)^* \qquad \text{(from **Example 13**)}$$
$$= {z_1}^*$$
$$\therefore \quad \left(\frac{z_1}{z_2}\right)^* = \frac{{z_1}^*}{{z_2}^*} \quad \{\text{on dividing both sides by } {z_2}^*\}$$

6 **a** Let $z = a+bi$ and $w = c+di$

$$\begin{aligned}\therefore \quad zw^* + z^*w &= (a+bi)(c-di) + (a-bi)(c+di)\\ &= ac-adi+bci+bd+ac+adi-bci+bd\\ &= ac+bd+ac+bd\\ &= 2ac+2bd \quad \text{which is a real number}\end{aligned}$$

b Let $z = a+bi$ and $w = c+di$

$$\begin{aligned}\therefore \quad zw^* - z^*w &= (a+bi)(c-di) - (a-bi)(c+di)\\ &= ac-adi+bci+bd-[ac+adi-bci+bd]\\ &= 2bci-2adi\\ &= (2bc-2ad)i \quad \text{which is purely imaginary or zero}\end{aligned}$$

7 **a**
$$\begin{aligned}\text{If} \quad z &= a+bi\\ \text{then} \quad z^2 &= (a+bi)(a+bi)\\ &= a^2+2abi+b^2i^2\\ &= (a^2-b^2)+2abi\end{aligned}$$

b From **a**,
$$\begin{aligned}\left(z^2\right)^* &= (a^2-b^2)-2abi\\ &= (a^2-2abi+b^2i^2)\\ &= (a-bi)^2\\ &= (z^*)^2 \quad \text{as required.}\end{aligned}$$

c
$$\begin{aligned}z^3 &= (z^2)z\\ &= ((a^2-b^2)+2abi)(a+bi)\\ &= a(a^2-b^2)+b(a^2-b^2)i+2a^2bi+2ab^2i^2\\ &= a^3-ab^2+a^2bi-b^3i+2a^2bi-2ab^2\\ &= (a^3-3ab^2)+(3a^2b-b^3)i\\ \therefore \quad \left(z^3\right)^* &= (a^3-3ab^2)-(3a^2b-b^3)i\end{aligned}$$

$$\begin{aligned}(z^*)^3 &= (z^*)^2\, z^*\\ &= \left[(a^2-b^2)-2abi\right](a-ib)\\ &= a(a^2-b^2)-b(a^2-b^2)i-2a^2bi+2ab^2i^2\\ &= a^3-ab^2-a^2bi+b^3i-2a^2bi-2ab^2\\ &= (a^3-3ab^2)-(3a^2b-b^3)i \quad \text{which is } \left(z^3\right)^* \quad \text{as required}\end{aligned}$$

8 $w = \dfrac{z-1}{z^*+1}$ where $z = a+bi$

$$\therefore \quad w = \frac{(a-1)+bi}{(a+1)-bi} \times \frac{(a+1)+bi}{(a+1)+bi}$$
$$= \frac{(a^2-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2}$$
$$= \frac{(a^2-b^2-1)+2abi}{(a+1)^2+b^2}$$

a w is real if $2ab = 0$
i.e., if $a = 0$ or $b = 0$, $a \neq 1$
However, if $b = 0$ and $a = -1$, w is undefined and hence is not real.
$\therefore$ $a = 0$ or $(b = 0,\ a \neq -1)$.

b w is purely imaginary if
$a^2 - b^2 - 1 = 0$
i.e., $a^2 - b^2 = 1$
and neither a nor b is zero.

EXERCISE 8B.5

1 **a** $(z_1z_2z_3)^* = [z_1 \times (z_2 \times z_3)]^*$
$= {z_1}^* (z_2 \times z_3)^*$ {as $(zw)^* = z^*\,w^*$)}
$= {z_1}^* \times {z_2}^* \times {z_3}^*$ {as $(zw)^* = z^*\,w^*$) again}

b $(z_1z_2z_3z_4)^* = (z_1z_2z_3)^* \times {z_4}^*$ {as $(zw)^* = z^*\,w^*$)}
$= {z_1}^* \times {z_2}^* \times {z_3}^* \times {z_4}^*$ {using **a**}

c $(z_1 \times z_2 \times z_3 .... z_n)^* = {z_1}^* \times {z_2}^* \times {z_3}^* \ ......\ {z_n}^*$

d $(z^n)^* = (z \times z \times z \times \ ......\ \times z)^*$
$= z^* \times z^* \times z^* \times z^* \times \ ......\ \times z^*$ {using **c**}
$= (z^*)^n$

EXERCISE 8C.1

1 **a** $3P(x)$
$= 3(x^2+2x+3)$
$= 3x^2+6x+9$

b $P(x)+Q(x)$
$= (x^2+2x+3)+(4x^2+5x+6)$
$= 5x^2+7x+9$

c $P(x)-2Q(x)$
$= (x^2+2x+3)-2(4x^2+5x+6)$
$= x^2+2x+3-8x^2-10x-12)$
$= -7x^2-8x-9$

d $P(x)\,Q(x)$
$= (x^2+2x+3)(4x^2+5x+6)$
$= 4x^4+8x^3+12x^2+5x^3+10x^2$
$+15x+6x^2+12x+18$
$= 4x^4+13x^3+28x^2+27x+18$

2 **a** $f(x)+g(x)$
$= (x^2-x+2)+(x^3-3x+5)$
$= x^3+x^2-4x+7$

b $g(x)-f(x)$
$= (x^3-3x+5)-(x^2-x+2)$
$= x^3-3x+5-x^2+x-2$
$= x^3-x^2-2x+3$

c $2f(x)+3g(x)$
$= 2(x^2-x+2)+3(x^3-3x+5)$
$= 2x^2-2x+4+3x^3-9x+15$
$= 3x^3+2x^2-11x+19$

d $g(x)+x\,f(x)$
$= (x^3-3x+5)+x(x^2-x+2)$
$= x^3-3x+5+x^3-x^2+2x$
$= 2x^3-x^2-x+5$

e $f(x)\,g(x)$
$= (x^2-x+2)(x^3-3x+5)$
$= x^5-x^4+2x^3-3x^3+3x^2$
$-6x+5x^2-5x+10$
$= x^5-x^4-x^3+8x^2-11x+10$

f $[f(x)]^2$
$= (x^2-x+2)(x^2-x+2)$
$= x^4-x^3+2x^2-x^3+x^2$
$-2x+2x^2-2x+4$
$= x^4-2x^3+5x^2-4x+4$

3 **a**

$$\begin{aligned} &(x^2-2x+3)(2x+1) \\ &= 2x^3-4x^2+6x+x^2-2x+3 \\ &= 2x^3-3x^2+4x+3 \end{aligned}$$

b

$$\begin{aligned} &(x-1)^2(x^2+3x-2) \\ &= (x^2-2x+1)(x^2+3x-2) \\ &= x^4-2x^3+x^2+3x^3-6x^2 \\ &\quad +3x-2x^2+4x-2 \\ &= x^4+x^3-7x^2+7x-2 \end{aligned}$$

c

$$\begin{aligned} &(x+2)^3 \\ &= (x+2)(x+2)^2 \\ &= (x+2)(x^2+4x+4) \\ &= x^3+2x^2+4x^2+8x+4x+8 \\ &= x^3+6x^2+12x+8 \end{aligned}$$

d

$$\begin{aligned} &(2x^2-x+3)^2 \\ &= (2x^2-x+3)(2x^2-x+3) \\ &= 4x^4-2x^3+6x^2-2x^3+x^2 \\ &\quad -3x+6x^2-3x+9 \\ &= 4x^4-4x^3+13x^2-6x+9 \end{aligned}$$

e

$$\begin{aligned} &(2x-1)^4 \\ &= (2x-1)^2(2x-1)^2 \\ &= (4x^2-4x+1)(4x^2-4x+1) \\ &= 16x^4-16x^3+4x^2-16x^3+16x^2 \\ &\quad -4x+4x^2-4x+1 \\ &= 16x^4-32x^3+24x^2-8x+1 \end{aligned}$$

f

$$\begin{aligned} &(3x-2)^2(2x+1)(x-4) \\ &= (9x^2-12x+4)(2x^2-7x-4) \\ &= 18x^4-24x^3+8x^2-63x^3+84x^2 \\ &\quad -28x-36x^2+48x-16 \\ &= 18x^4-87x^3+56x^2+20x-16 \end{aligned}$$

4 **a**

$$\begin{aligned} &(2x^2-3x+5)(3x-1) \\ &= 6x^3-11x^2+18x-5 \end{aligned}$$

as

$$\begin{array}{rrrr} & 2 & -3 & 5 \\ \times & & 3 & -1 \\ \hline & -2 & 3 & -5 \\ 6 & -9 & 15 & \\ \hline 6 & -11 & 18 & -5 \\ \hline \end{array}$$

b

$$\begin{aligned} &(4x^2-x+2)(2x+5) \\ &= 8x^3+18x^2-x+10 \end{aligned}$$

as

$$\begin{array}{rrrr} & 4 & -1 & 2 \\ \times & & 2 & 5 \\ \hline & 20 & -5 & 10 \\ 8 & -2 & 4 & \\ \hline 8 & 18 & -1 & 10 \\ \hline \end{array}$$

c

$$\begin{aligned} &(2x^2+3x+2)(5-x) \\ &= -2x^3+7x^2+13x+10 \end{aligned}$$

as

$$\begin{array}{rrrr} & 2 & 3 & 2 \\ \times & & -1 & 5 \\ \hline & 10 & 15 & 10 \\ -2 & -3 & -2 & \\ \hline -2 & 7 & 13 & 10 \\ \hline \end{array}$$

d

$$\begin{aligned} &(x-2)^2(2x+1) \\ &= (x^2-4x+4)(2x+1) \\ &= 2x^3-7x^2+4x+4 \end{aligned}$$

as

$$\begin{array}{rrrr} & 1 & -4 & 4 \\ \times & & 2 & 1 \\ \hline & 1 & -4 & 4 \\ 2 & -8 & 8 & \\ \hline 2 & -7 & 4 & 4 \\ \hline \end{array}$$

e

$$\begin{aligned} &(x^2-3x+2)(2x^2+4x-1) \\ &= 2x^4-2x^3-9x^2+11x-2 \end{aligned}$$

as

$$\begin{array}{rrrrr} & & 1 & -3 & 2 \\ & \times & 2 & 4 & -1 \\ \hline & & -1 & 3 & -2 \\ & 4 & -12 & 8 & \\ 2 & -6 & 4 & & \\ \hline 2 & -2 & -9 & 11 & -2 \\ \hline \end{array}$$

f

$$\begin{aligned} &(3x^2-x+2)(5x^2+2x-3) \\ &= 15x^4+x^3-x^2+7x-6 \end{aligned}$$

as

$$\begin{array}{rrrrr} & & 3 & -1 & 2 \\ & \times & 5 & 2 & -3 \\ \hline & & -9 & 3 & -6 \\ & 6 & -2 & 4 & \\ 15 & -5 & 10 & & \\ \hline 15 & 1 & -1 & 7 & -6 \\ \hline \end{array}$$

g

$$\begin{aligned} &(x^2-x+3)^2 \\ &= x^4-2x^3+7x^2-6x+9 \end{aligned}$$

as

$$\begin{array}{rrrrr} & & 1 & -1 & 3 \\ & \times & 1 & -1 & 3 \\ \hline & & 3 & -3 & 9 \\ & -1 & 1 & -3 & \\ 1 & -1 & 3 & & \\ \hline 1 & -2 & 7 & -6 & 9 \\ \hline \end{array}$$

h

$$\begin{aligned} &(2x^2+x-4)^2 \\ &= 4x^4+4x^3-15x^2-8x+16 \end{aligned}$$

as

$$\begin{array}{rrrrr} & & 2 & 1 & -4 \\ & \times & 2 & 1 & -4 \\ \hline & & -8 & -4 & 16 \\ & 2 & 1 & -4 & \\ 4 & 2 & -8 & & \\ \hline 4 & 4 & -15 & -8 & 16 \\ \hline \end{array}$$

i
$$\begin{aligned} &(2x+5)^3\\ &= (2x+5)^2\,(2x+5)\\ &= (4x^2+20x+25)(2x+5)\\ &= 8x^3+60x^2+150x+125 \end{aligned}$$

as
$$\begin{array}{rrrr} & 4 & 20 & 25\\ \times & & 2 & 5\\ \hline & 20 & 100 & 125\\ 8 & 40 & 50 & \\ \hline 8 & 60 & 150 & 125\\ \hline \end{array}$$

j
$$\begin{aligned} &(x^3+x^2-2)^2\\ &= x^6+2x^5+x^4-4x^3-4x^2+4 \end{aligned}$$

as
$$\begin{array}{rrrrrrr} & & & 1 & 1 & 0 & -2\\ & & \times & 1 & 1 & 0 & -2\\ \hline & & & -2 & -2 & 0 & 4\\ & & 0 & 0 & 0 & 0 & \\ & 1 & 1 & 0 & -2 & & \\ 1 & 1 & 0 & -2 & & & \\ \hline 1 & 2 & 1 & -4 & -4 & 0 & 4\\ \hline \end{array}$$

EXERCISE 8C.2

1 **a**
$$\begin{array}{r|l} & \qquad x\\ \cline{2-2} x+2 & x^2+2x-3\\ & \underline{x^2+2x}\\ & \qquad\quad -3 \end{array}$$
$\therefore\quad Q(x) = x$
$\qquad\quad R = -3$

b
$$\begin{array}{r|l} & \qquad x-4\\ \cline{2-2} x-1 & x^2-5x+1\\ & \underline{x^2-x}\\ & \quad -4x+1\\ & \quad \underline{-4x+4}\\ & \qquad\ -3 \end{array}$$
$\therefore\quad Q(x) = x-4$
$\qquad\quad R = -3$

c
$$\begin{array}{r|l} & \qquad 2x^2+10x+16\\ \cline{2-2} x-2 & 2x^3+6x^2-4x+3\\ & \underline{2x^3-4x^2}\\ & \qquad 10x^2-4x\\ & \qquad \underline{10x^2-20x}\\ & \qquad\qquad 16x+3\\ & \qquad\qquad \underline{16x-32}\\ & \qquad\qquad\qquad 35 \end{array}$$
$\therefore\quad Q(x) = 2x^2+10x+16$
$\qquad\quad R = 35$

2 **a**
$$\begin{array}{r|l} & \qquad x+1\\ \cline{2-2} x-4 & x^2-3x+6\\ & \underline{x^2-4x}\\ & \qquad x+6\\ & \qquad \underline{x-4}\\ & \qquad\quad 10 \end{array}$$
$$\therefore\quad D(x) = x+1+\frac{10}{x-4}$$

b
$$\begin{array}{r|l} & \qquad x+1\\ \cline{2-2} x+3 & x^2+4x-11\\ & \underline{x^2+3x}\\ & \qquad x-11\\ & \qquad \underline{x+\ 3}\\ & \qquad\ -14 \end{array}$$
$$\therefore\quad D(x) = x+1-\frac{14}{x+3}$$

c
$$\begin{array}{r|l} & \qquad 2x-3\\ \cline{2-2} x-2 & 2x^2-7x+2\\ & \underline{2x^2-4x}\\ & \quad -3x+2\\ & \quad \underline{-3x+6}\\ & \qquad\quad -4 \end{array}$$
$$\therefore\quad D(x) = 2x-3-\frac{4}{x-2}$$

d
$$\begin{array}{r|l} & \qquad x^2+\ x-2\\ \cline{2-2} 2x+1 & 2x^3+3x^2-3x-2\\ & \underline{2x^3+x^2}\\ & \qquad 2x^2-3x\\ & \qquad \underline{2x^2+x}\\ & \qquad\quad -4x-2\\ & \qquad\quad \underline{-4x-2}\\ & \qquad\qquad\quad 0 \end{array}$$
$$\therefore\quad D(x) = x^2+x-2$$

e

$$\begin{array}{r|l} & x^2+4x+4 \\ \hline 3x-1 & 3x^3+11x^2+8x+7 \\ & \underline{3x^3-x^2} \\ & \quad 12x^2+8x \\ & \quad \underline{12x^2-4x} \\ & \qquad 12x+7 \\ & \qquad \underline{12x-4} \\ & \qquad\quad 11 \end{array}$$

$$\therefore \quad D(x)=x^2+4x+4+\frac{11}{3x-1}$$

f

$$\begin{array}{r|l} & x^3-2x^2+\tfrac{5}{2}x-\tfrac{1}{4} \\ \hline 2x+3 & 2x^4-x^3-x^2+7x+4 \\ & \underline{2x^4+3x^3} \\ & \quad -4x^3-x^2 \\ & \quad \underline{-4x^3-6x^2} \\ & \qquad 5x^2+7x \\ & \qquad \underline{5x^2+\tfrac{15}{2}x} \\ & \qquad\quad -\tfrac{1}{2}x+4 \\ & \qquad\quad \underline{-\tfrac{1}{2}x-\tfrac{3}{4}} \\ & \qquad\qquad \tfrac{19}{4} \end{array}$$

$$\therefore \quad D(x)=x^3-2x^2+\tfrac{5}{2}x-\tfrac{1}{4}+\frac{\frac{19}{4}}{2x+3}$$

3 **a**

$$\begin{array}{r|l} & x+2 \\ \hline x-2 & x^2+0x+5 \\ & \underline{x^2-2x} \\ & \quad 2x+5 \\ & \quad \underline{2x-4} \\ & \qquad 9 \end{array}$$

$$\therefore \quad D(x)=x+2+\frac{9}{x-2}$$

b

$$\begin{array}{r|l} & 2x+1 \\ \hline x+1 & 2x^2+3x+0 \\ & \underline{2x^2+2x} \\ & \quad x+0 \\ & \quad \underline{x+1} \\ & \qquad -1 \end{array}$$

$$\therefore \quad D(x)=2x+1-\frac{1}{x+1}$$

c

$$\begin{array}{r|l} & 3x-4 \\ \hline x+2 & 3x^2+2x-5 \\ & \underline{3x^2+6x} \\ & \quad -4x-5 \\ & \quad \underline{-4x-8} \\ & \qquad +3 \end{array}$$

$$\therefore \quad D(x)=3x-4+\frac{3}{x+2}$$

d

$$\begin{array}{r|l} & x^2+3x-2 \\ \hline x-1 & x^3+2x^2-5x+2 \\ & \underline{x^3-x^2} \\ & \quad 3x^2-5x \\ & \quad \underline{3x^2-3x} \\ & \qquad -2x+2 \\ & \qquad \underline{-2x+2} \end{array}$$

$$\therefore \quad D(x)=x^2+3x-2$$

e

$$\begin{array}{r|l} & 2x^2-8x+31 \\ \hline x+4 & 2x^3+0x^2-x+0 \\ & \underline{2x^3+8x^2} \\ & \quad -8x^2-x \\ & \quad \underline{-8x^2-32x} \\ & \qquad 31x+0 \\ & \qquad \underline{31x+124} \\ & \qquad\quad -124 \end{array}$$

$$\therefore \quad D(x)=2x^2-8x+31-\frac{124}{x+4}$$

f

$$\begin{array}{r|l} & x^2+3x+6 \\ \hline x-2 & x^3+x^2+0x-5 \\ & \underline{x^3-2x^2} \\ & \quad 3x^2+0x \\ & \quad \underline{3x^2-6x} \\ & \qquad 6x-5 \\ & \qquad \underline{6x-12} \\ & \qquad\quad 7 \end{array}$$

$$\therefore \quad D(x)=x^2+3x+6+\frac{7}{x-2}$$

EXERCISE 8C.3

1 **a**

$$\begin{array}{r|l} & x+1 \\ \hline x^2+x+1 & x^3+2x^2+x-3 \\ & \underline{x^3+x^2+x} \\ & \quad x^2+0x-3 \\ & \quad \underline{x^2+x+1} \\ & \qquad -x-4 \end{array}$$

$$\therefore \quad \begin{aligned} Q(x) &= x+1 \\ R(x) &= -x-4 \end{aligned}$$

b

$$\begin{array}{r|l} & 3 \\ \hline x^2-1 & 3x^2-x+0 \\ & \underline{3x^2 \qquad -3} \\ & \quad -x+3 \end{array}$$

$$\therefore \quad \begin{aligned} Q(x) &= 3 \\ R(x) &= -x+3 \end{aligned}$$

c

$$\begin{array}{r|l}
\multicolumn{2}{r}{3x} \\ \cline{2-2}
x^2+1 & 3x^3+0x^2+x-1 \\
& \underline{3x^3 \qquad\quad +3x} \\
& \qquad\qquad -2x-1
\end{array}$$

$$\therefore \quad \begin{aligned} Q(x) &= 3x \\ R(x) &= -2x-1 \end{aligned}$$

d

$$\begin{aligned} Q(x) &= 0 \\ R(x) &= x-4 \end{aligned}$$

2 **a**

$$\begin{array}{r|l}
\multicolumn{2}{r}{1} \\ \cline{2-2}
x^2+x+1 & x^2-x+1 \\
& \underline{x^2+x+1} \\
& \quad -2x
\end{array}$$

$$\therefore \quad D(x) = 1 - \frac{2x}{x^2+x+1}$$

b

$$\begin{array}{r|l}
\multicolumn{2}{r}{x} \\ \cline{2-2}
x^2+2 & x^3+0x^2+0x+0 \\
& \underline{x^3 \qquad\quad +2x} \\
& \qquad\qquad -2x+0
\end{array}$$

$$\therefore \quad D(x) = x - \frac{2x}{x^2+2}$$

c

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^2+x+3} \\ \cline{2-2}
x^2-x+1 & x^4+0x^3+3x^2+x-1 \\
& \underline{x^4-x^3+x^2} \\
& \qquad x^3+2x^2+x \\
& \qquad \underline{x^3-x^2+x} \\
& \qquad\qquad 3x^2+0x-1 \\
& \qquad\qquad \underline{3x^2-3x+3} \\
& \qquad\qquad\qquad 3x-4
\end{array}$$

$$\therefore \quad D(x) = x^2+x+3 + \frac{3x-4}{x^2-x+1}$$

d

$$\frac{2x^3-x+6}{(x-1)^2} = \frac{2x^3-x+6}{x^2-2x+1}$$

$$\begin{array}{r|l}
\multicolumn{2}{r}{2x+4} \\ \cline{2-2}
x^2-2x+1 & 2x^3+0x^2-x+6 \\
& \underline{2x^3-4x^2+2x} \\
& \qquad 4x^2-3x+6 \\
& \qquad \underline{4x^2-8x+4} \\
& \qquad\qquad 5x+2
\end{array}$$

$$\therefore \quad D(x) = 2x+4 + \frac{5x+2}{(x-1)^2}$$

e

$$\frac{x^4}{(x+1)^2} = \frac{x^4}{x^2+2x+1}$$

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^2-2x+3} \\ \cline{2-2}
x^2+2x+1 & x^4+0x^3+0x^2+0x+0 \\
& \underline{x^4+2x^3+x^2} \\
& \quad -2x^3-x^2+0x \\
& \quad \underline{-2x^3-4x^2-2x} \\
& \qquad\qquad 3x^2+2x+0 \\
& \qquad\qquad \underline{3x^2+6x+3} \\
& \qquad\qquad\qquad -4x-3
\end{array}$$

$$\therefore \quad D(x) = x^2-2x+3 - \frac{4x+3}{(x+1)^2}$$

f

$$\frac{x^4-2x^3+x+5}{(x-1)(x+2)} = \frac{x^4-2x^3+x+5}{x^2+x-2}$$

$$\begin{array}{r|l}
\multicolumn{2}{r}{x^2-3x+5} \\ \cline{2-2}
x^2+x-2 & x^4-2x^3+0x^2+x+5 \\
& \underline{x^4+x^3-2x^2} \\
& \quad -3x^3+2x^2+x \\
& \quad \underline{-3x^3-3x^2+6x} \\
& \qquad\qquad 5x^2-5x+5 \\
& \qquad\qquad \underline{5x^2+5x-10} \\
& \qquad\qquad\quad -10x+15
\end{array}$$

$$\therefore \quad D(x) = x^2-3x+5 + \frac{15-10x}{(x-1)(x+2)}$$

3

$$\begin{aligned} \frac{P(x)}{x-2} &= \frac{(x-2)(x^2+2x+3)+7}{x-2} \\ &= x^2+2x+3+\frac{7}{x-2}, \quad \text{etc} \end{aligned}$$

$\therefore$ quotient is x^2+2x+3
remainder is 7

EXERCISE 8C.4

1 **a**

$$\begin{array}{r|rrr|}
+1 & 3 & -2 & -3 \\
& 0 & 3 & 1 \\ \hline
& 3 & 1 & \multicolumn{1}{|r}{-2}
\end{array}$$

$$\therefore \quad D(x) = 3x+1-\frac{2}{x-1}$$

b

$$\begin{array}{r|rrrr|}
-3 & 1 & 5 & 6 & 5 \\
& 0 & -3 & -6 & 0 \\ \hline
& 1 & 2 & 0 & \multicolumn{1}{|r}{5}
\end{array}$$

$$\therefore \quad D(x) = x^2+2x+\frac{5}{x+3}$$

c
$$\begin{array}{r|rr|r} -1 & 3 & -1 & 2 \\ & 0 & -3 & 4 \\ \hline & 3 & -4 & 6 \end{array}$$

$$\therefore \quad D(z) = 3z - 4 + \frac{6}{z+1}$$

d
$$\begin{array}{r|rrr|r} -3 & 1 & 0 & 0 & 27 \\ & 0 & -3 & 9 & -27 \\ \hline & 1 & -3 & 9 & 0 \end{array}$$

$$\therefore \quad D(x) = x^2 - 3x + 9$$

e
$$\begin{array}{r|rrrr|r} +3 & 1 & -2 & 1 & 0 & -4 \\ & 0 & 3 & 3 & 12 & 36 \\ \hline & 1 & 1 & 4 & 12 & 32 \end{array}$$

$$\therefore \quad D(z) = z^3 + z^2 + 4z + 12 + \frac{32}{z-3}$$

f
$$\begin{array}{r|rrrr|r} -1 & 1 & 0 & 1 & -1 & 0 \\ & 0 & -1 & 1 & -2 & 3 \\ \hline & 1 & -1 & 2 & -3 & 3 \end{array}$$

$$\therefore \quad D(z) = z^3 - z^2 + 2z - 3 + \frac{3}{z+1}$$

2 **a**
$$\begin{aligned} & P(x) \\ &= (x-2)(3x^2-2x+1)+4 \\ &= 3x^3 - 2x^2 + x - 6x^2 + 4x - 2 + 4 \\ &= 3x^3 - 8x^2 + 5x + 2 \end{aligned}$$

b
$$\begin{aligned} & P(x) \\ &= (x+3)(x^2-7x+8)-5 \\ &= x^3 - 7x^2 + 8x + 3x^2 - 21x + 24 - 5 \\ &= x^3 - 4x^2 - 13x + 19 \end{aligned}$$

EXERCISE 8D.1

1 **a** $2x^2 - 5x - 12$ has zeros

$$\begin{aligned} x &= \frac{5 \pm \sqrt{25 - 4(2)(-12)}}{4} \\ &= \frac{5 \pm \sqrt{121}}{4} \\ &= \frac{5 \pm 11}{4} \\ &= 4,\ -\tfrac{6}{4} \end{aligned}$$

$\therefore$ zeros are $4, -\frac{3}{2}$

b $x^2 + 6x + 10$ has zeros

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{36 - 4(1)(10)}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= -3 \pm i \end{aligned}$$

$\therefore$ zeros are $-3 \pm i$

c $z^2 - 6z + 6$ has zeros

$$\begin{aligned} z &= \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2} \\ &= \frac{6 \pm \sqrt{12}}{2} \\ &= 3 \pm \sqrt{3} \end{aligned}$$

$\therefore$ zeros are $3 \pm \sqrt{3}$

d
$$\begin{aligned} & x^3 - 4x \\ &= x(x^2 - 4) \\ &= x(x+2)(x-2) \end{aligned}$$

$\therefore$ zeros are $0, \pm 2$

e
$$\begin{aligned} & z^3 + 2z \\ &= z(z^2 + 2) \\ &= z(z^2 - 2i^2) \\ &= z(z + i\sqrt{2})(z - i\sqrt{2}) \end{aligned}$$

$\therefore$ zeros are $0, \pm i\sqrt{2}$

f
$$\begin{aligned} & z^4 + 4z^2 - 5 \\ &= (z^2+5)(z^2-1) \\ &= (z^2 - 5i^2)(z^2 - 1) \\ &= (z + i\sqrt{5})(z - i\sqrt{5})(z+1)(z-1) \end{aligned}$$

$\therefore$ zeros are $\pm i\sqrt{5}, \pm 1$

2 **a**
$$\begin{aligned} 5x^2 &= 3x + 2 \\ \therefore \quad 5x^2 - 3x - 2 &= 0 \\ \therefore \quad (5x+2)(x-1) &= 0 \end{aligned}$$

$\therefore$ roots are $1, -\frac{2}{5}$

b
$$\begin{aligned} (2x+1)(x^2+3) &= 0 \\ \therefore \quad (2x+1)(x^2 - 3i^2) &= 0 \\ \therefore \quad (2x+1)(x + i\sqrt{3})(x - i\sqrt{3}) &= 0 \end{aligned}$$

$\therefore$ roots are $-\frac{1}{2}, \pm i\sqrt{3}$

c
$$\begin{aligned} -2z(z^2-2z+2) &= 0 \\ z = 0 \text{ or } &\frac{2\pm\sqrt{4-4(1)(2)}}{2} \\ = 0 \text{ or } &\frac{2\pm\sqrt{-4}}{2} \\ = 0 \text{ or } &1\pm i \end{aligned}$$
$\therefore$ roots are $0,\ 1\pm i$

e
$$\begin{aligned} z^3+5z &= 0 \\ z(z^2+5) &= 0 \\ z(z^2-5i^2) &= 0 \\ z(z+i\sqrt{5})(z-i\sqrt{5}) &= 0 \end{aligned}$$
$\therefore$ roots are $0,\ \pm i\sqrt{5}$

d
$$\begin{aligned} x^3 &= 5x \\ \therefore\quad x^3-5x &= 0 \\ x(x^2-5) &= 0 \\ x(x+\sqrt{5})(x-\sqrt{5}) &= 0 \end{aligned}$$
$\therefore$ roots are $0,\ \pm\sqrt{5}$

f
$$\begin{aligned} z^4 &= 3z^2+10 \\ \therefore\quad z^4-3z^2-10 &= 0 \\ (z^2-5)(z^2+2) &= 0 \\ (z^2-5)(z^2-2i^2) &= 0 \\ (z+\sqrt{5})(z-\sqrt{5})(z+i\sqrt{2})(z-i\sqrt{2}) &= 0 \end{aligned}$$
$\therefore$ roots are $\pm\sqrt{5},\ \pm i\sqrt{2}$

3 **a**
$$\begin{aligned} &2x^2-7x-15 \\ &= (2x+3)(x-5) \end{aligned}$$

c
$$\begin{aligned} &x^3+2x^2-4x \\ &= x(x^2+2x-4) \end{aligned}$$
x^2+2x-4 is zero when
$$\begin{aligned} x &= \frac{-2\pm\sqrt{4+16}}{2} \\ &= -1\pm\sqrt{5} \end{aligned}$$
$$\begin{aligned} \therefore\quad &x^3+2x^2-4x \\ &= x(x+1+\sqrt{5})(x+1-\sqrt{5}) \end{aligned}$$

b $z^2-6z+16$ is zero when
$$\begin{aligned} z &= \frac{6\pm\sqrt{36-4(1)(16)}}{2} \\ &= 3\pm i\sqrt{7} \end{aligned}$$
$$\begin{aligned} \therefore\quad &z^2-6z+16 \\ &= (z-3+i\sqrt{7})(z-3-i\sqrt{7}) \end{aligned}$$

d
$$\begin{aligned} &6z^3-z^2-2z \\ &= z(6z^2-z-2) \\ &= z(2z+1)(3z-2) \end{aligned}$$

e
$$\begin{aligned} &z^4-6z^2+5 \\ &= (z^2-1)(z^2-5) \\ &= (z+1)(z-1)(z+\sqrt{5})(z-\sqrt{5}) \end{aligned}$$

f
$$\begin{aligned} &z^4-2z^2-2 \\ &= (z^2-2)(z^2+1) \\ &= (z+\sqrt{2})(z-\sqrt{2})(z+i)(z-i) \end{aligned}$$

4
$$\begin{aligned} P(x) &= a(x-\alpha)(x-\beta)(x-\gamma) \\ \therefore\quad P(\alpha) &= a\times 0\times(\alpha-\beta)(\alpha-\gamma)=0 \\ \text{and}\quad P(\beta) &= a(\beta-\alpha)\times 0\times(\beta-\gamma)=0 \\ \text{and}\quad P(\gamma) &= a(\gamma-\alpha)(\gamma-\beta)\times 0=0 \end{aligned}$$

$\therefore$ α, β and γ all satisfy $P(x)=0$

$\therefore$ α, β and γ are zeros of $P(x)$

5 **a** The zeros ± 2
have sum $= 0$ and product $= -4$
$\therefore$ come from quadratic factor z^2-4
and zero 3 comes from $(z-3)$
$\therefore$ $P(z)=a(z^2-4)(z-3),\ a\neq 0$

b The zeros $\pm i$
have sum $= 0$ and product $= 1$
$\therefore$ come from quadratic factor z^2+1
and zero -2 comes from $(z+2)$
$\therefore$ $P(z)=a(z^2+1)(z+2),\ a\neq 0$

c The zeros $-1\pm i$
have sum $= -2$ and product $= 2$
$\therefore$ come from quadratic factor z^2+2z+2
and zero 3 comes from $(z-3)$
$\therefore$ $P(z)=a(z-3)(z^2+2z+2),\ a\neq 0$

d The zeros $-2\pm\sqrt{2}$
have sum $= -4$ and product $= 2$
$\therefore$ come from quadratic factor z^2+4z+2
and zero -1 comes from $(z+1)$
$\therefore$ $P(z)=a(z+1)(z^2+4z+2),\ a\neq 0$

6 **a** For zeros of ± 1, sum $= 0$ and product $= -1$ $\therefore$ come from $z^2 - 1$

For zeros of $\pm\sqrt{2}$, sum $= 0$ and product $= -2$ $\therefore$ come from $z^2 - 2$

$\therefore \quad P(z) = a(z^2 - 1)(z^2 - 2), \quad a \neq 0$

b For zeros of $\pm i\sqrt{3}$, sum $= 0$ and product $= 3$ $\therefore$ come from $z^2 + 3$

zeros of $2, -1$ come from $(z - 2)(z + 1)$

$\therefore \quad P(z) = a(z - 2)(z + 1)(z^2 + 3), \quad a \neq 0$

c For zeros of $\pm\sqrt{3}$, sum $= 0$ and product $= -3$ $\therefore$ come from $z^2 - 3$

For zeros of $1 \pm i$, sum $= 2$ and product $= 2$ $\therefore$ come from $z^2 - 2z + 2$

$\therefore \quad P(z) = a(z^2 - 3)(z^2 - 2z + 2), \quad a \neq 0$

d For zeros of $2 \pm \sqrt{5}$, sum $= 4$ and product $= -1$ $\therefore$ come from $z^2 - 4z - 1$

For zeros of $-2 \pm 3i$, sum $= -4$ and product $= 13$ $\therefore$ come from $z^2 + 4z + 13$

$\therefore \quad P(z) = a(z^2 - 4z - 1)(z^2 + 4z + 13), \quad a \neq 0$

EXERCISE 8D.2

1 **a** $2x^2 + 4x + 5 = ax^2 + [2b - 6]x + c$

Equating coefficients gives

$a = 2, \quad 2b - 6 = 4,$ and $c = 5$

$\therefore \quad 2b = 10$

i.e., $b = 5$

$\therefore \quad a = 2 \quad b = 5 \quad c = 5$

b
$$\begin{aligned}
&2x^3 - x^2 + 6\\
&= (x - 1)^2(2x + a) + bx + c\\
&= (x^2 - 2x + 1)(2x + a) + bx + c\\
&= 2x^3 + [a - 4]x^2 + [2 - 2a]x + a + bx + c\\
&= 2x^3 + [a - 4]x^2 + [2 - 2a + b]x + [a + c]
\end{aligned}$$

Equating coefficients gives $\therefore \quad a - 4 = -1 \qquad 2 - 2a + b = 0 \qquad a + c = 6$

$\therefore \quad a = 3 \qquad b = 2a - 2 \qquad c = 6 - a$

$b = 4 \qquad c = 3$

2 **a**
$$\begin{aligned}
z^4 + 4 &= (z^2 + az + 2)(z^2 + bz + 2)\\
&= z^4 + [a + b]z^3 + [4 + ab]z^2 + [2a + 2b]z + 4
\end{aligned}$$

Equating coefficients gives:

$a + b = 0 \qquad 4 + ab = 0$

$\therefore \quad a = -b \qquad \therefore \quad ab = -4$

By inspection $a = 2$ and $b = -2$

or $a = -2$ and $b = 2$

$$\begin{array}{ccccc}
 & & 1 & a & 2\\
 & \times & 1 & b & 2\\
\hline
 & & 2 & 2a & 4\\
 & b & ab & 2b & \\
1 & a & 2 & & \\
\hline
1 & a + b & 4 + ab & 2a + 2b & 4\\
\hline
\end{array}$$

b
$$\begin{aligned}
&2z^4 + 5z^3 + 4z^2 + 7z + 6\\
&= (z^2 + az + 2)(2z^2 + bz + 3)\\
&= 2z^4 + [2a + b]z^3 + [ab + 7]z^2 + [3a + 2b]z + 6
\end{aligned}$$

Equating coefficients gives:

$$\begin{aligned}
2a + b &= 5 \text{ .... (1)}\\
3a + 2b &= 7 \text{ .... (2)}\\
ab + 7 &= 4 \text{ .... (3)}
\end{aligned}$$

$$\begin{array}{ccccc}
 & & 1 & a & 2\\
 & \times & 2 & b & 3\\
\hline
 & & 3 & 3a & 6\\
 & b & ab & 2b & \\
2 & 2a & 4 & & \\
\hline
2 & 2a + b & ab + 7 & 3a + 2b & 6\\
\hline
\end{array}$$

$\therefore \quad 4a + 2b = 10 \qquad \{(1) \times 2\}$

and $3a + 2b = 7$

and solving these two equations gives

$a = 3, \quad b = -1$

which checks with (3) as $ab + 7 = -3 + 7 = 4$ ✓

3 Consider

$$\begin{aligned} z^4+64 &= (z^2+az+8)(z^2+bz+8) \\ &= z^4+[a+b]z^3+[ab+16]z^2+[8a+8b]z+64 \end{aligned}$$

$$\begin{array}{rrrrr} & & 1 & a & 8 \\ & \times & 1 & b & 8 \\ \hline & & 8 & 8a & 64 \\ & b & ab & 8b & \\ 1 & a & 8 & & \\ \hline 1 & a+b & ab+16 & 8a+8b & 64 \\ \hline \end{array}$$

Equating coefficients gives:

$a+b=0$ and $ab+16=0$

$\therefore\ a=-b$ i.e., $ab=-16$

$\therefore$ by inspection $a=4$ and $b=-4$

or $a=-4$ and $b=4$

$\therefore\ z^4+64$ can be factorised into $(z^2+4z+8)(z^2-4z+8)$

Now consider

$$\begin{aligned} z^4+64 &= (z^2+az+16)(z^2+bz+4) \\ &= z^4+[a+b]z^3+[ab+20]z^2+[4a+16b]z+64 \end{aligned}$$

$$\begin{array}{rrrrr} & & 1 & a & 16 \\ & \times & 1 & b & 4 \\ \hline & & 4 & 4a & 64 \\ & b & ab & 16b & \\ 1 & a & 16 & & \\ \hline 1 & a+b & ab+20 & 4a+16b & 64 \\ \hline \end{array}$$

Equating coefficients gives:

$a+b=0$ .... (1) and $ab+20=0$

$4a+16b=0$ .... (2) i.e., $ab=-20$ .... (3)

Solution to (1), (2) is $a=b=0$

But this does not satisfy (3)

$\therefore$ no values of a and b exist which obey the original assumption

$\therefore$ cannot be factorised in this way.

4 Consider

$$\begin{aligned} &x^4-4x^2+8x-4 \\ &= (x^2+ax+2)(x^2+bx-2) \\ &= x^4+[a+b]x^3+[ab]x^2+[2b-2a]x-4 \end{aligned}$$

$$\begin{array}{rrrrr} & & 1 & a & 2 \\ & \times & 1 & b & -2 \\ \hline & & -2 & -2a & -4 \\ & b & ab & 2b & \\ 1 & a & 2 & & \\ \hline 1 & a+b & ab & 2b-2a & -4 \\ \hline \end{array}$$

Equating coefficients gives:

$a+b=0$ and $ab=-4$ and $-2a+2b=8$

$\therefore\ 2a+2b=0$ .... (1)

$-2a+2b=8$ .... (2)

Adding (1) and (2) gives $4b=8$ $\therefore\ b=2$ and hence $a=-2$

$$\therefore\ P(x)=(x^2-2x+2)(x^2+2x-2)$$

Now if $x^4+8x=4x^2+4$

then $x^4-4x^2+8x-4=0$

$\therefore\ (x^2-2x+2)(x^2+2x-2)=0$

$\therefore\ x^2-2x+2=0$ or $x^2+2x-2=0$

$$\therefore\ x=\frac{2\pm\sqrt{4-8}}{2}=1\pm i \quad \text{or} \quad x=-\frac{2\pm\sqrt{4+8}}{2}=-1\pm\sqrt{3}$$

$$\therefore\ x=1\pm i,\ -1\pm\sqrt{3}$$

5 **a**

$$\begin{aligned} P(z) &= 2z^3-z^2+az-3 \\ &= (2z-3)(z^2+bz+1) \quad \text{for some value } b \\ &= 2z^3+[2b-3]z^2+[2-3b]z-3 \end{aligned}$$

$$\begin{array}{rrrr} & 1 & b & 1 \\ \times & & 2 & -3 \\ \hline & -3 & -3b & -3 \\ 2 & 2b & 2 & \\ \hline 2 & 2b-3 & 2-3b & -3 \\ \hline \end{array}$$

Equating coefficients gives:

$\therefore\ 2b-3=-1$ and $2-3b=a$

$2b=2$ $\therefore\ a=2-3$

$b=1$ $a=-1$

$\therefore \quad P(z) = (2z-3)(z^2+z+1)$

$\downarrow$

this quadratic has zeros of $z = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore \quad a = -1$ and zeros are $\frac{3}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

b
$$\begin{aligned} P(z) &= 3z^3 - z^2 + [a+1]z + a \\ &= (3z+2)(z^2+bz+c) \\ &= 3z^3 + [2+3b]z^2 + [2b+3c]z + 2c \end{aligned}$$

	1	b	c
$\times$		3	2
	2	$2b$	$2c$
3	$3b$	$3c$	
3	$2+3b$	$2b+3c$	$2c$

Equating coefficients gives:

$\therefore \quad 2+3b = -1, \quad 2b+3c = a+1 \quad$ and $\quad 2c = a$

Now as
$$\begin{aligned} 2+3b &= -1 \\ \therefore \quad 3b &= -3 \\ \therefore \quad b &= -1 \end{aligned}$$

Substituting $b=-1$ and $a=2c$ into $2b+3c = a+1$ gives
$$\begin{aligned} 2(-1)+3c &= 2c+1 \\ \therefore \quad -2+3c &= 2c+1 \\ c &= 3 \\ \text{and so} \quad a &= 6 \end{aligned}$$

$\therefore \quad P(z) = (3z+2)\underbrace{(z^2-z+3)}$

$\downarrow$

with zeros $\dfrac{1 \pm \sqrt{1-4(3)(1)}}{2} = \dfrac{1 \pm i\sqrt{11}}{2}$

So, the zeros are $-\frac{2}{3}, \frac{1}{2} \pm i\frac{\sqrt{11}}{2}$ and $a = 6$

6 **a**
$$\begin{aligned} P(x) &= 2x^4 + ax^3 + bx^2 - 12x - 8 \\ &= (2x+1)(x-2)(x^2+cx+4) \\ &= (2x^2-3x-2)(x^2+cx+4) \end{aligned}$$

		2	-3	-2
	$\times$	1	c	4
		8	-12	-8
	$2c$	$-3c$	$-2c$	
2	-3	-2		
2	$2c-3$	$6-3c$	$-2c-12$	-8

Equating coefficients: $\quad 2c-3 = a,$

$6-3c = b \quad$ and $\quad -2c-12 = -12$

The last equation has solution $c = 0$, and consequently, $a = -3$ and $b = 6$

$\therefore \quad P(x) = (2x+1)(x-2)(x^2+4) = (2x+1)(x-2)(x+2i)(x-2i)$

$\therefore \quad$ zeros are $-\frac{1}{2}$, 2 and $\pm 2i$ and $a = -3, \ b = 6.$

b
$$\begin{aligned} P(x) &= 2x^4 + ax^3 + bx^2 + ax + 3 \\ &= (x+3)(2x-1)(x^2+cx-1) \\ &= (2x^2+5x-3)(x^2+cx-1) \end{aligned}$$

		2	5	-3
	$\times$	1	c	-1
		-2	-5	3
	$2c$	$5c$	$-3c$	
2	5	-3		
2	$2c+5$	$5c-5$	$-5-3c$	3

Equating coefficients: $\quad a = 2c+5,$

$b = 5c-5, \quad a = -5-3c$

$$\begin{aligned} \therefore \quad 2c+5 &= -5-3c \qquad \{\text{equating } a\text{'s}\} \\ \therefore \quad 5c &= -10 \\ c &= -2 \quad \text{and so,} \quad a = 1 \quad b = -15 \end{aligned}$$

$\therefore \quad P(x) = (x+3)(2x-1)(x^2-2x-1)$

$\uparrow$

this quadratic has zeros $\dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$\therefore \quad$ zeros are $-3, \frac{1}{2}, 1 \pm \sqrt{2}$ and $a = 1, \ b = -15$

7 **a**

$$\begin{aligned}&x^3+3x^2-9x+c\\&=(x+a)^2(x+b)\\&=(x^2+2ax+a^2)(x+b)\\&=x^3+[b+2a]x^2+[a^2+2ab]x+a^2b\end{aligned}$$

	1	$2a$	a^2
×		1	b
	b	$2ab$	a^2b
1	$2a$	a^2	
1	$b+2a$	a^2+2ab	a^2b

Equating coefficients gives

$2a+b=3$, $a^2+2ab=-9$ and $c=a^2b$

Substituting $b=3-2a$ into the second equation gives:

$$\begin{aligned}a^2+2a(3-2a)&=-9\\\therefore\quad a^2+6a-4a^2&=-9\\\therefore\quad -3a^2+6a&=-9\\\therefore\quad 3a^2-6a-9&=0\\\therefore\quad a^2-2a-3&=0\\\therefore\quad (a-3)(a+1)&=0\\\therefore\quad a=3\quad\text{or}\quad a&=-1\\\therefore\quad b=-3\quad\text{or}\quad b&=5\\\therefore\quad c=-27\quad\text{or}\quad c&=5\end{aligned}$$

If $c=-27$, $P(x)=(x+3)^2(x-3)$

If $c=5$, $P(x)=(x-1)^2(x+5)$

b

$$\begin{aligned}&3x^3+4x^2-x+m\\&=(x+a)^2(3x+b)\\&=(x^2+2ax+a^2)(3x+b)\\&=x^3+[6a+b]x^2+[3a^2+2ab]x+a^2b\end{aligned}$$

	1	$2a$	a^2
×		3	b
	b	$2ab$	a^2b
3	$6a$	$3a^2$	
3	$6a+b$	$3a^2+2ab$	a^2b

Equating coefficients gives

$6a+b=4$, $3a^2+2ab=-1$ and $a^2b=m$

Substituting $b=4-6a$ into the second equation gives:

$$\begin{aligned}3a^2+2a(4-6a)&=-1\\3a^2+8a-12a^2&=-1\\9a^2-8a-1&=0\\(9a+1)(a-1)&=0\\\therefore\quad a=-\tfrac{1}{9}\quad&\text{or}\quad a=1\end{aligned}$$

When $a=1$, $b=-2$ and $m=-2$. So, $P(x)=(x+1)^2(3x-2)$ $\therefore$ zeros are -1, $\frac{2}{3}$

When $a=-\frac{1}{9}$, $b=\frac{14}{3}$ and $m=\frac{14}{243}$. So, $P(x)=(x-\frac{1}{9})^2(3x+\frac{14}{3})$

$\therefore$ zeros are $\frac{1}{9}$, $-\frac{14}{9}$

EXERCISE 8D.3

1 **a** If $P(2)=7$, then $P(x)=(x-2)Q(x)+7$ and $P(x)$ divided by $(x-2)$ leaves a remainder of 7.

b If $P(x)=(x+3)Q(x)-8$, then $P(-3)=-8$ and $P(x)$ divided by $(x+3)$ leaves a remainder of -8.

c If $P(x)$ when divided by $(x-5)$ has a remainder of 11, then $P(5)=11$ and $P(x)=(x-5)Q(x)+11$.

2 **a**

$$\begin{aligned}P(x)&=x^3+2x^2-7x+5\\\therefore\quad R&=P(1)\quad\{\text{Remainder thm.}\}\\&=1^3+2(1)^2-7+5\\&=1\end{aligned}$$

b

$$\begin{aligned}P(x)&=x^4-2x^2+3x-1\\\therefore\quad R&=P(-2)\quad\{\text{Remainder thm.}\}\\&=(-2)^4-2(-2)^2+3(-2)-1\\&=16-8-6-1\\&=1\end{aligned}$$

3 **a**
$$\begin{aligned} P(x) &= x^3 - 2x + a \\ \text{Now } P(2) &= 7 \quad \{\text{Remainder thm.}\} \\ \therefore \quad 2^3 - 2(2) + a &= 7 \\ 4 + a &= 7 \\ \therefore \quad a &= 3 \end{aligned}$$

b
$$\begin{aligned} P(x) &= 2x^3 + x^2 + ax - 5 \\ \text{Now } P(-1) &= -8 \\ \therefore \quad 2(-1)^3 + (-1)^2 + a(-1) - 5 &= -8 \\ -2 + 1 - a - 5 &= -8 \\ \therefore \quad -a - 6 &= -8 \\ -a &= -2 \\ \therefore \quad a &= 2 \end{aligned}$$

4 $P(x) = x^3 + 2x^2 + ax + b$

Now $P(1) = 4$ and $P(-2) = 16$ {Remainder theorem}

If $P(1) = 4$ then $1 + 2 + a + b = 4$ and so $a + b = 1$ ...... (1)

If $P(-2) = 16$ then
$$\begin{aligned} (-2)^3 + 2(-2)^2 + a(-2) + b &= 16 \\ \therefore \quad -8 + 8 - 2a + b &= 16 \\ \therefore \quad -2a + b &= 16 \text{ ...... (2)} \end{aligned}$$

Solving (1) and (2)
$$\begin{aligned} -a - b &= -1 \\ -2a + b &= 16 \\ \therefore \quad -3a &= 15 \qquad \{\text{adding}\} \\ \therefore \quad a &= -5 \qquad \text{and so} \quad b = 6 \\ \text{i.e.,} \quad a &= -5 \quad \text{and} \quad b = 6 \end{aligned}$$

5 $P(x) = 2x^n + ax^2 - 6$

By the Remainder theorem, $P(1) = -7$
$$\begin{aligned} \therefore \quad 2(1)^n + a(1)^2 - 6 &= -7 \\ \therefore \quad 2 + a - 6 &= -7 \\ \therefore \quad a &= -3 \end{aligned}$$

So, $P(x) = 2x^n - 3x^2 - 6$

and since $P(-3) = 129$,
$$\begin{aligned} \therefore \quad 2(-3)^n - 3(-3)^2 - 6 &= 129 \\ 2(-3)^n - 27 - 6 &= 129 \\ 2(-3)^n &= 162 \\ (-3)^n &= 81 \\ \text{obviously} \quad n &= 4 \\ \therefore \quad a &= -3 \quad \text{and} \quad n = 4 \end{aligned}$$

6 $P(z) = Q(z)(z^2 - 3z + 2) + (4z - 7) = Q(z)(z - 2)(z - 1) + (4z - 7)$

a Remainder is $P(1)$ {Remainder thm.}
$$\begin{aligned} \therefore \quad R &= Q(1) \times 0 + (4 - 7) \\ &= -3 \end{aligned}$$

b Remainder is $P(2)$ {Remainder thm.}
$$\begin{aligned} \therefore \quad R &= Q(2) \times 0 + [4(2) - 7] \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

7 Suppose $P(z)$ is divided by $(z - 3)(z + 1)$

$$\therefore \quad P(z) = Q(z) \times (z - 3)(z + 1) + \underset{\substack{\uparrow \\ \text{the remainder must be of this form}}}{(Az + B)}$$

Now $P(-1) = -8$
$$\begin{aligned} \therefore \quad Q(-1) \times 0 + (-A + B) &= -8 \\ \therefore \quad -A + B &= -8 \quad \text{...... (1)} \end{aligned}$$

and $P(3) = 4$
$$\begin{aligned} \therefore \quad Q(3) \times 0 + (3A + B) &= 4 \\ \therefore \quad 3A + B &= 4 \quad \text{...... (2)} \end{aligned}$$

Solving (1) and (2)
$$\begin{aligned} -A + B &= -8 \\ -3A - B &= -4 \\ \hline -4A &= -12 \\ \therefore \quad A &= 3 \qquad \text{and so} \quad B = -5 \\ \therefore \quad R(z) &= 3z - 5 \end{aligned}$$

8 Suppose $P(x)$ is divided by $(x-a)(x-b)$ and has remainder $Ex+F$

hence $P(x) = Q(x)\times(x-a)(x-b)+Ex+F$

Now $P(a) = Ea+F$ ...... (1) and $P(b) = Eb+F$ ...... (2)

Subtracting (2) and (1), $P(b)-P(a) = Eb-Ea = E(b-a)$

$$\therefore \quad E = \frac{P(b)-P(a)}{b-a}$$

$$\therefore \text{ from (1)} \quad F = P(a)-Ea = P(a)-\left[\frac{P(b)-P(a)}{b-a}\right]a$$

$$\begin{aligned} \text{Now} \quad R(x) &= Ex+F \\ \therefore \quad R(x) &= \left[\frac{P(b)-P(a)}{b-a}\right]x+P(a)-\left[\frac{P(b)-P(a)}{b-a}\right]a \\ \therefore \quad R(x) &= \left[\frac{P(b)-P(a)}{b-a}\right](x-a)+P(a) \end{aligned}$$

9 Now suppose $P(x)$ is divided by $(x-a)^2$ and has remainder $Ex+F$

$$\begin{aligned} \therefore \quad P(x) &= Q(x)\times(x-a)^2+Ex+F \\ \text{Hence,} \quad P(a) &= Q(a)(0)^2+Ea+F \\ \therefore \quad P(a) &= Ea+F \quad \text{...... (1)} \\ \text{Now} \quad P'(x) &= [Q'(x)\times(x-a)^2+Q(x)\times 2(x-a)]+E \\ \therefore \quad P'(a) &= E \quad \text{...... (2)} \end{aligned}$$

$$\begin{aligned} \text{Now substituting (2) into (1)} \quad P(a) &= P'(a)\times a+F \\ \therefore \quad F &= P(a)-aP'(a) \\ \text{Now} \quad R(x) &= Ex+F \\ &= xP'(a)+(P(a)-aP'(a)) \\ &= xP'(a)-aP'(a)+P(a) \\ \therefore \quad R(x) &= (x-a)P'(a)+P(a) \end{aligned}$$

EXERCISE 8D.4

1 **a** $P(x) = 2x^3+x^2+kx-4$

$$\begin{array}{r|rrrr|} -2 & 2 & 1 & k & -4 \\ & 0 & -4 & 6 & -2k-12 \\ \hline & 2 & -3 & k+6 & -2k-16 \end{array}$$

if $x+2$ is a factor then $P(-2) = 0$

$$\begin{aligned} \therefore \quad -2k-16 &= 0 \\ \therefore \quad k &= -8 \\ \therefore \quad P(x) &= 2x^3+x^2-8x-4 \\ &= (x+2)(2x^2-3x-2) \quad \{\text{as when } k=-8,\ k+6=-2\} \\ \therefore \quad P(x) &= (x+2)(2x+1)(x-2) \quad \text{and} \quad k=-8 \end{aligned}$$

b $P(x) = x^4-3x^3-kx^2+6x$

$$\begin{array}{r|rrrrr|} 3 & 1 & -3 & -k & 6 & 0 \\ & 0 & 3 & 0 & -3k & 18-9k \\ \hline & 1 & 0 & -k & 6-3k & 18-9k \end{array}$$

if $x-3$ is a factor then $P(3) = 0$

$$\begin{aligned} \therefore \quad 18-9k &= 0 \\ \therefore \quad 9k &= 18 \\ \therefore \quad k &= 2 \end{aligned}$$

$$\begin{aligned} \therefore \quad P(x) &= x^4-3x^3-2x^2+6x \\ \therefore \quad P(x) &= (x-3)(x^3-2x) \quad \{\text{as when } k=2,\ -k=-2 \text{ and } 6-3k=0\} \\ &= x(x-3)(x^2-2) \\ &= x(x-3)(x+\sqrt{2})(x-\sqrt{2}) \quad \text{and} \quad k=2 \end{aligned}$$

2 $P(x) = 2x^3 + ax^2 + bx + 5$

if $x-1$ is a factor, $P(1) = 0$

$$\begin{aligned} \therefore\quad 2(1)^3 + a(1)^2 + b(1) + 5 &= 0 \\ 2 + a + b + 5 &= 0 \\ \therefore\quad a + b &= -7 \text{ ...... (1)} \end{aligned}$$

if $x+5$ is a factor, $P(-5) = 0$

$$\begin{aligned} 2(-5)^3 + a(-5)^2 + b(-5) + 5 &= 0 \\ -250 + 25a - 5b + 5 &= 0 \\ 25a - 5b &= 245 \\ \therefore\quad 5a - b &= 49 \text{ ...... (2)} \end{aligned}$$

Adding (1) and (2) gives: $6a = 42$

$\therefore\ a = 7$ and $b = -14$

3 **a** $P(z) = z^3 - z^2 + [k-5]z + [k^2-7]$

$$\begin{array}{r|rrr|r} 3 & 1 & -1 & k-5 & k^2-7 \\ & 0 & 3 & 6 & 3k+3 \\ \hline & 1 & 2 & k+1 & k^2+3k-4 \end{array}$$

if 3 is a zero, $R = P(3) = 0$

$$\begin{aligned} \therefore\quad k^2 + 3k - 4 &= 0 \\ (k+4)(k-1) &= 0 \\ \therefore\quad k &= -4 \quad\text{or}\quad k = 1 \end{aligned}$$

if $k = 1$

$P(z) = (z-3)(z^2+2z+2)$

the quadratic has zeros: $\dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$\therefore$ zeros are $3,\ -1 \pm i$

if $k = -4$

$$\begin{aligned} P(z) &= (z-3)(z^2+2z-3) \\ &= (z-3)(z+3)(z-1) \end{aligned}$$

$\therefore$ zeros are $3,\ -3$ and 1

b $P(z) = z^3 + mz^2 + (3m-2)z - 10m - 4$

$$\begin{array}{r|rrr|r} 2 & 1 & m & 3m-2 & -10m-4 \\ & 0 & 2 & 2m+4 & 10m+4 \\ \hline 2 & 1 & m+2 & 5m+2 & 0 \ldots (*) \\ & 0 & 2 & 2m+8 & \\ \hline & 1 & m+4 & 7m+10 & \end{array}$$

if $z-2$ is a factor, $P(2) = 0$

since $* = 0$ R_1 is always 0

$\therefore$ $z-2$ is always a factor

now for $(z-2)^2$ to be a factor

$7m + 10 = 0$ $\{R_2$ is also $0\}$ $\therefore$ $m = -\frac{10}{7}$

4 **a** **i**

$$\begin{aligned} P(x) &= x^3 - a^3 \\ P(a) &= a^3 - a^3 \\ &= 0 \end{aligned}$$

$\therefore$ $x-a$ is a linear factor of $P(x)$ for all a

ii

$$\begin{array}{r|rrr|r} a & 1 & 0 & 0 & -a^3 \\ & 0 & a & a^2 & a^3 \\ \hline & 1 & a & a^2 & 0 \end{array}$$

$\therefore$ $P(x) = (x-a)(x^2+ax+a^2)$

b **i**

$$\begin{aligned} P(x) &= x^3 + a^3 \\ P(-a) &= -a^3 + a^3 \\ &= 0 \end{aligned}$$

i.e. $x+a$ is a factor of $P(x)$ for all a

ii

$$\begin{array}{r|rrr|r} -a & 1 & 0 & 0 & a^3 \\ & 0 & -a & a^2 & -a^3 \\ \hline & 1 & -a & a^2 & 0 \end{array}$$

$\therefore$ $P(x) = (x+a)(x^2-ax+a^2)$

5 **a** Consider $P(x) = x^n + 1$

if $x+1$ is a factor then

$$\begin{aligned} P(-1) &= 0 \\ \therefore\quad (-1)^n + 1 &= 0 \\ \therefore\quad (-1)^n &= -1 \end{aligned}$$

which is only true if n is odd

if n is odd $(-1)^n = -1$

$\therefore$ $(-1)^n + 1 = 0$

then $P(-1) = 0$ if $P(x) = x^n + 1$

$\therefore$ $x = -1$ is a zero of $P(x)$

$\therefore$ $x+1$ is a factor of $P(x)$

$\therefore$ $x+1$ is a factor of $x^n + 1 \Leftrightarrow n$ is odd.

b $P(x) = x^3 - 3ax - 9$ and if $x - 1 - a$ is a factor then $P(1+a) = 0$

$$\begin{array}{r|rrr|r} 1+a & 1 & 0 & -3a & -9 \\ & 0 & 1+a & 1+2a+a^2 & a^3+1 \\ \hline & 1 & 1+a & a^2-a+1 & a^3-8 \end{array}$$

$\therefore$ $a^3 - 8 = 0$

$\therefore$ $a = 2$ {the only real soln.}

EXERCISE 8E.1

1 **a** A single factor such as $(x-\alpha)$ indicates that the graph *cuts* the x-axis at α.

b A squared factor such as $(x-\alpha)^2$ indicates that the graph *touches* the x-axis at α.

c A cubed factor such as $(x-\alpha)^3$ indicates that the graph *cuts* the x-axis at α, and at α the graph is horizontal.

2 **a** The x-intercepts are: -1, 2 and 3 $\therefore\ y = a(x+1)(x-2)(x-3),\ a \neq 0$

As the curve passes through (0, 12), $12 = a(1)(-2)(-3)\ \therefore\ a = 2$

$\therefore\ y = 2(x+1)(x-2)(x-3)$

b The x-intercepts are: -3, $-\frac{1}{2}$ and $\frac{1}{2}$ $\therefore\ y = a(x+3)(2x+1)(2x-1),\ a \neq 0$

As the curve passes through (0, 6), $6 = a(3)(1)(-1)\ \therefore\ a = -2$

$\therefore\ y = -2(x+3)(2x+1)(2x-1)$

c The x-intercepts are: -4, -4 and 3 $\therefore\ y = a(x+4)^2(x-3),\ a \neq 0$

As the curve passes through (0, -12), $-12 = a(4)^2(-3)\ \therefore\ a = \frac{1}{4}$

$\therefore\ y = \frac{1}{4}(x+4)^2(x-3)$

d The x-intercepts are: -5, -2 and 5 $\therefore\ y = a(x+5)(x+2)(x-5),\ a \neq 0$

As the curve passes through (0, -5), $-5 = a(5)(2)(-5)\ \therefore\ a = \frac{1}{10}$

$\therefore\ y = \frac{1}{10}(x+5)(x+2)(x-5)$

e The x-intercepts are: -4, 3 and 3 $\therefore\ y = a(x+4)(x-3)^2,\ a \neq 0$

As the curve passes through (0, 9), $9 = a(4)(-3)^2\ \therefore\ a = \frac{1}{4}$

$\therefore\ y = \frac{1}{4}(x+4)(x-3)^2$

f The x-intercepts are: -3, -2 and $-\frac{1}{2}$ $\therefore\ y = a(x+3)(x+2)(2x+1),\ a \neq 0$

As the curve passes through (0, -12), $-12 = a(3)(2)(1)\ \therefore\ a = -2$

$\therefore\ y = -2(x+3)(x+2)(2x+1)$

3 **a** $P(x) = a(x-3)(x-1)(x+2)$

Since $P(x)$ passes through (2, -4)

$-4 = a(-1)(1)(4)$

$\therefore\ -4 = -4a$

$\therefore\ a = 1$

$\therefore\ P(x) = (x-3)(x-1)(x+2)$

b $P(x) = ax(x+2)(2x-1)$

Since $P(x)$ passes through (-3, -21)

$-21 = -3a(-1)(-7)$

$\therefore\ -21 = -21a$

$\therefore\ a = 1$

$\therefore\ P(x) = x(x+2)(2x-1)$

c $P(x) = a(x-1)^2(x+2)$

Since $P(x)$ passes through (4, 54)

$54 = a(9)(6)$

$\therefore\ a = 1$

$\therefore\ P(x) = (x-1)^2(x+2)$

d $P(x) = a(3x+2)^2(x-4)$

Since $P(x)$ passes through (-1, -5)

$-5 = a(1)(-5)$

$\therefore\ a = 1$

$\therefore\ P(x) = (3x+2)^2(x-4)$

4 **a** $y = 2(x-1)(x+2)(x+4)$

has x-intercepts 1, -2, -4

has y-intercept $2(-1)(2)(4) = -16$

$\therefore$ matches graph **F**

b $y = -(x+1)(x-2)(x-4)$

has x-intercepts -1, 2, 4

has y-intercept $-(1)(-2)(-4) = -8$

$\therefore$ matches graph **C**

c $y = (x-1)(x-2)(x+4)$

has x-intercepts 1, 2, -4

has y-intercept $(-1)(-2)(4) = 8$

$\therefore$ matches graph **A**

d $y = -2(x-1)(x+2)(x+4)$

has x-intercepts 1, -2, -4

has y-intercept $-2(-1)(2)(4) = 16$

$\therefore$ matches graph **E**

e $y = -(x-1)(x+2)(x+4)$
has x-intercepts 1, -2, -4
has y-intercept $-(-1)(2)(4) = 8$
$\therefore$ matches graph **D**

f $y = 2(x-1)(x-2)(x+4)$
has x-intercepts 1, 2, -4
has y-intercept $2(-1)(-2)(4) = 16$
$\therefore$ matches graph **B**

5 **a** $\frac{1}{2}$ and -3 are zeros, and so $(2x-1)$ and $(x+3)$ are factors
$\therefore \quad P(x) = (2x-1)(x+3)(ax+b)$

But $P(0) = 30 \quad \therefore \quad b(-1)(3) = 30$ and so $b = -10$
$\therefore \quad P(x) = (2x-1)(x+3)(ax-10)$

Now $P(1) = -20 \quad (1)(4)(a-10) = -20$
$\therefore \quad a - 10 = -5$ and so $a = 5$
$\therefore \quad P(x) = (2x-1)(x+3)(5x-10)$
i.e., $P(x) = 5(x-2)(2x-1)(x+3)$

b 1 is a zero and so $(x-1)$ is a factor, touches at -2 indicates that $(x+2)^2$ is a factor
$\therefore \quad P(x) = k(x-1)(x+2)^2$
But $P(0) = 8 \quad \therefore \quad 8 = k(-1)(2)^2$ and so $k = -2$
$\therefore \quad P(x) = -2(x-1)(x+2)^2$

c cuts at (2, 0) and so $(x-2)$ is a factor
$\therefore \quad P(x) = (x-2)(ax^2+bx+c)$
But $P(0) = -4 \quad \therefore \quad -2c = -4$ and so $c = 2$ ...... (1)

Also $P(1) = -1 \quad \therefore \quad -1(a+b+2) = -1$
$\therefore \quad a+b+2 = 1 \quad \therefore \quad a+b = -1$ ...... (2)

Also $P(-1) = -21 \quad \therefore \quad -3(a-b+2) = -21$
$\therefore \quad a-b+2 = 7 \quad \therefore \quad a-b = 5$ ...... (3)

Adding (2) and (3) gives $\quad 2a = 4$
$\therefore \quad a = 2$ and so $b = -3$
$\therefore \quad P(x) = (x-2)(2x^2-3x+2)$

EXERCISE 8E.2

1 **a** $P(x) = a(x+1)^2(x-1)^2$
where $a \neq 0$, and passes through (0, 2)
$2 = a(1)(1)$
$\therefore \quad a = 2$
$\therefore \quad P(x) = 2(x+1)^2(x-1)^2$

b $P(x) = a(x+3)(x+1)^2(3x-2)$
where $a \neq 0$, and passes through (0, -6)
$-6 = a(3)(1)(-2)$
$\therefore \quad a = 1$
$\therefore \quad P(x) = (x+3)(x+1)^2(3x-2)$

c $P(x) = a(x+2)(x+1)(x-2)^2$
where $a \neq 0$, and passes through (0, -16)
$-16 = a(2)(1)(4)$
$\therefore \quad a = -2$
$\therefore \quad P(x) = -2(x+2)(x+1)(x-2)^2$

d $P(x) = a(x+3)(x+1)(2x-3)(x-3)$
where $a \neq 0$, and passes through (0, -9)
$-9 = a(3)(1)(-3)(-3)$
$\therefore \quad a = -\frac{1}{3}$
$\therefore \quad P(x) = -\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$

e $P(x) = a(x+1)(x-4)^3$
where $a \neq 0$, and passes through (0, -16)
$-16 = a(1)(-4)^3$
$\therefore \quad a = \frac{1}{4}$
$\therefore \quad P(x) = \frac{1}{4}(x+1)(x-4)^3$

f $P(x) = ax^2(x+2)(x-3)$
where $a \neq 0$, and passes through (-3, 54)
$54 = a(9)(-1)(-6)$
$\therefore \quad 54 = 54a$
$\therefore \quad a = 1$
$\therefore \quad P(x) = x^2(x+2)(x-3)$

2 **a** $y = (x-1)^2(x+1)(x+3)$
has x-intercepts -1, -3, touches at 1
has y-intercept $(-1)^2(1)(3) = 3 \ \ (> 0)$
$\therefore$ matches graph **C**

b $y = -2(x-1)^2(x+1)(x+3)$
has x-intercepts -1, -3, touches at 1
has y-intercept $-2(-1)^2(1)(3) = -6 \ \ (< 0)$
$\therefore$ matches graph **F**

c $y = (x-1)(x+1)^2(x+3)$
has x-intercepts 1, -3, touches at -1
has y-intercept $(-1)(1)^2(3) = -3 \ \ (< 0)$
$\therefore$ matches graph **A**

d $y = (x-1)(x+1)^2(x-3)$
has x-intercepts 1, 3, touches at -1
has y-intercept $(-1)(1)^2(-3) = 3 \ \ (> 0)$
$\therefore$ matches graph **E**

e $y = -\frac{1}{3}(x-1)(x+1)(x+3)^2$
has x-intercepts 1, -1, touches at -3
has y-intercept $-\frac{1}{3}(-1)(1)(3)^2 = 3 \ \ (> 0)$
$\therefore$ matches graph **B**

f $y = -(x-1)(x+1)(x-3)^2$
has x-intercepts 1, -1, touches at 3
has y-intercept $-(-1)(1)(3)^2 = 9 \ \ (> 0)$
$\therefore$ matches graph **D**

3 **a** $P(x) = a(x+4)(2x-1)(x-2)^2$
where $a \neq 0$, and passes through (1, 5)
$5 = a \times 5 \times 1 \times 1$
$\therefore \ a = 1$
$\therefore \ P(x) = (x+4)(2x-1)(x-2)^2$

b $P(x) = a(3x-2)^2(x+3)^2$
where $a \neq 0$, and passes through $(-4, 49)$
$49 = a(-14)^2(1)$
$\therefore \ a = \frac{1}{4}$
$\therefore \ P(x) = \frac{1}{4}(3x-2)^2(x+3)^2$

c $P(x) = a(2x+1)(2x-1)(x+2)(x-2)$
where $a \neq 0$, and passes through $(1, -18)$
$-18 = a(3)(1)(3)(-1)$
$\therefore \ a = 2$
$\therefore \ P(x) = 2(2x+1)(2x-1)(x+2)(x-2)$

d $P(x) = (x-1)^2(ax^2+bx+c)$
where $a \neq 0$, and cuts y-axis at $(0, -1)$
$-1 = 1 \times (0+0+c)$
$\therefore \ c = -1$
$\therefore \ P(x) = (x-1)^2(ax^2+bx-1)$

But $P(-1) = -4$
$\therefore \ -4 = 4(a-b-1)$
$\therefore \ a - b = 0$ ...... (1)
Also $P(2) = 15$
$\therefore \ 15 = 1(4a+2b-1)$
$\therefore \ 16 = 4a+2b$
$\therefore \ 2a+b = 8$ ...... (2)
Adding (1) and (2) we get:
$\therefore \ a = \frac{8}{3}$ and so $b = \frac{8}{3}$ also
$\therefore \ P(x) = (x-1)^2(\frac{8}{3}x^2 + \frac{8}{3}x - 1)$
$\therefore \ P(x) = \frac{1}{3}(x-1)^2(8x^2+8x-3)$

EXERCISE 8E.3

1 **a** $P(x) = x^3 - 3x^2 - 3x + 1$
From a calculator -1 is a zero.
Check: $P(-1) = -1 - 3 + 3 + 1 = 0$
$\therefore \ x+1$ is a factor

$$\begin{array}{r|rrr|r} -1 & 1 & -3 & -3 & 1 \\ & 0 & -1 & 4 & -1 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

From the division process $x^2 - 4x + 1$ is a quadratic factor

and it has zeros of $\dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore$ zeros are -1, $2 \pm \sqrt{3}$

b $P(x) = x^3 - 3x^2 + 4x - 2$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

From a calculator 1 is a zero.

Check: $P(1) = 1 - 3 + 4 - 2 = 0$

$\therefore \quad x - 1$ is a factor

From the division process $x^2 - 4x + 1$ is a quadratic factor

and it has zeros of $\dfrac{2 \pm \sqrt{4 - 4 \times 2}}{2} = \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$\therefore$ zeros are $1, 1 \pm i$

c $P(x) = 2x^3 - 3x^2 - 4x - 35$

$$\begin{array}{r|rrrr} \frac{7}{2} & 2 & -3 & -4 & -35 \\ & 0 & 7 & 14 & 35 \\ \hline & 2 & 4 & 10 & 0 \end{array}$$

From a calculator $\frac{7}{2}$ is a zero.

Check:
$$\begin{aligned} P(\tfrac{7}{2}) &= \tfrac{343}{4} - \tfrac{147}{4} - 14 - 35 \\ &= \tfrac{343-147-56-140}{4} \\ &= 0 \end{aligned}$$

From the division process $2x^2 + 4x + 10$ is a quadratic factor

$$\begin{aligned} \therefore \quad P(x) &= (x - \tfrac{7}{2})(2x^2 + 4x + 10) \\ &= (2x - 7)(x^2 + 2x + 5) \end{aligned}$$

where the quadratic has zeros $\dfrac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$

$\therefore$ zeros are $\frac{7}{2}, -1 \pm 2i$

d $P(x) = 2x^3 - x^2 + 20x - 10$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 20 & -10 \\ & 0 & 1 & 0 & 10 \\ \hline & 2 & 0 & 20 & 0 \end{array}$$

From a calculator $\frac{1}{2}$ is a zero.

Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{1}{4} + 10 - 10 = 0$

$$\begin{aligned} \therefore \quad P(x) &= (x - \tfrac{1}{2})(2x^2 + 20) \\ &= (2x - 1)(x^2 + 10) \end{aligned}$$

$\therefore$ zeros are $\frac{1}{2}, \pm i\sqrt{10}$

e $P(x) = 4x^4 - 4x^3 - 25x^2 + x + 6$

$$\begin{array}{r|rrrrr} -2 & 4 & -4 & -25 & 1 & 6 \\ & 0 & -8 & 24 & 2 & -6 \\ \hline 3 & 4 & -12 & -1 & 3 & 0 \\ & 0 & 12 & 0 & -3 & \\ \hline & 4 & 0 & -1 & 0 & \end{array}$$

From a calculator -2 and 3 are zeros

Check: $P(-2) = 64 + 32 - 100 - 2 + 6 = 0$

$P(3) = 324 - 108 - 225 + 3 + 6 = 0$

$\therefore \quad P(x) = (x + 2)(x - 3)(4x^2 - 1)$

$\therefore$ zeros are $-2, 3, \pm\frac{1}{2}$

f $P(x) = x^4 - 6x^3 + 22x^2 - 48x + 40$

$$\begin{array}{r|rrrrr} 2 & 1 & -6 & 22 & -48 & 40 \\ & 0 & 2 & -8 & 28 & -40 \\ \hline 2 & 1 & -4 & 14 & -20 & 0 \\ & 0 & 2 & -4 & 20 & \\ \hline & 1 & -2 & 10 & 0 & \end{array}$$

From a calculator 2 seems to be a double zero.
{Graph touches the x-axis at 2}

Check: $P(2) = 16 - 48 + 88 - 96 + 40 = 0$

$\therefore \quad P(x) = (x - 2)^2(x^2 - 2x + 10)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4 - 40}}{2}$

$= 1 \pm 3i$

$\therefore$ zeros are $2, 2, 1 \pm 3i$

2 **a** $P(x) = x^3+2x^2+3x+6$

From a calculator -2 is a zero.

Check: $P(-2) = -8+8-6+6=0$

$\therefore\quad P(x) = (x+2)(x^2+3)$

$\therefore\quad P(x) = (x+2)(x+i\sqrt{3})(x-i\sqrt{3})$

$\therefore$ roots of $P(x)=0$ are $x=-2$ and $x=\pm i\sqrt{3}$

$$\begin{array}{r|rrr|r} -2 & 1 & 2 & 3 & 6 \\ & 0 & -2 & 0 & -6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

b $P(x) = 2x^3+3x^2-3x-2$

From a calculator 1 is a zero.

Check: $P(1) = 2+3-3-2=0$

$\therefore\quad P(x) = (x-1)(2x^2+5x+2)$

$= (x-1)(2x+1)(x+2)$

$\therefore$ roots of $P(x)=0$ are $1, -\frac{1}{2}, -2$

$$\begin{array}{r|rrr|r} 1 & 2 & 3 & -3 & -2 \\ & 0 & 2 & 5 & 2 \\ \hline & 2 & 5 & 2 & 0 \end{array}$$

c $P(x) = x^3-6x^2+12x-8$

From a calculator 2 is a zero.

Check: $P(2) = 8-24+24-8=0$

$\therefore\quad P(x) = (x-2)(x^2-4x+4)$

$= (x-2)(x-2)(x-2)$

$\therefore$ only root is $x=2$, (a treble root)

$$\begin{array}{r|rrr|r} 2 & 1 & -6 & 12 & -8 \\ & 0 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

d $P(x) = 2x^3-5x^2-9x+18$

From a calculator 3 is a zero.

Check: $P(3) = 54-45-27+18=0$

$\therefore\quad P(x) = (x-3)(2x^2+x-6)$

$= (x-3)(2x-3)(x+2)$

$\therefore$ roots of $2x^3-5x^2-9x+18=0$ are $3, \frac{3}{2}$ and -2

$$\begin{array}{r|rrr|r} 3 & 2 & -5 & -9 & 18 \\ & 0 & 6 & 3 & -18 \\ \hline & 2 & 1 & -6 & 0 \end{array}$$

e $P(x) = x^4-x^3-9x^2+11x+6=0$

From a calculator 2 and -3 are zeros.

Check: $P(2) = 16-8-36+22+6=0$

$P(-3) = 81+27-81-33+6=0$

$\therefore\quad P(x) = (x-2)(x+3)(x^2-2x-1)$

where the quadratic has zeros of $\dfrac{2\pm\sqrt{4+4}}{2} = 1\pm\sqrt{2}$

$\therefore$ roots are $2, -3, 1\pm\sqrt{2}$

$$\begin{array}{r|rrrr|r} 2 & 1 & -1 & -9 & 11 & 6 \\ & 0 & 2 & 2 & -14 & -6 \\ \hline -3 & 1 & 1 & -7 & -3 & 0 \\ & 0 & -3 & 6 & 3 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

f $P(x) = 2x^4-13x^3+27x^2-13x-15$

From a calculator $-\frac{1}{2}$ and 3 are zeros.

Check:

$P(-\frac{1}{2}) = \frac{1}{8}+\frac{13}{8}+\frac{27}{4}+\frac{13}{2}-15=0$

$P(3) = 162-351+243-39-15=0$

$\therefore\quad P(x) = (x+\frac{1}{2})(x-3)(2x^2-8x+10)$

$= (2x+1)(x-3)(x^2-4x+5)$

where the quadratic has zeros of $\dfrac{4\pm\sqrt{16-20}}{2} = 2\pm i$

$\therefore\quad x=-\frac{1}{2},\ x=3$ or $x=2\pm i$

$$\begin{array}{r|rrrr|r} -\frac{1}{2} & 2 & -13 & 27 & -13 & -15 \\ & 0 & -1 & 7 & -17 & 15 \\ \hline 3 & 2 & -14 & 34 & -30 & 0 \\ & 0 & 6 & -24 & 30 & \\ \hline & 2 & -8 & 10 & 0 & \end{array}$$

3 **a** Consider $P(x) = x^3 - 3x^2 + 4x - 2$

$$\begin{array}{r|rrr|r} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

From a calculator 1 is a zero.

(*Check:* $P(1) = 1 - 3 + 4 - 2 = 0$)

Now $P(x) = (x-1)(x^2 - 2x + 2)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore \quad P(x) = (x-1)(x - [1+i])(x - [1-i])$

b Consider $P(x) = x^3 + 3x^2 + 4x + 12$

$$\begin{array}{r|rrr|r} -3 & 1 & 3 & 4 & 12 \\ & 0 & -3 & 0 & -12 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

From a calculator -3 is a zero.

(*Check:* $P(-3) = -27 + 27 - 12 + 12 = 0$)

Now $P(x) = (x+3)(x^2 + 4)$

$\therefore \quad P(x) = (x+3)(x-2i)(x+2i)$

c Consider $P(x) = 2x^3 - 9x^2 + 6x - 1$

$$\begin{array}{r|rrr|r} \frac{1}{2} & 2 & -9 & 6 & -1 \\ & 0 & 1 & -4 & 1 \\ \hline & 2 & -8 & 2 & 0 \end{array}$$

From a calculator $\frac{1}{2}$ is a zero.

(*Check:* $P(\frac{1}{2}) = \frac{2}{8} - \frac{9}{4} + 3 - 1 = 0$)

Now $P(x) = (x - \frac{1}{2})(2x^2 - 8x + 2)$
$= (2x-1)(x^2 - 4x + 1)$

where the quadratic has zeros of $\dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore \quad P(x) = (2x-1)(x - \left[2+\sqrt{3}\right])(x - \left[2-\sqrt{3}\right])$

d $P(x) = x^3 - 4x^2 + 9x - 10$

$$\begin{array}{r|rrr|r} 2 & 1 & -4 & 9 & -10 \\ & 0 & 2 & -4 & 10 \\ \hline & 1 & -2 & 5 & 0 \end{array}$$

From a calculator 2 is a zero.

(*Check:* $P(2) = 8 - 16 + 18 - 10 = 0$)

Now $P(x) = (x-2)(x^2 - 2x + 5)$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore \quad P(x) = (x-2)(x - [1+2i])(x - [1-2i])$

e $P(x) = 4x^3 - 8x^2 + x + 3$

$$\begin{array}{r|rrr|r} 1 & 4 & -8 & 1 & 3 \\ & 0 & 4 & -4 & -3 \\ \hline & 4 & -4 & -3 & 0 \end{array}$$

From a calculator 1 is a zero.

(*Check:* $P(1) = 4 - 8 + 1 + 3 = 0$)

Now $P(x) = (x-1)(4x^2 - 4x - 3)$
$= (x-1)(2x-3)(2x+1)$

$\therefore \quad P(x) = (x-1)(2x+1)(2x-3)$

f $P(x) = 3x^4 + 4x^3 + 5x^2 + 12x - 12$

$$\begin{array}{r|rrrr|r} -2 & 3 & 4 & 5 & 12 & -12 \\ & 0 & -6 & 4 & -18 & 12 \\ \cline{2-6} \frac{2}{3} & 3 & -2 & 9 & -6 & 0 \\ & 0 & 2 & 0 & 6 & \\ \cline{2-5} & 3 & 0 & 9 & 0 & \end{array}$$

From a calculator -2 and $\frac{2}{3}$ are zeros.

(*Check:* $P(-2) = 48 - 32 + 20 - 24 - 12 = 0$
$P(\frac{2}{3}) = \frac{16}{27} + \frac{32}{27} + \frac{20}{9} + 8 - 12 = 0$)

Now $P(x) = (x+2)(x - \frac{2}{3})(3x^2 + 9)$
$= (x+2)(3x-2)(x^2+3)$

i.e., $P(x) = (x+2)(3x-2)(x + i\sqrt{3})(x - i\sqrt{3})$

g $P(x) = 2x^4 - 3x^3 + 5x^2 + 6x - 4$

From a calculator -1 and $\frac{1}{2}$ are zeros.)

(*Check:* $P(-1) = 2 + 3 + 5 - 6 - 4 = 0$

$P(\frac{1}{2}) = \frac{1}{8} - \frac{3}{8} + \frac{5}{4} + 3 - 4 = 0)$

$$\begin{array}{r|rrrr|r}
-1 & 2 & -3 & 5 & 6 & -4 \\
 & 0 & -2 & 5 & -10 & 4 \\
\hline
\frac{1}{2} & 2 & -5 & 10 & -4 & 0 \\
 & 0 & 1 & -2 & 4 & \\
\hline
 & 2 & -4 & 8 & \multicolumn{1}{|r}{0} &
\end{array}$$

Now $P(x) = (x+1)(x-\frac{1}{2})(2x^2 - 4x + 8)$

$= (x+1)(2x-1)(x^2 - 2x + 4)$

where the quadratic has zeros $\dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$\therefore \quad P(x) = (x+1)(2x-1)(x - \left[1 + i\sqrt{3}\right])(x - \left[1 - i\sqrt{3}\right])$

h $P(x) = 2x^3 + 5x^2 + 8x + 20$

From a calculator $-\frac{5}{2}$ is a zero.

(*Check:* $P(-\frac{5}{2}) = -\frac{125}{4} + \frac{125}{4} - 20 + 20 = 0)$

$$\begin{array}{r|rrrr|}
-\frac{5}{2} & 2 & 5 & 8 & 20 \\
 & 0 & -5 & 0 & -20 \\
\hline
 & 2 & 0 & 8 & \multicolumn{1}{|r}{0}
\end{array}$$

Now $P(x) = (x + \frac{5}{2})(2x^2 + 8)$

$= (2x+5)(x^2+4)$

$\therefore \quad P(x) = (2x+5)(x-2i)(x+2i)$

4 **a** Using technology, $x^3 + 2x^2 - 6x - 6$ has zeros of -0.8596, 2.1326 and -3.2731

b Using technology, $x^3 + x^2 - 7x - 8$ has zeros of -2.5182, -1.1782 and 2.6964

EXERCISE 8F.1

1 Since it is a real polynomial, zeros must be $-\frac{1}{2}$, $1 - 3i$ and $1 + 3i$.

For $1 \pm 3i$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore$ factors are $(2x+1)$ and $(x^2 - 2x + 10)$

$\therefore \quad P(x) = k(2x+1)(x^2 - 2x + 10) \quad k \neq 0$

2 $p(1) = p(2+i) = 0$

Hence zeros of $p(x)$ must be $1, 2 \pm i$ {as $p(x)$ is real}

For $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

$\therefore$ factors must be $(x-1)$ and $(x^2 - 4x + 5)$

$\therefore \quad p(x) = k(x-1)(x^2 - 4x + 5)$

Since $p(0) = -20$ then $-20 = k(-1)(5)$

$\therefore \quad k = 4$

$\therefore \quad p(x) = 4(x-1)(x^2 - 4x + 5)$

$\therefore \quad p(x) = 4x^3 - 20x^2 + 36x - 20$

$$\begin{array}{rrrr}
 & 1 & -4 & 5 \\
\times & & 4 & -4 \\
\hline
 & -4 & 16 & -20 \\
4 & -16 & 20 & \\
\hline
4 & -20 & 36 & -20 \\
\hline
\end{array}$$

3 $2 - 3i$ is a zero of $z^3 + pz + q$ and as the cubic has real coefficients, $2 + 3i$ is also a zero.

For $2 \pm 3i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - 9i^2 = 13$

$\therefore \quad x^2 - 4x + 13$ is a factor

$\therefore \quad z^3 + pz + q = (x^2 - 4x + 13)(x + a)$ for some a.

$$\begin{array}{rrrr}
 & 1 & -4 & 13 \\
\times & & 1 & a \\
\hline
 & a & -4a & 13a \\
1 & -4 & 13 & \\
\hline
1 & a-4 & 13-4a & 13a \\
\hline
\end{array}$$

Equating coefficients:

$a - 4 = 0, \quad 13 - 4a = p \quad$ and $\quad 13a = q$

$\therefore \quad a = 4, \quad p = -3, \quad q = 52$

$\therefore$ the other zeros are -4 and $2 + 3i$

Check: Since $P(2-3i)=0,\ \ (2-3i)^3+p(2-3i)+q=0$

Expanding, $(-46-9i)+p(2-3i)+q = 0$

$\therefore\ \ (-46+2p+q)+(-9-3p)i = 0$

Equating real and imaginary parts, $-46+2p+q=0$ .....(1)

and $-9-3p=0$ .....(2)

From (2), $p=-3$, so in (1), $-46-6+q=0$ $\therefore$ $p=-3$, $q=52$

4 $3+i$ is a root of $z^4-2z^3+az^2+bz+10=0$ where the coefficients are real.

$\therefore$ $3-i$ is also a root

For $3\pm i$, $\alpha+\beta=6$ and $\alpha\beta=9-i^2=10$

$\therefore$ $z^2-6z+10$ is a factor

$\therefore$ $z^4-2z^3+az^2+bz+10$

$=(z^2-6z+10)(z^2+sz+1)$ for some s.

$$\begin{array}{ccccc}
 & & 1 & -6 & 10 \\
 & \times & 1 & s & 1 \\
\hline
 & & 1 & -6 & 10 \\
 & s & -6s & 10s & \\
1 & -6 & 10 & & \\
\hline
1 & s-6 & 11-6s & 10s-6 & 10 \\
\hline
\end{array}$$

Equating coefficients:

$s-6=-2$, $11-6s=a$ and $10s-6=b$

$\therefore$ $s=4$ $a=11-6(4)=-13$ $b=10(4)-6=34$

$\therefore$ the other factor is z^2+4z+1 which has zeros $\dfrac{-4\pm\sqrt{16-4}}{2}=-2\pm\sqrt{3}$

$\therefore$ $a=-13$ $b=34$ and the other roots are $3-i$, $-2\pm\sqrt{3}$.

5 Let the purely imaginary zero be bi. Since $P(z)$ is real, another zero is $-bi$, (b is real)

$\therefore$ z^2+b^2 is a factor of $P(z)$

$$\begin{aligned}\therefore\ \ z^3+az^2+3z+9 &= (z^2+b^2)(z+c)\\ &= z^3+cz^2+b^2z+b^2c\end{aligned}$$

Equating coefficients $b^2=3$, $b^2c=9$ and $a=c$

$\therefore$ $c=3$ and $a=3$ and $b=\pm\sqrt{3}$

$\therefore$ $P(z)=(z+3)(z^2+3)$ i.e., $P(z)=(z+3)(z+i\sqrt{3})(z-i\sqrt{3})$, $a=3$

6 Let ai be the purely imaginary zero of $3x^3+kx^2+15x+10$

$\therefore$ as $P(x)$ is real, $-ai$ is also a zero

For $\pm ai$, $\alpha+\beta=0$ and $\alpha\beta=-a^2i^2=a^2$

$\therefore$ x^2+a^2 is a factor

$$\begin{aligned}\therefore\ \ 3x^3+kx^2+15x+10 &= (x^2+a^2)(3x+b)\\ &= 3x^3+bx^2+3a^2x+a^2b\end{aligned}$$

Equating coefficients $k=b$

and $3a^2=15$ $\therefore$ $a^2=5$

and $a^2b=10$ $\therefore$ $b=2$ $\therefore$ $k=2$

$$\begin{array}{cccc}
 & 1 & 0 & a^2 \\
\times & & 3 & b \\
\hline
 & b & 0 & a^2b \\
3 & 0 & 3a^2 & \\
\hline
3 & b & 3a^2 & a^2b \\
\hline
\end{array}$$

$\therefore$ $P(x)=(x^2+5)(3x+2)$

$\therefore$ $P(x)=(3x+2)(x-i\sqrt{5})(x+i\sqrt{5})$, $k=2$

EXERCISE 8F.2

1 **a** $f(t)=kt(t-a)^2$

From the graph a is the t-value at the point where the graph touches the t-axis.

$\therefore$ $a=700$ milliseconds

i.e., the time the barrier has returned to its original position

b when $t=100$ ms $f(t)=85$ mm

$$\begin{aligned}\therefore\ \ 85 &= k\times 100(100-700)^2\\ 85 &= 100\times k\times 360\,000\\ k &= \frac{85}{36\,000\,000}\\ \therefore\ \ f(t) &= \frac{85t}{36\,000\,000}(t-700)^2\end{aligned}$$

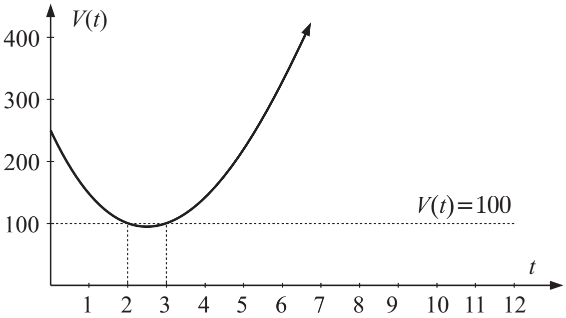
c Using technology to find the maximum, on $0 \leqslant t \leqslant 700$
the maximum occurs when $t = 233$ ms i.e., when $f(t) = 119.98 \doteqdot 120$ mm

2 $V(t) = -t^3 + 30t^2 - 131t + 250$ we graph $V(t)$ against t and add the graph of $V(t) = 100$.

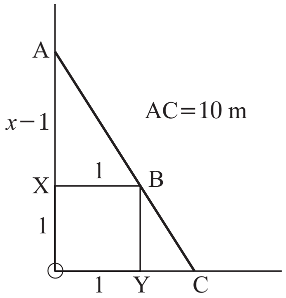
From the graph, the level drops below 100 ML when $t = 2$ and rises above 100 ML again when $t = 3$.

Now as $t = 0$ is Jan 1st, $0 \leqslant t < 1$ is January.

$\therefore$ as irrigation is prohibited for $2 < t < 3$, it is banned during March.



3



Let the height of the wall where the ladder touches be x m.

Using similar triangles AXB, AOC:

$$\frac{x-1}{1} = \frac{x}{\text{OC}}$$

$$\therefore \quad \text{OC} = \frac{x}{x-1}$$

but $x^2 + \text{OC}^2 = 10^2$

$$x^2 + \left(\frac{x}{x-1}\right)^2 = 100$$

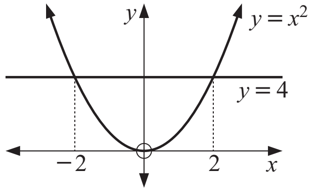
Using technology to find the intersection of $y = x^2 + \left(\dfrac{x}{x-1}\right)^2$ and $y = 100$

$x \doteqdot 1.112$ or 9.938

i.e., distance $\doteqdot 9.938$ m or 1.112 m

EXERCISE 8G.1

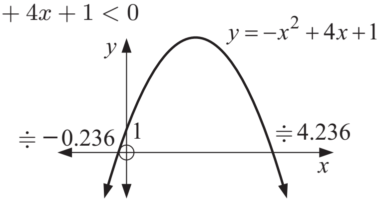
1 **a** $x^2 > 4$



So, $x^2 > 4$ when the graph of $y = x^2$ is above the graph of $y = 4$.

This is when $x < -2$ or $x > 2$
i.e., $x \in\,]-\infty,\, -2[$ or $x \in\,]2,\, \infty[$

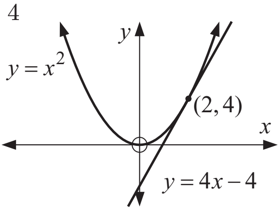
b $-x^2 + 4x + 1 < 0$



So, if $-x^2 + 4x + 1 < 0$, the graph will be below the x-axis.

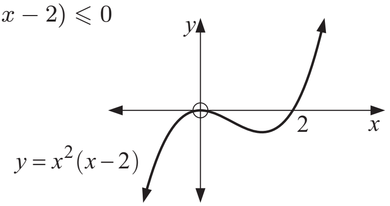
$\therefore$ $x < -0.236$ or $x > 4.236$
i.e., $x \in\,]-\infty,\, -0.236[$ or $x \in\,]4.236,\, \infty[$

c $x^2 \geqslant 4x - 4$



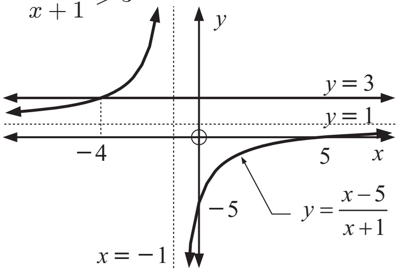
$x^2 \geqslant 4x - 4$ when $y = x^2$ is above or on $y = 4x - 4$.
This is true for all x in R.

d $x^2(x-2) \leqslant 0$



$x^2(x-2) \leqslant 0$ when $y = x^2(x-2)$ is below or on the x-axis.
$\therefore$ $x \leqslant 2$ i.e., $x \in\,]-\infty,\, 2]$

e $\dfrac{x-5}{x+1} \geqslant 3$

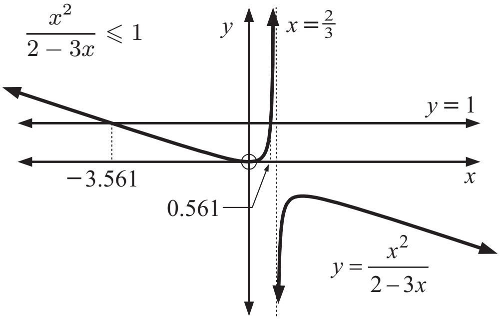


$\dfrac{x-5}{x+1} \geqslant 3$ when the graph of

$y = \dfrac{x-5}{x+1}$ is on or above

$y = 3$, $\therefore$ $x \in [-4, -1[$

f $\dfrac{x^2}{2-3x} \leqslant 1$



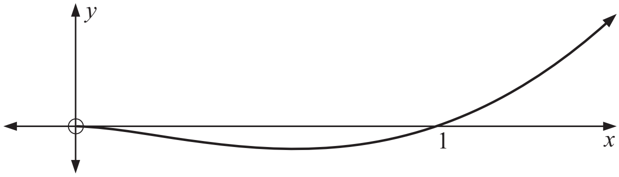
$\dfrac{x^2}{2-3x} \leqslant 1$ when the graph of $y = \dfrac{x^2}{2-3x}$

is on or below the graph of $y = 1$.

$\therefore$ $x \in [-3.561, 0.561]$ or $x \in]\frac{2}{3}, \infty[$

2 f is defined when $\ln x$ is defined. This is when $x > 0$.

So, the domain is $x \in]0, \infty[$



If $f(x) \leqslant 0$ then $x^2 \ln x \leqslant 0$

$\therefore$ the graph of $y = x^2 \ln x$

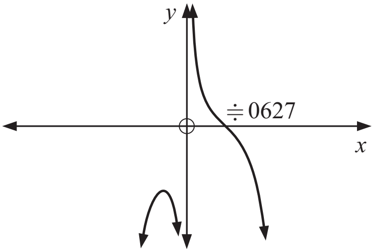
is on or below $y = 0$.

$\therefore$ $0 < x \leqslant 1$

i.e., $x \in]0, 1]$

3 $f(x) = \dfrac{2}{x} - e^{2x^2-x+1}$

a



b domain is $\{x : x \in R, \ x \neq 0\}$
range is $\{y : y \in R\}$

c If $e^{2x^2-x+1} > \dfrac{2}{x}$

then $\dfrac{2}{x} - e^{2x^2-x+1} < 0$

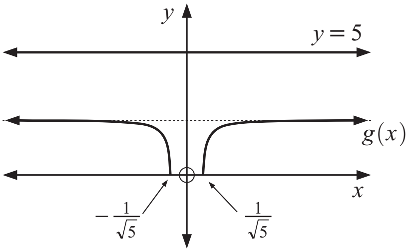
So, we want x such that $f(x) < 0$.

This is for $x < 0$ or $x > 0.627$

i.e., $x \in]-\infty, 0[$ or $x \in]0.627, \infty[$

4 $g(x) = \sqrt{5 - \dfrac{1}{x^2}}$

a



b g is finite is $x \neq 0$

and real if $5 - \dfrac{1}{x^2} \geqslant 0$

i.e., $5 \geqslant \dfrac{1}{x^2}$

$\therefore$ $x \leqslant -\frac{1}{\sqrt{5}}$ or $x \geqslant \frac{1}{\sqrt{5}}$

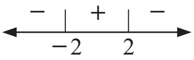
So, $5 \geqslant \dfrac{1}{x^2}$ when the graph exists i.e., $x \in]-\infty, -\frac{1}{\sqrt{5}}]$ or $x \in [\frac{1}{\sqrt{5}}, \infty[$

EXERCISE 8G.2

1 **a** $4-x^2$

$=(2+x)(2-x)$ which

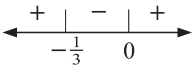
has sign diagram:



b $3x^2+x$

$=x(3x+1)$ which

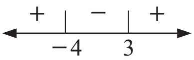
has sign diagram:



c x^2-x-12

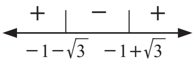
$=(x+4)(x-3)$ which

has sign diagram:



d x^2+2x-2 has zeros of

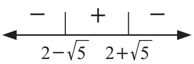
$$\frac{-2\pm\sqrt{4-4(1)(-2)}}{2}=-1\pm\sqrt{3}$$



e $-x^2+4x+1$ has zeros of

$$\frac{-4\pm\sqrt{16-4(-1)(1)}}{-2}$$

$$=\frac{-4\pm\sqrt{20}}{-2}=2\pm\sqrt{5}$$



f $-2x^2+x-2$

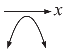
has $\Delta = 1^2-4(-2)(-2)$

$= 1-16$

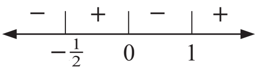
$= -15$ i.e., < 0

and "a" $= -2$ is < 0

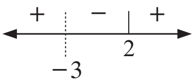
$\therefore$ is negative for all x



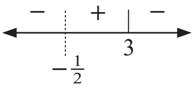
g



h



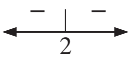
i



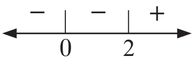
j $4x-4-x^2$

$=-(x^2-4x+4)$

$=-(x-2)^2$



k



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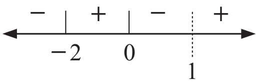


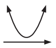
m $2x^3-5x^2+10x$

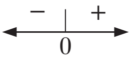
$=x(2x^2-5x+10)$

The quadratic has "a" > 0

and $\Delta = 25-4(2)(10)=-55$

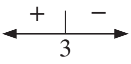
and as $\Delta<0$ the quadratic factor is always positive.





n $(3-x)(x^2+2)$

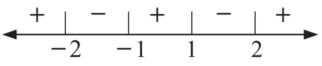
has x^2+2 always positive.



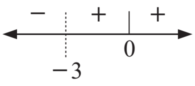
o x^4-5x^2+4

$=(x^2-1)(x^2-4)$

$=(x+1)(x-1)(x+2)(x-2)$



p $\dfrac{x^2}{x+3}$

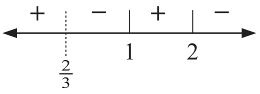


r $\dfrac{x^2}{2-3x}+1$

$=\dfrac{x^2}{2-3x}+\dfrac{2-3x}{2-3x}$

$=\dfrac{x^2-3x+2}{2-3x}$

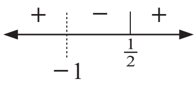
$=\dfrac{(x-1)(x-2)}{2-3x}$



q $\dfrac{x-5}{x+1}+3$

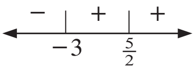
$=\dfrac{x-5}{x+1}+3\left(\dfrac{x+1}{x+1}\right)$

$=\dfrac{4x-2}{x+1}$



EXERCISE 8G.3

1 **a** $(2x-5)^2(x+3) < 0$



$\therefore \quad x < -3$

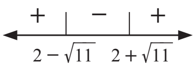
i.e., $\quad x \in \,]-\infty, \ -3[$

b $x^2 \geqslant 4x + 7$

$\therefore \quad x^2 - 4x - 7 \geqslant 0$

But $\quad x^2 - 4x - 7 = 0$

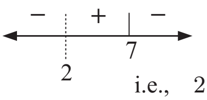
$$\text{when} \quad x = \frac{4 \pm \sqrt{16 - 4(1)(-7)}}{2} = \frac{4 \pm \sqrt{44}}{2} = 2 \pm \sqrt{11}$$



$\therefore \quad x < 2 - \sqrt{11} \quad \text{or} \quad x > 2 + \sqrt{11}$

So, $\ x \in \,]-\infty, 2-\sqrt{11}\,[\ \text{ or } \ ]2+\sqrt{11}, \infty[$

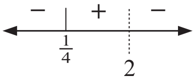
c
$$\frac{x+3}{x-2} > 2$$
$$\therefore \quad \frac{x+3}{x-2} - 2 > 0$$
$$\therefore \quad \frac{x+3}{x-2} - 2\left(\frac{x-2}{x-2}\right) > 0$$
$$\therefore \quad \frac{-x+7}{x-2} > 0$$



i.e., $\quad 2 < x < 7$

i.e., $\quad x \in \,]\ 2, 7\ [$

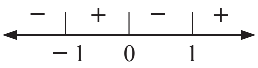
d
$$\frac{3x+1}{2-x} \leqslant 1$$
$$\therefore \quad \frac{3x+1}{2-x} - 1 \leqslant 0$$
$$\therefore \quad \frac{3x+1}{2-x} - 1\left(\frac{2-x}{2-x}\right) \leqslant 0$$
$$\therefore \quad \frac{4x-1}{2-x} \leqslant 0$$



$\therefore \quad x \leqslant \frac{1}{4} \quad \text{or} \quad x > 2$

i.e., $\quad x \in \,]\ -\infty, \frac{1}{4}\ ] \quad \text{or} \quad]\ 2, \infty\ [$

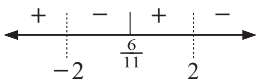
e
$$x^3 \geqslant x$$
$$\therefore \quad x^3 - x \geqslant 0$$
$$\therefore \quad x(x^2-1) \geqslant 0$$
$$\therefore \quad x(x+1)(x-1) \geqslant 0$$



$\therefore \quad -1 \leqslant x \leqslant 0 \quad \text{or} \quad x \geqslant 1$

$\therefore \quad x \in [\ -1, 0\ ] \quad \text{or} \quad [\ 1, \infty\ [$

f
$$\frac{2x-3}{x+2} < \frac{2x}{x-2}$$
$$\therefore \quad \frac{2x-3}{x+2} - \frac{2x}{x-2} < 0$$
$$\therefore \quad \left(\frac{2x-3}{x+2}\right)\left(\frac{x-2}{x-2}\right) - \left(\frac{2x}{x-2}\right)\left(\frac{x+2}{x+2}\right) < 0$$
$$\therefore \quad \frac{2x^2 - 7x + 6 - \left[2x^2 + 4x\right]}{(x+2)(x-2)} < 0$$
$$\therefore \quad \frac{-11x+6}{(x+2)(x-2)} < 0$$

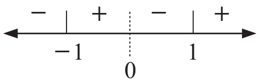


$\therefore \quad -2 < x < \frac{6}{11} \quad \text{or} \quad x > 2$

i.e., $\quad x \in \,]\ -2, \frac{6}{11}\ [\quad \text{or} \quad]\ 2, \infty\ [$

g

$$\begin{aligned} x &\geqslant \frac{1}{x} \\ \therefore\quad x-\frac{1}{x} &\geqslant 0 \\ \therefore\quad \frac{x^2}{x}-\frac{1}{x} &\geqslant 0 \\ \therefore\quad \frac{x^2-1}{x} &\geqslant 0 \\ \therefore\quad \frac{(x+1)(x-1)}{x} &\geqslant 0 \end{aligned}$$



$\therefore\quad -1 \leqslant x < 0 \quad\text{or}\quad x \geqslant 1$

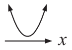
i.e., $x \in [-1, 0[\quad\text{or}\quad [1, \infty[$

h

$$\begin{aligned} x^2 &\leqslant \frac{8}{x} \\ \therefore\quad x^2-\frac{8}{x} &\leqslant 0 \\ \therefore\quad \frac{x^3}{x}-\frac{8}{x} &\leqslant 0 \\ \therefore\quad \frac{x^3-8}{x} &\leqslant 0 \\ \therefore\quad \frac{(x-2)(x^2+2x+4)}{x} &\leqslant 0 \end{aligned}$$

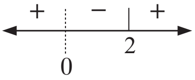
But the quadratic has "a" $= 1$ which is > 0

and $\Delta = 2^2 - 4(1)(4) = -12$



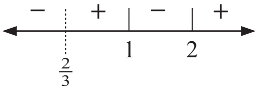
i.e., always positive

$\therefore\quad 0 < x \leqslant 2$ i.e., $x \in\ ]0, 2]$



i

$$\begin{aligned} \frac{x^2}{3x-2} &\leqslant 1 \\ \therefore\quad \frac{x^2}{3x-2}-1 &\leqslant 0 \\ \therefore\quad \frac{x^2}{3x-2}-\left(\frac{3x-2}{3x-2}\right) &\leqslant 0 \\ \therefore\quad \frac{x^2-3x+2}{(3x-2)} &\leqslant 0 \\ \therefore\quad \frac{(x-1)(x-2)}{3x-2} &\leqslant 0 \end{aligned}$$



$\therefore\quad x < \frac{2}{3} \quad\text{or}\quad 1 \leqslant x \leqslant 2$

$\therefore\quad x \in\ ]-\infty, \frac{2}{3}[\quad\text{or}\quad [\,1, 2\,]$

2 $kx^2 + 2x - (k+1) = 0$

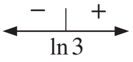
$$\begin{aligned} \text{has}\quad \Delta &= 4 + 4k(k+1) \\ &= 4 + 4k^2 + 4k \\ &= 4(k^2+k+1) \\ \Delta &= 0 \\ \text{when}\quad k &= \frac{-1 \pm \sqrt{1-4}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2} \end{aligned}$$

but k is real, so no values for k exist.

3

$$\begin{aligned} e^{2x} + 2e^x &\geqslant 6 + 3e^x \\ \therefore\quad e^{2x} - e^x - 6 &\geqslant 0 \\ \therefore\quad (e^x-3)(e^x+2) &\geqslant 0 \\ \text{Now}\quad (e^x-3)(e^x+2) &= 0 \\ \text{when}\quad e^x = 3 \quad\text{or}\quad e^x &= -2 \end{aligned}$$

i.e., when $x = \ln 3$ as $e^x > 0$

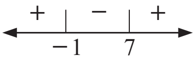


$\therefore\quad e^{2x} + 2e^x \geqslant 6 + 3e^x \quad$ when $x \geqslant \ln 3$

EXERCISE 8G.4

1 **a**

$$\begin{aligned} |x-3| &\leqslant 4 \\ \therefore\quad (x-3)^2 &\leqslant 4^2 \\ \therefore\quad (x-3)^2 - 4^2 &\leqslant 0 \\ \therefore\quad (x-3+4)(x-3-4) &\leqslant 0 \\ \therefore\quad (x+1)(x-7) &\leqslant 0 \end{aligned}$$

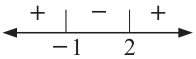


$\therefore\quad -1 \leqslant x \leqslant 7$

i.e., $x \in [-1, 7]$

b

$$\begin{aligned} |2x-1| &\leqslant 3 \\ \therefore\quad (2x-1)^2 &\leqslant 3^2 \\ \therefore\quad (2x-1)^2 - 3^2 &\leqslant 0 \\ \therefore\quad (2x-1+3)(2x-1-3) &\leqslant 0 \\ \therefore\quad (2x+2)(2x-4) &\leqslant 0 \end{aligned}$$

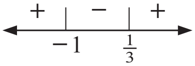


$\therefore\quad -1 \leqslant x \leqslant 2$

i.e., $x \in [\,-1, 2\,]$

c

$$\begin{aligned} |3x+1| &> 2 \\ \therefore\quad (3x+1)^2 &> 2^2 \\ \therefore\quad (3x+1)^2-2^2 &> 0 \\ \therefore\quad (3x+1+2)(3x+1-2) &> 0 \\ \therefore\quad (3x+3)(3x-1) &> 0 \end{aligned}$$

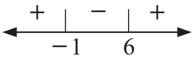


$$\therefore\quad x < -1 \quad\text{or}\quad x > \tfrac{1}{3}$$

i.e., $x \in\,]-\infty, -1[$ or $]\tfrac{1}{3}, \infty[$

d

$$\begin{aligned} |5-2x| &\geqslant 7 \\ \therefore\quad (5-2x)^2 &\geqslant 7^2 \\ \therefore\quad (5-2x)^2-7^2 &\geqslant 0 \\ \therefore\quad (5-2x+7)(5-2x-7) &\geqslant 0 \\ \therefore\quad (-2x+12)(-2x-2) &\geqslant 0 \end{aligned}$$

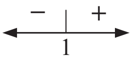


$$\therefore\quad x \leqslant -1 \quad\text{or}\quad x \geqslant 6$$

$\therefore\quad x \in\,]-\infty, -1]$ or $[6, \infty[$

e

$$\begin{aligned} |x| &\geqslant |2-x| \\ \therefore\quad x^2 &\geqslant (2-x)^2 \\ \therefore\quad x^2-(2-x)^2 &\geqslant 0 \\ \therefore\quad [x+(2-x)][x-(2-x)] &\geqslant 0 \\ \therefore\quad 2(2x-2) &\geqslant 0 \end{aligned}$$

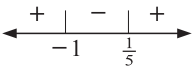


$$\therefore\quad x \geqslant 1$$

i.e., $x \in [1, \infty[$

f

$$\begin{aligned} 3|x| &\leqslant |1-2x| \\ \therefore\quad 9|x|^2 &\leqslant (1-2x)^2 \\ \therefore\quad 9x^2-(1-2x)^2 &\leqslant 0 \\ \therefore\quad (3x+[1-2x])(3x-[1-2x]) &\leqslant 0 \\ \therefore\quad (x+1)(5x-1) &\leqslant 0 \end{aligned}$$

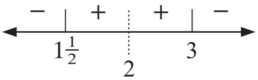


$$\therefore\quad -1 \leqslant x \leqslant \tfrac{1}{5}$$

i.e., $x \in [-1, \tfrac{1}{5}]$

g

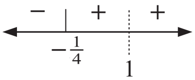
$$\begin{aligned} \left|\frac{x}{x-2}\right| &\geqslant 3 \\ \therefore\quad \left(\frac{x}{x-2}\right)^2 &\geqslant 3^2 \\ \therefore\quad \left(\frac{x}{x-2}\right)^2 - 3^2 &\geqslant 0 \\ \therefore\quad \left(\frac{x}{x-2}+3\right)\left(\frac{x}{x-2}-3\right) &\geqslant 0 \\ \therefore\quad \left(\frac{x+3x-6}{x-2}\right)\left(\frac{x-3x+6}{x-2}\right) &\geqslant 0 \\ \therefore\quad \frac{(4x-6)(-2x+6)}{(x-2)^2} &\geqslant 0 \end{aligned}$$



$\therefore\quad x \in [1\tfrac{1}{2}, 3]$, but $x \neq 2$

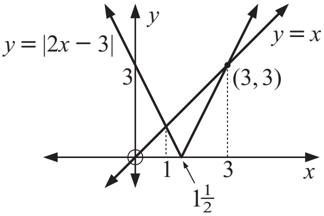
h

$$\begin{aligned} \left|\frac{2x+3}{x-1}\right| &\geqslant 2 \\ \therefore\quad \left(\frac{2x+3}{x-1}\right)^2 &\geqslant 2^2 \\ \therefore\quad \left(\frac{2x+3}{x-1}\right)^2 - 2^2 &\geqslant 0 \\ \therefore\quad \left(\frac{2x+3}{x-1}+2\right)\left(\frac{2x+3}{x-1}-2\right) &\geqslant 0 \\ \therefore\quad \left(\frac{2x+3+2x-2}{x-1}\right)\left(\frac{2x+3-2x+2}{x-1}\right) &\geqslant 0 \\ \therefore\quad \frac{(4x+1)(5)}{(x-1)^2} &\geqslant 0 \end{aligned}$$



$\therefore\quad x \in [-\tfrac{1}{4}, \infty[$, but $x \neq 1$

2 **a**

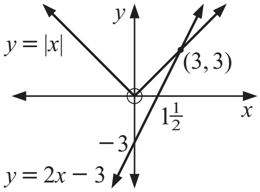


$|2x-3| < x$ when the modulus graph is below the line.

i.e., $1 < x < 3$

So, $x \in\,]1, 3[$

b

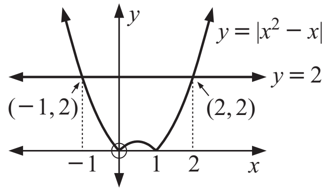


$2x-3 < |x|$ where the line is below the modulus graph.

$\therefore\quad x < 3$

So, $x \in\,]-\infty, 3[$

c

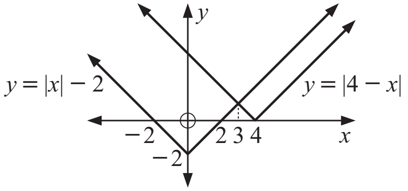


$\left|x^2 - x\right| > 2$ when the modulus graph is above the line.

$\therefore \quad x < -1 \quad$ or $\quad x > 2$

So, $\quad x \in \,]\, -\infty, -1 \,[\quad$ or $\quad]\, 2, \infty \,[$

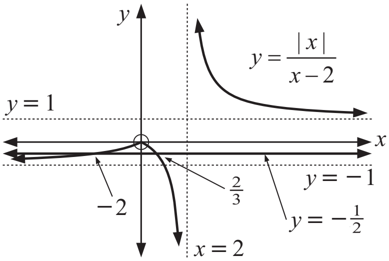
d



$|x| - 2 \geqslant |4 - x|$ when the graph of $y = |x| - 2$ is above or on the graph of $y = |4 - x|$.

$\therefore \quad x \geqslant 3 \quad$ So, $\quad x \in [\, 3, \infty \,[$

3



If $\quad \dfrac{|x|}{x-2} \geqslant -\frac{1}{2}$

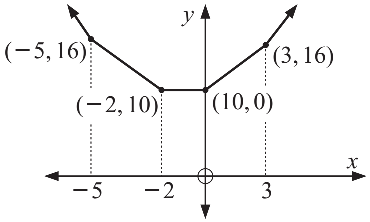
then the graph of $\quad y = f(x) \quad$ is above or on $y = -\frac{1}{2}$.

They intersect at -2 and $\frac{2}{3}$

$\therefore \quad -2 \leqslant x \leqslant \frac{2}{3} \quad$ or $\quad x > 2$

i.e., $\quad x \in [\, -2, \frac{2}{3} \,] \quad$ or $\quad]\, 2, \infty \,[$.

4 **a**



b **i** Suppose x is anywhere on AB, then

XP $= |x - (-5)| = |x + 5|$

XQ $= |x - (-2)| = |x + 2|$

XO $= |x - 0| = |x|$

XR $= |x - 3|$

So total length is

$|x + 5| + |x + 2| + |x| + |x - 3|$

ii The minimum length is 10 km when $\quad -2 \leqslant x \leqslant 0 \quad$ i.e., anywhere between O and Q.

iii We need to graph $\quad y = |x + 5| + |x + 2| + |x| + |x - 3| + |x - 7|$

From technology, the minimum cable length is 17 km when $\quad x = 0 \quad$ i.e., at O.

REVIEW SET 8A

1 **a** $a + bi = 4 = 4 + 0i, \quad \therefore \quad a = 4, \quad b = 0$

b $(1 - 2i)(a + bi) = -5 - 10i$

$$\therefore \quad a + bi = \frac{-5 - 10i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i}$$

$$= \frac{-5 - 10i - 10i - 20i^2}{1 - 4i^2}$$

$$= \frac{15 - 20i}{5}$$

$$= 3 - 4i$$

$\therefore \quad a = 3 \quad b = -4$

c

$$(a + 2i)(1 + bi) = 17 - 19i$$

$$\therefore \quad a + 2i + abi + 2i^2 b = 17 - 19i$$

$$\therefore \quad (a - 2b) + i(ab + 2) = 17 - 19i$$

Equating real and imaginary parts,

$a - 2b = 17 \qquad$ and $\qquad ab + 2 = -19$

$\therefore \quad a = 2b + 17 \qquad\qquad ab = -21$

$$\therefore \quad b(2b + 17) = -21$$

$$\therefore \quad 2b^2 + 17b + 21 = 0$$

$$\therefore \quad (2b + 3)(b + 7) = 0$$

$$\therefore \quad b = -\tfrac{3}{2} \quad \text{or} \quad b = -7$$

When $\quad b = -7, \quad a = 3 \quad$ and when $\quad b = -\frac{3}{2}, \quad a = 14$

2 $z = 3 + i \qquad w = -2 - i$

a

$$\begin{aligned} & 2z - 3w \\ &= 2(3+i) - 3(-2-i) \\ &= 6 + 2i + 6 + 3i \\ &= 12 + 5i \end{aligned}$$

b

$$\begin{aligned} & \frac{z^*}{w} \\ &= \frac{3-i}{-2-i} \times \frac{-2+i}{-2+i} \\ &= \frac{-6+2i+3i-i^2}{4-i^2} \\ &= \frac{-5+5i}{5} \\ &= -1 + i \end{aligned}$$

c

$$\begin{aligned} & z^3 \\ &= (3+i)^3 \\ &= 3^3 + 3(3^2)(i) + 3(3)(i^2) + i^3 \\ &= 27 + 27i + 9i^2 - i \\ &= 27 - 9 + 26i \\ &= 18 + 26i \end{aligned}$$

3

$$\begin{aligned} \text{Let} \quad z &= a + bi, \quad w = c + di \\ \therefore \quad zw^* - z^*w &= (a+bi)(c-di) - (a-bi)(c+di) \\ &= ac - adi + bci - bdi^2 - ac - adi + bci + bdi^2 \\ &= 2bci - 2adi \\ &= 2i(bc - ad) \end{aligned}$$

which is purely imaginary if $bc - ad \neq 0$ and zero if $bc - ad = 0$.

4 **a**

$$\begin{aligned} & (3x^3 + 2x - 5)(4x - 3) \\ &= 12x^4 - 9x^3 + 8x^2 - 6x - 20x + 15 \\ &= 12x^4 - 9x^3 + 8x^2 - 26x + 15 \end{aligned}$$

b $(2x^2 - x + 3)^2 = 4x^4 - 4x^3 + 13x^2 - 6x + 9$

$$\begin{array}{rrrrr} & & 2 & -1 & 3 \\ & \times & 2 & -1 & 3 \\ \hline & & 6 & -3 & 9 \\ & -2 & 1 & -3 & \\ 4 & -2 & 6 & & \\ \hline 4 & -4 & 13 & -6 & 9 \\ \hline \end{array}$$

5 **a**

$$\begin{array}{r|rrrr} & & x^2 & -\;2x & +\;4 \\ \cline{2-5} x+2 & x^3 & +0x^2 & +0x & +0 \\ & x^3 & +2x^2 & & \\ \cline{2-3} & & -2x^2 & +0x & \\ & & -2x^2 & -4x & \\ \cline{3-4} & & & 4x & +0 \\ & & & 4x & +8 \\ \cline{4-5} & & & & -8 \end{array}$$

$$\therefore \quad \frac{x^3}{x+2} = x^2 - 2x + 4 - \frac{8}{x+2}$$

b $(x+2)(x+3) = x^2 + 5x + 6$

$$\begin{array}{r|rrrr} & & & x & -5 \\ \cline{2-5} x^2 + 5x + 6 & x^3 & +0x^2 & +0x & +0 \\ & x^3 & +5x^2 & +6x & \\ \cline{2-4} & & -5x^2 & -6x & +0 \\ & & -5x^2 & -25x & -30 \\ \cline{3-5} & & & 19x & +30 \end{array}$$

$$\therefore \quad \frac{x^3}{(x+2)(x+3)} = x - 5 + \frac{19x + 30}{(x+2)(x+3)}$$

6 The Remainder theorem:

"When a polynomial $P(x)$ is divided by $x - \alpha$ until a constant remainder R is obtained then $R = P(\alpha)$."

Proof

$$\begin{aligned} \text{From the division process,} \quad P(x) &= (x - \alpha)Q(x) + R \\ \text{Now if} \quad x = \alpha \qquad P(\alpha) &= (\alpha - \alpha) \times Q(\alpha) + R \\ \therefore \quad P(\alpha) &= 0 \times Q(\alpha) + R \\ \therefore \quad P(\alpha) &= R \\ \text{i.e.,} \quad R &= P(\alpha) \end{aligned}$$

7 Let $P(z) = z^2 + az + [3 + a]$

if $-2 + bi$ is a zero then

$$\begin{aligned} P(-2+bi) &= 0 \\ \therefore\quad (-2+bi)^2 + a(-2+bi) + 3 + a &= 0 \\ 4 - 4bi + b^2i^2 - 2a + abi + 3 + a &= 0 \\ (4 - b^2 - 2a + 3 + a) + i(-4b + ab) &= 0 \end{aligned}$$

$$\therefore\quad 4 - b^2 - 2a + 3 + a = 0 \qquad \text{and} \qquad -4b + ab = 0$$

$$a = 7 - b^2 \qquad \therefore\quad b(a-4) = 0$$

$$\therefore\quad b = 0 \quad \text{or} \quad a = 4$$

If $b = 0$ then $a = 7 - 0 = 7.$

If $a = 4$ then $b^2 = 3$ and so $b = \pm\sqrt{3}.$

8 Let $P(z) = 2z^4 - 5z^3 + 13z^2 - 4z - 6$

From a calculator 1 and $-\frac{1}{2}$ are zeros.

(*Check:* $P(1) = 2 - 5 + 13 - 4 - 6 = 0$

$P(-\frac{1}{2}) = \frac{1}{8} + \frac{5}{8} + \frac{13}{4} + 2 - 6 = 0)$

$$\begin{array}{r|rrrr|r} 1 & 2 & -5 & 13 & -4 & -6 \\ & 0 & 2 & -3 & 10 & 6 \\ \hline -\frac{1}{2} & 2 & -3 & 10 & 6 & 0 \\ & 0 & -1 & 2 & -6 & \\ \hline & 2 & -4 & 12 & 0 & \end{array}$$

$$\begin{aligned} \therefore\quad P(z) &= (z-1)(z+\tfrac{1}{2})(2z^2 - 4z + 12) \\ &= (z-1)(2z+1)(z^2 - 2z + 6) \end{aligned}$$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-24}}{2} = 1 \pm i\sqrt{5}.$ $\therefore$ zeros are $1, -\frac{1}{2}, 1 \pm i\sqrt{5}.$

9 Let $P(z) = z^4 + 2z^3 - 2z^2 + 8$

From a calculator -2 seems to be a double zero.

(*Check:* $P(-2) = 16 - 16 - 8 + 8 = 0$

$\therefore\quad P(z) = (z+2)^2(z^2 - 2z + 2))$

$$\begin{array}{r|rrrr|r} -2 & 1 & 2 & -2 & 0 & 8 \\ & 0 & -2 & 0 & 4 & -8 \\ \hline -2 & 1 & 0 & -2 & 4 & 0 \\ & 0 & -2 & 4 & -4 & \\ \hline & 1 & -2 & 2 & 0 & \end{array}$$

where the quadratic has zeros of $\dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore\quad P(z) = (z+2)^2(z - [1+i])(z - [1-i])$

10 $2 - i\sqrt{3}$ and $\sqrt{2} + 1$

Since the quartic has real rational coefficients, $2 - i\sqrt{3}$ and $2 + i\sqrt{3}$ are zeros

$\sqrt{2} + 1$ and $-\sqrt{2} + 1$ are zeros

$\therefore$ the four zeros are: $2 \pm i\sqrt{3}$ and $\pm\sqrt{2} + 1$

For $2 \pm i\sqrt{3}$, $\alpha + \beta = 4$, $\alpha\beta = 4 - 3i^2 = 7$

For $\pm\sqrt{2} + 1$, $\alpha + \beta = 2$, $\alpha\beta = -2 + 1 = -1$

$\therefore\quad P(x) = (x^2 - 4x + 7)(x^2 - 2x - 1)$

$\therefore\quad P(x) = x^4 - 6x^3 + 14x^2 - 10x - 7$

$$\begin{array}{rrrrr} & & 1 & -4 & 7 \\ & \times & 1 & -2 & -1 \\ \hline & & -1 & 4 & -7 \\ & -2 & 8 & -14 & \\ 1 & -4 & 7 & & \\ \hline 1 & -6 & 14 & -10 & -7 \\ \hline \end{array}$$

11
$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + b \quad \text{......}\ (1) \\ &= (x-k)^2(x+a) \\ &= (x^2 - 2kx + k^2)(x+a) \\ &= x^3 + [a - 2k]x^2 + [k^2 - 2ak]x + ak^2 \quad \text{......}\ (2) \end{aligned}$$

$$\begin{array}{rrrr} & 1 & -2k & k^2 \\ \times & & 1 & a \\ \hline & a & -2ak & ak^2 \\ 1 & -2k & k^2 & \\ \hline 1 & a-2k & k^2-2ak & ak^2 \\ \hline \end{array}$$

Equating coefficients of (1) and (2) gives

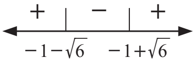
$\therefore\quad a - 2k = -3, \quad k^2 - 2ak = -9 \quad$ and $\quad ak^2 = b$

Since $a = 2k - 3$, then as
$$\begin{aligned} k^2 - 2ak &= -9 \\ k^2 - 2k(2k-3) &= -9 \\ k^2 - 4k^2 + 6k &= -9 \\ \therefore\ 3k^2 - 6k - 9 &= 0 \\ k^2 - 2k - 3 &= 0 \\ (k-3)(k+1) &= 0 \quad \therefore\ k = -1 \text{ or } k = 3 \end{aligned}$$

If $k = -1$, $a = -5$ and $b = ak^2 = -5$ and so $f(x) = (x+1)^2(x-5)$
and the roots of $f(x) = 0$ are k and $-a$, i.e., $-1, 5$

If $k = 3$, $a = 3$ and $b = ak^2 = 3 \times 9 = 27$ and so $f(x) = (x-3)^2(x+3)$
and the roots of $f(x) = 0$ are $3, -3$

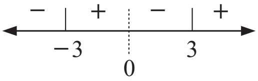
12 **a**
$$\begin{aligned} x^2 + 2x &\geqslant 5 \\ \therefore\ x^2 + 2x - 5 &\geqslant 0 \\ \therefore\ x^2 + 2x - 5 &= 0 \quad \text{when} \\ x &= \frac{-2 \pm \sqrt{4 - 4(1)(-5)}}{2} \\ &= \frac{-2 \pm 2\sqrt{6}}{2} \\ &= -1 \pm \sqrt{6} \end{aligned}$$



$\therefore\ x \leqslant -1 - \sqrt{6}$ or $x \geqslant -1 + \sqrt{6}$

i.e., $x \in\ ]-\infty, -1 - \sqrt{6}\,]$ or
$x \in [\,-1 + \sqrt{6}, \infty\,[$

b
$$\begin{aligned} x &< \frac{9}{x} \\ \therefore\ x - \frac{9}{x} &< 0 \\ \therefore\ \frac{x^2 - 9}{x} &< 0 \\ \therefore\ \frac{(x+3)(x-3)}{x} &< 0 \end{aligned}$$

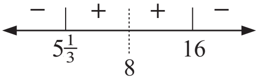


$\therefore\ x < -3$ or $0 < x < 3$

$\therefore\ x \in\ ]-\infty, -3\,[$
or $x \in\ ]\,0, 3\,[$

c
$$\begin{aligned} \left|\frac{x}{8-x}\right| &\leqslant 2 \\ \therefore\ \left(\frac{x}{8-x}\right)^2 &\leqslant 2^2 \\ \therefore\ \left(\frac{x}{8-x}\right)^2 - 2^2 &\leqslant 0 \\ \therefore\ \left(\frac{x}{8-x} + 2\right)\left(\frac{x}{8-x} - 2\right) &\leqslant 0 \end{aligned}$$

$$\therefore\ \frac{(16-x)(3x-16)}{(8-x)^2} \leqslant 0$$



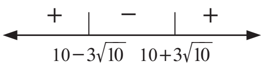
$\therefore\ x \leqslant 5\frac{1}{3}$ or $x \geqslant 16$

i.e., $x \in\ ]-\infty, 5\frac{1}{3}\,]$ or $x \in [\,16, \infty\,[$

13 They meet where
$$\begin{aligned} x^2 + (2x+k)^2 + 8x - 4(2x+k) + 2 &= 0 \\ \therefore\ x^2 + 4x^2 + 4kx + k^2 + 8x - 8x - 4k + 2 &= 0 \\ \text{i.e., } 5x^2 + 4kx + [k^2 - 4k + 2] &= 0 \end{aligned}$$

Now
$$\begin{aligned} \Delta &= (4k)^2 - 4(5)(k^2 - 4k + 2) \\ &= 16k^2 - 20k^2 + 80k - 40 \\ &= -4k^2 + 80k - 40 \end{aligned}$$

and $\Delta < 0$ when
$$\begin{aligned} -4k^2 + 80k - 40 &< 0 \\ \text{i.e., } k^2 - 20k + 10 &> 0 \\ \text{i.e., } k^2 - 20k + 10^2 + 10 - 10^2 &> 0 \\ (k-10)^2 - 90 &> 0 \\ \therefore\ [k - 10 + 3\sqrt{10}][k - 10 - 3\sqrt{10}] &> 0 \end{aligned}$$



$\therefore\ k \in\ ]-\infty, 10 - 3\sqrt{10}\,[$ or $k \in\ ]\,10 + 3\sqrt{10}, \infty\,[$

14 $P(x) = x^n + 3x^2 + kx + 6$

$P(-1) = 12$ {Remainder Theorem} $\therefore\ 12 = (-1)^n + 3 - k + 6$

$\therefore\ k = (-1)^n - 3$ ...... (1)

$P(1) = 8$ {Remainder Theorem} $\therefore\ 8 = 1^n + 3 + k + 6$

$\therefore\ 8 = 10 + k$

$\therefore\ k = -2$

$\therefore$ (1) becomes $-2 = (-1)^n - 3$

$(-1)^n = 1$ $\therefore$ n is even

If $34 < n < 38$, then $n = 36$.

15 Let
$$\begin{aligned} x^3 - x + 1 &= (x-\alpha)(x-\beta)(x-\gamma) \\ &= (x^2 - [\alpha+\beta]x + \alpha\beta)(x-\gamma) \\ &= x^3 - [\alpha+\beta+\gamma]x^2 + [\alpha\beta+\beta\gamma+\alpha\gamma] - \alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \alpha\beta+\beta\gamma+\alpha\gamma &= -1 \\ \therefore\ \alpha\beta\gamma &= -1 \end{aligned}$$

Now $\gamma = \dfrac{-1}{\alpha\beta}$ ...... (1) and

$$\begin{aligned} \alpha\beta + \gamma(\alpha+\beta) &= -1 \text{ ..... (2)} \\ \therefore\ \alpha\beta + \gamma(-\gamma) &= -1 \\ \therefore\ \alpha\beta - \gamma^2 &= -1 \\ \therefore\ \alpha\beta - \frac{1}{(\alpha\beta)^2} &= -1 \quad \{\text{using (1)}\} \\ \therefore\ (\alpha\beta)^3 - 1 &= -(\alpha\beta)^2 \\ \therefore\ (\alpha\beta)^3 + (\alpha\beta)^2 - 1 &= 0 \end{aligned}$$

$\therefore\ \alpha\beta$ is a root of $x^3 + x^2 - 1 = 0$

REVIEW SET 8B

1 If $z = a + bi$

then
$$\begin{aligned} z + z^* &= (a+bi) + (a-bi) \\ &= 2a \quad \text{which is real.} \end{aligned}$$

Also,
$$\begin{aligned} zz^* &= (a+bi)(a-bi) \\ &= a^2 + abi - abi - b^2i^2 \\ &= a^2 + b^2 \quad \text{which is real.} \end{aligned}$$

2 $z = 4 + i$ $w = 3 - 2i$
$$\begin{aligned} 2w^* - iz &= 2(3+2i) - i(4+i) \\ &= 6 + 4i - 4i - i^2 \\ &= 7 \end{aligned}$$

3 If $\dfrac{2-3i}{2a+bi} = 3 + 2i$ then $2a + bi = \dfrac{2-3i}{3+2i} \times \dfrac{3-2i}{3-2i}$

$$\therefore\ 2a + bi = \frac{6 - 4i - 9i + 6i^2}{9 - 4i^2} = \frac{-13i}{13}$$

$\therefore\ 2a + bi = 0 - i$ $\therefore\ a = 0$ and $b = -1$

4 If $a + ai$ is a root of $x^2 - 6x + b = 0$ then
$$\begin{aligned} (a+ai)^2 - 6(a+ai) + b &= 0 \\ \therefore\ a^2 + 2a^2i - a^2 - 6a - 6ai + b &= 0 \\ \therefore\ (b - 6a) + (2a^2 - 6a)i &= 0 \\ \therefore\ b = 6a \text{ and } 2a^2 - 6a &= 0 \\ \therefore\ b = 6a \text{ and } 2a(a-3) &= 0 \end{aligned}$$

$\therefore\ a = 0$ or 3, and when $a = 0$, $b = 0$

and when $a = 3$, $b = 18$

5 Let $P(x) = x^{47} - 3x^{26} + 5x^3 + 11$

$$\begin{aligned} \therefore \quad R &= P(-1) \quad \{\text{Remainder theorem}\} \\ &= (-1)^{47} - 3(-1)^{26} + 5(-1)^3 + 11 \\ &= -1 - 3 - 5 + 11 \\ \therefore \quad \text{remainder} &= 2 \end{aligned}$$

6 Touches the x-axis at $(-2, 0)$ and cuts it at $(1, 0)$.

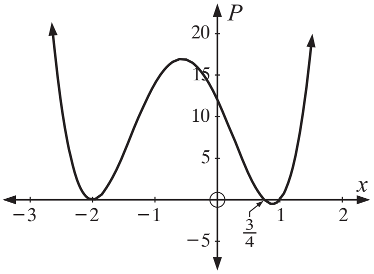
$\therefore \quad P(x) = (x+2)^2(x-1)(ax+b)$

But $P(0) = 12 \quad \therefore \quad 4(-1)b = 12 \quad \therefore \quad b = -3$

$\therefore \quad P(x) = (x+2)^2(x-1)(ax-3)$

Also $P(2) = 80 \quad \therefore \quad 80 = 16(1)(2a-3)$

$$\begin{aligned} \therefore \quad 2a - 3 &= 5 \\ \therefore \quad 2a &= 8 \\ \therefore \quad a &= 4 \\ \therefore \quad P(x) &= (x+2)^2(x-1)(4x-3) \end{aligned}$$



7 As $(x-3)(x+2) = x^2 - x - 6$,

$= Q(x)[x^2 - x - 6] + (Ax + B)$, where $Q(x)$ is the quotient and $Ax + B$ is the remainder.

Now $P(x)$ has remainder 2 when divided by $x - 3$ and so $P(3) = 2$ {Rem.Theorem}

$$\begin{aligned} \therefore \quad Q(3)[9 - 3 - 6] + (3A + B) &= 2 \\ \therefore \quad 3A + B &= 2 \quad \text{...... (1)} \end{aligned}$$

Also $P(x)$ has remainder -13 when divided by $x + 2$ and so $P(-2) = -13$

$$\begin{aligned} \therefore \quad Q(-2)[4 + 2 - 6] + [-2A + B] &= -13 \\ \therefore \quad -2A + B &= -13 \quad \text{...... (2)} \end{aligned}$$

Solving (1) and (2): $5A = 15$

$$\begin{aligned} \therefore \quad A &= 3 \quad \text{and} \quad B = -7 \\ \therefore \quad R(x) &= 3x - 7 \end{aligned}$$

8 Since the coefficients are rational, $3 + i\sqrt{2}$ and $1 + \sqrt{2}$ also have to be zeros.

For zeros of $3 \pm i\sqrt{2}$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - 2i^2 = 11$

For zeros of $1 \pm \sqrt{2}$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$\therefore \quad P(x) = a(x^2 - 6x + 11)(x^2 - 2x - 1), \quad a \neq 0$

$\therefore \quad P(x) = a(x^4 - 8x^3 + 22x^2 - 16x - 11), \quad a \neq 0$

$$\begin{array}{rrrrr} & & 1 & -6 & 11 \\ & \times & 1 & -2 & -1 \\ \hline & & -1 & 6 & -11 \\ & -2 & 12 & -22 & \\ 1 & -6 & 11 & & \\ \hline 1 & -8 & 22 & -16 & -11 \\ \hline \end{array}$$

9 $P(z) = 2z^3 + z^2 + 10z + 5$

From a calculator, $-\frac{1}{2}$ is a zero.

Check: $P(-\frac{1}{2}) = -\frac{1}{4} + \frac{1}{4} - 5 + 5 = 0$

$$\begin{array}{r|rrr|r} -\frac{1}{2} & 2 & 1 & 10 & 5 \\ & 0 & -1 & 0 & -5 \\ \hline & 2 & 0 & 10 & 0 \end{array}$$

$$\begin{aligned} \therefore \quad P(z) &= (z + \tfrac{1}{2})(2z^2 + 10) \\ P(z) &= (2z + 1)(z^2 + 5) \\ P(z) &= (2z + 1)(z - i\sqrt{5})(z + i\sqrt{5}) \end{aligned}$$

10 Zeros are $2 + i$ and $-1 + 3i$.

Since we have a real polynomial, the other zeros are $2 - i$ and $-1 - 3i$.

For zeros of $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

For zeros of $-1 \pm 3i$, $\alpha + \beta = -2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore \quad P(x) = a(x^2 - 4x + 5)(x^2 + 2x + 10), \quad a \neq 0$

REVIEW SET 8C

1 Since $3-2i$ is a zero, so is $3+2i$. These have $\alpha+\beta=6$ and $\alpha\beta=9-4i^2=13$.

$\therefore\quad z^2-6z+13$ is a factor

$$\begin{aligned}\therefore\quad P(z)&=(z^2-6z+13)(z^2+Az+B)\\&=z^4+kz^3+32z+3k-1\end{aligned}$$

$$\begin{array}{ccccc} & & 1 & -6 & 13\\ & \times & 1 & A & B\\ \hline & & B & -6B & 13B\\ & A & -6A & 13A & \\ 1 & -6 & 13 & & \\ \hline 1 & A-6 & B-6A+13 & 13A-6B & 13B\\ \hline\end{array}$$

Equating coefficients gives

$$\begin{aligned}A-6&=k\\B-6A+13&=0\\13A-6B&=32\\3k-1&=13B\end{aligned}$$

$$\begin{aligned}-6A+B&=-13 \quad\text{......}\ (1)\\13A-6B&=32 \quad\ \ \text{......}\ (2)\\(1)\times 6\qquad -36A+6B&=-78 \quad\text{......}\ (3)\end{aligned}$$

Adding (2) and (3) gives: $-23A=-46 \quad\therefore\quad A=2$

$\therefore\quad k=A-6=2-6=-4$ and $B=6A-13=12-13=-1$

$\therefore\quad P(z)=(z^2-6z+13)(z^2+2z-1)$

$$\uparrow$$

this quadratic has zeros $\dfrac{-2\pm\sqrt{4+4}}{2}=-1\pm\sqrt{2}$

$\therefore$ zeros are $3\pm 2i,\quad -1\pm\sqrt{2}$

2 Consider $P(z)=z^4+2z^3+6z^2+8z+8$.

If one zero is purely imaginary, for example, ai, then $-ai$ is also a zero (a is real).

$\pm ai$ have $\alpha+\beta=0$ and $\alpha\beta=-a^2i^2=a^2$

$\therefore\quad z^2+a^2$ is a factor, i.e., z^2+A is a factor

$$\begin{aligned}\therefore\quad P(z)&=(z^2+A)(z^2+Bz+C)\\&=z^4+Bz^3+[A+C]z^2+ABz+AC\end{aligned}$$

$$\begin{array}{ccccc} & & 1 & B & C\\ & \times & 1 & 0 & A\\ \hline & & A & AB & AC\\ 1 & B & C & & \\ \hline 1 & B & A+C & AB & AC\\ \hline\end{array}$$

Equating coefficients gives:

$B=2,\quad A+C=6,\quad AB=8$ and $AC=8$

$\therefore\quad B=2 \qquad\therefore\quad A=4 \qquad\therefore\quad C=2$

$\therefore\quad P(z)=(z^2+4)(z^2+2z+2)$

$$\downarrow$$

has zeros of $\dfrac{-2\pm\sqrt{4-8}}{2}=-1\pm i$

$\therefore$ zeros are $\pm 2i,\ -1\pm i$

3 $$\frac{P(x)}{x^2-3x+2}=Q(x)+\frac{2x+3}{x^2-3x+2}$$

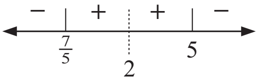
$$\begin{aligned}\therefore\quad P(x)&=Q(x)(x^2-3x+2)+2x+3\\\text{and}\quad P(2)&=Q(2)[4-6+2]+4+3\\&=Q(2)\times[0]+7\\&=7,\quad\text{so the remainder is 7.}\end{aligned}$$

4 $$\begin{aligned}\left|\frac{2x-1}{2-x}\right|&\geqslant 3\\\therefore\quad\left(\frac{2x-1}{2-x}\right)^2&\geqslant 3^2\\\therefore\quad\left(\frac{2x-1}{2-x}\right)^2-3^2&\geqslant 0\end{aligned}$$

$$\therefore \quad \left(\frac{2x-1}{2-x}+3\right)\left(\frac{2x-1}{2-x}-3\right) \geqslant 0$$

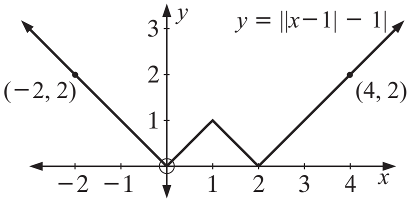
$$\therefore \quad \left(\frac{2x-1+6-3x}{2-x}\right)\left(\frac{2x-1-6+3x}{2-x}\right) \geqslant 0$$

$$\therefore \quad \frac{(5-x)(5x-7)}{(2-x)^2} \geqslant 0$$



$$\therefore \quad x \in \left[\,\tfrac{7}{5}, 5\,\right], \quad x \neq 2$$

5 **a**

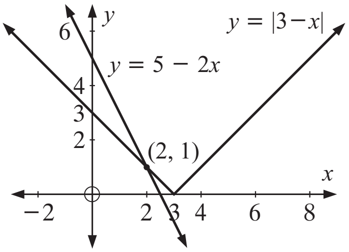


b $||x-1|-1| < 2$

for $-2 \leqslant x \leqslant 4$

i.e., $x \in [-2, 4]$

6 **a**



b

$$|3-x| > 5-2x$$

If $x \leqslant 3$, $3-x > 0$

$\therefore \quad 3-x > 5-2x$

$\therefore \quad x > 2$

If $x > 3$, $3-x < 0$

then $-(3-x) > 5-2x$

$\therefore \quad -3+x > 5-2x$

$\therefore \quad 3x > 8$

$\therefore$ over all $x > \frac{8}{3}$ which is true

{as $|3-x| > 5-2x$ when $x > 2$}

$\therefore$ over all $x > 2$

7 **a** If $x+iy=0$ then $x=0$ and $y=0$ {equating real and imaginary parts}

b

$$(3-2i)(x+i) = 17+yi$$

$$3x+3i-2xi-2i^2 = 17+yi$$

$$(3x+2)+i(3-2x) = 17+yi$$

$\therefore \quad 3x+2 = 17$ and $y = 3-2x$

$\therefore \quad 3x = 15$ and so $\therefore \quad y = 3-10$

$\therefore \quad x = 5$ $\qquad \therefore \quad y = -7$

c

$$(x+iy)^2 = x-iy$$

$$x^2+i^2y^2+2xyi = x-iy$$

$x^2-y^2 = x$ and $2xy = -y$

Now if $2xy+y = 0$

then $y(2x+1) = 0$

$\therefore \quad y = 0$ or $x = -\frac{1}{2}$

If $y=0$, then $x^2=x$ and so $x = 0$ or 1

If $x = -\frac{1}{2}$, then $\frac{1}{4}-y^2 = -\frac{1}{2}$, $\quad \therefore \quad y^2 = \frac{3}{4}$

and so $y = \pm\frac{\sqrt{3}}{2}$

Possible solutions are:

x	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
y	0	0	$\frac{\sqrt{3}}{2}$	$\frac{-\sqrt{3}}{2}$

8 Let $z = a + bi$ and $w = c + di$ where $b \neq 0$ and $d \neq 0$ ...... (1)

Now $z + w = (a + c) + (b + d)i$ and
$$\begin{aligned} zw &= (a + bi)(c + di) \\ &= (ac - bd) + i(bc + ad) \end{aligned}$$

As $z + w$ is real, $b + d = 0$ $\quad \therefore \quad b = -d$ ...... (2)

As zw is real, $bc + ad = 0$ ...... (3)

Substituting (2) into (3)
$$\begin{aligned} -dc + ad &= 0 \\ \therefore \quad d(a - c) &= 0 \end{aligned}$$

Since $d \neq 0$ {from (1)}

$\therefore \quad a = c$ and $b = -d$

$\therefore \quad z^* = a - bi = c + di = w$

9 If $\sqrt{z} = \dfrac{2}{3 - 2i} + 2 + 5i$, using technology $z = \dfrac{-3737}{169} + \dfrac{4416}{169}i$.

10 Let $P(x) = 2x^{17} + 5x^{10} - 7x^3 + 6$

Now
$$\begin{aligned} R &= P(2) \qquad \text{\{Remainder theorem\}} \\ &= 2^{18} + 5 \times 2^{10} - 7 \times 2^3 + 6 \\ &= 267\,214 \end{aligned}$$

REVIEW SET 8D

1 Let $P(z) = 2z^3 + az^2 + 62z + [a - 5]$

If $5 - i$ is a zero, then so is $5 + i$ {as $P(z)$ is a real polynomial}

and for these zeros
$$\begin{aligned} \alpha + \beta &= 10 \\ \alpha\beta &= 25 - i^2 = 26 \end{aligned}$$

$\therefore \quad P(z) = (z^2 - 10z + 26)(2z + b)$

	1	-10	26
$\times$		2	b
	b	$-10b$	$26b$
2	-20	52	
2	$b - 20$	$52 - 10b$	$26b$

Equating coefficients gives:

$a = b - 20$, $52 - 10b = 62$ and $a - 5 = 26b$

From
$$\begin{aligned} 52 - 10b &= 62 \\ -10b &= 10 \\ \therefore \quad b &= -1 \end{aligned}$$

and as $a = b - 20$ then $a = -21$

(*Check:* $a - 5 = -26$ and $26b = -26$ ✓)

$$\begin{aligned} \therefore \quad P(z) &= 2z^3 - 21z^2 + 62z - 26 \\ &= (z^2 - 10z + 26)(2z - 1) \end{aligned}$$

$\therefore$ other two zeros are $\frac{1}{2}$, $5 + i$ and $a = -21$

2 **a** zeros are $i\sqrt{2}$ and $\frac{1}{2}$, $\therefore$ another zero must be $-i\sqrt{2}$

Now $\pm i\sqrt{2}$ have $\alpha + \beta = 0$ and $\alpha\beta = -2i^2 = 2$

$\therefore \quad x^2 + 2$ is a factor

$\therefore \quad P(x) = k(2x - 1)(x^2 + 2), \quad k \neq 0$

b zeros are $1 - i$ and $-3 - i$

$\therefore$ other zeros must be $1 + i$ and $-3 + i$

$1 \pm i$ have $\alpha + \beta = 2$ and $\alpha\beta = 1 - i^2 = 2$ and $-3 \pm i$ have $\alpha + \beta = -6$ and $\alpha\beta = 9 - i^2 = 10$

$\therefore \quad P(x) = k(x^2 - 2x + 2)(x^2 + 6x + 10), \quad k \neq 0$

3
$$\begin{aligned} P(x) &= 2x^3+7x^2+kx-k \quad \text{......} \; (1)\\ &= (x+a)^2(2x+b)\\ &= (x^2+2ax+a^2)(2x+b)\\ &= x^3[b+4a]x^2+[2ab+2a^2]x+a^2b \quad \text{......} \; (2) \end{aligned}$$

$$\begin{array}{cccc} & 1 & 2a & a^2 \\ \times & & 2 & b \\ \hline & b & 2ab & a^2b \\ 2 & 4a & 2a^2 & \\ \hline 2 & b+4a & 2ab+2a^2 & a^2b \\ \hline \end{array}$$

Equating coefficients gives in (1) and (2):

$b+4a=7, \quad 2ab+2a^2=k \quad$ and $\quad a^2b=-k$

$$\begin{aligned} \therefore \quad 2ab+2a^2 &= -a^2b \qquad \{\text{equating } k\text{'s}\}\\ \therefore \quad 2a(7-4a)+2a^2 &= -a^2(7-4a)\\ \therefore \quad 14a-8a^2+2a^2+7a^2-4a^3 &= 0\\ \therefore \quad 4a^3-a^2-14a &= 0\\ \therefore \quad a(4a^2-a-14) &= 0\\ \therefore \quad a(4a+7)(a-2) &= 0\\ \therefore \quad a &= 0, -\tfrac{7}{4} \text{ or } 2 \end{aligned}$$

If $\quad a=0, \quad b=7 \quad$ and $\quad k=0.$

If $\quad a=2, \quad b=-1 \quad$ and $\quad k=4.$

If $\quad a=-\frac{7}{4}, \quad b=14 \quad$ and $\quad k=-\frac{343}{8}$

$\therefore$ largest value of k is $k=4$ and $P(x)=(x+2)^2(2x-1).$

4
$$\begin{aligned} 2z^4-3z^3+2z^2 &= 6z+4\\ \therefore \quad 2z^4-3z^3+2z^2-6z-4 &= 0 \end{aligned}$$

From a calculator we see that 2 and $-\frac{1}{2}$ are solutions

(*Check:* $\quad P(2)=32-24+8-12-4 = 0$

$P(-\frac{1}{2})=\frac{1}{8}+\frac{3}{8}+\frac{1}{2}+3-4 = 0$)

$$\begin{array}{c|ccccc|} 2 & 2 & -3 & 2 & -6 & -4 \\ & 0 & 4 & 2 & 8 & 4 \\ \hline -\frac{1}{2} & 2 & 1 & 4 & 2 & 0 \\ & 0 & -1 & 0 & -2 & \\ \hline & 2 & 0 & 4 & 0 & \end{array}$$

$$\begin{aligned} \therefore \quad P(z) &= (z-2)(z+\tfrac{1}{2})(2z^2+4)\\ &= (z-2)(2z+1)(z^2+2) \qquad \therefore \text{ roots are } \; 2, -\tfrac{1}{2}, \pm i\sqrt{2} \end{aligned}$$

5 $P(z)=z^3+az^2+kz+ka$

$P(-a)=-a^3+a^3-ka+ka=0$

$\therefore \quad z+a$ is a factor

$$\begin{array}{c|cccc|} -a & 1 & a & k & ka \\ & 0 & -a & 0 & -ka \\ \hline & 1 & 0 & k & 0 \end{array}$$

$\therefore \quad P(z)=(z+a)(z^2+k)$

a $\therefore \quad P(z)=0 \quad$ has one real root if $\quad k>0 \quad a\in R,$

b and 3 real roots if $\quad k\leqslant 0 \quad a\in R$

or
$$\begin{aligned} P(z) &= z^3+az^2+kz+ka\\ &= z^2(z+a)+k(z+a)\\ &= (z+a)(z^2+k), \quad \text{etc} \end{aligned}$$

6 Let $\quad P(x)=6x^3+ax^2-4ax+b$

$3x+2 \quad$ and $\quad x-2 \quad$ are factors

$$\begin{aligned} \therefore \quad 6x^3+ax^2-4ax+b &= (3x+2)(x-2)(2x+c)\\ &= (3x^2-4x-4)(2x+c)\\ &= 6x^3-8x^2-8x+3cx^2-4cx-4c\\ &= 6x^3+[3c-8]x^2+[-8-4c]-4c \end{aligned}$$

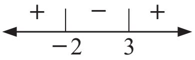
Equating coefficients gives: $\quad a=3c-8, \quad -4a=-8-4c \quad$ and $\quad b=-4c$

Equating a's gives:
$$\begin{aligned} 3c-8 &= 2+c\\ \therefore \quad 2c &= 10\\ \therefore \quad c &= 5 \end{aligned}$$

Consequently, $\; a=3(5)-8=7 \quad$ and $\quad b=-20.$

7 **a**

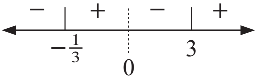
$$\begin{aligned} |1-2x| &\leqslant 5 \\ \therefore\quad (1-2x)^2 &\leqslant 5^2 \\ \therefore\quad (1-2x)^2-5^2 &\leqslant 0 \\ \therefore\quad [(1-2x)+5]\,[1-2x-5] &\leqslant 0 \\ \therefore\quad [6-2x]\,[-2x-4] &\leqslant 0 \end{aligned}$$



$$\therefore\quad -2 \leqslant x \leqslant 3$$
i.e., $x \in [-2, 3]$

b

$$\begin{aligned} \frac{x^2-1}{x} &\geqslant \frac{8}{3} \\ \therefore\quad \frac{x^2-1}{x}-\frac{8}{3} &\geqslant 0 \\ \therefore\quad \frac{3(x^2-1)-8x}{3x} &\geqslant 0 \\ \therefore\quad \frac{3x^2-8x-3}{3x} &\geqslant 0 \\ \therefore\quad \frac{(3x+1)(x-3)}{3x} &\geqslant 0 \end{aligned}$$



$$\therefore\quad -\tfrac{1}{3} \leqslant x < 0 \quad\text{or}\quad x \geqslant 3$$
i.e., $x \in [-\frac{1}{3}, 0[$ or $[3, \infty[$

8 $y = x-k$ meets $(x-2)^2+(y+3)^2=4$ where

$$\begin{aligned} (x-2)^2+(x-k+3)^2 &= 4 \\ \therefore\quad x^2-4x+4+(x-k)^2+6(x-k)+9-4 &= 0 \\ \therefore\quad x^2-4x+4+x^2-2kx+k^2+6x-6k+5 &= 0 \\ \therefore\quad 2x^2+[-4-2k+6]\,x+\left[4+k^2-6k+5\right] &= 0 \\ \text{i.e.,}\quad 2x^2+[2-2k]x+[k^2-6k+9] &= 0 \end{aligned}$$

which is a quadratic in x and in the tangent case $\Delta = 0$

$$\begin{aligned} \therefore\quad [2-2k]^2-4(2)(k^2-6k+9) &= 0 \\ \therefore\quad 4-8k+4k^2-8k^2+48k-72 &= 0 \\ \therefore\quad -4k^2+40k-68 &= 0 \\ \text{i.e.,}\quad k^2-10k+17 &= 0 \\ \therefore\quad k &= 5 \pm 2\sqrt{2} \end{aligned}$$

9

$$\begin{array}{r|rrrrr} & & & x^2 & +\,3x & -\,9 \\ \hline x^2+2 & x^4 & +3x^3 & -7x^2 & +11x & -1 \\ & x^4 & & +2x^2 & & \\ \hline & & 3x^3 & -9x^2 & +11x & \\ & & 3x^3 & & +6x & \\ \hline & & & -9x^2 & +5x & -1 \\ & & & -9x^2 & & -18 \\ \hline & & & & 5x & +17 \end{array}$$

$\therefore\quad Q(x) = x^2+3x-9 \qquad R(x) = 5x+17$ and the new function would be divisible by x^2+2 if $x^4+3x^3-7x^2+(2+a)x+b = P(x)-R(x)$

$$\begin{aligned} \text{and as}\quad P(x)-R(x) &= x^4+3x^3-7x^2+11x-1-(5x+17) \\ &= x^4+3x^3-7x^2+6x-18 \end{aligned}$$

then $2+a = 6$ and $b=-18$ {equating coefficients}

$\therefore\quad a = 4$ and $b = -18$

Chapter 9

COUNTING AND BINOMIAL THEOREM

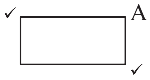
EXERCISE 9A

1 There are 3 paths from P to Q
4 paths from Q to R
2 paths from R to S

$\therefore$ number of routes possible
$= 3 \times 4 \times 2$ {product principle}
$= 24$

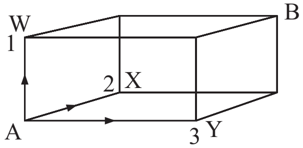
2 **a** There are 4 choices for A. But once A is located, there is 1 choice for B, 1 for C and 1 for D
$\therefore$ there are $4 \times 1 \times 1 \times 1 = 4$ ways.

b There are 4 choices for A. But once A is located there are 2 choices for B. Once B is located there is 1 choice for C and 1 for D
$\therefore$ there are $4 \times 2 \times 1 \times 1 = 8$ ways.



c There are 4 choices for A. Once A is located there are 3 choices for B. Once B is located there are 2 choices for C and then 1 for D
$\therefore$ there are $4 \times 3 \times 2 \times 1 = 24$ ways.

3



From A there are 3 possible first leg paths, to W, X or Y. Then there are 2 second leg paths to B
$\therefore$ total number $= 3 \times 2 = 6$ paths.

4 Any of the 7 teams could be in 'top' position. Then there are 6 left which could be in the 'second' position.
So, there are $7 \times 6 = 42$ possible ways.

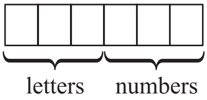
5 Any of the 8 teams could be 'top'.
Any of the remaining 7 could be second.
Any of the remaining 6 could be third.
Any of the remaining 5 could be fourth
$\therefore$ there are $8 \times 7 \times 6 \times 5$
$= 1680$ ways.

6 There are 5 digits to choose from.

a Number of ways $= 5 \times 5 \times 5 = 125$

b Number of ways $= 5 \times 4 \times 3 = 60$

7



Repetitions are allowed.
$\therefore$ total number of ways
$= 26 \times 26 \times 26 \times 10 \times 10 \times 10$
$= 17\,576\,000$

8 **a** The 1st letter could go into either box, i.e., 2 ways and the second could go into either box, i.e., 2 ways $\therefore$ there are $2 \times 2 = 4$ ways.
These are:

Box X	Box Y
A, B	-
A	B
B	A
-	A, B

b There are $3 \times 3 = 9$ ways.

c There are $3 \times 3 \times 3 \times 3 = 81$ ways.

EXERCISE 9B

1 **a** There are $2 \times 2 + 3 \times 3$
$= 13$ different paths

b There are $4 \times 2 + 3 \times 2 \times 2$
$= 20$ different paths

c There are $2 + 4 \times 2 + 3 \times 3$
$= 19$ different paths

d There are $2 \times 2 + 2 \times 2 + 2 \times 3 \times 4$
$= 32$ different paths

EXERCISE 9C

1
$0! = 1$
$1! = 1$
$2! = 2 \times 1 = 2$
$3! = 3 \times 2 \times 1 = 6$
$4! = 4 \times 3 \times 2 \times 1 = 24$
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
$6! = 6 \times 5! = 6 \times 120 = 720$
$7! = 7 \times 6! = 7 \times 720 = 5040$
$8! = 8 \times 7! = 8 \times 5040 = 40\,320$
$9! = 9 \times 8! = 9 \times 40\,320 = 362\,880$
$10! = 10 \times 9! = 10 \times 362\,880 = 3\,628\,800$

2 **a** $\dfrac{6!}{5!} = \dfrac{6 \times 5!}{5!} = 6$

b $\dfrac{6!}{4!} = \dfrac{6 \times 5 \times 4!}{4!} = 30$

c $\dfrac{6!}{7!} = \dfrac{6!}{7 \times 6!} = \frac{1}{7}$

d $\dfrac{4!}{6!} = \dfrac{4!}{6 \times 5 \times 4!} = \frac{1}{30}$

e $\dfrac{100!}{99!} = \dfrac{100 \times 99!}{99!} = 100$

f $\dfrac{7!}{5! \times 2!} = \dfrac{7 \times 6 \times 5!}{5! \times 2} = 21$

3 **a** $\dfrac{n!}{(n-1)!} = \dfrac{n \times (n-1)!}{(n-1)!} = n$

b $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

c $\dfrac{(n+1)!}{(n-1)!} = \dfrac{(n+1)(n)(n-1)!}{(n-1)!} = n(n+1)$

4 **a** $7 \times 6 \times 5 = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \dfrac{7!}{4!}$

b $10 \times 9 = \dfrac{10 \times 9 \times 8!}{8!} = \dfrac{10!}{8!}$

c $11 \times 10 \times 9 \times 8 \times 7 = \dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6!} = \dfrac{11!}{6!}$

d $\dfrac{13 \times 12 \times 11}{3 \times 2 \times 1} = \dfrac{13 \times 12 \times 11 \times 10!}{10! \times 3 \times 2 \times 1} = \dfrac{13!}{10!3!}$

e $\dfrac{1}{6 \times 5 \times 4} = \dfrac{3!}{6 \times 5 \times 4 \times 3!} = \dfrac{3!}{6!}$

f $\dfrac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17} = \dfrac{4! \times 16!}{20 \times 19 \times 18 \times 17 \times 16!} = \dfrac{4! \times 16!}{20!}$

5 **a**
$5! + 4!$
$= 5 \times 4! + 4!$
$= 4![5+1]$
$= 6 \times 4!$

b
$11! - 10!$
$= 11 \times 10! - 10!$
$= 10![11-1]$
$= 10 \times 10!$

c
$6! + 8!$
$= 6! + 8 \times 7 \times 6!$
$= 6![1 + 8 \times 7]$
$= 6! \times 57$
$= 57 \times 6!$

d
$12! - 10!$
$= 12 \times 11 \times 10! - 10!$
$= 10![12 \times 11 - 1]$
$= 10! \times 131$
$= 131 \times 10!$

e
$9! + 8! + 7!$
$= 9 \times 8 \times 7! + 8 \times 7! + 7!$
$= 7![72 + 8 + 1]$
$= 7! \times 81$
$= 81 \times 7!$

f
$7! - 6! + 8!$
$= 7 \times 6! - 6! + 8 \times 7 \times 6!$
$= 6![7 - 1 + 56]$
$= 6! \times 62$
$= 62 \times 6!$

g
$12! - 2 \times 11!$
$= 12 \times 11! - 2 \times 11!$
$= 11![12 - 2]$
$= 10 \times 11!$

h
$3 \times 9! + 5 \times 8!$
$= 3 \times 9 \times 8! + 5 \times 8!$
$= 8![27 + 5]$
$= 32 \times 8!$

6 **a**
$$\frac{12!-11!}{11} = \frac{12\times 11!-11!}{11} = \frac{11![12-1]}{11} = \frac{11!\times 11}{11} = 11!$$

b
$$\frac{10!+9!}{11} = \frac{10\times 9!+9!}{11} = \frac{9![10+1]}{11} = \frac{9!\times 11}{11} = 9!$$

c
$$\frac{10!-8!}{89} = \frac{10\times 9\times 8!-8!}{89} = \frac{8![90-1]}{89} = \frac{8!\times 89}{89} = 8!$$

d
$$\frac{10!-9!}{9!} = \frac{10\times 9!-9!}{9!} = \frac{9!(10-1)}{9!} = 9$$

e
$$\frac{6!+5!-4!}{4!} = \frac{6\times 5\times 4!+5\times 4!-4!}{4!} = \frac{4![30+5-1]}{4!} = 34$$

f
$$\frac{n!+(n-1)!}{(n-1)!} = \frac{n\times(n-1)!+(n-1)!}{(n-1)!} = \frac{(n-1)![n+1]}{(n-1)!} = n+1$$

g
$$\frac{n!-(n-1)!}{n-1} = \frac{n\times(n-1)!-(n-1)!}{n-1} = \frac{(n-1)![n-1]}{n-1} = (n-1)!$$

h
$$\frac{(n+2)!+(n+1)!}{n+3} = \frac{(n+2)(n+1)!+(n+1)!}{n+3} = \frac{(n+1)![n+2+1]}{n+3} = (n+1)!$$

EXERCISE 9D

1 **a** W, X, Y, Z

b WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY

c WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZX, YZW, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX

2 **a** AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED

b ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

2 at a time: 20 3 at a time: 60

3 **a** There are $5! = 120$ different orderings.

b There are $8\times 7\times 6 = 336$ different orderings.

c There are $10\times 9\times 8\times 7 = 5040$ different signals.

4 **a** $\begin{array}{|c|c|}\hline 4 & 3 \\ \hline \end{array}$ i.e., $4\times 3 = 12$ different signals

b $\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline \end{array}$ i.e., $4\times 3\times 2 = 24$ different signals

c $12+24 = 36$ different signals {using **a** and **b**}

5 There are 6 different letters $\therefore$ $6! = 720$ permutations.

a $\begin{array}{|c|c|c|c|c|c|}\hline 4 & 3 & 2 & 1 & 1 & 1 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\text{E}} & \multicolumn{1}{c}{\text{D}} \end{array}$ i.e., $4\times 3\times 2\times 1\times 1\times 1 = 24$ permutations

b

1	4	3	2	1	1
F					A

i.e., $1 \times 4 \times 3 \times 2 \times 1 \times 1 = 24$ permutations

c

2	4	3	2	1	1
A or E					the other one

i.e., $2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$ permutations

6 **a**

7	7	7

i.e., $7^3 = 343$ different numbers

b

7	6	5

i.e., $7 \times 6 \times 5 = 210$ different numbers

c

6	5	4
6 remain		fill 1st with any of 4 odds

i.e., $6 \times 5 \times 4 = 120$ different numbers

7 There are no restrictions $\therefore$ $6! = 720$ different ways

3	3	2	2	1	1
B	G	B	G	B	G

or

3	3	2	2	1	1
G	B	G	B	G	B

i.e., $3 \times 3 \times 2 \times 2 \times 1 \times 1 + 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 72$ different ways

8 **a**

9	9	8
cannot use 0		

i.e., $9 \times 9 \times 8 = 648$ numbers

b

8	8	1
cannot use 0 or 5		a 5

i.e., $8 \times 8 \times 1 = 64$ numbers

c

9	8	1
		0

i.e., $9 \times 8 \times 1 = 72$ different numbers

d $64 + 72 = 136$ different numbers {using **b** and **c**}

9 **a** As there are no restrictions, the number of ways is $5! = 120$.

b X and Y are together in 2! ways {XY or YX}

They, together with the other three books can be permutated in 4! ways.

$\therefore$ total number $= 2! \times 4! = 48$ ways.

c $120 - 48 = 72$ ways {using **a** and **b**}

10 **a** As there are no restrictions, the number of ways is $10! = 3\,628\,800$.

b A, B and C are together in 3! ways {ABC, ACB, BAC, BCA, CAB, CBA}

They, together with the other 7 can be permutated in 8! ways.

$\therefore$ total number is $3! \times 8! = 241\,920$ ways.

EXERCISE 9E

1 **a** $C_1^8 = \dfrac{8}{1} = 8$

b $C_2^8 = \dfrac{8 \times 7}{2 \times 1} = 28$

c $C_3^8 = \dfrac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$

d $C_6^8 = \dfrac{8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3}}{\cancel{6 \times 5 \times 4 \times 3} \times 2 \times 1} = 28$

e $C_8^8 = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= 1,$ as expected

2 $C^n_{n-r} = \dfrac{n!}{(n-r)!(n-[n-r])!} = \dfrac{n!}{(n-r)!r!} = C^n_r$

3 ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE and $C^5_3 = 10$ ✓

4 There are $C^{17}_{11} = 12\,376$ different teams.

5 There are $C^9_5 = 126$ different possible selections.
If question **1** is compulsory there are $C^1_1 C^8_4 = 1 \times 70 = 70$ possible selections.

6 If no restrictions, there are $C^{13}_3 = 286$ different committees.
$C^1_1 C^{12}_2$ consist of the president and two others, i.e., 66 of them.

7 If no restrictions, there are $C^{12}_5 = 792$ different teams.

- **a** Those containing the captain and vice-captain number $C^2_2 C^{10}_3 = 1 \times 120 = 120.$
- **b** Those containing exactly one of the captain and vice captain number $C^2_1 C^{10}_4 = 2 \times 210 = 420.$

8 Number of different teams $= C^3_3 C^1_0 C^{11}_6 = 1 \times 1 \times 462 = 462.$

9
- **a** If 1 person is always in the selection, number of ways $= C^1_1 C^9_3 = 84$
- **b** If 2 are always excluded, the number of ways $= C^2_0 C^8_4 = 70$
- **c** If 1 is always 'in' and 2 are always 'out', the number of ways is $C^1_1 C^2_0 C^7_3 = 35$

10
- **a** If there are no restrictions the number of ways $= C^{16}_5 = 4368$
- **b** The three men can be chosen in C^{10}_3 ways and the 2 women in C^6_2 ways.
  $\therefore$ total number of ways $= C^{10}_3 \times C^6_2 = 120 \times 15 = 1800$ ways.
- **c** If it contains all men, the number of ways $= C^{10}_5 \times C^6_0 = 252$
- **d** If it contains at least 3 men it would contain
  3 men and 2 women or 4 men and 1 woman or 5 men and 0 women
  and this can be done in $C^{10}_3 C^6_2 + C^{10}_4 C^6_1 + C^{10}_5 C^6_0$ ways $= 3312$ ways.
- **e** If it contains at least one of each sex, the total number of ways
  $= C^{10}_1 C^6_4 + C^{10}_2 C^6_3 + C^{10}_3 C^6_2 + C^{10}_4 C^6_1 = 4110$ *or* $C^{16}_5 - C^{10}_0 C^6_5 - C^{10}_5 C^6_0 = 4110$

11
- **a** The 2 doctors can be chosen in C^6_2 ways
  The 1 dentist can be chosen in C^3_1 ways
  The 2 others can be chosen in C^7_2 ways
  $\therefore$ the total number of ways $= C^6_2 \times C^3_1 \times C^7_2 = 945$
- **b** If it contains 2 doctors, 3 must be chosen from the other 10 i.e., in $C^6_2 C^{10}_3 = 1800$ ways
- **c** If it contains at least one of the two professions this can be done in
  $C^9_1 C^7_4 + C^9_2 C^7_3 + C^9_3 C^7_2 + C^9_4 C^7_1 = 4347$ *or* $C^{16}_5 - C^9_0 C^7_5 = 4347$

12 There are 20 points (for vertices) to choose from and any 2 form a line.
This can be done in C^{20}_2 ways. But this count includes the 20 lines joining the vertices.
$\therefore$ the number of diagonals $= C^{20}_2 - 20 = 190 - 20 = 170$

13
- **a**
  - **i** $C^{12}_2 = 66$ lines can be determined.
  - **ii** Of the lines in **a i** $C^1_1 C^{11}_1 = 11$ pass through B.
- **b**
  - **i** $C^{12}_3 = 220$ triangles can be determined.
  - **ii** Of the triangles in **b i** $C^1_1 C^{11}_2 = 55$ have vertex B.

14 If the digits are in ascending order they must be from 1 to 9 i.e., 9 of them, and we want any 4.

This can be done in $C^9_4 = 126$ ways.

Once they have been selected they can be put in one ascending order

$\therefore$ total number $= 126 \times 1 = 126$.

15 **a** The different committees of 4, consisting of selections from 5 men and 6 women *in all possible ways* are

(4 men, 0 women) or (3 men, 1 woman) or (2 men, 2 women) or (1 man, 3 women) or (0 men, 4 women)

i.e., $C^5_4 C^6_0 + C^5_3 C^6_1 + C^5_2 C^6_2 + C^5_1 C^6_3 + C^5_0 C^6_4 = C^{11}_4 \longleftarrow$ total number unrestricted.

b The generalisation is:

$$C^m_0 C^n_r + C^m_1 C^n_{r-1} + C^m_2 C^n_{r-2} + \text{......} + C^m_{r-2} C^n_2 + C^m_{r-1} C^n_1 + C^m_r C^n_0 = C^{m+n}_r$$

16 **a** Consider a simpler case of 4 people (A, B, C and D) going into two equal groups.

AB CD (1) (1) and (6) are the same division.
AC BD (2) (2) and (5) are the same division
AD BC (3) (3) and (4) are the same division
BC AD (4)
BD AC (5) So, the number of ways is $\frac{1}{2}$ of C^6_3.
CD AB (6)

So, for 2 equal groups of 6, the number of ways $\frac{1}{2}$ of $C^{12}_6 C^6_6$

$= \frac{1}{2} \times 924$

$= 462$

b For 3 equal groups of 4, the number of ways $= \dfrac{1}{3!} \times C^{12}_4 \times C^8_4 \times C^4_4$

$= 5775$

EXERCISE 9F

1 **a**
$$\begin{aligned}&(x+1)^3\\ &= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3\\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

b
$$\begin{aligned}&(x+2)^3\\ &= x^3 + 3x^2(2)^1 + 3x(2)^2 + (2)^3\\ &= x^3 + 6x^2 + 12x + 8\end{aligned}$$

c
$$\begin{aligned}&(x-4)^3\\ &= x^3 + 3x^2(-4)^1 + 3x(-4)^2 + (-4)^3\\ &= x^3 - 12x^2 + 48x - 64\end{aligned}$$

d
$$\begin{aligned}&(2x+1)^3\\ &= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3\\ &= 8x^3 + 12x^2 + 6x + 1\end{aligned}$$

e
$$\begin{aligned}&(2x-1)^3\\ &= (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3\\ &= 8x^3 - 12x^2 + 6x - 1\end{aligned}$$

f
$$\begin{aligned}&(3x-1)^3\\ &= (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3\\ &= 27x^3 - 27x^2 + 9x - 1\end{aligned}$$

g
$$\begin{aligned}&(2x+5)^3\\ &= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3\\ &= 8x^3 + 60x^2 + 150x + 125\end{aligned}$$

h
$$\begin{aligned}&\left(2x + \frac{1}{x}\right)^3\\ &= (2x)^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3\\ &= 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}\end{aligned}$$

2 **a** $(x+2)^4 = x^4 + 4x^3(2)^1 + 6x^2(2)^2 + 4x(2)^3 + 2^4$
$= x^4 + 8x^3 + 24x^2 + 32x + 16$

b $(x-2)^4 = x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$
$= x^4 - 8x^3 + 24x^2 - 32x + 16$

c $(2x+3)^4 = (2x)^4 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)(3)^3 + (3)^4$
$= 16x^4 + 12 \times 8x^3 + 54 \times 4x^2 + 108 \times 2x + 81$
$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$

d $(3x-1)^4 = (3x)^4 + 4(3x)^3(-1) + 6(3x)^2(-1)^2 + 4(3x)(-1)^3 + (-1)^4$
$= 81x^4 - 4 \times 27x^3 + 6 \times 9x^2 - 4 \times 3x + 1$
$= 81x^4 - 108x^3 + 54x^2 - 12x + 1$

e $\left(x+\frac{1}{x}\right)^4 = x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4$
$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

f $\left(2x-\frac{1}{x}\right)^4 = (2x)^4 + 4(2x)^3\left(-\frac{1}{x}\right) + 6(2x)^2\left(-\frac{1}{x}\right)^2 + 4(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4$
$= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4}$

3 **a** $(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5$
$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

b $(x-2)^5 = x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5$
$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

c $(2x+1)^5 = (2x)^5 + 5(2x)^4(1) + 10(2x)^3(1)^2 + 10(2x)^2(1)^3 + 5(2x)(1)^4 + (1)^5$
$= 32x^5 + 5 \times 16x^4 + 10 \times 8x^3 + 10 \times 4x^2 + 10x + 1$
$= 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

d $\left(2x-\frac{1}{x}\right)^5$
$= (2x)^5 + 5(2x)^4\left(-\frac{1}{x}\right) + 10(2x)^3\left(-\frac{1}{x}\right)^2 + 10(2x)^2\left(-\frac{1}{x}\right)^3 + 5(2x)\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$
$= 32x^5 - 80x^3 + 80x - \frac{40}{x} + \frac{10}{x^3} - \frac{1}{x^5}$

4 **a**
1 5 10 10 5 1 ⟵ the 5th row
1 6 15 20 15 6 1 ⟵ the 6th row

b **i** $(x+2)^6 = x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6$
$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $(2x-1)^6 = (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4$
$+ 6(2x)(-1)^5 + (-1)^6$
$= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1$
$= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $\left(x+\frac{1}{x}\right)^6$
$= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$
$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

5 **a**
$$\begin{aligned}\left(1+\sqrt{2}\right)^3 &= (1)^3+3(1)^2\left(\sqrt{2}\right)+3(1)\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^3\\ &= 1+3\sqrt{2}+3\times 2+2\times\sqrt{2}\\ &= 1+3\sqrt{2}+6+2\sqrt{2}\\ &= 7+5\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\left(1+\sqrt{5}\right)^4 &= (1)^4+4(1)^3\left(\sqrt{5}\right)+6(1)^2\left(\sqrt{5}\right)^2+4(1)\left(\sqrt{5}\right)^3+\left(\sqrt{5}\right)^4\\ &= 1+4\sqrt{5}+30+20\sqrt{5}+25\\ &= 56+24\sqrt{5}\end{aligned}$$

c
$$\begin{aligned}&\left(2-\sqrt{2}\right)^5\\ &= (2)^5+5(2)^4\left(-\sqrt{2}\right)+10(2)^3\left(-\sqrt{2}\right)^2+10(2)^2\left(-\sqrt{2}\right)^3+5(2)^1\left(-\sqrt{2}\right)^4+\left(-\sqrt{2}\right)^5\\ &= 32-80\sqrt{2}+160-80\sqrt{2}+40-4\sqrt{2}\\ &= 232-164\sqrt{2}\end{aligned}$$

6 **a**
$$\begin{aligned}(2+x)^6 &= (2)^6+6(2)^5x+15(2)^4x^2+20(2)^3x^3+15(2)^2x^4+6(2)x^5+x^6\\ &= 64+192x+240x^2+160x^3+60x^4+12x^5+x^6\end{aligned}$$

b $(2.01)^6$ is obtained by letting $x=0.01$

$$\begin{aligned}\therefore\quad (2.01)^6 &= 64+192\times(0.01)+240\times(0.01)^2+160\times(0.01)^3+60\times(0.01)^4\\ &\qquad +12\times(0.01)^5+(0.01)^6\\ &= 65.944\,160\,601\,201\end{aligned}$$

$$\begin{array}{l} 64\\ \;\;1.92\\ \;\;0.024\\ \;\;0.000\,16\\ \;\;0.000\,000\,6\\ \;\;0.000\,000\,001\,2\\ \;\;0.000\,000\,000\,001\\ \hline 65.944\,160\,601\,201\end{array}$$

7
$$\begin{aligned}&(2x+3)(x+1)^4\\ &= (2x+3)(x^4+4x^3+6x^2+4x+1)\\ &= 2x^5+8x^4+12x^3+8x^2+2x\\ &\qquad +3x^4+12x^3+18x^2+12x+3\\ &= 2x^5+11x^4+24x^3+26x^2+14x+3\end{aligned}$$

8 **a** $(3a+b)^5=(3a)^5+5(3a)^4b+10(3a)^3b^2+\ldots\ldots$

$\therefore$ the coefficient of a^3b^2 is $10\times 3^3=270$

b $(2a+3b)^6=(2a)^6+6(2a)^5(3b)+15(2a)^4(3b)^2+20(2a)^3(3b)^3+\ldots\ldots$

$\therefore$ the coefficient of a^3b^3 is $20\times 2^3\times 3^3=4320$

EXERCISE 9G

1 **a**
$$\begin{aligned}(1+2x)^{11} &= 1^{11}+\binom{11}{1}1^{10}(2x)^1+\binom{11}{2}1^9(2x)^2+\ldots..+\binom{11}{10}1^1(2x)^{10}+\binom{11}{11}(2x)^{11}\\ &= 1+\binom{11}{1}(2x)+\binom{11}{2}(2x)^2+\ldots..+\binom{11}{10}(2x)^{10}+\binom{11}{11}(2x)^{11}\end{aligned}$$

b
$$\begin{aligned}&\left(3x+\frac{2}{x}\right)^{15}\\ &= (3x)^{15}+\binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right)+\binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2+\ldots..+\binom{15}{14}(3x)\left(\frac{2}{x}\right)^{14}+\binom{15}{15}\left(\frac{2}{x}\right)^{15}\end{aligned}$$

c
$$\begin{aligned}&\left(2x-\frac{3}{x}\right)^{20}\\ &= (2x)^{20}+\binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right)+\binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2+\ldots..+\binom{20}{19}(2x)\left(-\frac{3}{x}\right)^{19}+\binom{20}{20}\left(-\frac{3}{x}\right)^{20}\end{aligned}$$

2 **a** For $(2x+5)^{15}$, $a=(2x)$, $b=5$ and $n=15$

Now $T_{r+1}=\binom{n}{r}a^{n-r}b^r$ and letting $r=5$ gives $T_6=\binom{15}{5}(2x)^{10}5^5$.

b For $\left(x^2+\dfrac{5}{x}\right)^9$, $a=(x^2)$, $b=\left(\dfrac{5}{x}\right)$ and $n=9$

Now $T_{r+1}=\binom{n}{r}a^{n-r}b^r$ and letting $r=3$ gives $T_4=\binom{9}{3}(x^2)^6\left(\dfrac{5}{x}\right)^3$.

c For $\left(x-\dfrac{2}{x}\right)^{17}$, $a=x$, $b=\left(-\dfrac{2}{x}\right)$ and $n=17$

Now $T_{r+1}=\binom{n}{r}a^{n-r}b^r$ and letting $r=9$ gives $T_{10}=\binom{17}{9}x^8\left(-\dfrac{2}{x}\right)^9$.

d For $\left(2x^2-\dfrac{1}{x}\right)^{21}$, $a=(2x^2)$, $b=\left(-\dfrac{1}{x}\right)$ and $n=21$

Now $T_{r+1}=\binom{n}{r}a^{n-r}b^r$ and letting $r=8$ gives $T_9=\binom{21}{8}(2x^2)^{13}\left(-\dfrac{1}{x}\right)^8$.

3 **a** In $\left(3+2x^2\right)^{10}$, $a=3$, $b=(2x^2)$ and $n=10$

$$\begin{aligned}\text{Now}\quad T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{10}{r}3^{10-r}(2x^2)^r\\&=\binom{10}{r}3^{10-r}2^rx^{2r}\end{aligned}$$

We now let $2r=10$

$\therefore\ r=5$

So, $T_6=\underbrace{\binom{10}{5}3^52^5}x^{10}$

$\therefore$ the coefficient is $\binom{10}{5}3^52^5$.

b In $\left(2x^2-\dfrac{3}{x}\right)^6$, $a=(2x^2)$, $b=\left(-\dfrac{3}{x}\right)$ and $n=6$

$$\begin{aligned}\text{Now}\quad T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{6}{r}\left(2x^2\right)^{6-r}\left(-\frac{3}{x}\right)^r\\&=\binom{6}{r}2^{6-r}x^{12-2r}\frac{(-3)^r}{x^r}\\&=\binom{6}{r}2^{6-r}(-3)^rx^{12-3r}\end{aligned}$$

We now let $12-3r=3$

$\therefore\ 3r=9$

$\therefore\ r=3$

So, $T_4=\underbrace{\binom{6}{3}2^3(-3)^3}x^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3(-3)^3$.

c In $\left(2x^2-\dfrac{1}{x}\right)^{12}$, $a=(2x^2)$, $b=\left(-\dfrac{1}{x}\right)$ and $n=12$

$$\begin{aligned}\text{Now}\quad T_{r+1}&=\binom{n}{r}a^{n-r}b^r\\&=\binom{12}{r}\left(2x^2\right)^{12-r}\left(-\frac{1}{x}\right)^r\\&=\binom{12}{r}2^{12-r}x^{24-2r}\frac{(-1)^r}{x^r}\\&=\binom{12}{r}2^{12-r}(-1)^rx^{24-3r}\end{aligned}$$

We now let $24-3r=12$

$\therefore\ 3r=12$

$\therefore\ r=4$

So, $T_5=\underbrace{\binom{12}{4}2^8(-1)^4}x^{12}$

$\therefore$ the coefficient is $\binom{12}{4}2^8$.

4 **a** For $\left(x+\dfrac{2}{x^2}\right)^{15}$, $a=x$, $b=\dfrac{2}{x^2}$ and $n=15$

Now $T_{r+1}=\binom{n}{r}a^{n-r}b^r=\binom{15}{r}x^{15-r}\left(\dfrac{2}{x^2}\right)^r=\binom{15}{r}x^{15-r}\dfrac{2^r}{x^{2r}}=\binom{15}{r}2^rx^{15-3r}$

The constant term does not contain x. $\therefore$ $15-3r=0$ $\therefore$ $r=5$

$\therefore$ the constant term is $\binom{15}{5}2^5$.

b For $\left(x-\dfrac{3}{x^2}\right)^9$, $a=x$, $b=\left(-\dfrac{3}{x^2}\right)$ and $n=9$

$$\begin{aligned}\text{Now}\quad T_{r+1} &= \binom{n}{r}a^{n-r}b^r\\ &= \binom{9}{r}x^{9-r}\left(-\frac{3}{x^2}\right)^r\\ &= \binom{9}{r}x^{9-r}\frac{(-3)^r}{x^{2r}}\\ &= \binom{9}{r}(-3)^r x^{9-3r}\end{aligned}$$

The constant term does not contain x.

$$\begin{aligned}\therefore\quad 9-3r &= 0\\ \therefore\quad r &= 3\end{aligned}$$

and $T_4 = \binom{9}{3}(-3)^3x^0$

$\therefore$ the constant term is $\binom{9}{3}(-3)^3$.

5 **a**, **b**

$$\begin{array}{llll}\text{Row 1} & 1\ \ 1 \longleftarrow & \text{sum} = 1+1=2 & =2^1\\ \text{Row 2} & 1\ \ 2\ \ 1 \longleftarrow & \text{sum} = 1+2+1=4 & =2^2\\ \text{Row 3} & 1\ \ 3\ \ 3\ \ 1 \longleftarrow & \text{sum} = 1+3+3+1=8 & =2^3\\ \text{Row 4} & 1\ \ 4\ \ 6\ \ 4\ \ 1 \longleftarrow & \text{sum} = 1+4+6+4+1=16 & =2^4\\ \text{Row 5} & 1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 \longleftarrow & \text{sum} = 1+5+10+10+5+1=32 & =2^5\end{array}$$

c It seems that the sum of the numbers in row n of Pascal's triangle is 2^n.

d
$$\begin{aligned}(1+x)^n &= \binom{n}{0}1^n+\binom{n}{1}1^{n-1}x+\binom{n}{2}1^{n-2}x^2+\binom{n}{3}1^{n-3}x^3+.....+\binom{n}{n-1}1^1x^{n-1}+\binom{n}{n}x^n\\ &= \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+.....+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n\end{aligned}$$

{as all powers of 1 are 1}

Now letting $x=1$ gives LHS $=(1+1)^n=2^n$

and RHS $=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+.....+\binom{n}{n-1}+\binom{n}{n}$

$$\therefore\quad \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+.....+\binom{n}{n-1}+\binom{n}{n}=2^n$$

6 **a**
$$\begin{aligned}&(x+2)(x^2+1)^8\\ &=(x+2)\left[(x^2)^8+\binom{8}{1}(x^2)^7 1+\binom{8}{2}(x^2)^6 1^2+.....+\binom{8}{6}(x^2)^2 1^6+\binom{8}{7}(x^2)^1 1^7+\binom{8}{8}1^8\right]\end{aligned}$$

only terms which when multiplied give an x^5

$\therefore$ coefficient of x^5 is $1\times\binom{8}{6}=\binom{8}{6}=28.$

b
$$\begin{aligned}&(2-x)(3x+1)^9\\ &=(2-x)\left[(3x)^9+\binom{9}{1}(3x)^8+\binom{9}{2}(3x)^7+\binom{9}{3}(3x)^6+\binom{9}{4}(3x)^5+.....\right]\end{aligned}$$

$\therefore$ coefficient of x^6 is $2\times\binom{9}{3}\times3^6+(-1)\times\binom{9}{4}\times3^5=2\binom{9}{3}3^6-\binom{9}{4}3^5=91\,854$

7 **a** $\binom{n}{1}=C_1^n=\dfrac{n}{1}=n$ and $\binom{n}{2}=C_2^n=\dfrac{n(n-1)}{2\times1}=\dfrac{n(n-1)}{2}$

b $(1+x)^n$ has $T_3=\binom{n}{2}1^{n-2}x^2=\binom{n}{2}x^2$ and $n\geqslant 2$

But this term is $36x^2$ $\therefore\ \binom{n}{2}=36$

$$\begin{aligned}\therefore\quad \frac{n(n-1)}{2} &= 36\\ \therefore\quad n(n-1) &= 72\\ \therefore\quad n^2-n-72 &= 0\\ \therefore\quad (n-9)(n+8) &= 0\end{aligned}$$

$\therefore\ n=9$ or -8

But $n\geqslant2$ $\therefore\ n=9$

$$\begin{aligned}\text{and}\quad T_4 &= \binom{n}{3}1^{n-3}x^3\\ &= \binom{9}{3}x^3\\ &= 84x^3\end{aligned}$$

c
$$\begin{aligned}(1+kx)^n &= 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + \,.....\\ &= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + \,.....\end{aligned}$$

$$\therefore\ \binom{n}{1}k = -12 \qquad \text{and} \qquad \binom{n}{2}k^2 = 60$$

$$\therefore\ nk = -12 \quad \text{and} \quad \frac{n(n-1)}{2}k^2 = 60$$

$$\therefore\ n(n-1)k^2 = 120$$

$$\text{But} \quad k = -\frac{12}{n} \quad \therefore\ n(n-1)\,\frac{144}{n^2} = 120$$

$$\begin{aligned}\therefore\ 144(n-1) &= 120n \quad \{n \geqslant 2\}\\ \therefore\ 144n - 120n &= 144\\ \therefore\ 24n &= 144\\ \therefore\ n &= 6\end{aligned}$$

$$\text{and so, as} \quad k = -\frac{12}{n}, \quad k = -2$$

8
$$\begin{aligned}T_{r+1} &= \binom{n}{r}a^{n-r}b^r \quad \text{where} \quad n = 10, \quad a = (x^2), \quad b = \left(\frac{1}{ax}\right)\\ &= \binom{10}{r}(x^2)^{10-r}\left(\frac{1}{ax}\right)^r\\ &= \binom{10}{r}x^{20-2r} \times \frac{1}{a^r x^r}\\ &= \binom{10}{r}x^{20-3r} \times \frac{1}{a^r}\end{aligned}$$

We let $20 - 3r = 11 \quad \therefore\ 3r = 9 \quad \therefore\ r = 3$

and $T_4 = \binom{10}{3}x^{11} \times \dfrac{1}{a^3} = \dfrac{\binom{10}{3}}{a^3}\,x^{11}$

Thus $\dfrac{\binom{10}{3}}{a^3} = 15$ i.e., $\dfrac{120}{a^3} = 15$

$$\begin{aligned}\therefore\ a^3 &= 8\\ \therefore\ a &= 2\end{aligned}$$

REVIEW SET 9A

1 **a**

26	26	10	10	10	10

i.e., $26^2 \times 10^4 = 6\,760\,000$ if no restrictions

b

5	26	10	10	10	10

↑
a vowel

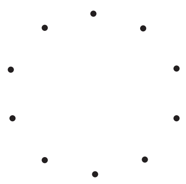
i.e., $5 \times 26 \times 10^4 = 1\,300\,000$
if the first letter is a vowel.

c

26	25	10	9	8	7

i.e., $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3\,276\,000$, if no repetitions

2



a To form a line we need to select any two points from the 10.

$\therefore$ total is $C_2^{10} = 45$ lines.

b To form a triangle we need to select any three points from the 10.

$\therefore$ total is $C_3^{10} = 120$ triangles.

3 **a** $\dfrac{n!}{(n-1)!} = \dfrac{n(n-1)(n-2)!}{(n-2)!}$
$= n(n-1)$

b $\dfrac{n!+(n+1)!}{n!} = \dfrac{n!+(n+1)n!}{n!}$
$= \dfrac{n![1+n+1]}{n!}$
$= n+2$

4 Total number, unrestricted $= C_5^{8+7} = C_5^{15} = 3003$

a Total with 2 men and 3 women $= C_2^8 C_3^7 = 980$

b Total with at least one man $=$ total unrestricted $-$ total with all women
$= 3003 - C_0^8 C_5^7$
$= 2982$

5 There are $C_2^8 = 28$ handshakes possible.

6 **a** With no restrictions there are $C_5^{10} = 252$ different committees.

b Those consisting of at least one of each sex
$= C_5^{10} - C_5^6 C_0^4$ {there are no committees consisting of 5 women}
$= 246$

7 **a** $$\begin{array}{|c|c|c|c|c|}\hline 4&3&2&1&1\\\hline \multicolumn{4}{c}{}&\multicolumn{1}{c}{\uparrow}\\ \multicolumn{4}{c}{}&\multicolumn{1}{c}{\text{T}}\end{array}$$ i.e., $4 \times 3 \times 2 \times 1 \times 1 = 24$ end in T

b $$\begin{array}{|c|c|c|c|c|}\hline 1&3&2&1&1\\\hline \multicolumn{1}{c}{\uparrow}&\multicolumn{3}{c}{}&\multicolumn{1}{c}{\uparrow}\\ \multicolumn{1}{c}{\text{P}}&\multicolumn{3}{c}{}&\multicolumn{1}{c}{\text{T}}\end{array}$$ i.e., $1 \times 3 \times 2 \times 1 \times 1 = 6$ begin with P and end with T

8 **a** $(x-2y)^3 = x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3$
$= x^3 - 6x^2y + 12xy^2 - 8y^3$

b $(3x+2)^4 = (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4$
$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$

9 In the expansion of $(2x+5)^6$, $a = (2x)$, $b = 5$, $n = 6$

$T_{r+1} = \binom{n}{r}a^{n-r}b^r$
$= \binom{6}{r}(2x)^{6-r}5^r$
$= \binom{6}{r}2^{6-r}x^{6-r}5^r$

For the coefficient of x^3 we let $6 - r = 3$
$\therefore\ r = 3$

and $T_4 = \underbrace{\binom{6}{3}2^3 5^3}\, x^3$

$\therefore$ the coefficient is $\binom{6}{3}2^3 5^3 = 20\,000$.

10 The three sisters can sit together in 3! ways. They as a group plus the other 5 people make 6 items which can be permutated in 6! ways.

$\therefore$ total number of orderings $= 3! \times 6!$
$= 4320$

11 **a** $$\begin{array}{|c|c|c|}\hline 9&10&10\\\hline \multicolumn{1}{c}{\uparrow}&\multicolumn{2}{c}{}\end{array}$$ can't use 0

if no restrictions i.e., $9 \times 10 \times 10 = 900$

b To be divisible by 5 the last digit must be 0 or 5.

i.e., $\begin{array}{|c|c|c|}\hline 9&10&1\\\hline \multicolumn{2}{c}{}&\multicolumn{1}{c}{\uparrow}\\ \multicolumn{2}{c}{}&\multicolumn{1}{c}{0}\end{array}$ or $\begin{array}{|c|c|c|}\hline 9&10&1\\\hline \multicolumn{2}{c}{}&\multicolumn{1}{c}{\uparrow}\\ \multicolumn{2}{c}{}&\multicolumn{1}{c}{5}\end{array}$ i.e., $9 \times 10 \times 1 + 9 \times 10 \times 1 = 180$ of them.

REVIEW SET 9B

1 **a** If no restrictions there are $C_8^{18} = 43\,758$ different teams possible.

b If 4 of each sex are needed there are $C_4^{11} \times C_4^7 = 11\,550$ different teams.

c If at least 2 women are needed, total number
$$\begin{aligned} &= C_8^{18} - C_8^{11}C_0^7 - C_7^{11}C_1^7 \\ &= 41\,283 \quad \text{different teams} \end{aligned}$$

2 **a** $\underset{\substack{\uparrow \\ \text{can't use } 0}}{\boxed{9}}\boxed{9}\boxed{8}\boxed{7}$ i.e., $9 \times 9 \times 8 \times 7 = 4536$ numbers are possible

b Either they end in 0 or in 5

$\boxed{9}\boxed{8}\boxed{7}\underset{\substack{\uparrow \\ 0}}{\boxed{1}}$ or $\underset{\substack{\uparrow \\ \text{can't use } 0}}{\boxed{8}}\boxed{8}\boxed{7}\underset{\substack{\uparrow \\ 5}}{\boxed{1}}$ i.e., $9 \times 8 \times 7 \times 1 + 8 \times 8 \times 7 \times 1$
$= 952$ numbers are possible.

3 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$\therefore \quad (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

a
$$\begin{aligned} (x-3)^6 &= x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6 \\ &= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729 \end{aligned}$$

b
$$\begin{aligned} &\left(1+\frac{1}{x}\right)^6 \\ &= (1)^6 + 6(1)^5\left(\frac{1}{x}\right) + 15(1)^4\left(\frac{1}{x}\right)^2 + 20(1)^3\left(\frac{1}{x}\right)^3 + 15(1)^2\left(\frac{1}{x}\right)^4 + 6(1)\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= 1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6} \end{aligned}$$

4
$$\begin{aligned} \left(\sqrt{3}+2\right)^5 &= \left(\sqrt{3}\right)^5 + 5\left(\sqrt{3}\right)^4(2) + 10\left(\sqrt{3}\right)^3(2)^2 + 10\left(\sqrt{3}\right)^2(2)^3 + 5\left(\sqrt{3}\right)^1(2)^4 + 2^5 \\ &= 9\sqrt{3} + 90 + 120\sqrt{3} + 240 + 80\sqrt{3} + 32 \\ &= 362 + 209\sqrt{3} \end{aligned}$$

5
$$\begin{aligned} (4+x)^3 &= 4^3 + 3(4)^2x^1 + 3(4)^1x^2 + x^3 \\ &= 64 + 48x + 12x^2 + x^3 \end{aligned}$$

Letting $x = 0.02$ gives
$$\begin{aligned} (4.02)^3 &= 64 + 48(0.02) + 12(0.02)^2 + (0.02)^3 \\ &= 64 + 0.96 + 0.0048 + 0.000\,008 \\ &= 64.964\,808 \end{aligned}$$

6 For $\left(3x^2 + \frac{1}{x}\right)^8$, $a = (3x^2)$, $b = \left(\frac{1}{x}\right)$, $n = 8$

$$\begin{aligned} T_{r+1} &= \binom{n}{r}a^{n-r}b^r \\ &= \binom{8}{r}\left(3x^2\right)^{8-r}\left(\frac{1}{x}\right)^r \\ &= \binom{8}{r}3^{8-r}x^{16-2r-r} \\ &= \binom{8}{r}3^{8-r}x^{16-3r} \end{aligned}$$

Now a constant term does not contain x

$$\begin{aligned} \therefore \quad 16 - 3r &= 0 \\ \therefore \quad 3r &= 16 \\ \therefore \quad r &= 5\tfrac{1}{3} \end{aligned}$$

which is impossible as r is in $\mathbf{Z}$

$\therefore$ no constant term exists.

7 In $\left(2x-\dfrac{3}{x^2}\right)^{12}$, $a=(2x)$, $b=\left(-\dfrac{3}{x^2}\right)$, $n=12$

$$\begin{aligned}T_{r+1} &= \tbinom{n}{r}a^{n-r}b^r\\ &= \tbinom{12}{r}(2x)^{12-r}\left(-\frac{3}{x^2}\right)^r\\ &= \tbinom{12}{r}2^{12-r}x^{12-r}\frac{(-3)^r}{x^{2r}}\\ &= \tbinom{12}{r}2^{12-r}(-3)^r\,x^{12-3r}\end{aligned}$$

For the coefficient of x^{-6} we let

$$\begin{aligned}12-3r &= -6\\ \therefore\ 3r &= 18\\ \therefore\ r &= 6\end{aligned}$$

So, $T_7=\underbrace{\tbinom{12}{6}2^6(-3)^6}x^{-6}$

$\therefore$ the coefficient is $\binom{12}{6}2^6(-3)^6$.

8

$$\begin{aligned}&(2x+3)(x-2)^6\\ &=(2x+3)\left[x^6+6x^5(-2)+15x^4(-2)^2+.....\right]\end{aligned}$$

$\therefore$ coefficient of x^5 is $2\times15\times(-2)^2+3\times6\times(-2)=120-36=84$

9

$$\begin{aligned}T_{r+1} &= C^9_r(2x)^{9-r}\left(\frac{1}{ax^2}\right)^r\\ &= C^9_r\,2^{9-r}x^{9-r}\times\frac{1}{a^rx^{2r}}\\ &= C^9_r\,2^{9-r}a^{-r}x^{9-3r}\end{aligned}$$

Letting $r=2$, $T_3=\underbrace{C^9_2\,2^7a^{-2}}\,x^3$

$$\begin{aligned}\therefore\quad \frac{C^9_2\,2^7}{2} &= 288\\ \therefore\quad a^2 &= \frac{C^9_2\,2^7}{288}=16\\ \therefore\quad a &= \pm4\end{aligned}$$

10

$$\begin{aligned}T_{r+1} &= C^6_r\,(3x)^{6-r}\left(\frac{-2}{x^2}\right)^r\\ &= C^6_r\,3^{6-r}x^{6-r}(-2)^rx^{-2r}\\ &= C^6_r\,3^{6-r}x^{6-3r}(-2)^r\end{aligned}$$

and we let

$$\begin{aligned}6-3r &= 0\\ \therefore\ r &= 2\end{aligned}$$

$\therefore\quad T_3=\underbrace{C^6_2\,3^4(-2)^2}\,x^3$

$\therefore$ coefficient of $x^3=C^6_2\,3^42^2=4860$

11 $\left(x^3+\dfrac{q}{x^3}\right)^8$ has

$$\begin{aligned}T_{r+1} &= C^8_r(x^3)^{8-r}\left(\frac{q}{x^3}\right)^r\\ &= C^8_r\,x^{24-3r}\,\frac{q^r}{x^{3r}}\\ &= C^8_r\,x^{24-6r}q^r\end{aligned}$$

which has constant term $C^8_4q^4$ $\{24-6r=0$ when $r=4\}$

$\left(x^3+\dfrac{q}{x^3}\right)^4$ has

$$\begin{aligned}T_{r+1} &= C^4_r(x^3)^{4-r}\left(\frac{q}{x^3}\right)^r\\ &= C^4_r\,x^{12-3r}\,q^r\,x^{-3r}\\ &= C^4_r\,x^{12-6r}q^r\end{aligned}$$

which has constant term $C^4_2\,q^2$ $\{12-6r=0$ when $r=2\}$

$$\begin{aligned}\therefore\quad C^8_4\,q^4 &= C^4_2\,q^2\\ \text{i.e.,}\quad 70q^4-6q^2 &= 0\\ \therefore\quad 2q^2(35q^2-3) &= 0\\ \therefore\quad 35q^2-3 &= 0\quad\text{as}\quad q\neq0\\ \therefore\quad q^2 &= \tfrac{3}{35}\\ \therefore\quad q &= \pm\sqrt{\tfrac{3}{35}}\end{aligned}$$

Chapter 10

MATHEMATICAL INDUCTION

EXERCISE 10A

1 **a**

$$\begin{aligned} 3 &= 4\times 1-1\\ 7 &= 4\times 2-1\\ 11 &= 4\times 3-1\\ 15 &= 4\times 4-1\\ 19 &= 4\times 5-1 \quad \text{etc.}\end{aligned}$$

Our proposition is:

The nth term of the sequence $3, 7, 11, 15, 19, ...$ is $4n-1$ for $n = 1, 2, 3, 4, ...$

b

$$\begin{aligned} 3^1 &= 3 & 1+2(1) &= 3\\ 3^2 &= 9 & 1+2(2) &= 5\\ 3^3 &= 27 & 1+2(3) &= 7\\ 3^4 &= 81 & 1+2(4) &= 9\end{aligned}$$

Our proposition is:

$3^n > 1+2n$ for $n = 2, 3, 4, 5, ...$

i.e., $n \geqslant 2$, $n \in Z^+$

c

$$\begin{aligned} 11^1-1 &= 10\\ 11^2-1 &= 121-1 = 120\\ 11^3-1 &= 1331-1 = 1330\\ 11^4-1 &= 14\,641-1 = 14\,640\end{aligned}$$

Our proposition is:

11^n-1 is divisible by 10 for all $n \in Z^+$

d

$$\begin{aligned} &2 = 2 = 1\times 2\\ &2+4 = 6 = 2\times 3\\ &2+4+6 = 12 = 3\times 4\\ &2+4+6+8 = 20 = 4\times 5\\ &2+4+6+8+10 = 30 = 5\times 6\end{aligned}$$

Our proposition is:

$$2+4+6+8+10+\,.....\,+\underset{\substack{\uparrow\\ n\text{th term}}}{2n} = n(n+1) \quad \text{for all } n \in Z^+$$

e

$$\begin{aligned} &1! = 1\\ &1!+2\times 2! = 1+2(2) = 5\\ &1!+2\times 2!+3\times 3! = 1+4+18 = 23\\ &1!+2\times 2!+3\times 3!+4\times 4! = 1+4+18+96 = 119\end{aligned}$$

where each number result is 1 less than a factorial number

$$\begin{aligned} 1 &= 2!-1\\ 5 &= 3!-1\\ 23 &= 4!-1\\ 119 &= 5!-1\end{aligned}$$

Our proposition is:

$1!+2\times 2!+3\times 3!+4\times 4!+\,......\,+n\times n! = (n+1)!-1$ for all $n \in Z^+$

f

$$\frac{1}{2!} = \frac{1}{2} = \frac{2!-1}{2!}$$

$$\frac{1}{2!}+\frac{2}{3!} = \frac{1}{2}+\frac{2}{6} = \frac{5}{6} = \frac{3!-1}{3!}$$

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!} = \frac{1}{2}+\frac{2}{6}+\frac{3}{24} = \frac{23}{24} = \frac{4!-1}{4!}$$

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!} = \frac{1}{2}+\frac{2}{6}+\frac{3}{24}+\frac{4}{120} = \frac{119}{120} = \frac{5!-1}{5!}$$

Our proposition is: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\dfrac{4}{5!}+\,......\,+\dfrac{n}{(n+1)!} = \dfrac{(n+1)!-1}{(n+1)!}$, $n \in Z^+$

g

$$\begin{aligned} &7^1+2 = 7+2 = 9 = 3\times 3\\ &7^2+2 = 49+2 = 51 = 3\times 17\\ &7^3+2 = 343+2 = 345 = 3\times 115\\ &7^4+2 = 2401+2 = 2403 = 3\times 801\end{aligned}$$

Our proposition is:

7^n+2 is divisible by 3 for all $n \in Z^+$

h $(1-\frac{1}{2}) = \frac{1}{2}$

$(1-\frac{1}{2})(1-\frac{1}{3}) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

$(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$

$(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})(1-\frac{1}{5}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$

Our proposition is: $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})......\left(1-\dfrac{1}{n+1}\right) = \dfrac{1}{n+1}, \quad n \in Z^+$

i $\dfrac{1}{2\times 5} = \dfrac{1}{10}$

$\dfrac{1}{2\times 5} + \dfrac{1}{5\times 8} = \dfrac{1}{10} + \dfrac{1}{40} = \dfrac{5}{40} = \dfrac{1}{8} = \dfrac{2}{16}$

$\dfrac{1}{2\times 5} + \dfrac{1}{5\times 8} + \dfrac{1}{8\times 11} = \dfrac{1}{10} + \dfrac{1}{40} + \dfrac{1}{88} = \dfrac{3}{22}$

$\dfrac{1}{2\times 5} + \dfrac{1}{5\times 8} + \dfrac{1}{8\times 11} + \dfrac{1}{11\times 14} = \dfrac{1}{7} = \dfrac{4}{28}$

10, 16, 22, 28 is arithmetic with $u_1 = 10, \quad d = 6$

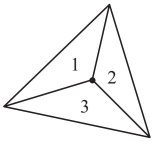
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 10 + 6(n-1) \\ &= 6n+4 \end{aligned}$$

Our proposition is:

$$\frac{1}{2\times 5} + \frac{1}{5\times 8} + \frac{1}{8\times 11} + \frac{1}{11\times 14} + ... + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4} \quad \text{for all } n \in Z^+$$

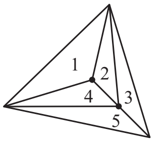
{2, 5, 8, 11 are arithmetic with $u_1 = 2, \quad d = 3 \quad \therefore \quad u_n = 2 + (n-1)3 = 3n-1$}

2 For $n = 1$



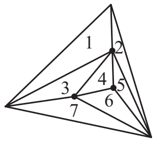
$T_1 = 3$
$= 2 \times 1 + 1$

For $n = 2$



$T_2 = 5$
$= 2 \times 2 + 1$

For $n = 3$



$T_3 = 7$
$= 2 \times 3 + 1$

Our proposition is:

The number of triangular partitions for n dots within the triangle is $T_n = 2n+1, \quad n \in Z^+$.

EXERCISE 10B

1 **a** P_n is "$1+2+3+4+......+n = \dfrac{n(n+1)}{2}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= \dfrac{1(2)}{2} = 1$, $\therefore$ P_1 is true

(2) If P_k is true then $1+2+3+4+......+k = \dfrac{k(k+1)}{2}$ ...... $(*)$

Thus $1+2+3+4+......+k+(k+1)$

$$\begin{aligned} &= \frac{k(k+1)}{2} + k + 1 \quad \{\text{using } (*)\} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \quad \{\text{to equalise denominators}\} \\ &= \frac{(k+1)[k+2]}{2} \quad \left\{\text{common factor of } \frac{(k+1)}{2}\right\} \\ &= \frac{(k+1)([k+1]+1)}{2} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

b P_n is "$1\times 2+2\times 3+3\times 4+ ...... +n(n+1)=\dfrac{n(n+1)(n+2)}{3}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1\times 2=2$ and RHS $=\dfrac{1(2)(3)}{3}=2$, $\therefore$ P_1 is true

(2) If P_k is true then

$$1\times 2+2\times 3+3\times 4+ ...... +k(k+1)=\frac{k(k+1)(k+2)}{3} \quad ......\ (*)$$

$$\begin{aligned}\therefore\quad & 1\times 2+2\times 3+3\times 4+ ...... +k(k+1)+(k+1)(k+2)\\ &=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \quad \{\text{using } (*)\}\\ &=\frac{k(k+1)(k+2)}{3}+\frac{3(k+1)(k+2)}{3} \quad \{\text{to equalise denominators}\}\\ &=\frac{(k+1)(k+2)[k+3]}{3} \quad \{\text{common factor of } (k+1)(k+2)\}\\ &=\frac{[k+1]([k+1]+1)([k+1]+2)}{3}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

c P_n is "$3\times 5+6\times 6+9\times 7+12\times 8+ ...... +3n(n+4)=\dfrac{n(n+1)(2n+13)}{2}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=3\times 5=15$, RHS $=\dfrac{1\times 2\times(2+13)}{2}=15$, $\therefore$ P_1 is true

(2) If P_k is true, then

$$3\times 5+6\times 6+9\times 7+ ...... +3k(k+4)=\frac{k(k+1)(2k+13)}{2} \quad ......\ (*)$$

$$\begin{aligned}\text{Now}\quad & 3\times 5+6\times 6+9\times 7+ ...... +3k(k+4)+3(k+1)(k+5)\\ &=\frac{k(k+1)(2k+13)}{2}+3(k+1)(k+5) \quad \{\text{using } (*)\}\\ &=\frac{k(k+1)(2k+13)}{2}+\frac{6(k+1)(k+5)}{2} \quad \{\text{to equalise denominators}\}\\ &=\frac{(k+1)[k(2k+13)+6(k+5)]}{2} \quad \{\text{common factor}\}\\ &=\frac{(k+1)[2k^2+19k+30]}{2}\\ &=\frac{(k+1)(k+2)(2k+15)}{2}\\ &=\frac{(k+1)([k+1]+1)(2[k+1]+13)}{2}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

d P_n is "$1^3+2^3+3^3+4^3+ ...... +n^3=\dfrac{n^2(n+1)^2}{4}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1^3=1$, RHS $=\dfrac{1^2(2)^2}{4}=1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^3+2^3+3^3+\ldots\ldots+k^3=\dfrac{k^2(k+1)^2}{4}$ ...... $(*)$

Now $1^3+2^3+3^3+\ldots\ldots+k^3+(k+1)^3$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3 \quad \{\text{using } (*)\}$$

$$=\frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} \quad \{\text{equalising denominators}\}$$

$$=\frac{(k+1)^2[k^2+4(k+1)]}{4} \quad \{\text{common factor}\}$$

$$=\frac{(k+1)^2[k^2+4k+4]}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

e P_n is "$1+2\times2+3\times2^2+4\times2^3+\ldots\ldots+n\times2^{n-1}=(n-1)\times2^n+1$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1$, RHS $=(0)2^0+1=1$, $\therefore$ P_1 is true

(2) If P_k is true, then

$$1+2\times2+3\times2^2+4\times2^3+\ldots\ldots+k\times2^{k-1}=(k-1)2^k+1 \quad \ldots\ldots (*)$$

Now $1+2\times2+3\times2^2+4\times2^3\ldots\ldots+k\,2^{k-1}+(k+1)2^k$

$$=(k-1)2^k+1+(k+1)2^k \quad \{\text{using } (*)\}$$

$$=2^k[k-1+k+1]+1$$

$$=2^k[2k]+1$$

$$=k\,2^{k+1}+1$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

2 **a** P_n is "$\dfrac{1}{1\times2}+\dfrac{1}{2\times3}+\dfrac{1}{3\times4}+\ldots\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times2}=\dfrac{1}{2}$, RHS $=\dfrac{1}{1+1}=\dfrac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\dfrac{1}{1\times2}+\dfrac{1}{2\times3}+\dfrac{1}{3\times4}+\ldots\ldots+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$ ...... $(*)$

Now $\dfrac{1}{1\times2}+\dfrac{1}{2\times3}+\dfrac{1}{3\times4}+\ldots\ldots+\dfrac{1}{k(k+1)}+\dfrac{1}{(k+1)(k+2)}$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \{\text{using } (*)\}$$

$$=\frac{k}{k+1}\left(\frac{k+2}{k+2}\right)+\frac{1}{(k+1)(k+2)} \quad \{\text{equalising denominators}\}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{k+1}{[k+1]+1}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

and $$\frac{1}{10\times 11}+\frac{1}{11\times 12}+\frac{1}{12\times 13}+ ...... +\frac{1}{20\times 21} = S_{20}-S_9$$
$$= \tfrac{20}{21}-\tfrac{9}{10}$$
$$= \tfrac{11}{210}$$

b P_n is "$\dfrac{1}{1\times 2\times 3}+\dfrac{1}{2\times 3\times 4}+\dfrac{1}{3\times 4\times 5}+ ...... +\dfrac{1}{n(n+1)(n+2)}=\dfrac{n(n+3)}{4(n+1)(n+2)}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times 2\times 3}=\dfrac{1}{6}$, RHS $=\dfrac{1(4)}{4(2)(3)}=\dfrac{1}{6}$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\frac{1}{3\times 4\times 5}+ ..... +\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \quad ...... (*)$$

Now $$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+ ..... +\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \quad \{\text{using } (*)\}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)}\left(\frac{k+3}{k+3}\right)+\frac{4}{4(k+1)(k+2)(k+3)} \quad \{\text{equalising denominators}\}$$

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$\begin{array}{r|rrr|r} -1 & 1 & 6 & 9 & 4 \\ & 0 & -1 & -5 & -4 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

$$= \frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

3 **a** P_n is "$1\times 1!+2\times 2!+3\times 3!+ ...... +n\times n!=(n+1)!-1$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $= 1\times 1! = 1\times 1 = 1$ RHS $= 2!-1 = 2-1 = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1\times 1!+2\times 2!+3\times 3!+ ...... +k\times k!=(k+1)!-1$ ...... $(*)$

Now $$1\times 1!+2\times 2!+3\times 3!+ ...... +k\times k!+(k+1)(k+1)!$$

$$= (k+1)!-1+(k+1)(k+1)! \quad \{\text{using } (*)\}$$

$$= (k+1)![1+k+1]-1$$

$$= (k+1)!(k+2)-1$$

$$= (k+2)!-1$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

b P_n is "$\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+......+\dfrac{n}{(n+1)!}=\dfrac{(n+1)!-1}{(n+1)!}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=\dfrac{1}{2!}=\dfrac{1}{2}$, RHS $=\dfrac{2!-1}{2!}=\dfrac{2-1}{2}=\dfrac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+......+\frac{k}{(k+1)!}=\frac{(k+1)!-1}{(k+1)!} \quad ......(*)$$

Now $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+......+\dfrac{k}{(k+1)!}+\dfrac{k+1}{(k+2)!}$

$$=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!} \quad \{\text{using } (*)\}$$

$$=\left(\frac{k+2}{k+2}\right)\left[\frac{(k+1)!-1}{(k+1)!}\right]+\frac{k+1}{(k+2)!} \quad \{\text{equalising denominators}\}$$

$$=\frac{(k+2)!-(k+2)+k+1}{(k+2)!}$$

$$=\frac{(k+2)!-k-2+k+1}{(k+2)!}$$

$$=\frac{(k+2)!-1}{(k+2)!}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

and $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+......+\dfrac{9}{10!}=\dfrac{10!-1}{10!}=\dfrac{3\,628\,799}{3\,628\,800}$

4 P_n is "$1\times n+2\times(n-1)+3\times(n-2)+......+(n-1)\times 2+n\times 1=\dfrac{n(n+1)(n+2)}{6}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1\times 1=1$, RHS $=\dfrac{1(2)(3)}{6}=1$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1\times k+2(k-1)+3(k-2)+......+(k-1)2+k\times 1=\frac{k(k+1)(k+2)}{6} \quad ......(*)$$

Now $1(k+1)+2(k)+3(k-1)+......+k\,2+(k+1)1$

$$=1(k)+2(k-1)+3(k-2)+......+k\,1+1+2+3+......+k+k+1$$

{using the hint}

$$=\frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2} \quad \{\text{using } (*) \text{ and the sum of an Arith. series}\}$$

$$=\frac{k(k+1)(k+2)}{6}+\frac{3(k+1)(k+2)}{6} \quad \{\text{equalising denominators}\}$$

$$=\frac{(k+1)(k+2)[k+3]}{6}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

5 **a** P_n is "$n^3 + 2n$ is divisible by 3 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $1^3 + 2(1) = 3$ which is divisible by 3

(2) If P_k is true, then $k^3 + 2k = 3A$ where $A \in Z$ ...... $(*)$

Now
$$\begin{aligned}&(k+1)^3 + 2(k+1)\\&= k^3 + 3k^2 + 3k + 1 + 2k + 2\\&= (k^3 + 2k) + 3k^2 + 3k + 3\\&= 3A + 3k^2 + 3k + 3 \quad \{\text{using } (*)\}\\&= 3[A + k^2 + k + 1] \quad \text{where } A + k^2 + k + 1 \text{ is an integer}\\&\qquad\qquad\qquad\qquad \text{as } A \text{ and } k \text{ are integers}\end{aligned}$$

$\therefore$ $(k+1)^3 + 2(k+1)$ is divisible by 3

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

b P_n is "$n(n^2 + 5)$ is divisible by 6 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $1(1^2 + 5) = 1 \times 6 = 6$ which is divisible by 6 $\therefore$ P_1 is true

(2) If P_k is true, then $k(k^2 + 5) = 6A$ where A is an integer ...... $(*)$

Now
$$\begin{aligned}(k+1)[(k+1)^2 + 5] &= (k+1)(k^2 + 2k + 1 + 5)\\&= (k+1)(k^2 + 2k + 6)\\&= k^3 + 2k^2 + 6k + k^2 + 2k + 6\\&= k^3 + 5k + [3k^2 + 3k + 6]\\&= k(k^2 + 5) + 3k(k+1) + 6\\&= 6A + 6 + 3k(k+1)\end{aligned}$$

We notice that $k(k+1)$ is the product of consecutive integers, one of which must be even $\therefore$ $k(k+1) = 2B$ where $B \in Z$

$$\begin{aligned}\therefore\quad (k+1)[(k+1)^2 + 5] &= 6A + 6 + 3(2B)\\&= 6(A + 1 + B) \quad \text{where } A + 1 + B \in Z\end{aligned}$$

$\therefore$ $(k+1)[(k+1)^2 + 5]$ is divisible by 6

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

c P_n is "$6^n - 1$ is divisible by 5 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $6^1 - 1 = 5$ which is divisible by 5 $\therefore$ P_1 is true

(2) If P_k is true, then $6^k - 1 = 5A$ where $A \in Z$ ...... $(*)$

Now
$$\begin{aligned}&6^{k+1} - 1\\&= 6^k \times 6 - 1\\&= 6[5A + 1] - 1 \quad \{\text{using } (*)\}\\&= 30A + 6 - 1\\&= 30A + 5\\&= 5(6A + 1) \quad \text{where } 6A + 1 \in Z\end{aligned}$$

i.e., $6^{k+1} - 1$ is divisible by 5

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

d P_n is "$7^n - 4^n - 3^n$ is divisible by 12 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $7^1 - 4^1 - 3^1 = 0$ which is divisible by 12 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k - 4^k - 3^k = 12A$ where $A \in Z$ ...... $(*)$

Now
$$\begin{aligned} & 7^{k+1} - 4^{k+1} - 3^{k+1} \\ &= 7(7^k) - 4(4^k) - 3(3^k) \\ &= 7[12A + 4^k + 3^k] - 4(4^k) - 3(3^k) \quad \{\text{using } (*)\} \\ &= 84A + 7(4^k) + 7(3^k) - 4(4^k) - 3(3^k) \\ &= 84A + 3(4^k) + 4(3^k) \\ &= 84A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1} \\ &= 12(7A + 4^{k-1} + 3^{k-1}) \quad \text{where } k \geqslant 2,\ k \in Z^+ \\ &= 12 \times \text{an integer} \quad \{\text{as } 4^{k-1} \text{ and } 3^{k-1} \text{ are integers}\} \end{aligned}$$

$\therefore$ $7^{k+1} - 4^{k+1} - 3^{k+1}$ is divisible by 12

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

6 P_n is "$\dfrac{2^n - (-1)^n}{3}$ is an odd number for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\dfrac{2^1 - (-1)^1}{3} = \dfrac{3}{3} = 1$ which is odd $\therefore$ P_1 is true

(2) If P_k is true, then $\dfrac{2^k - (-1)^k}{3} = 2A + 1$ where $A \in Z$ ...... $(*)$

Now
$$\begin{aligned} \frac{2^{k+1} - (-1)^{k+1}}{3} &= \frac{2(2^k) - (-1)^{k+1}}{3} \\ &= \frac{2[6A + 3 + (-1)^k] - (-1)^{k+1}}{3} \\ &= \frac{12A + 6 + 2(-1)^k - (-1)(-1)^k}{3} \\ &= \frac{12A + 6 + 2(-1)^k + (-1)^k}{3} \\ &= \frac{12A + 6 + 3(-1)^k}{3} \\ &= 4A + 2 + (-1)^k \end{aligned}$$

Now $4A + 2$ is always even and $(-1)^k$ is either $+1$ or -1

$\therefore$ $4A + 2 + (-1)^k$ is odd

$\therefore$ $\dfrac{2^{k+1} - (-1)^{k+1}}{3}$ is odd

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

7 **a** P_n is "$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \ldots\ldots \left(1 - \dfrac{1}{n+1}\right) = \dfrac{1}{n+1}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS $= \left(1 - \frac{1}{2}\right) = \frac{1}{2}$, RHS $= \dfrac{1}{1+1} = \dfrac{1}{2}$ $\therefore$ P_1 is true

(2) If P_k is true, then $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\ldots\left(1-\frac{1}{k+1}\right)=\frac{1}{k+1}$ ...... $(*)$

$$\therefore \quad \left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{4}\right)\ldots\ldots\left(1-\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right)$$

$$=\frac{1}{k+1}\left(1-\frac{1}{k+2}\right) \quad \{\text{using } (*)\}$$

$$=\frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right)$$

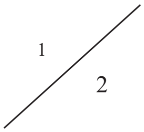
$$=\frac{1}{k+1}\left(\frac{k+1}{k+2}\right)$$

$$=\frac{1}{k+2}$$

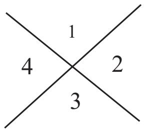
Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

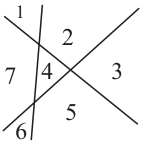
b



1 line, $R_1 = 2$



2 lines, $R_2 = 4$



3 lines, $R_3 = 7$, etc.

P_n is "For n lines as described, $R_n=\frac{n(n+1)}{2}+1$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, $R_1=\frac{1(2)}{2}+1=1+1=2$ $\therefore$ P_1 is true

(2) If P_k is true, then $R_k=\frac{k(k+1)}{2}+1$ ...... $(*)$

Now the addition of another line creates another $k+1$ regions

$$\therefore \quad R_{k+1}=\frac{k(k+1)}{2}+1+k+1$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}+1$$

$$=\frac{k^2+k+2k+2}{2}+1$$

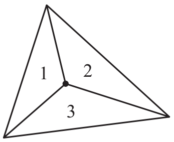
$$=\frac{k^2+3k+2}{2}+1$$

$$=\frac{(k+1)(k+2)}{2}+1$$

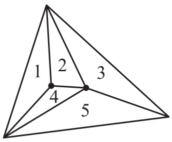
Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

c



$n = 1$
$T_1 = 3$



$n = 2$
$T_2 = 5$

P_n is "For n points inside the triangle (as described) there are $T_n = 2n+1$ triangular partitions for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

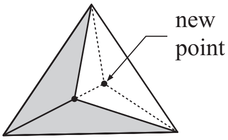
(1) If $n=1$, $T_1=2(1)+1=3$ $\therefore$ P_1 is true

(2) If P_k is true, then $T_k = 2k + 1$ ...... $(*)$

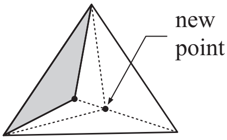
Adding an extra point within the triangle gives the $k+1^{\text{th}}$ case.

This point could be either

- in an existing triangle or
- on an existing line between 2 triangles



So, 1 triangle becomes 3, a net increase of 2.



So, 2 triangles become 4, a net increase of 2.

In each case 2 triangles are added

$$\begin{aligned}\therefore\quad T_{k+1} &= T_k + 2\\ &= 2k+1+2 \quad \{\text{using } (*)\}\\ &= 2(k+1)+1\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

d P_n is "$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)......\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all $n \geqslant 2$, $n \in Z$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 2$, LHS $= 1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$ RHS $= \frac{2+1}{2(2)} = \frac{3}{4}$ $\therefore$ P_2 is true

(2) If P_k is true, then $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right).....\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$ ..... $(*)$

Now $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right).....\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$

$$\begin{aligned}&= \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \quad \{\text{using } (*)\}\\ &= \frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\ &= \frac{k+1}{2k}\left(\frac{k^2+2k+1-1}{(k+1)^2}\right)\\ &= \frac{k^2+2k}{2k(k+1)}\\ &= \frac{k(k+2)}{2k(k+1)}\\ &= \frac{k+2}{2(k+1)}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_2 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

EXERCISE 10C

1 **Proof** (by contradiction)

Let x be the positive number, so we need to prove that $x+9\left(\frac{1}{x}\right) \geqslant 6$

So, we suppose the opposite, i.e., $x+\frac{9}{x} < 6$ for all x

$$\therefore\ x\left(x+\frac{9}{x}\right) < 6x \quad \text{as } x>0$$
$$\therefore\ x^2+9 < 6x$$
$$\therefore\ x^2-6x+9 < 0$$
$$\therefore\ (x-3)^2 < 0$$

which is false as no perfect square can be negative

Hence, the supposition is false and its opposite is true i.e., $x+9\left(\frac{1}{x}\right) \geqslant 6$ for all x

i.e., the sum of a positive number and 9 times its reciprocal is at least 6.

2 **Proof** (by contradiction)

Suppose the solution of $3^x = 4$ is rational, i.e., $x=\frac{p}{q}$, $q \neq 0$ where p and q are integers having no common factors.

$$\therefore\ 3^{\frac{p}{q}} = 4$$
$$\therefore\ (3^{\frac{p}{q}})^q = 4^q$$
$$\therefore\ 3^p = 4^q$$

which is a contradiction as 3^p is odd and 4^q is even

$\therefore$ the supposition is false and so the solution of $3^x = 4$ is irrational.

3 **Proof** (by contradiction)

Suppose $\log_2 5$ is rational, i.e., $\log_2 5 = \frac{p}{q}$ where p and q are integers, $q \neq 0$ and p and q have no common factors.

$$\therefore\ 5 = 2^{\frac{p}{q}}$$
$$\therefore\ 5^q = (2^{\frac{p}{q}})^q$$
$$\therefore\ 5^q = 2^p$$

which is a contradiction as 5^q is odd and 2^p is even

$\therefore$ the supposition is false $\therefore$ $\log_2 5$ is irrational.

4 $\sqrt{2}$ is irrational **Proof** (by contradiction)

Suppose $\sqrt{2}$ is rational $\therefore$ $\sqrt{2} = \frac{p}{q}$ where p and q are $\in Z$, $q \neq 0$ and p and q have no common factors. ...... (1)

$\therefore$ $2 = \frac{p^2}{q^2}$ and so $p^2 = 2q^2$ ...... (2)

$\therefore$ p^2 is even

$\therefore$ p is even {the only even perfect squares come from squaring evens}

$\therefore$ $p = 2a$ where $a \in Z$

Back into (2) $(2a)^2 = 2q^2$

$\therefore$ $4a^2 = 2q^2$

$\therefore$ $q^2 = 2a^2$ $\therefore$ q^2 is even

$\therefore$ q is even

So, on the supposition, p and q are even and so have a common factor of 2.

This contradicts (1), so the supposition is false $\therefore$ $\sqrt{2}$ is irrational.

REVIEW SET 10A

1 P_n is "$1+3+5+7+ ...... +(2n-1)=n^2$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1$ and RHS $=1^2=1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1+3+5+7+ ...... +(2k-1)=k^2$ ...... $(*)$

$$\therefore\ 1+3+5+7+ ...... +(2k-1)+(2[k+1]-1)$$
$$=k^2+2k+2-1 \quad \{\text{using } (*)\}$$
$$=k^2+2k+1$$
$$=(k+1)^2$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

2 P_n is "7^n+2 is divisible by 3 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, $7^1+2=9$ which is divisible by 3 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k+2=3A$ where $A \in Z$

$$\begin{aligned}\therefore\ 7^{k+1}+2 &= 7\times 7^k+2\\ &= 7[3A-2]+2\\ &= 21A-14+2\\ &= 21A-12\\ &= 3(7A-4) \quad \text{where } 7A-4 \text{ is an integer as } A \text{ is an integer}\end{aligned}$$

$\therefore$ $7^{k+1}+2$ is divisible by 3

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

3 P_n is "$1\times2\times3+2\times3\times4+3\times4\times5+ ...... +n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1\times2\times3=6$, RHS $=\dfrac{1\times2\times3\times4}{4}=6$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1\times2\times3+2\times3\times4+ ...... +k(k+1)(k+2)=\frac{k(k+1)(k+2)(k+3)}{4} \ ......\ (*)$$

$$\therefore\ 1\times2\times3+2\times3\times4+ ...... +k(k+1)(k+2)+(k+1)(k+2)(k+3)$$
$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3) \quad \{\text{using } (*)\}$$
$$=\frac{k(k+1)(k+2)(k+3)}{4}+\frac{4(k+1)(k+2)(k+3)}{4} \quad \{\text{equalising denominators}\}$$
$$=\frac{(k+1)(k+2)(k+3)[k+4]}{4}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

4 P_n is "$1+r+r^2+r^3+ ...... +r^{n-1}=\dfrac{1-r^n}{1-r}$ for all $n \in Z^+$, $r \neq 1$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1$ and RHS $=\dfrac{1-r}{1-r}=1$ as $r \neq 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1+r+r^2+r^3+......r^{k-1} = \dfrac{1-r^k}{1-r}$ ...... $(*)$

Now $1+r+r^2+r^3+......+r^{k-1}+r^k$

$$= \frac{1-r^k}{1-r} + r^k \quad \{\text{using } (*)\}$$

$$= \frac{1-r^k}{1-r} + r^k\left(\frac{1-r}{1-r}\right) \quad \{\text{equalising denominators}\}$$

$$= \frac{1-r^k+r^k-r^{k+1}}{1-r}$$

$$= \frac{1-r^{k+1}}{1-r}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

5 P_n is "$5^{2n}-1$ is divisible by 24 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, $5^2-1=25-1=24$ is divisible by 24 $\therefore$ P_1 is true

(2) If P_k is true, then $5^{2k}-1=24A$ where $A \in Z$

Now $5^{2(k+1)}-1 = 5^{2k}5^2-1$

$$= 25[24A+1]-1$$
$$= 25 \times 24A + 25 - 1$$
$$= 25 \times 24A + 24$$
$$= 24(25A+1) \quad \text{where } 25A+1 \text{ is an integer}$$

$\therefore$ $5^{2(k+1)}-1$ is divisible by 24

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

REVIEW SET 10B

1 P_n is "$1^2+3^2+5^2+7^2+......+(2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1^2=1$, RHS $= \dfrac{1\times3\times1}{3} = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then $1^2+3^2+5^2+......+(2k-1)^2 = \dfrac{k(2k+1)(2k-1)}{3}$ ...... $(*)$

$\therefore$ $1^2+3^2+5^2+......+(2k-1)^2+(2k+1)^2$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2 \quad \{\text{using } (*)\}$$

$$= \frac{k(2k+1)(2k-1)}{3} + \frac{3(2k+1)^2}{3} \quad \{\text{equalising denominators}\}$$

$$= \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$= \frac{(2k+1)[2k^2-k+6k+3]}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{[k+1](2[k+1]+1)(2[k+1]-1)}{3}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true $\therefore$ P_n is true {Princ. of Math. Induction}

2 P_n is "$3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $3^4 - 8 - 9 = 81 - 17 = 64$ which is divisible by 64 $\therefore$ P_1 is true

(2) If P_k is true, then $3^{2k+2} - 8k - 9 = 64A$ where $A \in Z$

Now
$$\begin{aligned} & 3^{2(k+1)+2} - 8(k+1) - 9 \\ &= 3^{2k+2} \times 3^2 - 8k - 8 - 9 \\ &= 9[64A + 8k + 9] - 8k - 17 \\ &= 9 \times 64A + 72k + 81 - 8k - 17 \\ &= 9 \times 64A + 64k + 64 \\ &= 64(9A + k + 1) \quad \text{where } 9A + k + 1 \text{ is } \in Z \text{ as } A, k \in Z \end{aligned}$$

$\therefore$ $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 64

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

3 P_n is "$3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2n+1)2^{n-1} = 1 + (2n-1) \times 2^n$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS $= 3$ and RHS $= 1 + 1 \times 2^1 = 1 + 2 = 3$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2k+1)2^{k-1} = 1 + (2k-1) \times 2^k \;......\; (*)$$

$\therefore$ $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + ... + (2k+1)2^{k-1} + (2k+3)2^k$
$$\begin{aligned} &= 1 + (2k-1)2^k + (2k+3)2^k \quad \{\text{using } (*)\} \\ &= 1 + 2^k[2k - 1 + 2k + 3] \\ &= 1 + 2^k[4k + 2] \\ &= 1 + 2^k(2)(2k+1) \\ &= 1 + (2k+1)2^{k+1} \\ &= 1 + (2[k+1] - 1)2^{[k+1]} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

4 P_n is "$5^n + 3$ is divisible by 4 for all $n \in Z$, $n \geqslant 0$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 0$, $5^0 + 3 = 4$ which is divisible by 4 $\therefore$ P_0 is true

(2) If P_k is true, then $5^k + 3 = 4A$ where A is in Z ...... $(*)$

Now
$$\begin{aligned} 5^{k+1} + 3 &= 5 \times 5^k + 3 \\ &= 5[4A - 3] + 3 \quad \{\text{using } (*)\} \\ &= 20A - 15 + 3 \\ &= 20A - 12 \\ &= 4(5A - 3) \quad \text{where } 5A - 3 \text{ is in } Z, \text{ as } A \text{ is in } Z \end{aligned}$$

So, $5^{k+1} + 3$ is divisible by 4

Thus P_{k+1} is true whenever P_k is true and P_0 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

5 P_n is "$1\times 2^2+2\times 3^2+3\times 4^2+......+n(n+1)^2=\dfrac{n(n+1)(n+2)(3n+5)}{12}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1\times 2^2=4$ and RHS $=\dfrac{1\times 2\times 3\times 8}{12}=\dfrac{48}{12}=4$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1\times 2^2+2\times 3^2+3\times 4^2+......+k(k+1)^2=\frac{k(k+1)(k+2)(3k+5)}{12} \quad ......\ (*)$$

$$\therefore\quad 1\times 2^2+2\times 3^2+3\times 4^2+......+k(k+1)^2+(k+1)(k+2)^2$$

$$=\frac{k(k+1)(k+2)(3k+5)}{12}+(k+1)(k+2)^2 \quad \{\text{using } (*)\}$$

$$=\frac{k(k+1)(k+2)(3k+5)}{12}+\frac{12(k+1)(k+2)^2}{12} \quad \{\text{equalising denominators}\}$$

$$=\frac{(k+1)(k+2)[k(3k+5)+12(k+2)]}{12}$$

$$=\frac{(k+1)(k+2)[3k^2+5k+12k+24]}{12}$$

$$=\frac{(k+1)(k+2)(3k^2+17k+24)}{12}$$

$$=\frac{(k+1)(k+2)(k+3)(3k+8)}{12}$$

$$=\frac{[k+1]([k+1]+1)([k+1]+2)(3[k+1]+5)}{12}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

REVIEW SET 10C

1 P_n is "$1\times 3+2\times 4+3\times 5+4\times 6+......+n(n+2)=\dfrac{n(n+1)(2n+7)}{6}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=1\times 3=3$ and RHS $=\dfrac{1\times 2\times 9}{6}=\dfrac{18}{6}=3$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1\times 3+2\times 4+3\times 5+......+k(k+2)=\frac{k(k+1)(2k+7)}{6} \quad ......\ (*)$$

$$\therefore\quad 1\times 3+2\times 4+3\times 5+......+k(k+2)+(k+1)(k+3)$$

$$=\frac{k(k+1)(2k+7)}{6}+(k+1)(k+3) \quad \{\text{using } (*)\}$$

$$=\frac{k(k+1)(2k+7)}{6}+\frac{6(k+1)(k+3)}{6} \quad \{\text{equalising denominators}\}$$

$$=\frac{(k+1)[k(2k+7)+6(k+3)]}{6}$$

$$=\frac{(k+1)[2k^2+13k+18]}{6}$$

$$=\frac{(k+1)(k+2)(2k+9)}{6}$$

$$=\frac{[k+1]([k+1]+1)(2[k+1]+7)}{6}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

2 P_n is "$7^n - 1$ is divisible by 6 for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $7^1 - 1 = 6$ which is divisible by 6 $\therefore$ P_1 is true

(2) If P_k is true, then $7^k - 1 = 6A$ where $A \in Z$

Now
$$\begin{aligned} 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 7[6A+1] - 1 \\ &= 42A + 7 - 1 \\ &= 42A + 6 \\ &= 6(7A+1) \quad \text{where } 7A+1 \text{ is in } Z \end{aligned}$$

Thus $7^{k+1} - 1$ is divisible by 6

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

3 P_n is "$1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2 - 1)$ for all $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS $= 1^3 = 1$ and RHS $= 1^2(2-1) = 1 \times 1 = 1$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$1^3 + 3^3 + 5^3 + ... + (2k-1)^3 = k^2(2k^2 - 1) \quad ...... (*)$$

$\therefore$ $1^3 + 3^3 + 5^3 + ... + (2k-1)^3 + (2k+1)^3$

$$\begin{aligned} &= k^2(2k^2 - 1) + (2k+1)^3 \quad \{\text{using } (*)\} \\ &= 2k^4 - k^2 + (2k)^3 + 3(2k)^2 1 + 3(2k)1^2 + 1^3 \\ &= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\ &= (k+1)^2(2k^2 + 4k + 1) \\ &= (k+1)^2(2[k^2 + 2k + 1] - 1) \\ &= (k+1)^2(2[k+1]^2 - 1) \end{aligned}$$

$$\begin{array}{r|rrrr|r} -1 & 2 & 8 & 11 & 6 & 1 \\ & 0 & -2 & -6 & -5 & -1 \\ \hline -1 & 2 & 6 & 5 & 1 & 0 \\ & 0 & -2 & -4 & -1 & \\ \hline & 2 & 4 & 1 & 0 & \end{array}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

4 P_n is "$3^n - 1 - 2n$ is divisible by 4 for all n in Z^+, $n \geqslant 0$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 0$, $3^0 - 1 - 2(0) = 1 - 1 - 0 = 0$ which is divisible by 4 $\therefore$ P_0 is true

(2) If P_k is true, then $3^k - 1 - 2k = 4A$ where $A \in Z$ ...... $(*)$

Now $3^{k+1} - 1 - 2(k+1)$

$$\begin{aligned} &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3[4A + 1 + 2k] - 1 - 2k - 2 \quad \{\text{using } (*)\} \\ &= 12A + 3 + 6k - 1 - 2k - 2 \\ &= 12A + 4k \\ &= 4(3A + k) \quad \text{where } 3A + k \text{ is in } Z \end{aligned}$$

$\therefore$ $3^{k+1} - 1 - 2(k+1)$ is divisible by 4

Thus P_{k+1} is true whenever P_k is true and P_0 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

5 P_n is "$\dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+\dfrac{1}{5\times 7}+......+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$ for all $n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=\dfrac{1}{1\times 3}=\dfrac{1}{3}$, RHS $=\dfrac{1}{2+1}=\dfrac{1}{3}$ $\therefore$ P_1 is true

(2) If P_k is true, then

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+......+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \quad ......\ (*)$$

$$\therefore \quad \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+......+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \{\text{using } (*)\}$$

$$=\frac{k}{2k+1}\left(\frac{2k+3}{2k+3}\right)+\frac{1}{(2k+1)(2k+3)} \quad \{\text{equalising denominators}\}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$$

$$=\frac{[k+1]}{2[k+1]+1}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore$ P_n is true {Principle of Mathematical Induction}

Background knowledge

TRIGONOMETRY WITH RIGHT ANGLED TRIANGLES

EXERCISE A

1 **a**

$$\begin{aligned} 1^2 + x^2 &= 1.2^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x^2 &= 1.2^2 - 1^2 \\ \therefore\ x &= \sqrt{1.2^2 - 1^2} \\ \therefore\ x &\doteq 0.663 \end{aligned}$$

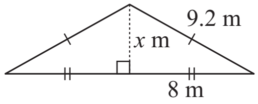
b

$$\begin{aligned} x^2 &= 3.8^2 + 2.1^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{3.8^2 + 2.1^2} \\ \therefore\ x &\doteq 4.34 \end{aligned}$$

c

$$\begin{aligned} x^2 &= 1.8^2 + 1.32^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{1.8^2 + 1.32^2} \\ \therefore\ x &\doteq 2.23 \end{aligned}$$

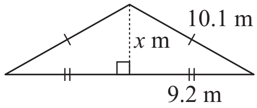
2 **a**



$$\begin{aligned} x^2 + 8^2 &= 9.2^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x^2 &= 9.2^2 - 8^2 \\ \therefore\ x &= \sqrt{9.2^2 - 8^2} \\ \therefore\ x &\doteq 4.54 \end{aligned}$$

$\therefore$ the height is 4.54 m

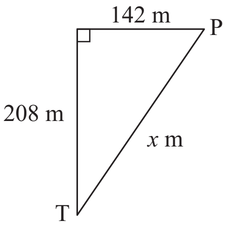
b



$$\begin{aligned} x^2 + 9.2^2 &= 10.1^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x^2 &= 10.1^2 - 9.2^2 \\ \therefore\ x &= \sqrt{10.1^2 - 9.2^2} \\ \therefore\ x &\doteq 4.17 \end{aligned}$$

$\therefore$ the height is 4.17 m

3

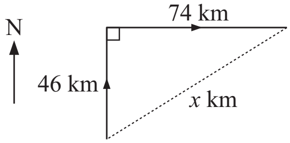


$$\begin{aligned} x^2 &= 208^2 + 142^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{208^2 + 142^2} \\ \therefore\ x &\doteq 252 \text{ m} \end{aligned}$$

$\therefore$ must hit the ball (in the air)

$$\begin{aligned} &(252 - 15)\text{ m} \\ &= 237\text{ m} \end{aligned}$$

4 **a**



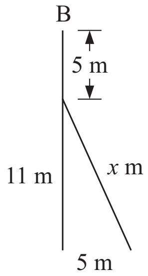
b

$$\begin{aligned} x^2 &= 46^2 + 74^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{46^2 + 74^2} \\ \therefore\ x &\doteq 87.1 \end{aligned}$$

$\therefore$ is 87.1 km from the start.

EXERCISE B

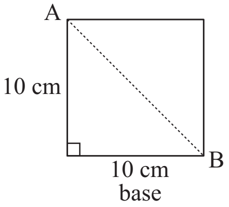
1



$$\begin{aligned} x^2 &= 5^2 + 11^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{5^2 + 11^2} \\ \therefore\ x &\doteq 12.083 \end{aligned}$$

But $4x + 2$ m is needed
i.e., $(4 \times 12.083 + 2)$ m
$\doteq 50.3$ m

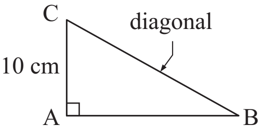
2



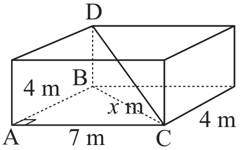
$$\begin{aligned} AB^2 &= 10^2 + 10^2 \quad \{\text{Pythagoras}\} \\ &= 100 + 100 \\ &= 200 \end{aligned}$$

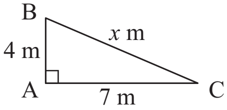
$$\begin{aligned} \text{But}\quad BC^2 &= AB^2 + AC^2 \quad \{\text{Pythagoras, again}\} \\ \therefore\quad BC^2 &= 200 + 10^2 \\ \therefore\quad BC &= \sqrt{300} \doteqdot 17.3 \end{aligned}$$

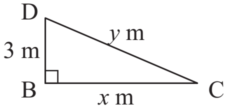
$\therefore$ the diagonal is 17.3 cm long.



3





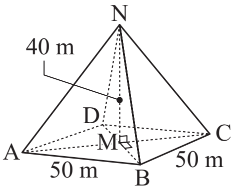


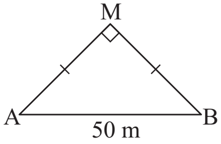
$$x^2 = 4^2 + 7^2 \quad \{\text{Pythagoras}\}$$

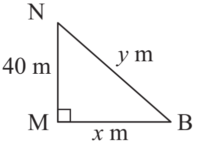
$$\begin{aligned} y^2 &= x^2 + 3^2 \quad \{\text{Pythagoras}\} \\ \therefore\quad y^2 &= 4^2 + 7^2 + 3^2 \\ \therefore\quad y &= \sqrt{4^2 + 7^2 + 3^2} \\ \therefore\quad y &\doteqdot 8.60 \end{aligned}$$

So, the distance is about 8.60 m

4







$$\begin{aligned} x^2 + x^2 &= 50^2 \quad \{\text{Pythagoras}\} \\ \therefore\quad 2x^2 &= 50^2 \\ \therefore\quad x^2 &= \tfrac{50^2}{2} \end{aligned}$$

$$\begin{aligned} y^2 &= 40^2 + x^2 \quad \{\text{Pythagoras}\} \\ \therefore\quad y^2 &= 40^2 + \tfrac{50^2}{2} \\ \therefore\quad y &= \sqrt{40^2 + \tfrac{50^2}{2}} \doteqdot 53.4 \end{aligned}$$

$\therefore$ each slant edge is 53.4 m long.

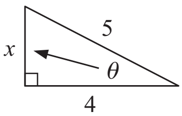
5

$$\begin{aligned} a^2 + 10^2 &= 22.5^2 \quad \text{and} \quad b^2 + 10^2 = 40.8^2 \quad \{\text{Pythagoras}\} \\ \therefore\quad a^2 + b^2 &= 22.5^2 - 10^2 + 40.8^2 - 10^2 \\ \therefore\quad \sqrt{a^2 + b^2} &= \sqrt{22.5^2 + 40.8^2 - 200} \doteqdot 44.4 \end{aligned}$$

$\therefore$ AB $\doteqdot$ 44.4 km, i.e., 44.4 km apart.

EXERCISE C

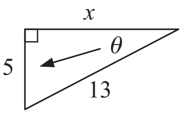
1 **a**



$$\begin{aligned} x^2 + 4^2 &= 5^2 \quad \{\text{Pythagoras}\} \\ \therefore\quad x &= \sqrt{5^2 - 4^2} \\ \therefore\quad x &= 3 \end{aligned}$$

$$\begin{aligned} \text{So,}\quad \sin\theta &= \tfrac{3}{5} \\ \cos\theta &= \tfrac{4}{5} \\ \tan\theta &= \tfrac{3}{4} \end{aligned}$$

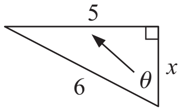
b



$$\begin{aligned} x^2 + 5^2 &= 13^2 \quad \{\text{Pythagoras}\} \\ \therefore\quad x &= \sqrt{13^2 - 5^2} \\ \therefore\quad x &= 12 \end{aligned}$$

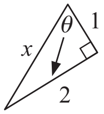
$$\begin{aligned} \text{So,}\quad \sin\theta &= \tfrac{5}{13} \\ \cos\theta &= \tfrac{12}{13} \\ \tan\theta &= \tfrac{5}{12} \end{aligned}$$

c



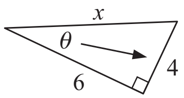
$$\begin{aligned} x^2+5^2 &= 6^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x &= \sqrt{6^2-5^2} \\ \therefore \quad x &= \sqrt{11} \\ \text{So,} \quad \sin\theta &= \tfrac{5}{6} \\ \cos\theta &= \tfrac{\sqrt{11}}{6} \\ \tan\theta &= \tfrac{5}{\sqrt{11}} \end{aligned}$$

d



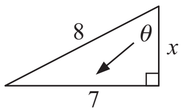
$$\begin{aligned} x^2 &= 1^2+2^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x &= \sqrt{1+4} \\ \therefore \quad x &= \sqrt{5} \\ \text{So,} \quad \sin\theta &= \tfrac{2}{\sqrt{5}} \\ \cos\theta &= \tfrac{1}{\sqrt{5}} \\ \tan\theta &= \tfrac{2}{1}=2 \end{aligned}$$

e



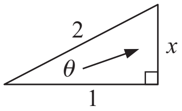
$$\begin{aligned} x^2 &= 4^2+6^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x &= \sqrt{52} \\ \text{So,} \quad \sin\theta &= \tfrac{4}{\sqrt{52}} \\ \cos\theta &= \tfrac{6}{\sqrt{52}} \\ \tan\theta &= \tfrac{4}{6}=\tfrac{2}{3} \end{aligned}$$

f



$$\begin{aligned} x^2+7^2 &= 8^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x^2 &= 8^2-7^2 \\ \therefore \quad x &= \sqrt{64-49} \\ \therefore \quad x &= \sqrt{15} \\ \text{So,} \quad \sin\theta &= \tfrac{7}{8} \\ \cos\theta &= \tfrac{\sqrt{15}}{8} \\ \tan\theta &= \tfrac{7}{\sqrt{15}} \end{aligned}$$

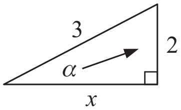
2 **a** as $\cos\theta=\tfrac{1}{2}$



$$\begin{aligned} x^2+1^2 &= 2^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad x^2 &= 4-1 \\ \therefore \quad x &= \sqrt{3} \\ \text{So,} \quad \sin\theta &= \frac{x}{2}=\tfrac{\sqrt{3}}{2} \\ \tan\theta &= \frac{x}{1}=\sqrt{3} \end{aligned}$$

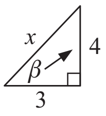
b

$$\begin{aligned} \text{as} \quad \sin\alpha &= \tfrac{2}{3} \\ \therefore \quad x^2+2^2 &= 3^2 \\ \therefore \quad x^2 &= 9-4 \\ \therefore \quad x &= \sqrt{5} \end{aligned}$$



$$\begin{aligned} \text{So,} \quad \cos\alpha &= \frac{x}{3}=\frac{\sqrt{5}}{3} \\ \tan\alpha &= \frac{2}{x}=\frac{2}{\sqrt{5}} \end{aligned}$$

c



as $\tan\beta=\tfrac{4}{3}$

$x=5 \quad \{3\text{-}4\text{-}5\ \Delta\}$

$$\begin{aligned} \text{So,} \quad \sin\beta &= \frac{4}{x}=\frac{4}{5} \\ \cos\beta &= \frac{3}{x}=\frac{3}{5} \end{aligned}$$

3 **a**

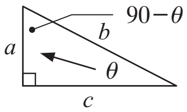
$$\begin{aligned} \sin\theta &= \frac{b}{c} \\ \cos\theta &= \frac{a}{c} \\ \tan\theta &= \frac{b}{a} \end{aligned}$$

b

$$\frac{\sin\theta}{\cos\theta}=\frac{\frac{b}{c}}{\frac{a}{c}}=\frac{b}{c}\times\frac{c}{a}=\frac{b}{a}$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta}=\tan\theta$$

4 **a**

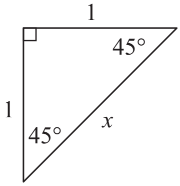


i $\sin\theta = \frac{a}{b}$ **ii** $\cos\theta = \frac{c}{b}$ **iii** $\sin(90-\theta) = \frac{c}{b}$ **iv** $\cos(90-\theta) = \frac{a}{b}$

b **i** The sine of an angle is the cosine of its complement.

ii The cosine of an angle is the sine of its complement.

5



a $x^2 = 1^2 + 1^2 = 2$

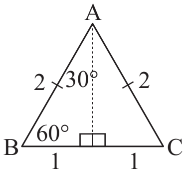
$\therefore\ x = \sqrt{2}$ {Pythagoras}

b $\sin 45^o = \frac{1}{x} = \frac{1}{\sqrt{2}}$

$\cos 45^o = \frac{1}{x} = \frac{1}{\sqrt{2}}$

$\tan 45^o = \frac{1}{1} = 1$

6



a $\measuredangle\text{ABN} = 60^o$

$\measuredangle\text{BAN} = 30^o$

b ΔABC is equilateral and AN is perpendicular to BC.

$\therefore$ N is the midpoint of BC.

$\therefore\ \text{BN} = 1$

$\therefore\ \text{AN}^2 = 2^2 - 1^2$ {Pythagoras}

$\therefore\ \text{AN} = \sqrt{3}$

c **i** $\sin 60^o = \frac{\text{AN}}{2} = \frac{\sqrt{3}}{2}$

$\cos 60^o = \frac{1}{2}$

$\tan 60^o = \frac{\text{AN}}{1} = \sqrt{3}$

ii $\sin 30^o = \frac{1}{2}$

$\cos 30^o = \frac{\text{AN}}{2} = \frac{\sqrt{3}}{2}$

$\tan 30^o = \frac{1}{\text{AN}} = \frac{1}{\sqrt{3}}$

EXERCISE D

1 **a** $\sin 35^o = \frac{x}{30}$

$\therefore\ 30 \times \sin 35^o = x$

$\therefore\ x \doteq 17.2$

b $\cos 50^o = \frac{x}{400}$

$\therefore\ 400 \times \cos 50^o = x$

$\therefore\ x \doteq 257$

c $\tan 60^o = \frac{x}{8.7}$

$\therefore\ \tan 60^o \times 8.7 = x$

$\therefore\ x \doteq 15.1$

d $\cos 65^o = \frac{3}{x}$

$\therefore\ x = \frac{3}{\cos 65^o}$

$\therefore\ x \doteq 7.10$

e $\tan 36.7^o = \frac{413}{x}$

$\therefore\ x = \frac{413}{\tan 36.7}$

$\therefore\ x \doteq 554$

f $\sin 53.9 = \frac{369}{x}$

$\therefore\ x = \frac{369}{\sin 53.9}$

$\therefore\ x \doteq 457$

2 **a** $\sin\theta = 0.9364$

$\therefore\ \theta = \sin^{-1}(0.9364)$

$\therefore\ \theta \doteq 69.5^o$

b $\cos\theta = 0.2381$

$\therefore\ \theta = \cos^{-1}(0.2381)$

$\therefore\ \theta \doteq 76.2^o$

c $\tan\theta = 1.7321$

$\therefore\ \theta = \tan^{-1}(1.7321)$

$\therefore\ \theta \doteq 60.0^o$

d $\cos\theta = \frac{2}{7}$

$\therefore\ \theta = \cos^{-1}\left(\frac{2}{7}\right)$

$\therefore\ \theta \doteq 73.4^o$

e $\sin\theta = \frac{1}{3}$

$\therefore\ \theta = \sin^{-1}\left(\frac{1}{3}\right)$

$\therefore\ \theta \doteq 19.5^o$

f $\tan\theta = \frac{14}{3}$

$\therefore\ \theta = \tan^{-1}\left(\frac{14}{3}\right)$

$\therefore\ \theta \doteq 77.9^o$

g $\sin\theta = \frac{\sqrt{3}}{11}$

$\therefore\ \theta = \sin^{-1}\left(\frac{\sqrt{3}}{11}\right)$

$\therefore\ \theta \doteq 9.06^o$

h $\cos\theta = \frac{5}{\sqrt{37}}$

$\therefore\ \theta = \cos^{-1}\left(\frac{5}{\sqrt{37}}\right)$

$\therefore\ \theta \doteq 34.7^o$

3 **a** $\sin\theta = \frac{5}{6}$

$\therefore\ \theta = \sin^{-1}\left(\frac{5}{6}\right)$

$\therefore\ \theta \doteq 56.4$

b $\tan\alpha = \frac{1}{12}$

$\therefore\ \alpha = \tan^{-1}\left(\frac{1}{12}\right)$

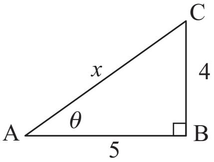
$\therefore\ \alpha \doteq 4.76$

c $\cos\beta = \frac{4}{6}$

$\therefore\ \beta = \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore\ \beta \doteq 48.2$

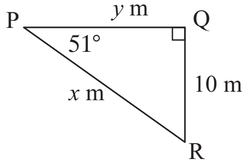
4 **a**



$$\begin{aligned} x^2 &= 4^2+5^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{4^2+5^2} \\ \therefore\ x &= \sqrt{41} \\ \therefore\ \text{AC} &= \sqrt{41}\text{ m} \\ \text{or} &\doteqdot 6.40\text{ m} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \tfrac{4}{5} \\ \therefore\ \theta &= \tan^{-1}\left(\tfrac{4}{5}\right) \\ \therefore\ \theta &\doteqdot 38.7 \\ \therefore\ \measuredangle\text{A} &\doteqdot 38.7^o, \quad \measuredangle\text{C} \doteqdot 51.3^o \end{aligned}$$

b



$$\begin{aligned} \measuredangle\text{R} &= 90^o - 51^o \\ &= 39^o \end{aligned}$$

$$\begin{aligned} \sin 51^o &= \frac{10}{x} \\ \therefore\ x &= \frac{10}{\sin 51^o} \\ \therefore\ x &\doteqdot 12.9 \\ \text{So,}\quad \text{PR} &\doteqdot 12.9\text{ m} \end{aligned}$$

$$\begin{aligned} \tan 51^o &= \frac{10}{y} \\ \therefore\ y &= \frac{10}{\tan 51^o} \\ \therefore\ y &\doteqdot 8.10 \\ \text{So,}\quad \text{PQ} &\doteqdot 8.10\text{ m} \end{aligned}$$

5 **a**

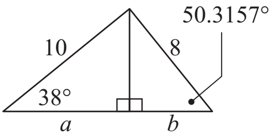
$$\begin{aligned} x^2+3^2 &= 4^2 \quad \{\text{Pythagoras}\} \\ \therefore\ x &= \sqrt{4^2-3^2} \\ \therefore\ x &= \sqrt{7} \quad (\text{or } 2.65) \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{2}{x} = \frac{2}{\sqrt{7}} \\ \therefore\ \theta &= \tan^{-1}\left(\tfrac{2}{\sqrt{7}}\right) \quad \text{and so} \quad \theta \doteqdot 37.1 \end{aligned}$$

b

$$\begin{aligned} \sin 38^o &= \frac{x}{10} \\ \therefore\ 10\times\sin 38^o &= x \\ \therefore\ x &\doteqdot 6.16 \end{aligned}$$

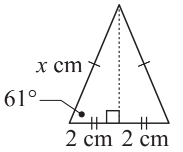
$$\begin{aligned} \sin\theta &= \frac{x}{8} \\ \therefore\ \sin\theta &\doteqdot \frac{6.1566}{8} \\ \therefore\ \theta &\doteqdot \sin^{-1}\left(\tfrac{6.1566}{8}\right) \\ \therefore\ \theta &\doteqdot 50.3 \end{aligned}$$



$$\cos 38^o = \frac{a}{10} \quad \text{and} \quad \cos 50.3157^o = \frac{b}{8}$$

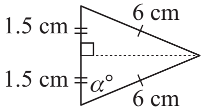
$$\begin{aligned} \text{Now}\quad y &= a+b \\ &= 10\cos 38^o + 8\cos 50.3157^o \\ &\doteqdot 13.0 \end{aligned}$$

6 **a**



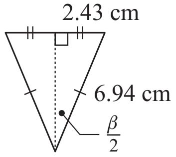
$$\begin{aligned} \cos 61^o &= \frac{2}{x} \\ \therefore\ x &= \frac{2}{\cos 61^o} \\ \therefore\ x &\doteqdot 4.13 \end{aligned}$$

b



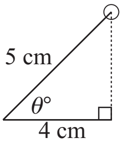
$$\begin{aligned} \cos\alpha &= \frac{1.5}{6} \\ \therefore\ \alpha &= \cos^{-1}\left(\frac{1.5}{6}\right) \\ \therefore\ \alpha &\doteqdot 75.5 \end{aligned}$$

c



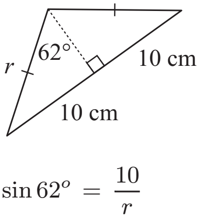
$$\begin{aligned} \sin\left(\frac{\beta}{2}\right) &= \tfrac{2.43}{6.94} \\ \therefore\ \beta &= 2\times\sin^{-1}\left(\tfrac{2.43}{6.94}\right) \\ \therefore\ \beta &\doteqdot 41.0 \end{aligned}$$

7 **a**



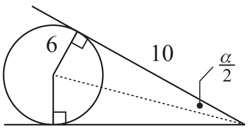
$$\begin{aligned} \cos\theta &= \tfrac{4}{5} \\ \therefore\ \theta &= \cos^{-1}\left(\tfrac{4}{5}\right) \\ \therefore\ \theta &\doteqdot 36.9 \end{aligned}$$

b



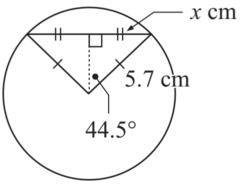
$$\begin{aligned} \sin 62^o &= \frac{10}{r} \\ \therefore\ r &= \frac{10}{\sin 62^o} \\ \therefore\ r &\doteqdot 11.3 \end{aligned}$$

c



$$\begin{aligned} \tan\left(\frac{\alpha}{2}\right) &= \tfrac{6}{10} = 0.6 \\ \therefore\ \alpha &= 2\times\tan^{-1}(0.6) \\ \therefore\ \alpha &\doteqdot 61.9 \end{aligned}$$

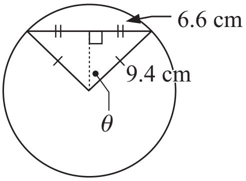
8



$$\begin{aligned}\sin 44.5^o &= \frac{x}{5.7}\\ \therefore\quad 5.7\times\sin 44.5^o &= x\\ \therefore\quad x &\doteqdot 3.995\\ \therefore\quad 2x &\doteqdot 7.99\end{aligned}$$

$\therefore$ chord is 7.99 cm long.

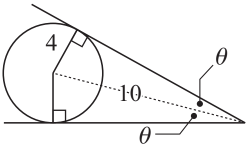
9



$$\begin{aligned}\sin\theta &= \tfrac{6.6}{9.4}\\ \therefore\quad \theta &= \sin^{-1}\left(\tfrac{6.6}{9.4}\right)\\ \therefore\quad \theta &\doteqdot 44.6\\ \therefore\quad 2\theta &\doteqdot 89.2\end{aligned}$$

$\therefore$ angle is 89.2^o

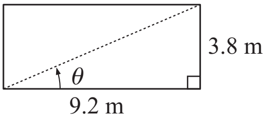
10



$$\begin{aligned}\sin\theta &= \tfrac{4}{10} = 0.4\\ \therefore\quad \theta &= \sin^{-1}(0.4)\\ \therefore\quad 2\theta &= 2\times\sin^{-1}(0.4)\\ &\doteqdot 47.2\end{aligned}$$

$\therefore$ angle is 47.2^o

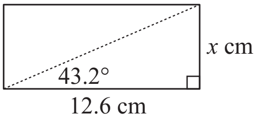
11



$$\begin{aligned}\tan\theta &= \frac{3.8}{9.2}\\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{3.8}{9.2}\right)\\ \therefore\quad \theta &\doteqdot 22.4\end{aligned}$$

$\therefore$ the angle is 22.4^o

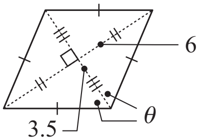
12



$$\begin{aligned}\tan 43.2^o &= \frac{x}{12.6}\\ \therefore\quad 12.6\times\tan 43.2^o &= x\\ \therefore\quad x &\doteqdot 11.8\end{aligned}$$

$\therefore$ the shorter side is 11.8 cm long

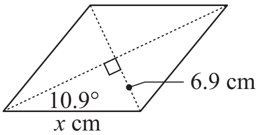
13



$$\begin{aligned}\tan\theta &= \frac{6}{3.5}\\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{6}{3.5}\right)\\ \therefore\quad 2\theta &= 2\times\tan^{-1}\left(\tfrac{6}{3.5}\right)\doteqdot 119.49\end{aligned}$$

So, the larger angle is $\doteqdot 119^o$

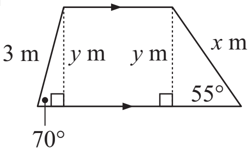
14



$$\begin{aligned}\sin 10.9^o &= \frac{6.9}{x}\\ \therefore\quad x &= \frac{6.9}{\sin 10.9^o}\doteqdot 36.5\end{aligned}$$

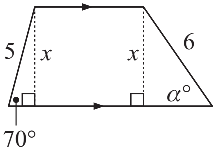
$\therefore$ sides are 36.5 cm long

15 a



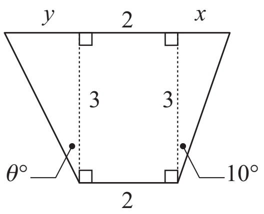
$$\begin{aligned}\sin 70^o &= \frac{y}{3}\\ \therefore\quad y &= 3\times\sin 70^o\\ \text{But}\quad \sin 55^o &= \frac{y}{x}\\ \therefore\quad \sin 55^o &= \frac{3\times\sin 70^o}{x}\\ \therefore\quad x &= \frac{3\times\sin 70^o}{\sin 55^o}\doteqdot 3.44\end{aligned}$$

b



$$\begin{aligned}\sin 70^o &= \frac{x}{5}\\ \therefore\quad 5\times\sin 70^o &= x\\ \text{Now}\quad \sin\alpha &= \frac{x}{6}\\ \therefore\quad \sin\alpha &= \frac{5\times\sin 70^o}{6}\\ \therefore\quad \alpha &= \sin^{-1}\left(\frac{5\times\sin 70^o}{6}\right)\doteqdot 51.5\end{aligned}$$

16

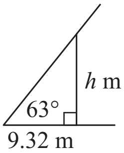


$$\tan 10^o = \frac{x}{3}$$
$$\therefore\quad 3 \times \tan 10^o = x$$
$$\therefore\quad x \doteqdot 0.5290$$
$$\text{But}\quad y + 2 + x = 5$$
$$\therefore\quad y \doteqdot 3 - 0.5290$$
$$\therefore\quad y \doteqdot 2.471$$

$$\text{Now}\quad \tan\theta = \frac{y}{3}$$
$$\therefore\quad \tan\theta = 0.8237$$
$$\therefore\quad \theta \doteqdot 39.48^o$$
$$\text{So,}\quad \beta \doteqdot 90 + 39.48^o$$
$$\therefore\quad \beta \doteqdot 129^o$$

EXERCISE E

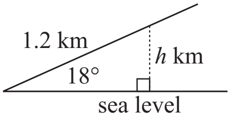
1



$$\tan 63^o = \frac{h}{9.32}$$
$$\therefore\quad \tan 63^o \times 9.32 = h$$
$$\therefore\quad h \doteqdot 18.3$$

$\therefore$ the height is 18.3 m

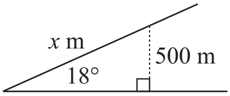
2 **a**



$$\sin 18^o = \frac{h}{1.2}$$
$$\therefore\quad 1.2 \times \sin 18^o = h$$
$$\therefore\quad h \doteqdot 0.371$$

$\therefore$ height is 371 m above sea level.

b



$$\sin 18^o = \frac{500}{x}$$
$$\therefore\quad x = \frac{500}{\sin 18^o}$$
$$\therefore\quad x \doteqdot 1618$$

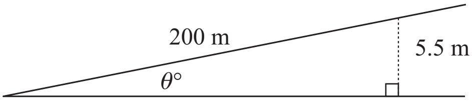
$\therefore$ have walked 1.62 km up the hill.

3

$$\text{Let}\quad \text{AB} = x \text{ m}$$
$$\text{Now}\quad \tan 37^o = \frac{120}{x}$$
$$\therefore\quad x = \frac{120}{\tan 37^o}$$
$$\therefore\quad x \doteqdot 159$$

$\therefore$ the canal is 159 m wide

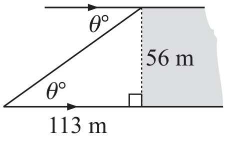
4



$$\sin\theta = \tfrac{5.5}{200}$$
$$\therefore\quad \theta = \sin^{-1}\left(\tfrac{5.5}{200}\right)$$
$$\therefore\quad \theta \doteqdot 1.58$$

i.e., an incline of 1.58^o

5



$$\tan\theta = \tfrac{56}{113}$$
$$\therefore\quad \theta \doteqdot \tan^{-1}\left(\tfrac{56}{113}\right)$$
$$\therefore\quad \theta \doteqdot 26.4$$

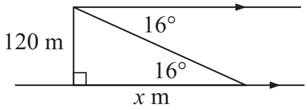
$\therefore$ angle of elevation is about 26.4^o

The angle of depression is also θ^o,

i.e., about 26.4^o

{equal alternate angles}

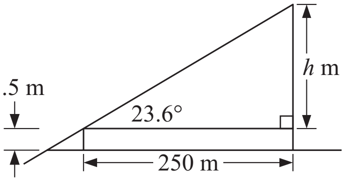
6



$$\tan 16^o = \frac{120}{x}$$
$$\therefore\quad x = \frac{120}{\tan 16^o}$$
$$\therefore\quad x \doteqdot 418$$

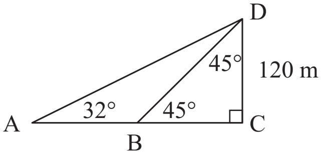
$\therefore$ boat is 418 m out from the base of the cliff.

7



$$\begin{aligned} \tan 23.6^o &= \frac{h}{250} \\ \therefore \quad 250 \times \tan 23.6^o &= h \\ \therefore \quad h &= 109.2 \\ \therefore \quad \text{tree height} &\doteqdot 109.2 + 1.5 \\ &\doteqdot 110.7 \text{ m} \\ &\doteqdot 111 \text{ m} \end{aligned}$$

8

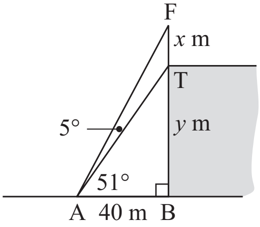


ΔBCD is right-angled isosceles

$$\begin{aligned} \therefore \quad \text{BC} &= 120 \text{ m also} \\ \text{In } \Delta\text{ACD}, \quad \tan 32^o &= \frac{120}{\text{AC}} \\ \therefore \quad \text{AC} &= \frac{120}{\tan 32^o} \\ \therefore \quad \text{AC} &\doteqdot 192 \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{she walks} \quad & (192 - 120) \text{ m} \\ & \doteqdot 72 \text{ m} \end{aligned}$$

9



$$\begin{aligned} \tan 51^o &= \frac{y}{40} \\ \therefore \quad 40 \times \tan 51^o &= y \\ \therefore \quad y &\doteqdot 49.396 \\ \tan 56^o &= \frac{x+y}{40} \\ \therefore \quad 40 \times \tan 56^o &= x + y \\ \therefore \quad x + y &\doteqdot 59.302 \\ \therefore \quad x &\doteqdot 59.302 - 49.396 \\ \therefore \quad x &\doteqdot 9.91 \end{aligned}$$

i.e., it is 9.91 m high

10 **a**

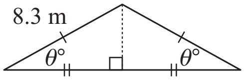
$$\begin{aligned} v &= \sqrt{gr \tan\theta} \\ &= \sqrt{9.8 \times 100 \times \tan 15^o} \\ &\doteqdot 16.2 \text{ m/s} \end{aligned}$$

b

$$\begin{aligned} v &= \sqrt{gr \tan\theta} \\ \therefore \quad 20 &= \sqrt{9.8 \times 200 \times \tan\theta} \\ \therefore \quad 400 &= 9.8 \times 200 \times \tan\theta \\ \therefore \quad \tfrac{2}{9.8} &= \tan\theta \\ \therefore \quad \theta &= \tan^{-1}(\tfrac{2}{9.8}) \\ \therefore \quad \theta &\doteqdot 11.5 \end{aligned}$$

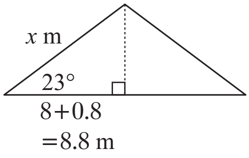
i.e., a banked angle of 11.5^o

11



$$\begin{aligned} 7.5 + 0.6 &= 8.1 \text{ m} \\ \cos\theta &= \frac{8.1}{8.3} \\ \therefore \quad \theta &= \cos^{-1}\left(\frac{8.1}{8.3}\right) \\ \therefore \quad \theta &\doteqdot 12.6 \end{aligned}$$

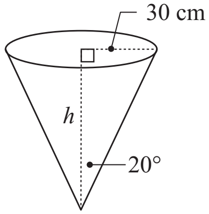
12



$$\begin{aligned} \cos 23^o &= \frac{8.8}{x} \\ \therefore \quad x &= \frac{8.8}{\cos 23^o} \\ \therefore \quad x &\doteqdot 9.56 \end{aligned}$$

$\therefore$ a beam is 9.56 m long

13



$$\tan 20^o = \frac{30}{h}$$

$$\therefore \quad h = \frac{30}{\tan 20^o}$$

$$\therefore \quad h \doteqdot 82.42 \text{ cm}$$

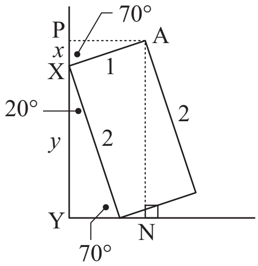
$$V = \tfrac{1}{3}\pi r^2 h$$

$$= \tfrac{1}{3}\pi \times 30^2 \times 82.42$$

$$\doteqdot 77\,683 \text{ cm}^3$$

$\therefore$ capacity is 77 683 mL
i.e., $\doteqdot$ 77.7 L

14



Height AN = PY also

$$\text{Now} \quad \cos 70^o = \frac{x}{1}$$

$$\therefore \quad x = \cos 70^o$$

$$\text{and} \quad \sin 70^o = \frac{y}{2}$$

$$\therefore \quad y = 2\sin 70^o$$

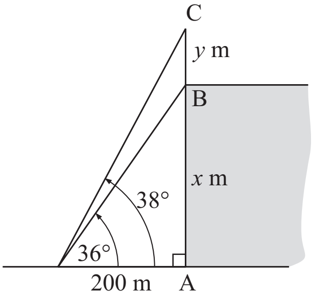
$$\text{Now} \quad \text{PY} = x + y$$

$$= \cos 70^o + 2\sin 70^o$$

$$\doteqdot 2.22 \text{ m}$$

$\therefore$ A is 2.22 m above the floor

15



$$\tan 36^o = \frac{x}{200}$$

$$\therefore \quad 200 \times \tan 36^o = x$$

$$\therefore \quad x \doteqdot 145.31 \quad \text{...... (1)}$$

$$\tan 38^o = \frac{x+y}{200}$$

$$\therefore \quad 200 \times \tan 38^o = x + y$$

$$\therefore \quad x + y \doteqdot 156.26 \quad \text{...... (2)}$$

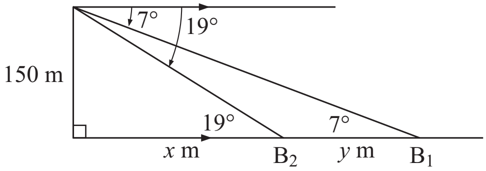
Using (1) and (2)

$$y \doteqdot 156.26 - 145.31$$

$$\therefore \quad y \doteqdot 10.9$$

$\therefore$ the pole is about 10.9 m long.

16



$$\tan 19^o = \frac{150}{x}$$

$$\therefore \quad x = \frac{150}{\tan 19^o} \doteqdot 435.63 \quad \text{...... (1)}$$

$$\tan 7^o = \frac{150}{x+y}$$

$$\therefore \quad x + y = \frac{150}{\tan 7^o} \doteqdot 1221.65 \quad \text{...... (2)}$$

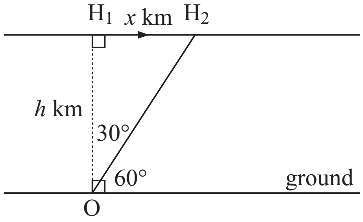
From (1) and (2)

$$y \doteqdot 1221.65 - 435.63$$

$$\therefore \quad y \doteqdot 786$$

i.e., needs to be 786 m closer in.

17



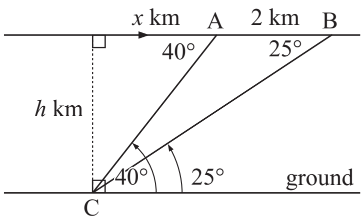
$$\begin{aligned} \text{speed} &= \frac{\text{distance}}{\text{time}} \\ \therefore \quad 100 &= \frac{x}{\frac{20}{3600}\text{ hours}} \\ \therefore \quad x &= 100 \times \tfrac{20}{3600} \\ \therefore \quad x &= \tfrac{20}{36} = \tfrac{5}{9} \end{aligned}$$

$$\text{Now} \quad \tan 30^o = \frac{\frac{5}{9}}{h}$$

$$\begin{aligned} \therefore \quad h &= \frac{\frac{5}{9}}{\tan 30^o} \\ \therefore \quad h &\doteqdot 0.962 \text{ km} \end{aligned}$$

∴ is about 962 m above ground level.

18



Notice that

$$\tan 40^o = \frac{h}{x}$$

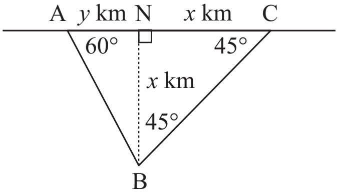
$$\text{and} \quad \tan 25^o = \frac{h}{x+2}$$

$$\begin{aligned} \therefore \quad h &= x \tan 40^o \\ \text{and} \quad h &= (x+2)\tan 25^o \\ \therefore \quad x\tan 40^o &= (x+2)\tan 25^o \\ \therefore \quad \frac{x+2}{x} &= \frac{\tan 40^o}{\tan 25^o} \\ \therefore \quad \frac{x+2}{x} &\doteqdot 1.799\,45 \\ \therefore \quad x+2 &\doteqdot 1.79945x \\ \therefore \quad 2 &\doteqdot 0.79945x \\ \therefore \quad x &\doteqdot 2.502 \end{aligned}$$

$$\begin{aligned} \text{So} \quad h &= 2.502 \times \tan 40^o \\ \therefore \quad h &\doteqdot 2.10 \end{aligned}$$

i.e., 2.10 km about ground level

19



ΔNBC is right angled isosceles

$$\therefore \quad \text{BN} = x \text{ km}$$

$$\text{Now} \quad \tan 60^o = \frac{x}{y}$$

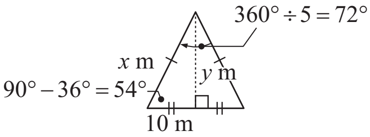
$$\therefore \quad \sqrt{3} = \frac{x}{5-x}$$

$$\{\text{as} \quad x + y = 5\}$$

$$\begin{aligned} \therefore \quad 5\sqrt{3} - x\sqrt{3} &= x \\ \therefore \quad 5\sqrt{3} &= x(1+\sqrt{3}) \\ \therefore \quad \frac{5\sqrt{3}}{1+\sqrt{3}} &= x \\ \therefore \quad x &\doteqdot 3.17 \end{aligned}$$

∴ is 3.17 km from the shore

20 Each triangle is:



$$\cos 54^o = \frac{10}{x} \qquad \therefore \quad x = \frac{10}{\cos 54^o}$$

$$\therefore \quad x \doteqdot 17.01$$

$$\tan 54^o = \frac{y}{10} \qquad \therefore \quad y = 10 \tan 54^o$$

$$\therefore \quad y \doteqdot 13.76$$

$$\begin{aligned} \text{But} \quad d &= x + y \\ &\doteqdot 17.01 + 13.76 \\ &\doteqdot 30.8 \end{aligned}$$

i.e., width of land is 30.8 m.

EXERCISE F

1 **a**

$$\begin{aligned}\tan\theta &= \frac{4-5}{-1-2}\\ \therefore\quad \tan\theta &= \tfrac{-1}{-3} = \tfrac{1}{3}\\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{1}{3}\right)\\ \therefore\quad \theta &\doteqdot 18.4^o\end{aligned}$$

b

$$\begin{aligned}\tan\theta &= \frac{-4--2}{-1-3}\\ &= \tfrac{-2}{-4}\\ &= \tfrac{1}{2}\\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{1}{2}\right)\\ \therefore\quad \theta &\doteqdot 26.6^o\end{aligned}$$

c

$$\begin{aligned}\tan\theta &= \frac{-5-1}{1--2}\\ &= -\tfrac{6}{3}\\ &= -2\\ \therefore\quad \theta &= 180^o - \tan^{-1}(2)\\ \therefore\quad \theta &\doteqdot 116.6^o\end{aligned}$$

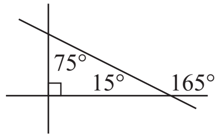
d

$$\begin{aligned}\tan\theta &= \frac{-1-4}{-2--7}\\ &= -\tfrac{5}{5}\\ &= -1\\ \therefore\quad \theta &= 135^o\end{aligned}$$

2

$$\begin{aligned}m &= \tan 60^o\\ &\doteqdot 1.73 \quad\text{or}\quad \sqrt{3}\end{aligned}$$

b



$$\begin{aligned}m &= \tan 165^o\\ &\doteqdot -0.268\end{aligned}$$

3 **a**

$$\begin{aligned}\theta &= 60^o\\ \therefore\quad \tan\theta &= \sqrt{3} = m\\ \therefore\quad y &= \sqrt{3}x + 2\end{aligned}$$

b

$$\begin{aligned}\theta &= 120^o\\ \therefore\quad m &= \tan 120^o\\ \therefore\quad m &= -\sqrt{3}\\ \therefore\quad y &= -\sqrt{3}x\end{aligned}$$

c

$$\begin{aligned}\theta &= 30^o\\ \therefore\quad \tan\theta &= \tfrac{1}{\sqrt{3}}\\ \text{So,}\quad y &= \tfrac{1}{\sqrt{3}}x + c\end{aligned}$$

But $(2\sqrt{3},\ 0)$ lies on the line

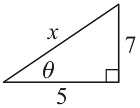
$$\begin{aligned}\therefore\quad 0 &= \tfrac{1}{\sqrt{3}}(2\sqrt{3}) + c\\ \therefore\quad 0 &= 2 + c\\ \therefore\quad c &= -2\end{aligned}$$

i.e., line is $y = \frac{1}{\sqrt{3}}x - 2$

REVIEW SET A

1 $\sin\theta = \frac{7}{11}$

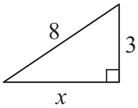
2



$$\begin{aligned}x^2 &= 5^2 + 7^2 \quad \{\text{Pythagoras}\}\\ \therefore\quad x &= \sqrt{5^2+7^2}\\ \therefore\quad x &= \sqrt{74}\end{aligned}$$

So, $\sin\theta = \frac{7}{\sqrt{74}}$, $\cos\theta = \frac{5}{\sqrt{74}}$

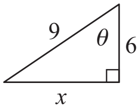
3



$$\begin{aligned}x^2 + 3^2 &= 8^2 \quad \{\text{Pythagoras}\}\\ \therefore\quad x &= \sqrt{8^2-3^2}\\ \therefore\quad x &= \sqrt{55}\end{aligned}$$

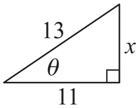
So, $\tan\theta = \frac{3}{\sqrt{55}}$

4



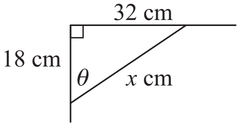
$$\begin{aligned}x^2 + 6^2 &= 9^2 \quad \{\text{Pythagoras}\}\\ \therefore\quad x &= \sqrt{9^2-6^2}\\ \therefore\quad x &= \sqrt{45}\\ \text{So,}\quad \sin\theta &= \tfrac{\sqrt{45}}{9}\\ \cos\theta &= \tfrac{6}{9} = \tfrac{2}{3}\\ \tan\theta &= \tfrac{\sqrt{45}}{6}\end{aligned}$$

5



$$\begin{aligned}x^2 + 11^2 &= 13^2\\ \therefore\quad x &= \sqrt{13^2-11^2}\\ \therefore\quad x &= \sqrt{48}\\ \text{So,}\quad \sin\theta &= \tfrac{\sqrt{48}}{13}\\ \cos\theta &= \tfrac{11}{13}\\ \tan\theta &= \tfrac{\sqrt{48}}{11}\end{aligned}$$

6



a

$$\begin{aligned}x^2 &= 18^2 + 32^2\\ \therefore\quad x &= \sqrt{18^2+32^2}\\ \therefore\quad x &\doteqdot 36.715\end{aligned}$$

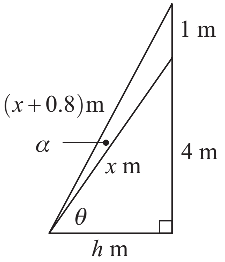
So, the support is 36.7 cm long

b

$$\begin{aligned}\tan\theta &= \tfrac{32}{18}\\ \therefore\quad \theta &= \tan^{-1}\left(\tfrac{32}{18}\right) \doteqdot 60.6\end{aligned}$$

$\therefore$ makes 60.6^o with the wall.

7



a

$$\begin{aligned} x^2 &= h^2+4^2 \text{ .... (1)}\\ \text{and}\quad (x+0.8)^2 &= h^2+5^2\\ \therefore\quad (x+0.8)^2-25 &= x^2-16\\ \therefore\quad x^2+1.6x+0.64-25 &= x^2-16\\ \therefore\quad 1.6x &= 8.36\\ \therefore\quad x &\doteqdot 5.225\\ \text{and}\quad x+0.8 &\doteqdot 6.025 \end{aligned}$$

∴ the extended ladder is 6.025 m long

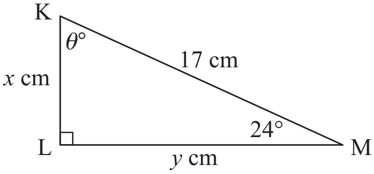
b

$$\sin\theta = \frac{4}{5.225} \qquad \sin(\theta+\alpha) = \frac{5}{6.025}$$

$$\therefore\quad \theta \doteqdot 49.956 \qquad \therefore\quad \theta+\alpha \doteqdot 56.086$$

$\therefore\quad \alpha \doteqdot 56.086 - 49.956 \doteqdot 6.13$ ∴ angle increases by 6.13^o

8



$$\theta = 90^o - 24^o = 66^o$$

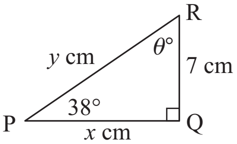
$$\cos 24^o = \frac{y}{17}, \quad \sin 24^o = \frac{x}{17}$$

$$\therefore\quad y = 17\cos 24^o \qquad \therefore\quad x = 17\sin 24^o$$

$$\therefore\quad y \doteqdot 15.5 \qquad \therefore\quad x \doteqdot 6.91$$

So, $\text{KL} \doteqdot 6.91$ cm
$\text{LM} \doteqdot 15.5$ cm

9



$$\theta = 90-38 = 52$$

$$\therefore\quad \measuredangle\text{PRQ} = 52^o$$

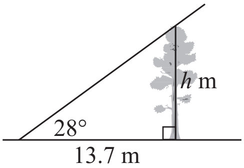
$$\tan 38^o = \frac{7}{x} \quad\text{and}\quad \sin 38^o = \frac{7}{y}$$

$$\therefore\quad x = \frac{7}{\tan 38^o} \qquad \therefore\quad y = \frac{7}{\sin 38^o}$$

$$\therefore\quad x \doteqdot 8.96 \qquad \therefore\quad y \doteqdot 11.4$$

So $\text{PQ} \doteqdot 8.96$ cm $\quad \text{PR} \doteqdot 11.4$ cm

10



$$\tan 28^o = \frac{h}{13.7}$$

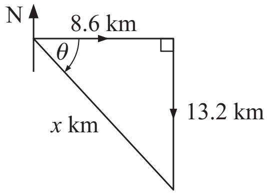
$$\therefore\quad 13.7\times\tan 28 = h$$

$$\therefore\quad h \doteqdot 7.28$$

∴ the height is 7.28 m

REVIEW SET B

1



$$x^2 = 8.6^2 + 13.2^2$$

$$\therefore\quad x = \sqrt{8.6^2+13.2^2}$$

$$\therefore\quad x \doteqdot 15.75$$

So, distance is 15.75 km

$$\tan\theta = \frac{13.2}{8.6}$$

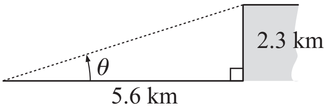
$$\therefore\quad \theta = \tan^{-1}\left(\frac{13.2}{8.6}\right)$$

$$\therefore\quad \theta \doteqdot 56.9^o$$

and $90^o + 56.9^o \doteqdot 147^o$

∴ bearing is 147^o

2



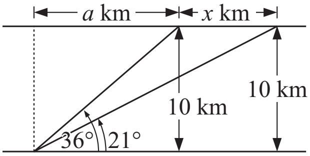
$\tan\theta = \frac{2.3}{5.6}$ {assuming a vertical mountain}

$\therefore \quad \theta = \tan^{-1}\left(\frac{2.3}{5.6}\right)$

$\therefore \quad \theta \doteqdot 22.3^o$

So, the angle of elevation is 22.3^o

3



$\tan 36^o = \frac{10}{a}$ and $\tan 21^o = \frac{10}{a+x}$

$\therefore \quad a = \frac{10}{\tan 36^o}$ and $a + x = \frac{10}{\tan 21^o}$

$\therefore \quad a \doteqdot 13.764$ and $a + x \doteqdot 26.051$

$\therefore \quad x \doteqdot 26.051 - 13.764$

$\therefore \quad x \doteqdot 12.287$

$\text{speed} = \dfrac{12.287 \text{ km}}{\frac{2}{60} \text{ hours}} \doteqdot 369 \text{ km/h}$

4 **a** $\sin\theta = 0.8147$
$\therefore \quad \theta = \sin^{-1}(0.8147)$
$\therefore \quad \theta \doteqdot 54.6^o$

b $\cos\theta = 0.0917$
$\therefore \quad \theta = \cos^{-1}(0.0917)$
$\therefore \quad \theta \doteqdot 84.7^o$

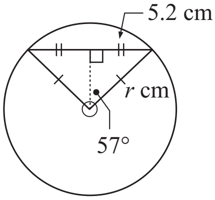
c $\tan\theta = 5.23$
$\therefore \quad \theta = \tan^{-1}(5.23)$
$\therefore \quad \theta \doteqdot 79.2^o$

5 **a** $\sin\theta = \frac{\sqrt{11}}{5}$
$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{11}}{5}\right)$
$\therefore \quad \theta \doteqdot 41.6^o$

b $\cos\theta = \frac{5}{7}$
$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{7}\right)$
$\therefore \quad \theta \doteqdot 44.4^o$

c $\tan\theta = 0.7452$
$\therefore \quad \theta = \tan^{-1}(0.7452)$
$\therefore \quad \theta \doteqdot 36.7^o$

6



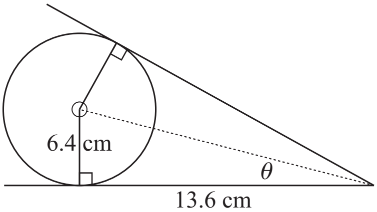
$\sin 57^o = \frac{5.2}{r}$

$\therefore \quad r = \frac{5.2}{\sin 57^o}$

$\therefore \quad r \doteqdot 6.20$

$\therefore$ the radius is 6.20 cm

7



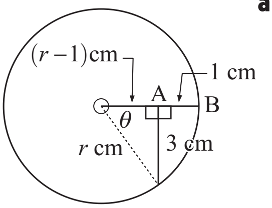
$\tan\theta = \frac{6.4}{13.6}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{6.4}{13.6}\right)$

$\therefore \quad \theta \doteqdot 25.2$

So, the angle measures 25.2^o

8



a If the radius OC is r cm

$$\begin{aligned} \text{then OA} &= (r-1) \text{ cm} \\ \therefore \quad (r-1)^2 + 3^2 &= r^2 \quad \{\text{Pythagoras}\} \\ \therefore \quad r^2 - 2r + 1 + 9 &= r^2 \\ \therefore \quad -2r + 10 &= 0 \\ \therefore \quad r &= 5 \end{aligned}$$

So, the radius is 5 cm

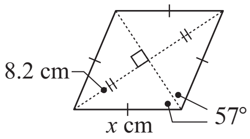
b $\sin\theta = \frac{3}{r} = \frac{3}{5}$

$\therefore \quad \theta = \sin^{-1}(0.6)$

$\therefore \quad \theta \doteqdot 36.9$

So, BC subtends 36.9^o at O.

9

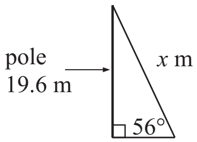


$$\sin 57^o = \frac{8.2}{x}$$

$$\therefore \quad x = \frac{8.2}{\sin 57^o} \doteqdot 9.78$$

i.e., sides are 9.78 cm long

10



$$\sin 56^o = \frac{19.6}{x}$$

$$\therefore \quad x = \frac{19.6}{\sin 56^o}$$

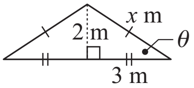
$$\therefore \quad x \doteqdot 23.64$$

and $3x \doteqdot 70.9$

So, the total length is 70.9 m

REVIEW SET C

1



a $x^2 = 2^2 + 3^2$ {Pythagoras}

$\therefore \quad x = \sqrt{2^2+3^2}$

$\therefore \quad x \doteqdot 3.61$

$\therefore$ beam AB is 3.61 m long.

b $\tan\theta = \frac{2}{3}$

$\therefore \quad \theta = \tan^{-1}(\frac{2}{3})$

$\therefore \quad \theta \doteqdot 33.7$

$\therefore$ beam is inclined at 33.7^o to the horizontal.

2 $\tan\theta = \dfrac{6-(-4)}{1-7} = \frac{10}{-6} = -\frac{5}{3}$

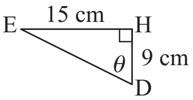
$\therefore \quad \theta = 180^o - \tan^{-1}\left(\frac{5}{3}\right)$

$\therefore \quad \theta \doteqdot 121^o$

3 $m = \tan 30^o = \frac{1}{\sqrt{3}}$

$\therefore$ equation is $y = \frac{1}{\sqrt{3}}x - 3$

4 a



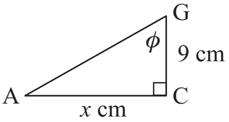
b $\tan\theta = \frac{15}{9} = \frac{5}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{5}{3}\right)$

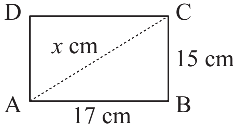
$\therefore \quad \theta \doteqdot 59.0$

So, $\measuredangle$HDE is 59.0^o

c



d



By Pythagoras $x^2 = 15^2 + 17^2$

$\therefore \quad x = \sqrt{15^2+17^2}$

$\therefore \quad x \doteqdot 22.67$

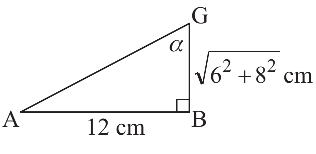
and $\tan\phi = \dfrac{x}{9}$

$\therefore \quad \phi = \tan^{-1}\left(\frac{22.67}{9}\right)$

$\therefore \quad \phi \doteqdot 68.3$

i.e., $\measuredangle$AGC is 68.3^o

5 a



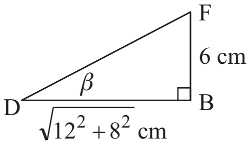
$$\tan\alpha = \frac{12}{\sqrt{6^2+8^2}}$$

$\therefore \quad \tan\alpha = \frac{12}{10}$

$\therefore \quad \alpha = \tan^{-1}\left(\frac{6}{5}\right) \doteqdot 50.2$

So, the angle measures 50.2^o

b

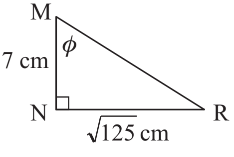


$$\tan\beta = \frac{6}{\sqrt{12^2+8^2}} = \frac{6}{\sqrt{208}}$$

$$\therefore \quad \beta = \tan^{-1}\left(\frac{6}{\sqrt{208}}\right) \doteqdot 22.6$$

So, the angle measures 22.6^o

6 a



b RN $= \sqrt{5^2+10^2}$

$= \sqrt{125}$

$\doteqdot 11.180......$

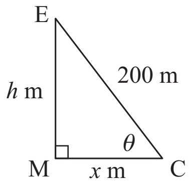
$\doteqdot 11.2$ cm

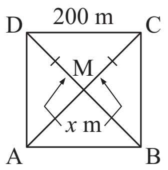
c $\tan\phi = \frac{\sqrt{125}}{7}$

$\therefore \quad \phi = \tan^{-1}\left(\frac{\sqrt{125}}{7}\right) \doteqdot 57.9$

i.e., $\measuredangle$RMN is 57.9^o

7 **a**





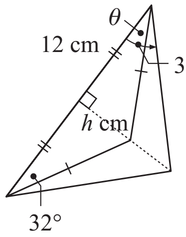
$$x^2 + x^2 = 200^2 \quad \{\text{Pythgoras}\}$$
$$\therefore \quad 2x^2 = 40\,000$$
$$\therefore \quad x^2 = 20\,000$$
$$\text{Also} \quad h^2 + x^2 = 200^2$$
$$\therefore \quad h^2 = 200^2 - 20\,000$$
$$\therefore \quad h = \sqrt{200^2 - 20\,000} \doteq 141$$

i.e., 141 m high

b

$$\sin\theta = \frac{h}{200} \doteq \frac{141.42}{200}$$
$$\therefore \quad \theta = \sin^{-1}\left(\frac{141.42}{200}\right) \doteq 45^o$$

8



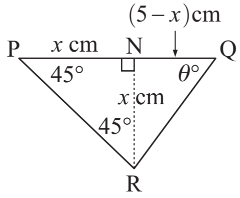
$$\tan 32^o = \frac{h}{12}$$
$$\therefore \quad 12 \times \tan 32^o = h$$
$$\therefore \quad h \doteq 7.4984$$
$$\text{Area} = \tfrac{1}{2} \text{ base} \times \text{height}$$
$$= \tfrac{1}{2} \times 24 \times 7.4984$$
$$\doteq 89.981$$
$$\therefore \quad \text{new area} = 2 \times \text{old area}$$
$$\doteq 179.96 \text{ cm}^2$$

Now if the new height is H cm

$$\tfrac{1}{2} \times 24 \times H \doteq 179.96$$
$$\therefore \quad 12H \doteq 179.96$$
$$\therefore \quad H \doteq 14.997$$
$$\therefore \quad \tan\theta = \frac{14.997}{12}$$
$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{14.997}{12}\right)$$
$$\doteq 51.3$$

So, the new base angles are 51.3^o.

9



Let NR $= x$ cm

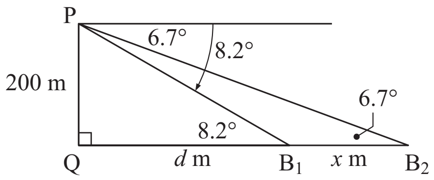
Now ΔPNR is right angled isosceles

$\therefore$ PN $= x$ cm also

and so QN $= (5 - x)$ cm

$$\text{Now} \quad \tan\theta = \frac{x}{5-x} \quad \{\text{in } \Delta\text{RQN}\}$$
$$\therefore \quad (5-x)\tan\theta = x$$
$$\therefore \quad 5\tan\theta - x\tan\theta = x$$
$$\therefore \quad 5\tan\theta = x + x\tan\theta$$
$$\therefore \quad 5\tan\theta = x(1+\tan\theta)$$
$$\therefore \quad \frac{5\tan\theta}{1+\tan\theta} = x$$
$$\text{i.e.,} \quad \text{RN} = \frac{5\tan\theta}{1+\tan\theta} \text{ cm}$$

10



Let $\text{QB}_1 = d$ m and $\text{B}_1\text{B}_2 = x$ m

$$\therefore \quad \tan 8.2^o = \frac{200}{d} \quad \text{and} \quad \tan 6.7^o = \frac{200}{d+x}$$
$$\therefore \quad d = \frac{200}{\tan 8.2} \doteq 1387.9$$
$$\text{and} \quad d + x = \frac{200}{\tan 6.7} \doteq 1702.5$$
$$\text{So} \quad x = 1702.5 - 1387.9 \doteq 315$$

i.e., the boats are 315 m apart.

Chapter 11

THE UNIT CIRCLE AND RADIAN MEASURE

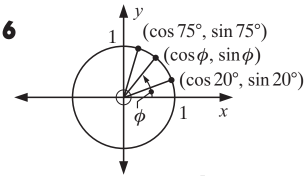
EXERCISE 11A

1 **a** The y-coordinate at 0^o is 0, $\therefore$ $\sin 0^o = 0$
b The y-coordinate at 15^o is $\doteqdot 0.26$, $\therefore$ $\sin 15^o = 0.26$
c The y-coordinate at 25^o is $\doteqdot 0.42$, $\therefore$ $\sin 25^o = 0.42$
d The y-coordinate at 30^o is $\doteqdot 0.5$, $\therefore$ $\sin 30^o = 0.50$
e The y-coordinate at 45^o is $\doteqdot 0.71$, $\therefore$ $\sin 45^o = 0.71$
f The y-coordinate at 60^o is $\doteqdot 0.87$, $\therefore$ $\sin 60^o = 0.87$
g The y-coordinate at 75^o is $\doteqdot 0.97$, $\therefore$ $\sin 75^o = 0.97$
h The y-coordinate at 90^o is 1, $\therefore$ $\sin 90^o = 1$

3 **a** The x-coordinate at 0^o is 1, $\therefore$ $\cos 0^o = 1$
b The x-coordinate at 15^o is $\doteqdot 0.97$, $\therefore$ $\cos 15^o \doteqdot 0.97$
c The x-coordinate at 25^o is $\doteqdot 0.91$, $\therefore$ $\cos 25^o \doteqdot 0.91$
d The x-coordinate at 30^o is $\doteqdot 0.87$, $\therefore$ $\cos 30^o \doteqdot 0.87$
e The x-coordinate at 45^o is $\doteqdot 0.71$, $\therefore$ $\cos 45^o \doteqdot 0.71$
f The x-coordinate at 60^o is $\doteqdot 0.5$, $\therefore$ $\cos 60^o \doteqdot 0.50$
g The x-coordinate at 75^o is $\doteqdot 0.26$, $\therefore$ $\cos 75^o \doteqdot 0.26$
h The x-coordinate at 90^o is 0, $\therefore$ $\cos 90^o = 0$

5 The coordinates
$(\cos 55^o, \sin 55^o)$
i.e., $(0.57, 0.82)$

6



7 **a**
$$\begin{aligned} \text{As}\quad & \cos^2\theta + \sin^2\theta = 1, \\ \text{then}\quad \sin\theta &= \sqrt{1-\cos^2\theta} \quad \text{as } \sin\theta > 0 \\ &= \sqrt{1-(0.8)^2} \\ &= 0.6 \end{aligned}$$

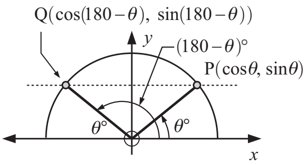
b
$$\begin{aligned} \text{As}\quad & \cos^2\theta + \sin^2\theta = 1, \\ \text{then}\quad \cos\theta &= \sqrt{1-\sin^2\theta} \quad \text{as } \cos\theta > 0 \\ &= \sqrt{1-(0.7)^2} \\ &\doteqdot 0.714 \end{aligned}$$

EXERCISE 11B

1 **a** 0.98 **b** 0.98 **c** 0.87 **d** 0.87 **e** 0.5 **f** 0.5 **g** 0 **h** 0

2 **a** It appears that
$\sin(180-\theta)^o = \sin\theta^o$

b

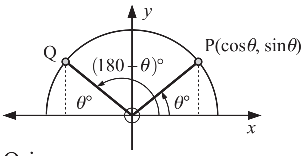


P and Q have the same y-coordinate.
$\therefore$ $\sin(180-\theta)^o = \sin\theta^o$

3 **a** -0.34
b 0.34
c -0.64
d 0.64
e -0.77
f 0.77
g -1
h 1

4 **a** $\cos(180-\theta)^o = -\cos\theta^o$

b



Q is
$(\cos(180-\theta))^o, \sin(180-\theta)^o)$

The x-coordinate of Q is the negative of the x-coordinate of P.
$\therefore$ $\cos(180-\theta)^o = -\cos\theta^o$

5 **a**
$$\begin{aligned} & \sin 45^o \\ &= \sin(180-45)^o \\ &= \sin 135^o \\ & \text{i.e.,}\quad \theta = 135^o \end{aligned}$$

b
$$\begin{aligned} & \sin 51^o \\ &= \sin(180-51)^o \\ &= \sin 129^o \\ & \text{i.e.,}\quad \theta = 129^o \end{aligned}$$

c
$$\begin{aligned} & \sin 74^o \\ &= \sin(180-74)^o \\ &= \sin 106^o \\ & \text{i.e.,}\quad \theta = 106^o \end{aligned}$$

d
$$\begin{aligned} & \sin 82^o \\ &= \sin(180-82)^o \\ &= \sin 98^o \\ & \text{i.e.,}\quad \theta = 98^o \end{aligned}$$

6 **a** $\sin 130^o$
$= \sin(180-130)^o$
$= \sin 50^o$
i.e., $\theta = 50^o$

b $\sin 146^o$
$= \sin(180-146)^o$
$= \sin 34^o$
i.e., $\theta = 34^o$

c $\sin 162^o$
$= \sin(180-162)^o$
$= \sin 18^o$
i.e., $\theta = 18^o$

d $\sin 171^o$
$= \sin(180-171)^o$
$= \sin 9^o$
i.e., $\theta = 9^o$

7 **a** $\sin 137^o$
$= \sin(180-137)^o$
$= \sin 43^o$
$\doteqdot 0.6820$

b $\sin 59^o$
$= \sin(180-59)^o$
$= \sin 121^o$
$\doteqdot 0.8572$

c $\cos 143^o$
$= -\cos(180-143)^o$
$= -\cos 37^o$
$\doteqdot -0.7986$

d $\cos 24^o$
$= -\cos(180-24)^o$
$= -\cos 156^o$
$= 0.9135$

e $\sin 115^o$
$= \sin(180-115)^o$
$= \sin 65^o$
$\doteqdot 0.9063$

f $\cos 132^o$
$= -\cos(180-132)^o$
$= -\cos 48^o$
$= -0.6691$

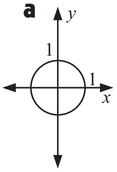
8 **a** $\measuredangle AOQ + \measuredangle BOQ = 180^o$
$\therefore\ \theta^o + \measuredangle BOQ = 180^o$
$\measuredangle BOQ = (180-\theta)^o$

b OQ is a reflection of OP in the y-axis and so Q has coordinates $(-\cos\theta, \sin\theta)$

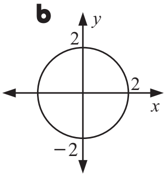
c $\cos(180-\theta)^o = -\cos\theta^o, \quad \sin(180-\theta)^o = \sin\theta^o$

EXERCISE 11C

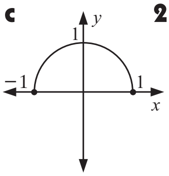
1 **a**



b



c



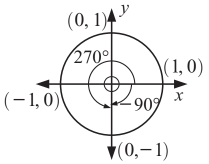
2 **a** **i** A$(\cos 26^o, \sin 26^o)$ B$(\cos 146^o, \sin 146^o)$ C$(\cos 199^o, \sin 199^o)$

ii A$(0.899, 0.438)$ B$(-0.829, 0.559)$ C$(-0.946, -0.326)$

b **i** A$(\cos 123^o, \sin 123^o)$ B$(\cos 251^o, \sin 251^o)$ C$(\cos(-35^o), \sin(-35^o))$

ii A$(-0.545, 0.839)$ B$(-0.326, -0.946)$ C$(0.819, -0.574)$

3



a $\cos 0^o = 1, \quad \sin 0^o = 0$

b $\cos 90^o = 0, \quad \sin 90^o = 1$

c $\cos 180^o = -1, \quad \sin 180^o = 0$

d $\cos 270^o = 0, \quad \sin 270^o = -1$

e $\cos(-90^o) = 0, \quad \sin(-90)^o = -1$

f $\cos 450^o = 0, \quad \sin 450^o = 1$

EXERCISE 11D.1

1 **a** $\frac{\pi}{4}$
$= \frac{180^o}{4}$
$= 45^o$

b $\frac{\pi}{6}$
$= \frac{180^o}{6}$
$= 30^o$

c $\frac{2\pi}{3}$
$= \dfrac{2 \times 180^o}{3}$
$= 120^o$

d $\frac{3\pi}{2}$
$= \dfrac{3 \times 180^o}{2}$
$= 270^o$

e $\frac{5\pi}{3}$
$= \dfrac{5 \times 180^o}{3}$
$= 300^o$

2 **a** 30^o
$= \frac{1}{6}$ of 180^o
$= \frac{\pi}{6}$
$\therefore$ arc length $= \frac{\pi}{6}$ units

b 60^o
$= \frac{1}{3}$ of 180^o
$= \frac{\pi}{3}$
$\therefore$ arc length $= \frac{\pi}{3}$ units

c 90^o
$= \frac{1}{2}$ of 180^o
$= \frac{\pi}{2}$
$\therefore$ arc length $= \frac{\pi}{2}$ units

d 120^o
$= \frac{2}{3}$ of 180^o
$= \frac{2\pi}{3}$
$\therefore$ arc length $= \frac{2\pi}{3}$ units

e 135^o
$= \frac{3}{4}$ of 180^o
$= \frac{3\pi}{4}$
$\therefore$ arc length $= \frac{3\pi}{4}$ units

f 150^o
$= \frac{5}{6}$ of 180^o
$= \frac{5\pi}{6}$
$\therefore$ arc length $= \frac{5\pi}{6}$ units

g 225^o
$= 5 \times 45^o$
$= 5 \times \frac{1}{4}$ of 180^o
$= \frac{5}{4} \times \pi$
$= \frac{5\pi}{4}$ $\quad\therefore$ arc length $= \frac{5\pi}{4}$ units

h 270^o
$= 3 \times 90^o$
$= 3 \times \frac{\pi}{2}$
$= \frac{3\pi}{2}$
$\therefore$ arc length $= \frac{3\pi}{2}$ units

3 **a** ΔOAP is isosceles, so $\angle$AOP $= 60^o$ $\therefore$ $\theta = 60$.

b 60^o is $\frac{1}{6}$ of 360^o, $\therefore$ AP $= \frac{1}{6} \times 2\pi$ i.e., AP $= \frac{\pi}{3}$.

c As AP < arc AP, then AP < 1 $\therefore$ $\theta < 60^o$, i.e., θ decreases.

d When AP $= \pi$, $\theta = 180^o$ So, when AP $= 1$, $\theta = \dfrac{180^o}{\pi} \doteqdot 57.3$

EXERCISE 11D.2

1 **a** $180^o = \pi$ radians
$\therefore$ $90^o = \frac{\pi}{2}$ radians

b $180^o = \pi$ radians
$\therefore$ $60^o = \frac{\pi}{3}$ radians

c $180^o = \pi$ radians
$\therefore$ $30^o = \frac{\pi}{6}$ radians

d $180^o = \pi$ radians
$\therefore$ $18^o = \frac{\pi}{10}$ radians

e $180^o = \pi$ radians
$\therefore$ $9^o = \frac{\pi}{20}$ radians

f $180^o = \pi$ radians
$\therefore$ $45^o = \frac{\pi}{4}$ radians
$\therefore$ $135^o = \frac{3\pi}{4}$ radians

g $180^o = \pi$ radians
$\therefore$ $45^o = \frac{\pi}{4}$ radians
$\therefore$ $225^o = \frac{5\pi}{4}$ radians

h $180^o = \pi$ radians
$\therefore$ $90^o = \frac{\pi}{2}$ radians
$\therefore$ $270^o = \frac{3\pi}{2}$ radians

i $360^o = 2 \times 180^o$
$= 2\pi$ radians

j $720^o = 4 \times 180^o$
$= 4\pi$ radians

k $180^o = \pi$ radians
$\therefore$ $45^o = \frac{\pi}{4}$ radians
$\therefore$ $315^o = \frac{7\pi}{4}$ radians

l $180^o = \pi$ radians
$\therefore$ $540^o = 3\pi$ radians

m $180^o = \pi$ radians
$\therefore$ $36^o = \frac{\pi}{5}$ radians

n $180^o = \pi$ radians
$\therefore$ $10^o = \frac{\pi}{18}$ radians
$\therefore$ $80^o = \frac{8\pi}{18}$ radians
$= \frac{4\pi}{9}$ radians

o $180^o = \pi$ radians
$\therefore$ $10^o = \frac{\pi}{18}$ radians
$\therefore$ $230^o = \frac{23\pi}{18}$ radians

2 **a** 36.7^o
$= 36.7 \times \frac{\pi}{180}$ radians
$\doteqdot 0.641$ radians

b 137.2^o
$= 137.2 \times \frac{\pi}{180}$ radians
$\doteqdot 2.39$ radians

c 317.9^o
$= 317.9 \times \frac{\pi}{180}$ radians
$\doteqdot 5.55$ radians

d 219.6^o
$= 219.6 \times \frac{\pi}{180}$ radians
$\doteqdot 3.83$ radians

e 396.7^o
$= 396.7 \times \frac{\pi}{180}$ radians
$\doteqdot 6.92$ radians

3 **a** $\frac{\pi}{5}$
$= \frac{180^o}{5}$
$= 36^o$

b $\frac{3\pi}{5}$
$= \frac{3 \times 180^o}{5}$
$= 108^o$

c $\frac{3\pi}{4}$
$= \frac{3 \times 180^o}{4}$
$= 135^o$

d $\frac{\pi}{18}$
$= \frac{180^o}{18}$
$= 10^o$

e $\frac{\pi}{9}$
$= \frac{180^o}{9}$
$= 20^o$

f $\frac{7\pi}{9}$
$= \frac{7 \times 180^o}{9}$
$= 140^o$

g $\frac{\pi}{10}$
$= \frac{180^o}{10}$
$= 18^o$

h $\frac{3\pi}{20}$
$= \frac{3 \times 180^o}{20}$
$= 27^o$

i $\frac{5\pi}{6}$
$= \frac{5 \times 180^o}{6}$
$= 150^o$

j $\frac{\pi}{8}$
$= \frac{180^o}{8}$
$= 22\frac{1}{2}^o$

4 **a** 2^c
$= 2 \times \frac{180}{\pi}$ degrees
$\doteqdot 114.59^o$

b 1.53^c
$= 1.53 \times \frac{180}{\pi}$ degrees
$\doteqdot 87.66^o$

c 0.867^c
$= 0.867 \times \frac{180}{\pi}$ degrees
$\doteqdot 49.68^o$

d 3.179^c
$= 3.179 \times \frac{180}{\pi}$ degrees
$\doteqdot 182.14^o$

e 5.267^c
$= 5.267 \times \frac{180}{\pi}$ degrees
$\doteqdot 301.78^o$

5 **a**

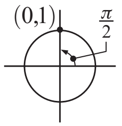
Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

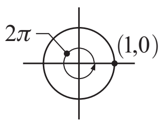
EXERCISE 11E.1

1 **a**



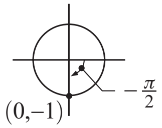
$\cos(\frac{\pi}{2}) = 0$
$\sin(\frac{\pi}{2}) = 1$

b



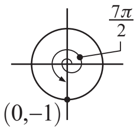
$\cos 2\pi = 1$
$\sin 2\pi = 0$

c



$\cos(-\frac{\pi}{2}) = 0$
$\sin(-\frac{\pi}{2}) = -1$

d



$\cos(\frac{7\pi}{2}) = 0$
$\sin(\frac{7\pi}{2}) = -1$

2 **a** $\cos^2\theta + \sin^2\theta = 1$
$\therefore \cos^2\theta + \frac{1}{4} = 1$
$\therefore \cos^2\theta = \frac{3}{4}$
$\therefore \cos\theta = \pm\frac{\sqrt{3}}{2}$

b $\cos^2\theta + \sin^2\theta = 1$
$\therefore \cos^2\theta + \frac{1}{9} = 1$
$\therefore \cos^2\theta = \frac{8}{9}$
$\therefore \cos\theta = \pm\frac{\sqrt{8}}{3}$
$= \pm\frac{2\sqrt{2}}{3}$

c $\cos^2\theta + \sin^2\theta = 1$
$\therefore \cos^2\theta + 0 = 1$
$\therefore \cos\theta = \pm 1$

d $\cos^2\theta + \sin^2\theta = 1$
$\therefore \cos^2\theta + 1 = 1$
$\therefore \cos\theta = 0$

3 **a** $\cos^2\theta + \sin^2\theta = 1$
$\therefore \frac{16}{25} + \sin^2\theta = 1$
$\therefore \sin^2\theta = \frac{9}{25}$
$\therefore \sin\theta = \pm\frac{3}{5}$

b $\cos^2\theta + \sin^2\theta = 1$
$\therefore \frac{9}{16} + \sin^2\theta = 1$
$\therefore \sin^2\theta = \frac{7}{16}$
$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$

c $\cos^2\theta + \sin^2\theta = 1$
$\therefore 1 + \sin^2\theta = 1$
$\therefore \sin^2\theta = 0$
$\therefore \sin\theta = 0$

d $\cos^2\theta + \sin^2\theta = 1$
$\therefore 0 + \sin^2\theta = 1$
$\therefore \sin^2\theta = 1$
$\therefore \sin\theta = \pm 1$

4 **a**

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve
2	$90 < \theta < 180$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve
3	$180 < \theta < 270$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve
4	$270 < \theta < 360$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve

b **i** 1 and 4
ii 2 and 3
iii 3
iv 2

5 **a**

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \quad \tfrac{4}{9} + \sin^2\theta = 1$$
$$\therefore \quad \sin^2\theta = \tfrac{5}{9}$$
$$\therefore \quad \sin\theta = \pm\tfrac{\sqrt{5}}{3}$$

But θ is in quadrant 1

where $\sin\theta > 0$

$$\therefore \quad \sin\theta = \tfrac{\sqrt{5}}{3}$$

b

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \quad \cos^2\theta + \tfrac{4}{25} = 1$$
$$\therefore \quad \cos^2\theta = \tfrac{21}{25}$$
$$\therefore \quad \cos\theta = \pm\tfrac{\sqrt{21}}{5}$$

But θ is in quadrant 2

where $\cos\theta < 1$

$$\therefore \quad \cos\theta = -\tfrac{\sqrt{21}}{5}$$

c

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \quad \cos^2\theta + \tfrac{9}{25} = 1$$
$$\therefore \quad \cos^2\theta = \tfrac{16}{25}$$
$$\therefore \quad \cos\theta = \pm\tfrac{4}{5}$$

But θ is in quadrant 4

where $\cos\theta > 0$

$$\therefore \quad \cos\theta = \tfrac{4}{5}$$

d

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \quad \tfrac{25}{169} + \sin^2\theta = 1$$
$$\therefore \quad \sin^2\theta = \tfrac{144}{169}$$
$$\therefore \quad \sin\theta = \pm\tfrac{12}{13}$$

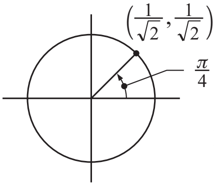
But θ is in quadrant 3

where $\sin\theta < 0$

$$\therefore \quad \sin\theta = -\tfrac{12}{13}$$

EXERCISE 11E.2

1



So $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

Draw separate unit circle diagrams for each case.

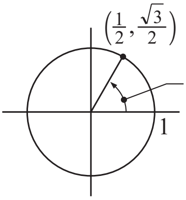
	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$

2 **Likewise, draw separate unit circle diagrams for each angle.**

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

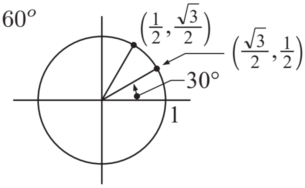
3 **a**

$$\sin^2 60^o$$
$$= \sin 60^o \times \sin 60^o$$
$$= \tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{3}}{2}$$
$$= \tfrac{3}{4}$$



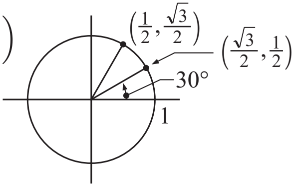
b

$$\sin 30^o \cos 60^o$$
$$= \tfrac{1}{2} \times \tfrac{1}{2}$$
$$= \tfrac{1}{4}$$



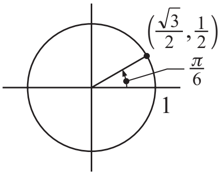
c

$$4 \sin 60^o \cos 30^o$$
$$= 4\left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)$$
$$= 3$$



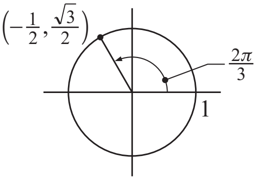
d

$$1 - \cos^2(\tfrac{\pi}{6})$$
$$= 1 - \left(\tfrac{\sqrt{3}}{2}\right)^2$$
$$= 1 - \tfrac{3}{4}$$
$$= \tfrac{1}{4}$$



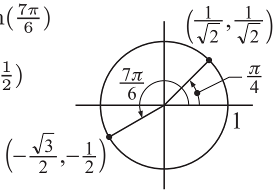
e

$$\sin^2(\tfrac{2\pi}{3}) - 1$$
$$= \left(\tfrac{\sqrt{3}}{2}\right)^2 - 1$$
$$= \tfrac{3}{4} - 1$$
$$= -\tfrac{1}{4}$$



f

$$\cos^2(\tfrac{\pi}{4}) - \sin(\tfrac{7\pi}{6})$$
$$= \left(\tfrac{1}{\sqrt{2}}\right)^2 - (-\tfrac{1}{2})$$
$$= \tfrac{1}{2} + \tfrac{1}{2}$$
$$= 1$$

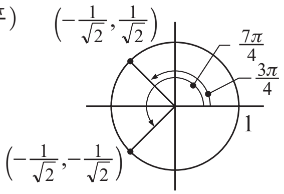


g $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$

$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$ or $\sqrt{2}$

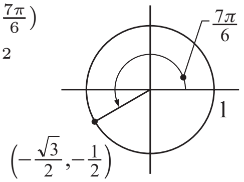


h $1 - 2\sin^2(\frac{7\pi}{6})$

$= 1 - 2(-\frac{1}{2})^2$

$= 1 - 2 \times \frac{1}{4}$

$= \frac{1}{2}$

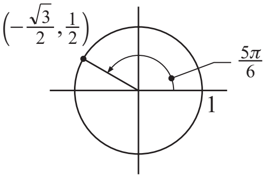


i $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$

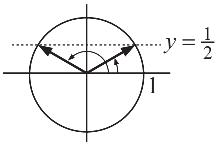
$= \left(-\frac{\sqrt{3}}{2}\right)^2 - (\frac{1}{2})^2$

$= \frac{3}{4} - \frac{1}{4}$

$= \frac{1}{2}$

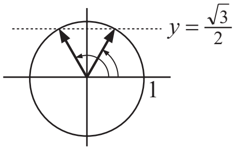


4 **a**



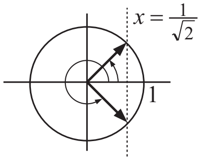
$\theta = 30^o$ or 150^o

b



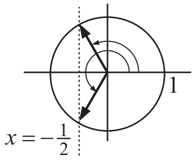
$\theta = 60^o$ or 120^o

c



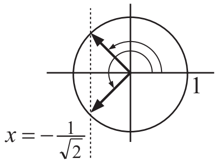
$\theta = 45^o$ or 315^o

d



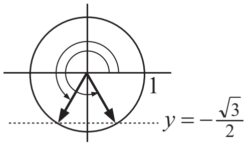
$\theta = 120^o$ or 240^o

e



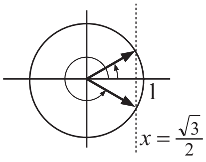
$\theta = 135^o$ or 225^o

f



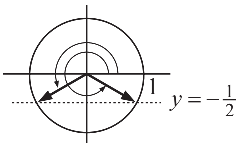
$\theta = 240^o$ or 300^o

5 **a**



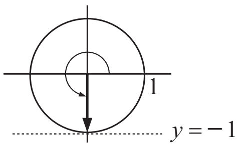
$\theta = 30^o, 330^o, 390^o, 690^o$

b



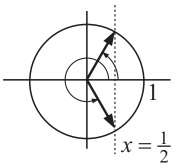
$\theta = 210^o, 330^o, 570^o, 690^o$

c



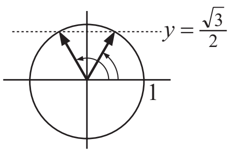
$\theta = 270^o, 630^o$

6 **a** $\cos\theta = \frac{1}{2}$



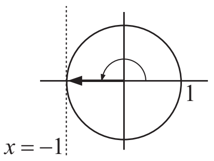
$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

b $\sin\theta = \frac{\sqrt{3}}{2}$



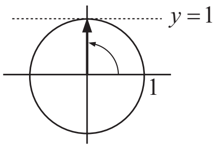
$\therefore \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

c $\cos\theta = -1$



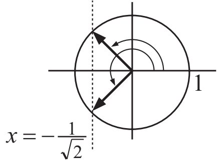
$\therefore \quad \theta = \pi$

d $\sin\theta = 1$



$\therefore \quad \theta = \frac{\pi}{2}$

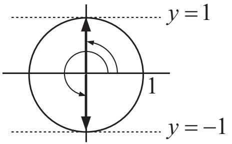
e $\cos\theta = -\frac{1}{\sqrt{2}}$



$\therefore \quad \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$

f $\sin^2\theta = 1$

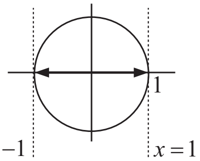
$\therefore \quad \sin\theta = \pm 1$



$\therefore \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

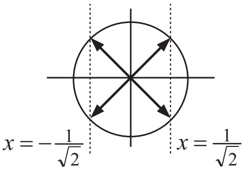
g $\cos^2\theta = 1$

$\therefore \quad \cos\theta = \pm 1$



$\therefore \quad \theta = 0,\ \pi,\ 2\pi$

h $\cos^2\theta = \frac{1}{2} \quad \therefore \quad \cos\theta = \pm\frac{1}{\sqrt{2}}$



$\therefore \quad \theta = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

EXERCISE 11F

1 **a** Area

$= \frac{1}{2} \times 9 \times 10 \times \sin 40^o$

$\doteq 28.9 \text{ cm}^2$

b Area

$= \frac{1}{2} \times 25 \times 31 \times \sin 82^o$

$\doteq 384 \text{ km}^2$

c Area

$= \frac{1}{2} \times 10.2 \times 6.4 \times \sin 125^o$

$\doteq 26.7 \text{ cm}^2$

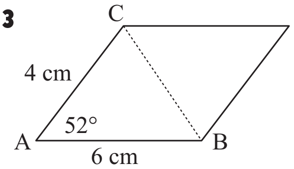
2 Area $= 150 \text{ cm}^2$

$\therefore \quad \frac{1}{2} \times 17 \times x \times \sin 68^o = 150$

$\therefore \quad x = \dfrac{2 \times 150}{17 \times \sin 68^o}$

$\therefore \quad x \doteq 19.0$

3



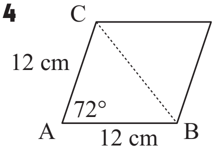
Area

$= 2\times$ area ΔABC

$= 2 \times \frac{1}{2} \times 4 \times 6 \times \sin 52^o$

$\doteq 18.9 \text{ cm}^2$

4

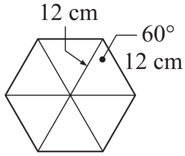


Area $= 2 \times$ area ΔABC

$= 2 \times \frac{1}{2} \times 12^2 \times \sin 72^o$

$\doteq 137 \text{ cm}^2$

5



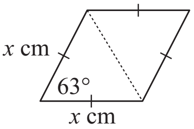
Area

$= 6 \times$ area of Δ

$= 6 \times \frac{1}{2} \times 12^2 \times \sin 60^o$

$\doteq 374 \text{ cm}^2$

6



Area $= 2 \times \frac{1}{2}x^2 \sin 63^o$

$\therefore \quad x^2 \sin 63^o = 50$

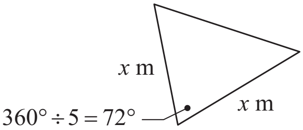
$\therefore \quad x^2 = \dfrac{50}{\sin 63^o}$

$\therefore \quad x = \sqrt{\dfrac{50}{\sin 63^o}}$

$\therefore \quad x \doteq 7.49$

So, sides are 7.49 cm long.

7



Area of $\Delta = \frac{338}{5}$

$\therefore \quad \frac{1}{2}x^2 \sin 72^o = \frac{338}{5}$

$\therefore \quad x^2 = \dfrac{2 \times 338}{5 \times \sin 72^o}$

$\therefore \quad x = \sqrt{\dfrac{2 \times 338}{5 \times \sin 72^o}}$

$\therefore \quad x \doteq 11.9$

So, OA $\doteq$ 11.9 m long.

8 **a** If the included angle is θ

then $\frac{1}{2} \times 5 \times 8 \times \sin\theta = 15$

$\therefore \quad 20\sin\theta = 15$

$\therefore \quad \sin\theta = \frac{3}{4}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{3}{4}\right)$

$\therefore \quad \theta \doteq 48.6^o$ or $(180 - 48.6)^o$

i.e., $\theta \doteq 48.6^o$ or 131.4^o

b Likewise,

$\frac{1}{2} \times 45 \times 53 \times \sin\theta = 800$

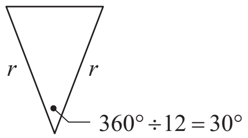
$\therefore \quad \sin\theta = \frac{800\times 2}{45\times 53}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1600}{45\times 53}\right)$

$\therefore \quad \theta \doteq 42.1^o$ or $(180 - 42.1)^o$

$\therefore \quad \theta \doteq 42.1^o$ or 137.9^o

9



Total area of 8 coins

$= 8 \times 12 \times \frac{1}{2}r^2 \sin 30^o$

$= 48r^2 \left(\frac{1}{2}\right)$

$= 24r^2$

Area of \$10 note

$= 8r \times 4r$

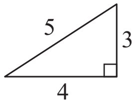
$= 32r^2$

Fraction covered

$= \dfrac{24r^2}{32r^2}$

$= \frac{3}{4} \quad \therefore \quad \frac{1}{4}$ is uncovered

10 **a** **i**



Area $= \frac{1}{2}$ base $\times$ alt

$= \frac{1}{2} \times 4 \times 3$

$= 6 \text{ cm}^2$

ii $s = \dfrac{3+4+5}{2} = 6$

$\therefore$ area $= \sqrt{6(6-3)(6-4)(6-5)}$

$= \sqrt{6 \times 3 \times 2 \times 1}$

$= 6 \text{ cm}^2$

b **i** $s = \dfrac{6+8+12}{2} = 13$

$\therefore \; A = \sqrt{13(13-6)(13-8)(13-12)}$

$= \sqrt{13 \times 7 \times 5 \times 1}$

$\doteqdot 21.3 \text{ cm}^2$

ii $s = \dfrac{7.2+8.9+9.7}{2} = 12.9$

$\therefore \; A = \sqrt{12.9(12.9-7.2)(12.9-8.9)(12.9-9.7)}$

$= \sqrt{12.9 \times 5.7 \times 4 \times 3.2}$

$\doteqdot 30.7 \text{ cm}^2$

EXERCISE 11G

1 **a** **i** arc length

$= \left(\frac{41.6}{360}\right) \times 2\pi \times 9$

$\doteqdot 6.53$ cm

ii area

$= \left(\frac{41.6}{360}\right) \times \pi \times 9^2$

$\doteqdot 29.4 \text{ cm}^2$

b **i** arc length

$= \left(\frac{122}{360}\right) \times 2\pi \times 4.93$

$\doteqdot 10.5$ cm

ii area

$= \left(\frac{122}{360}\right) \times \pi \times 4.93^2$

$\doteqdot 25.9 \text{ cm}^2$

2 **a** $\theta = 107.9^o, \quad l = 5.92$

$\therefore \left(\frac{107.9}{360}\right) \times 2\pi \times r = 5.92$

$\therefore \; r = \dfrac{5.92 \times 360}{107.9 \times 2 \times \pi}$

$\therefore \; r \doteqdot 3.14$ m

b Area $= \left(\frac{107.9}{360}\right) \times \pi \times (3.1436)^2$

$\doteqdot 9.30 \text{ m}^2$

3 **a** Area $= \left(\frac{\theta}{360}\right) \times \pi r^2$

$\therefore \; 20.8 = \left(\frac{68.2}{360}\right) \times \pi r^2$

$\therefore \; \dfrac{20.8 \times 360}{68.2 \times \pi} = r^2$

$\therefore \; r = \sqrt{\dfrac{20.8 \times 360}{68.2 \times \pi}}$

$\therefore \; r \doteqdot 5.91$ cm

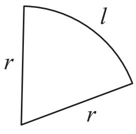
b Perimeter

$= l + 2r$

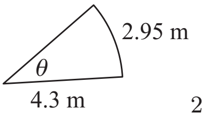
$= \left(\frac{68.2}{360}\right) \times 2\pi \times 5.912 +$

$+ 2 \times 5.912$

$\doteqdot 18.9$ cm



4 **a**



$l = \left(\frac{\theta}{360}\right) \times 2\pi \times r$

$\therefore \; 2.95 = \left(\frac{\theta}{360}\right) \times 2\pi \times 4.3$

$\therefore \; \dfrac{2.95 \times 360}{2 \times \pi \times 4.3} = \theta$

$\therefore \; \theta \doteqdot 39.3^o$

b Area $= \left(\frac{\theta}{360}\right) \times \pi r^2$

$\therefore \; 30 = \left(\frac{\theta}{360}\right) \times \pi \times 10^2$

$\therefore \; \dfrac{30 \times 360}{\pi \times 100} = \theta$

$\therefore \; \theta \doteqdot 34.4^o$

5 **a**

$$l = r\theta$$
$$\therefore\quad 6 = 8\theta$$
$$\therefore\quad \theta = \tfrac{3}{4}^{c}$$

b

$$l = r\theta$$
$$\therefore\quad 8.4 = 5\theta$$
$$\therefore\quad \theta = \tfrac{8.4}{5} = 1.68^{c}$$

c

$$l = r(2\pi - \theta)$$
$$\therefore\quad 31.7 = 8(2\pi - \theta)$$
$$\therefore\quad 2\pi - \theta = \tfrac{31.7}{8}$$
$$\therefore\quad \theta = 2\pi - \tfrac{31.7}{8}$$
$$\therefore\quad \theta \doteqdot 2.32^{c}$$

6 **a** Area $= \tfrac{1}{2}r^2\theta$

$= \tfrac{1}{2} \times 5^2 \times 0.7$

$= 8.75\ \text{cm}^2$

b Shaded area $=$ area of sector $-$ area of Δ

$= \tfrac{1}{2} \times 12^2 \times 1.5 - \tfrac{1}{2} \times 12 \times 12 \times \sin 1.5^{c}$

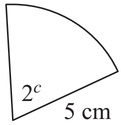
$\doteqdot 36.2\ \text{cm}^2$

c Area $=$ area of Δ $-$ area of sector

$= \tfrac{1}{2} \times 12 \times 30 \times \sin(0.66) - \tfrac{1}{2} \times 12^2 \times 0.66$

$\doteqdot 62.8\ \text{cm}^2$

7



a Arc length

$= r\theta$

$= 5 \times 2$

$= 10$ cm

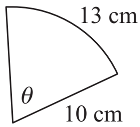
b Area

$= \tfrac{1}{2}r^2\theta$

$= \tfrac{1}{2} \times 5^2 \times 2$

$= 25\ \text{cm}^2$

8



a Arc length $= r\theta$

$\therefore\quad 13 = 10\theta$

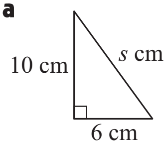
$\therefore\quad 1.3 = \theta$

and area $= \tfrac{1}{2}r^2\theta$

$= \tfrac{1}{2} \times 10^2 \times 1.3$

$= 65\ \text{cm}^2$

9 **a**



$s^2 = 6^2 + 10^2$ {Pythagoras}

$\therefore\quad s = \sqrt{6^2 + 10^2}$

$\therefore\quad s \doteqdot 11.6619$

$\therefore\quad s \doteqdot 11.7$ cm

b $r = s \doteqdot 11.7$ cm

c arc length $= 2\pi \times 6 \quad \doteqdot 37.6991 .... \doteqdot 37.7$ cm

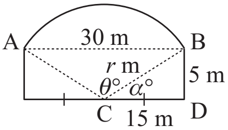
d arc length $= \left(\tfrac{\theta}{360}\right) \times 2\pi r$

$\therefore\quad \tfrac{\theta}{360} \times 2 \times \pi \times 11.6619 \doteqdot 37.6991$

$$\therefore\quad \frac{37.6991 \times 360}{2 \times \pi \times 11.6619} \doteqdot \theta$$

$\therefore\quad \theta \doteqdot 185^{o}$

10



a $\tan\alpha = \tfrac{5}{15}$

$\therefore\quad \alpha = \tan^{-1}\left(\tfrac{1}{3}\right)$

$\therefore\quad \alpha \doteqdot 18.43$

b $\theta + 2\alpha = 180$ {angles on a line}

$\therefore\quad \theta = 180 - 2 \times 18.43$

$\therefore\quad \theta \doteqdot 143.1$

Note: $r^2 = 5^2 + 15^2$

$\therefore\quad r^2 = 250$

c Area $= 2 \times$ area ΔCDB $+$ area sector

$= 2 \times \tfrac{1}{2} \times 15 \times 5 + \left(\tfrac{143.1}{360}\right) \times \pi \times 250$

$\doteqdot 387.3\ \text{m}^2$

11 **a** $l = \left(\tfrac{\theta}{360}\right) \times 2\pi r$

$= \frac{\frac{1}{60}}{360} \times 2 \times \pi \times 6370$ km

$\doteqdot 1.852\,957$ ...... km

$\doteqdot 1.853$ km

b speed $= \dfrac{\text{distance}}{\text{time}}$ $\quad\therefore\quad$ time $= \dfrac{\text{distance}}{\text{speed}}$

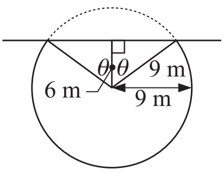
$= \dfrac{2130 \text{ km}}{480 \text{ n miles/h}}$

$= \dfrac{2130 \text{ km}}{480 \times 1.853 \text{ km/h}}$

$\doteqdot 2.3947$ ..... hours

$\doteqdot 2$ hours 24 min

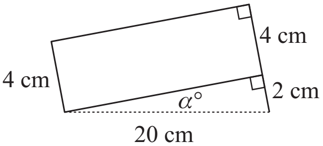
12



$$\cos\theta = \tfrac{6}{9} = \tfrac{2}{3}$$
$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{2}{3}\right)$$
$$\therefore \quad \theta \doteqdot 48.19^o$$
So, $360 - 2\theta \doteqdot 263.62^o$

Area
$= \text{area of } \Delta + \text{area of sector}$
$= \frac{1}{2} \times 9^2 \times \sin 96.38^o + \left(\frac{263.62}{360}\right) \times \pi \times 9^2$
$\doteqdot 227 \text{ m}^2$

13



a $\sin\alpha = \frac{2}{20} = 0.1$
$\therefore \quad \alpha = \sin^{-1}(0.1)$
$\therefore \quad \alpha \doteqdot 5.7392.....$
$\therefore \quad \alpha \doteqdot 5.739$

b $\theta + 90 + 90 + 2\alpha = 360$
$\therefore \quad \theta = 180 - 2\alpha$
$\doteqdot 180 - 2 \times 5.739$
$\doteqdot 168.5$

c $\phi + \theta = 360$
$\therefore \quad \phi \doteqdot 360 - 168.5$
$\therefore \quad \phi \doteqdot 191.5$

d length of belt
$= 2 \times \sqrt{20^2 - 2^2} + \frac{\theta}{360} \times 2\pi \times 4 + \frac{\phi}{360} \times 2\pi \times 6$
$\doteqdot 71.62$ cm

REVIEW SET 11A

1 **a** $\sin 70^o \doteqdot 0.94$ **b** $\cos 35^o \doteqdot 0.82$

2 $\text{M}(\cos 73^o, \sin 73^o) \doteqdot (0.292, 0.956)$ $\quad \text{N}(\cos 190^o, \sin 190^o) \doteqdot (-0.985, -0.174)$
$\text{P}(\cos 307^o, \sin 307^o) \doteqdot (0.602, -0.799)$

3 The x-coordinate of A $= -0.222$
$\therefore \quad \cos\theta = -0.222$
$\therefore \quad \theta = \cos^{-1}(-0.222)$
$\therefore \quad \theta \doteqdot 102.8^o$

4 **a** $\sin 120^o = \sin(180 - 120)^o = \sin 60^o$
$\therefore \quad \theta = 60^o$

b $\sin 165^o = \sin(180 - 165)^o = \sin 15^o$
$\therefore \quad \theta = 15^o$

c $\sin 95^o = \sin(180 - 95)^o = \sin 85^o$
$\therefore \quad \theta = 85^o$

5 **a** $\sin 47^o$
$= \sin(180 - 47)^o$
$= \sin 133^o$
$\therefore \quad \theta = 133^o$

b $\sin 8^o$
$= \sin(180 - 8)^o$
$= \sin 172^o$
$\therefore \quad \theta = 172^o$

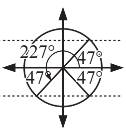
c $\sin 86^o$
$= \sin(180 - 86)^o$
$= \sin 94^o$
$\therefore \quad \theta = 94^o$

6 **a** $\sin 159^o$
$= \sin(180 - 159)^o$
$= \sin 21^o$
$\doteqdot 0.358$

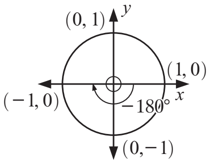
b $\cos 92^o$
$= -\cos(180 - 92)^o$
$= -\cos 88^o$
$\doteqdot -0.035$

c $\cos 75^o$
$= -\cos(180 - 75)^o$
$= -\cos 105^o$
$\doteqdot 0.259$

d $\sin 227^o = \sin(-47)^o$
$= -\sin 47^o$
$\doteqdot -0.731$



7



a $\cos 360^o = 1, \quad \sin 360^o = 0$

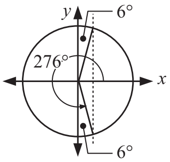
b $\cos(-180^o) = -1, \quad \sin(-180^o) = 0$

8 **a** $\sin 101^o$
$= \sin(180 - 101)^o$
$= \sin 79^o$
$\therefore \quad \theta = 79^o$

b $\sin 127^o$
$= \sin(180 - 127)^o$
$= \sin 53^o$
$\therefore \quad \theta = 53^o$

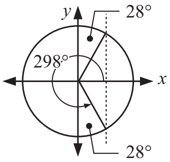
c $\sin 168^o$
$= \sin(180 - 168)^o$
$= \sin 12^o$
$\therefore \quad \theta = 12^o$

9 **a**



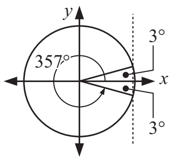
$\therefore \quad \theta = (90-6)^o = 84^o$

b



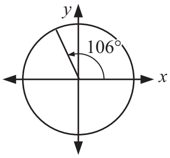
$\therefore \quad \theta = (90-28)^o = 62^o$

c



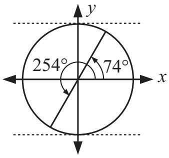
$\therefore \quad \theta = 3^o$

10 **a**



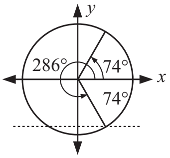
$$\begin{aligned} &\sin 106^o \\ &= \sin(180-106)^o \\ &= \sin 74^o \\ &\doteqdot 0.961 \end{aligned}$$

b



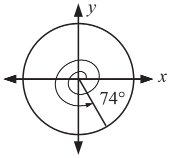
$$\begin{aligned} 254^o &= 74^o + 180^o \\ \therefore \quad \sin 254^o &= -\sin 74^o \\ &\doteqdot -0.961 \end{aligned}$$

c



$$\begin{aligned} \sin 286^o &= -\sin 74^o \\ &= -0.961 \end{aligned}$$

d

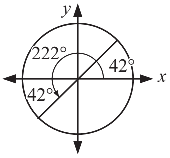


$$\begin{aligned} 646^o &= 360^o + 286^o \\ \therefore \quad \sin 646^o &= \sin 286^o \\ &= -0.961 \quad \{\text{from } \mathbf{c}\} \end{aligned}$$

11 **a**

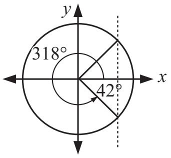
$$\begin{aligned} &\cos 138^o \\ &= -\cos(180-138)^o \\ &= -\cos 42^o \\ &\doteqdot -0.743 \end{aligned}$$

b



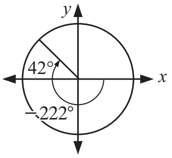
$$\begin{aligned} &\cos 222^o \\ &= -\cos 42^o \\ &= -0.743 \end{aligned}$$

c



$$\begin{aligned} &\cos 318^o \\ &= \cos 42^o \\ &= 0.743 \end{aligned}$$

d



$$\begin{aligned} &\cos(-222)^o \\ &= -\cos 42^o \\ &= -0.743 \end{aligned}$$

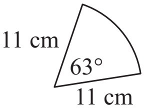
REVIEW SET 11B

1 $\text{Area} = \frac{1}{2} \times 7.3 \times 9.4 \times \sin 38^o$
$\doteqdot 21.1 \text{ km}^2$

2 **a** $\text{Area} = \left(\frac{80}{360}\right) \times T \times 13^2 \doteqdot 118 \text{ cm}^2$

b $\text{Area} = \frac{1}{2} \times 11 \times 9 \times \sin 65^o \doteqdot 44.9 \text{ cm}^2$

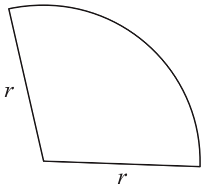
3



a $\text{Perimeter} = 2 \times 11 + \left(\frac{63}{360}\right) \times 2\pi \times 11$
$\doteqdot 34.1 \text{ cm}$

b $\text{Area} = \left(\frac{63}{360}\right) \times \pi \times 11^2$
$\doteqdot 66.5 \text{ cm}^2$

4



$$\begin{aligned} \text{Perimeter} &= 2r + \left(\tfrac{120}{360}\right) \times 2\pi r \\ \therefore \quad 36 &= 2r + \tfrac{1}{3} \times 2\pi r \\ \therefore \quad 36 &= r\left(2 + \tfrac{2\pi}{3}\right) \\ \therefore \quad r &= \frac{36}{2 + \frac{2\pi}{3}} \text{ cm} \\ \therefore \quad r &\doteqdot 8.7925..... \\ \therefore \quad r &\doteqdot 8.79 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{Area} &= \left(\tfrac{120}{360}\right) \times \pi r^2 \\ &= \tfrac{1}{3} \times \pi \times (8.7925)^2 \\ &\doteqdot 81.0 \text{ cm}^2 \end{aligned}$$

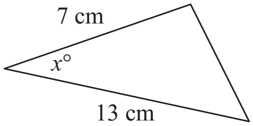
5 Area $= 42\text{ cm}^2$ $\quad\therefore\quad \frac{1}{2}\times 7\times 13\times\sin x = 42$

$$\therefore\quad \sin x = \frac{42\times 2}{7\times 13}$$

$$\therefore\quad x = \sin^{-1}\left(\tfrac{84}{91}\right)$$

$$\therefore\quad x \doteq 67.4 \quad\text{or}\quad 180-67.4$$

$$\therefore\quad x = 67.4 \quad\text{or}\quad 112.6$$



i.e., the included angle is 67.4^o or 112.6^o {assuming the figure is not drawn accurately}

6

$$\text{Area} = 80\text{ cm}^2$$

$$\therefore\quad \tfrac{1}{2}\times 11.3\times 19.2\sin x^o = 80$$

$$\therefore\quad \sin x^o = \frac{160}{11.3\times 19.2}$$

$$\therefore\quad x = \sin^{-1}\left(\frac{160}{11.3\times 19.2}\right)$$

$$\therefore\quad x \doteq 47.5 \quad\text{or}\quad 180-47.5$$

$$\therefore\quad x \doteq 47.5 \quad\text{or}\quad 132.5$$

When $x \doteq 47.5$, $\text{AC} \doteq \sqrt{19.2^2+11.3^2-2\times 19.2\times 11.3\times\cos 47.5^o} \doteq 14.3$ cm

When $x \doteq 132.5$, $\text{AC} \doteq \sqrt{19.2^2+11.3^2-2\times 19.2\times 11.3\times\cos 132.5^o} \doteq 28.1$ cm

7 Non shaded area

$= \text{area of } \Delta + \text{area of sector}$

$= \frac{1}{2}\times 7\times 7\times\sin 130^o + \left(\frac{230}{360}\right)\times\pi\times 7^2$

$\doteq 117.12\text{ cm}^2$

$\therefore$ shaded area

$\doteq \pi\times 7^2 - 117.12\text{ cm}^2$

$\doteq 36.8\text{ cm}^2$

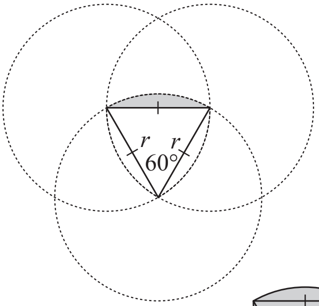
8 **a** $\text{BD} = \sqrt{125^2+120^2-2\times 125\times 120\times\cos 75^o} \doteq 149.2$

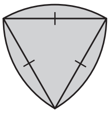
$$\therefore\quad \text{area} = \tfrac{1}{2}\times 120\times 125\sin 75^o + \tfrac{1}{2}\times 149.2\times 90\times\sin 30^o$$

$$\doteq 10\,600\text{ m}^2$$

b $\doteq 1.06$ ha $\quad(10\,000\text{ m}^2 = 1\text{ ha})$

9





shaded area of sector

$= \text{area of sector} - \text{area of }\Delta$

$= \frac{1}{6}\pi r^2 - \frac{1}{2}\times r\times r\times\sin 60^o$

$= \frac{\pi}{6}r^2 - \frac{1}{2}r^2\left(\frac{\sqrt{3}}{2}\right)$

$\therefore$ shaded area of figure

$= 3\left[\frac{\pi}{6}r^2 - \frac{\sqrt{3}}{4}r^2\right] + \frac{1}{2}r^2\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{2}r^2 - \frac{3\sqrt{3}}{4}r^2 + \frac{\sqrt{3}}{4}r^2$

$= \frac{\pi}{2}r^2 - \frac{1}{2}\sqrt{3}r^2$

$= \dfrac{r^2}{2}\left(\pi-\sqrt{3}\right)$

10 **a** 120^o

$= \left(120\times\frac{\pi}{180}\right)^c$

$= \frac{2\pi}{3}^c$

b 225^o

$= 5\times 45^o$

$= 5\times\frac{\pi}{4}^c$

$= \frac{5\pi}{4}^c$

c 150^o

$= 5\times 30^o$

$= 5\times\frac{\pi}{6}^c$

$= \frac{5\pi}{6}^c$

d 540^o

$= 3\times 180^o$

$= 3\pi^c$

11 **a** 71^o

$= \left(71\times\frac{\pi}{180}\right)^c$

$\doteq 1.239^c$

b 224.6^o

$= \left(124.6\times\frac{\pi}{180}\right)^c$

$\doteq 2.175^c$

c -142^o

$= \left(-142\times\frac{\pi}{180}\right)^c$

$\doteq -2.478^c$

d -25.3^o

$= \left(-25.3\times\frac{\pi}{180}\right)^c$

$\doteq -0.4416^c$

12 **a** $\frac{2\pi}{5} = \frac{2\times180^o}{5} = 72^o$ **b** $\frac{5\pi}{4} = \frac{5\times180^o}{4} = 225^o$ **c** $\frac{7\pi}{9} = \frac{7\times180^o}{9} = 140^o$ **d** $\frac{11\pi}{6} = \frac{11\times180^o}{6} = 330^o$

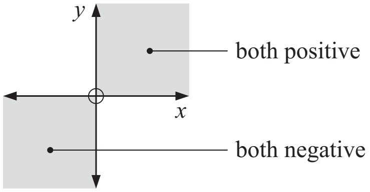
13 **a** $3^c = \left(3 \times \frac{180}{\pi}\right)^o \doteqdot 171.89^o$ **b** $1.46^c = \left(1.46 \times \frac{180}{\pi}\right)^o \doteqdot 83.65^o$ **c** $0.435^o = \left(0.435 \times \frac{180}{\pi}\right)^o \doteqdot 24.92^o$ **d** $-5.271^c = \left(-5.271 \times \frac{180}{\pi}\right)^o \doteqdot -302.01^o$

REVIEW SET 11C

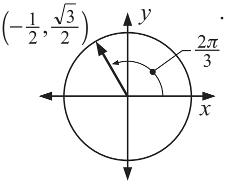
1 **a** The point is $(\cos 320^o, \sin 320^o)$, i.e., $(0.766, -0.643)$ approximately.

b The point is $(\cos 163^o, \sin 163^o)$, i.e., $(-0.956, 0.292)$ approximately.

2



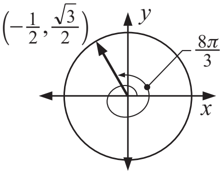
3 **a**



$\therefore \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

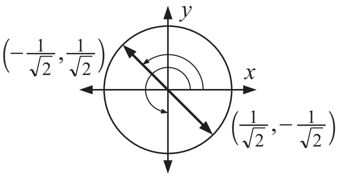
b



$\therefore \cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}$

$\sin\left(\frac{8\pi}{3}\right) = \frac{\sqrt{3}}{2}$

4

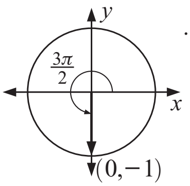


when $\cos\theta = -\sin\theta$

$\frac{\sin\theta}{\cos\theta} = -1$ and this only occurs at the two points shown.

$\therefore \tan\theta = -1$ So, θ can be found.

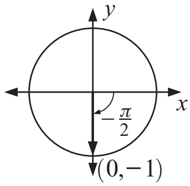
5 **a**



$\therefore \cos\left(\frac{3\pi}{2}\right) = 0$

$\sin\left(\frac{3\pi}{2}\right) = -1$

b



$\therefore \cos\left(-\frac{\pi}{2}\right) = 0$

$\sin\left(-\frac{\pi}{2}\right) = -1$

6 $\cos^2\theta + \sin^2\theta = 1$

$\therefore \frac{9}{16} + \sin^2\theta = 1$

$\therefore \sin^2\theta = \frac{7}{16}$

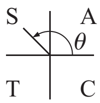
$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$

7 $\cos^2\theta + \sin^2\theta = 1$

$\therefore \frac{9}{16} + \sin^2\theta = 1$

$\therefore \sin^2\theta = \frac{7}{16}$

$\therefore \sin\theta = \pm\frac{\sqrt{7}}{4}$

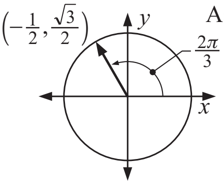


But θ is obtuse

$\therefore \sin\theta$ is positive

$\therefore \sin\theta = \frac{\sqrt{7}}{4}$

8 **a**

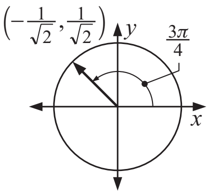


As $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin^2\left(\frac{2\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{3}{4}$

b



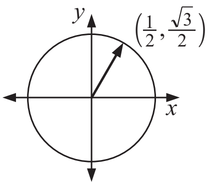
$\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right)$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= -\frac{2}{\sqrt{2}}$

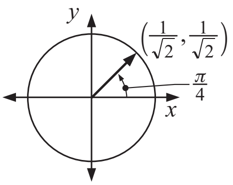
$= -\sqrt{2}$

9 **a**



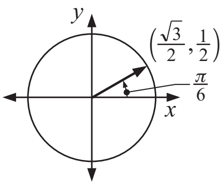
$$\begin{aligned} &2\sin\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{\pi}{3}\right) \\ &= 2 \times \tfrac{\sqrt{3}}{2} \times \tfrac{1}{2} \\ &= \tfrac{\sqrt{3}}{2} \end{aligned}$$

b



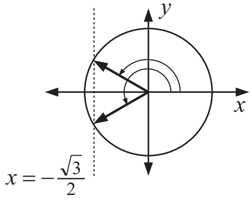
$$\begin{aligned} &\sin^2\left(\tfrac{\pi}{4}\right) - 1 \\ &= \left(\tfrac{1}{\sqrt{2}}\right)^2 - 1 \\ &= \tfrac{1}{2} - 1 \\ &= -\tfrac{1}{2} \end{aligned}$$

c



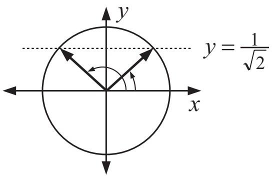
$$\begin{aligned} &\cos^2\left(\tfrac{\pi}{6}\right) - \sin^2\left(\tfrac{\pi}{6}\right) \\ &= \left(\tfrac{\sqrt{3}}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 \\ &= \tfrac{3}{4} - \tfrac{1}{4} \\ &= \tfrac{2}{4} \\ &= \tfrac{1}{2} \end{aligned}$$

10 **a**



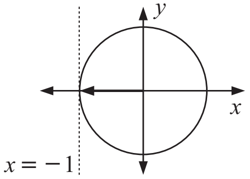
$\therefore \quad \theta = 150^o$ or 210^o

b



$\therefore \quad \theta = 45^o$ or 135^o

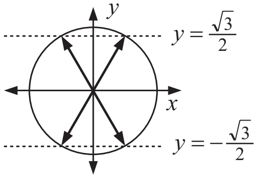
11 **a**



$x = -1$

$\therefore \quad \theta = \pi + k2\pi, \quad k \in Z$

b $\sin^2\theta = \frac{3}{4} \quad \therefore \quad \sin\theta = \pm\frac{\sqrt{3}}{2}$



$$\therefore \quad \theta = \left.\begin{matrix} \frac{\pi}{3} \\ \frac{2\pi}{3} \end{matrix}\right\} + k\pi, \quad k \in Z$$

Chapter 12

NON RIGHT ANGLED TRIANGLE TRIGONOMETRY

EXERCISE 12A

1 **a**
$$\text{BC}^2 = 21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^o$$
$$\therefore \quad \text{BC} = \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^o} \doteq 28.8 \text{ cm}$$

b
$$\text{PQ}^2 = 6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^o$$
$$\therefore \quad \text{PQ} = \sqrt{6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^o} \doteq 3.38 \text{ km}$$

c
$$\text{KM}^2 = 6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^o$$
$$\therefore \quad \text{KM} = \sqrt{6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^o} \doteq 14.2 \text{ m}$$

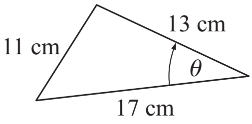
2
$$\cos A = \frac{12^2 + 13^2 - 11^2}{2 \times 12 \times 13}$$
$$\therefore \quad A = \cos^{-1}\left(\tfrac{192}{312}\right)$$
$$\therefore \quad A \doteq 52.0^o$$

$$\cos B = \frac{13^2 + 11^2 - 12^2}{2 \times 13 \times 11}$$
$$\therefore \quad B = \cos^{-1}\left(\tfrac{146}{286}\right)$$
$$\therefore \quad B \doteq 59.3^o$$

$$C = (180 - A - B)^o$$
$$\doteq 68.7^o$$

3
$$\cos Q = \frac{5^2 + 7^2 - 10^2}{2 \times 5 \times 7}$$
$$\therefore \quad Q = \cos^{-1}\left(\tfrac{-26}{70}\right)$$
$$\therefore \quad Q \doteq 112^o$$

4 **a**

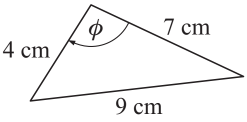


The smallest angle is opposite the shortest side.

$$\cos\theta = \frac{13^2 + 17^2 - 11^2}{2 \times 13 \times 17}$$
$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{337}{442}\right)$$
$$\therefore \quad \theta \doteq 40.3$$

So, the smallest angle measures 40.3^o.

b



The largest angle is opposite the longest side.

$$\cos\phi = \frac{4^2 + 7^2 - 9^2}{2 \times 4 \times 7}$$
$$\therefore \quad \phi = \cos^{-1}\left(-\tfrac{16}{56}\right)$$
$$\therefore \quad \phi \doteq 106.60$$

So, the largest angle measures 107^o, approx.

5 **a**
$$\cos\theta = \frac{2^2 + 5^2 - 4^2}{2 \times 2 \times 5}$$
$$= \tfrac{13}{20}$$
$$= 0.65$$

b
$$x^2 = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos\theta$$
$$\therefore \quad x = \sqrt{5^2 + 3^2 - 2 \times 5 \times 3 \times 0.65}$$
$$\therefore \quad x \doteq 3.81$$

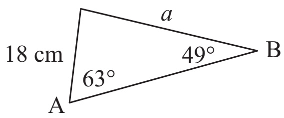
EXERCISE 12B.1

1 **a** By the sine rule,
$$\frac{x}{\sin 48^o} = \frac{23}{\sin 37^o}$$
$$\therefore \quad x = \frac{23 \times \sin 48^o}{\sin 37^o}$$
$$\therefore \quad x \doteq 28.4$$

b By the sine rule,
$$\frac{x}{\sin 115^o} = \frac{11}{\sin 48^o}$$
$$\therefore \quad x = \frac{11 \times \sin 115^o}{\sin 48^o}$$
$$\therefore \quad x \doteq 13.4$$

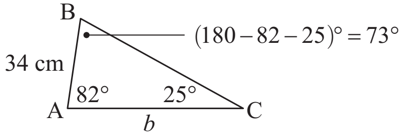
c By the sine rule,
$$\frac{x}{\sin 51^o} = \frac{4.8}{\sin 80^o}$$
$$\therefore \quad x = \frac{4.8 \times \sin 51^o}{\sin 80^o}$$
$$\therefore \quad x \doteq 3.79$$

2 **a**



$$\begin{aligned} \frac{a}{\sin 63^\circ} &= \frac{18}{\sin 49^o} \quad \{\text{sine rule}\} \\ \therefore\ a &= \frac{18 \times \sin 63^o}{\sin 49^o} \\ \therefore\ a &\doteqdot 21.25 \text{ cm} \end{aligned}$$

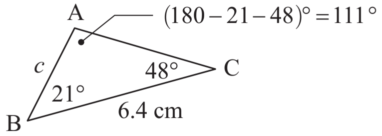
b



By the sine rule, $\dfrac{b}{\sin 73^\circ} = \dfrac{34}{\sin 25^\circ}$

$$\begin{aligned} \therefore\ b &= \frac{34 \times \sin 73^o}{\sin 25^o} \\ \therefore\ b &\doteqdot 76.9 \text{ cm} \end{aligned}$$

c



By the sine rule, $\dfrac{c}{\sin 48^\circ} = \dfrac{6.4}{\sin 111^\circ}$

$$\begin{aligned} \therefore\ c &= \frac{6.4 \times \sin 48^o}{\sin 111^o} \\ \therefore\ c &\doteqdot 5.09 \text{ cm} \end{aligned}$$

EXERCISE 12B.2

1 By the sine rule

$$\begin{aligned} \frac{\sin C}{11} &= \frac{\sin 40^o}{8} \\ \therefore\ \sin C &= \frac{11 \times \sin 40^o}{8} \\ \therefore\ C &= \sin^{-1}\left(\frac{11 \times \sin 40^o}{8}\right) \\ \therefore\ C &\doteqdot 62.1^o \quad \text{or} \quad (180 - 62.1)^o \\ \therefore\ C &\doteqdot 62.1^o \quad \text{or} \quad 117.9^o \end{aligned}$$

2 **a**

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin B}{b} \\ \therefore\ \sin A &= \frac{14.6 \times \sin 65^o}{17.4} \\ \therefore\ A &= \sin^{-1}\left(\frac{14.6 \times \sin 65^o}{17.4}\right) \\ \therefore\ A &\doteqdot 49.5^o \quad \text{or} \quad 180^o - 49.5^o \\ \therefore\ A &\doteqdot 49.5^o \quad \text{or} \quad 130.5^o \end{aligned}$$

Check: $A = 130.5^o$ is impossible as $A + B = 130.5^o + 65^o$ is already over 180^o. $\therefore\ A \doteqdot 49.5^o$

b

$$\begin{aligned} \frac{\sin B}{43.8} &= \frac{\sin 43^o}{31.4} \\ \therefore\ \sin B &= \frac{43.8 \times \sin 43^o}{31.4} \\ \therefore\ B &= \sin^{-1}\left(\frac{43.8 \times \sin 43^o}{31.4}\right) \\ \therefore\ B &\doteqdot 72.0^o \quad \text{or} \quad 108^o \end{aligned}$$

both of which are possible as $108 + 43 = 151$ which is < 180.

c

$$\begin{aligned} \frac{\sin C}{4.8} &= \frac{\sin 71^o}{6.5} \\ \therefore\ \sin C &= \frac{4.8 \times \sin 71^o}{6.5} \\ \therefore\ C &= \sin^{-1}\left(\frac{4.8 \times \sin 71^o}{6.5}\right) \\ \therefore\ C &\doteqdot 44.3^o \quad \text{or} \quad 135.7^o \end{aligned}$$

But $135.7 + 71$ is already > 180

$\therefore$ this case is impossible $\therefore\ C \doteqdot 44.3^o$

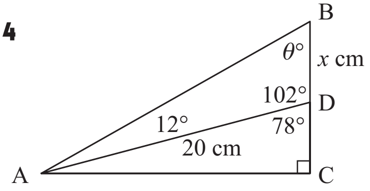
3 The third angle is $180^o - 85^o - 68^o = 27^o$

$\dfrac{\sin 85^o}{11.4} = 0.087\,38....$ and $\dfrac{\sin 27^o}{9.8} = 0.046\,32....$

$\therefore$ it is not possible as $\dfrac{\sin 85^o}{11.4} \neq \dfrac{\sin 27^o}{9.8}$

i.e., the sine rule is violated.

4



In ΔABD,

$$\begin{aligned} \theta &= (180 - 12 - 102)^o \\ \therefore\ \theta &= 66^o \end{aligned}$$

Now $\dfrac{x}{\sin 12^\circ} = \dfrac{20}{\sin 66^\circ}$

$$\begin{aligned} \therefore\ x &= \frac{20 \times \sin 12^o}{\sin 66^o} \\ \therefore\ x &\doteqdot 4.55 \end{aligned}$$

$\therefore$ BD is 4.55 cm long

5 First we find the length of the diagonal, d m.

$$\frac{d}{\sin 118^o} = \frac{22}{\sin 30^o}$$

$$\therefore \quad d = \frac{22 \times \sin 118^o}{\sin 30^o}$$

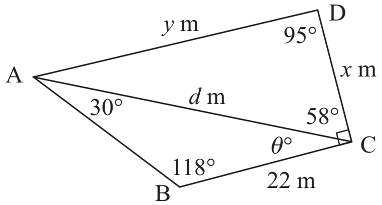
$$\therefore \quad d \doteq 38.85$$

Using the sine rule

$$\frac{y}{\sin 58^o} = \frac{38.85}{\sin 95^o}$$

$$\therefore \quad y = \frac{38.85 \times \sin 58^o}{\sin 95^o}$$

$$\therefore \quad y \doteq 33.1$$



$$\theta = 180^o - 30^o - 118^o = 32^o$$

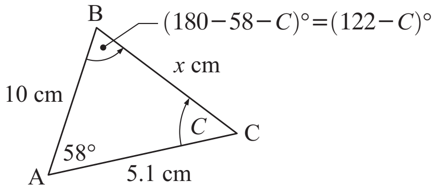
$$\therefore \quad \measuredangle ACD = 58^o$$

and $$\frac{x}{\sin(180-95-58)} = \frac{38.85}{\sin 95^o}$$

$$\therefore \quad x = \frac{38.85 \times \sin 27^o}{\sin 95^o}$$

$$\therefore \quad x \doteq 17.7$$

6 **a**



$$\frac{\sin C}{10} = \frac{\sin(122-C)}{5.1}$$

$$\therefore \quad 5.1 \sin C = 10 \sin(122-C)$$

Using technology,

$$\measuredangle C \doteq 88.7^o \quad \text{or} \quad (180-88.7)^o$$

$$\text{i.e.,} \quad \measuredangle C \doteq 88.7^o \quad \text{or} \quad 91.3^o$$

b Let BC $= x$ cm $\quad \therefore \quad x^2 = 10^2 + 5.1^2 - 2 \times 10 \times 5.1 \cos 58^o$

$$\therefore \quad x = \sqrt{10^2 + 5.1^2 - 20 \times 5.1 \times \cos 58^o}$$

$$\therefore \quad x \doteq 8.4828$$

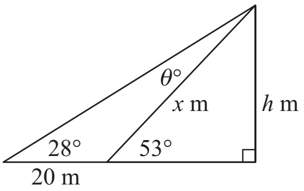
and $$\cos C = \frac{5.1^2 + 8.4828^2 - 10^2}{2 \times 5.1 \times 8.4828} \doteq -0.02309$$

$$\therefore \quad \widehat{C} \doteq \arccos(-0.02309) \doteq 91.3^o$$

c "When faced with using either the sine rule or the cosine rule it is better to use the *cosine rule* as it avoids *the ambiguous case*.

EXERCISE 12C

1



$$\theta^o + 28^o = 53^o$$

{exterior angle of a Δ theorem}

$$\therefore \quad \theta = 25^o$$

By the Sine Rule,

$$\frac{x}{\sin 28^o} = \frac{20}{\sin 25^o}$$

$$\therefore \quad x = \frac{20 \times \sin 28^o}{\sin 25^o}$$

$$\therefore \quad x \doteq 22.22$$

and $$\sin 53^o = \frac{h}{x}$$

$$\therefore \quad h = x \sin 53^o$$

$$= 22.22 \times \sin 53^o$$

$$\doteq 17.7 \text{ m}$$

$\therefore$ the pole is 17.7 m high

2 $$\text{PR}^2 = 63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^o$$

$$\therefore \quad \text{PR} = \sqrt{63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^o}$$

$$\therefore \quad \text{PR} \doteq 207 \text{ m}$$

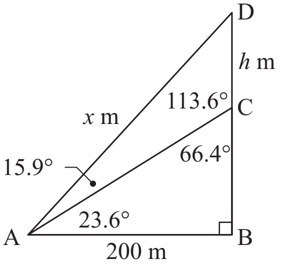
3 $$\cos T = \frac{220^2 + 340^2 - 165^2}{2 \times 220 \times 340}$$

$$\therefore \quad T = \cos^{-1}\left(\frac{136\,775}{149\,600}\right)$$

$$\therefore \quad T \doteq 23.9$$

$\therefore$ was 23.9^o off line.

4



In ΔABD,

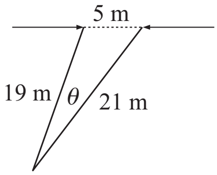
$$\cos(23.6+15.9)^o = \frac{200}{x}$$
$$\therefore \quad x = \frac{200}{\cos 39.5^o}$$
$$\therefore \quad x \doteq 259.2$$

In ΔACD,

$$\frac{h}{\sin 15.9^o} = \frac{259.2}{\sin 113.6^o}$$
$$\therefore \quad h = \frac{259.2 \times \sin 15.9^o}{\sin 113.6^o}$$
$$\therefore \quad h \doteq 77.5$$

∴ is 77.5 m high.

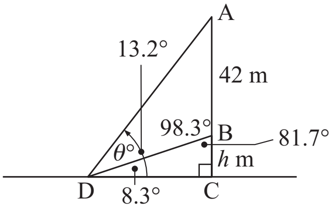
5



$$\cos\theta = \frac{19^2+21^2-5^2}{2\times 19\times 21}$$
$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{777}{798}\right)$$
$$\therefore \quad \theta = 13.2$$

∴ angle of view is 13.2^o

6



$$\theta = 13.2^o - 8.3^o = 4.9^o$$

In ΔABD,

$$\frac{\text{AD}}{\sin 98.3^o} = \frac{42}{\sin 4.9^o}$$
$$\therefore \quad \text{AD} = \frac{42\times\sin 98.3^o}{\sin 4.9^o}$$
$$\therefore \quad \text{AD} \doteq 486.56 \text{ m}$$

In ΔADC,

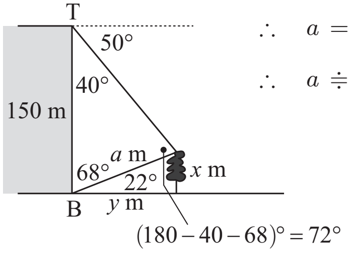
$$\sin 13.2^o = \frac{h+42}{486.56}$$
$$\therefore \quad h+42 = 486.56\times\sin 13.2^o$$
$$\therefore \quad h+42 \doteq 111.1$$
$$\therefore \quad h \doteq 69.1$$

∴ the hill is 69.1 m high

7 **a**

By the sine rule

$$\frac{a}{\sin 40^o} = \frac{150}{\sin 72^o}$$
$$\therefore \quad a = \frac{150\times\sin 40^o}{\sin 72^o}$$
$$\therefore \quad a \doteq 101.38$$



Now
$$\sin 22^o = \frac{x}{101.38}$$
$$\therefore \quad x = 101.38\times\sin 22^o$$
$$\therefore \quad x \doteq 38.0$$

b and
$$\cos 22^o = \frac{y}{101.38}$$
$$\therefore \quad y = 101.38\times\cos 22^o$$
$$\therefore \quad y \doteq 94.0$$

∴ the tree is 38.0 m high and 94.0 m from the building.

8 Using Pythagoras' theorem

$\text{RQ} = \sqrt{4^2+7^2} = \sqrt{65}$ m

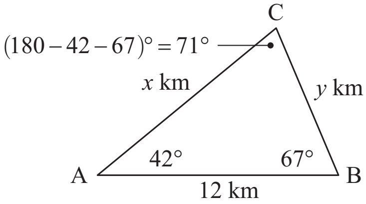
$\text{PQ} = \sqrt{8^2+7^2} = \sqrt{113}$ cm

$\text{PR} = \sqrt{8^2+4^2} = \sqrt{80}$ cm

Now
$$\cos Q = \frac{(\sqrt{113})^2+(\sqrt{65})^2-(\sqrt{80})^2}{2\times\sqrt{113}\times\sqrt{65}}$$
$$\therefore \quad \cos\theta \doteq \left(\tfrac{98}{171.4}\right)$$
$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{98}{171.4}\right)$$
$$\therefore \quad \theta \doteq 55.1$$

So ∠PQR measures 55.1^o

9



$$\frac{x}{\sin 67^o} = \frac{12}{\sin 71^o} = \frac{y}{\sin 42^o}$$

$$\therefore \quad x = \frac{12 \times \sin 67^o}{\sin 71^o} \quad \text{and} \quad y = \frac{12 \times \sin 42^o}{\sin 71^o}$$

$$\therefore \quad x \doteq 11.7 \qquad \therefore \quad y \doteq 8.49$$

So, C is 11.7 km from A and 8.49 km from B.

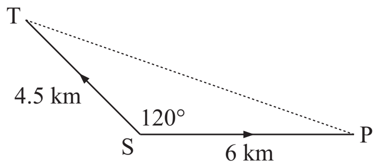
10 **a**

$$\text{QS} = \sqrt{8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70^o}$$
$$\doteq 11.93$$
$$\therefore \quad \text{area} = \tfrac{1}{2} \times 8 \times 12 \times \sin 70^o + \tfrac{1}{2} \times 10 \times 11.93 \times \sin 30^o$$
$$\doteq 74.9 \text{ km}^2$$

b

$$1 \text{ ha is } 100 \text{ m} \times 100 \text{ m}$$
$$= 0.1 \text{ km} \times 0.1 \text{ km}$$
$$= 0.01 \text{ km}^2$$
$$\therefore \quad 1 \text{ km}^2 = 100 \text{ ha}$$
$$\therefore \quad \text{area} \doteq 7490 \text{ ha}$$

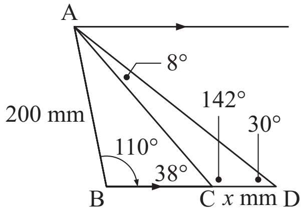
11



$$\text{Distance} = \text{speed} \times \text{time}$$
$$\text{So, after 45 min} = 0.75 \text{ h}$$
$$\text{ST} = 6 \times 0.75 = 4.5 \text{ km}$$
$$\text{SP} = 8 \times 0.75 = 6 \text{ km}$$
$$\text{Now} \quad \text{PT} = \sqrt{4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120^o}$$
$$\therefore \quad \text{PT} \doteq 9.12$$

So, they are 9.12 km apart.

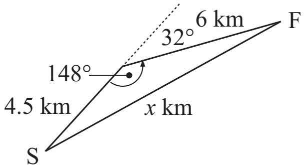
12



$$\text{In } \Delta\text{ABC} \quad \frac{\text{AC}}{\sin 110^o} = \frac{200}{\sin 38^o}$$
$$\therefore \quad \text{AC} = \frac{200 \times \sin 110^o}{\sin 38^o} \doteq 305.26$$
$$\text{and in } \Delta\text{ACD} \quad \frac{x}{\sin 8^o} = \frac{305.26}{\sin 30^o}$$
$$\therefore \quad x = \frac{305.26 \times \sin 8^o}{\sin 30^o} \doteq 84.968$$

$\therefore$ the metal strip is 85.0 mm wide.

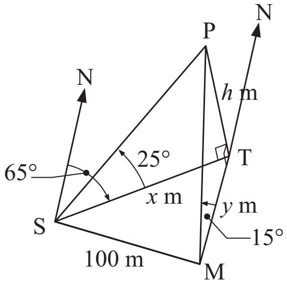
13



$$x = \sqrt{6^2 + (4.5)^2 - 2 \times 6 \times 4.5 \times \cos 148^o}$$
$$\therefore \quad x \doteq 10.1$$

$\therefore$ is 10.1 km from the start.

14

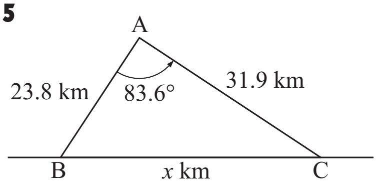


$$\text{In } \Delta\text{PST}, \quad \tan 25^o = \frac{h}{x} \qquad \text{In } \Delta\text{PMT}, \quad \tan 15^o = \frac{h}{y}$$
$$\therefore \quad x = \frac{h}{\tan 25^o} \qquad \therefore \quad y = \frac{h}{\tan 15^o}$$
$$\doteq 2.145h \qquad \doteq 3.732h$$

$$\text{But} \quad \measuredangle\text{STM} = 65^o \quad \{\text{equal alternate angles}\}$$
$$\text{and} \quad 100^2 = x^2 + y^2 - 2xy\cos 65^o$$
$$\therefore \quad 10\,000 = (2.145h)^2 + (3.732h)^2 - 2 \times (2.145)(3.732)h^2 \cos 65^o$$
$$\therefore \quad 10\,000 \doteq 11.763h^2$$
$$\therefore \quad h^2 \doteq 850.15$$
$$\therefore \quad h \doteq 29.2$$

So, the tree is 29.2 m high.

15



By the cosine rule

$$x^2 = 23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6^o$$
$$\therefore \quad x = \sqrt{23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6^o}$$
$$\therefore \quad x \doteq 37.6$$

$\therefore$ B and C are 37.6 km apart

Note: The helicopter's altitude of 4 km must be incorrect. It was not used, however.

REVIEW SET 12

1 **a**

$$\cos x = \frac{13^2 + 19^2 - 11^2}{2 \times 13 \times 19}$$
$$\therefore \quad \cos x = \tfrac{409}{494}$$
$$\therefore \quad x = \cos^{-1}\left(\tfrac{409}{494}\right)$$
$$\therefore \quad x \doteq 34.1$$

b

$$x = \sqrt{15^2 + 17^2 - 2 \times 15 \times 17 \times \cos 72^o}$$
$$\therefore \quad x \doteq 18.9$$

2 **a**

$$\cos x = \frac{11^2 + 19^2 - 13^2}{2 \times 11 \times 19}$$
$$\therefore \quad \cos x = \tfrac{313}{418}$$
$$\therefore \quad x = \cos^{-1}\left(\tfrac{313}{418}\right)$$
$$\therefore \quad x \doteq 41.5$$

b

$$x = \sqrt{14^2 + 21^2 - 2 \times 14 \times 21 \times \cos 47^o}$$
$$\therefore \quad x \doteq 15.4$$

3

$$\text{AC} = \sqrt{11^2 + 9.8^2 - 2 \times 11 \times 9.8 \times \cos 74^o}$$
$$\therefore \quad \text{AC} \doteq 12.554.....$$
$$\therefore \quad \text{AC} \doteq 12.6 \text{ cm}$$

Now $\dfrac{\sin C}{11} = \dfrac{\sin 74^o}{12.554}$

$$\therefore \quad \sin C = \frac{11 \times \sin 74^o}{12.554}$$
$$\therefore \quad C = \sin^{-1}\left(\frac{11 \times \sin 74^o}{12.554}\right)$$
$$\therefore \quad C = 57.4^o \quad \text{or} \quad \underset{\substack{\uparrow \\ \text{impossible as } 122.6 + 74 > 180}}{122.6^o}$$

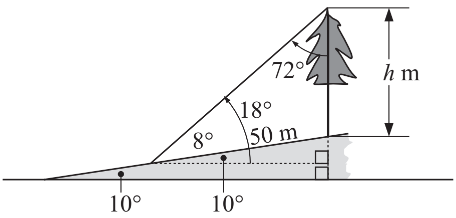
$\therefore$ ∡C measures 57.4^o

$\therefore$ ∡A measures 48.6^o

4

$$\text{DB} = \sqrt{7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 110^o}$$
$$\doteq 14.922$$
$$\therefore \quad \text{total area} = \tfrac{1}{2} \times 7 \times 11 \times \sin 110^o + \tfrac{1}{2} \times 16 \times 14.922 \times \sin 40^o$$
$$\doteq 113 \text{ cm}^2$$

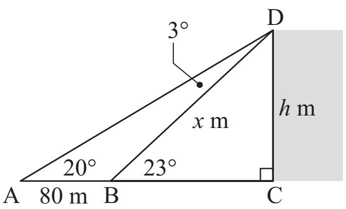
5



$$\frac{h}{\sin 8^o} = \frac{50}{\sin 72^o}$$
$$\therefore \quad h = \frac{50 \times \sin 8^o}{\sin 72^o}$$
$$\therefore \quad h \doteq 7.32$$

i.e., the tree is 7.32 m high

6



$$\text{In } \Delta\text{ABD}, \quad \frac{x}{\sin 20^o} = \frac{80}{\sin 3^o}$$

$$\therefore \quad x = \frac{80 \times \sin 20^o}{\sin 3^o} \doteqdot 522.8$$

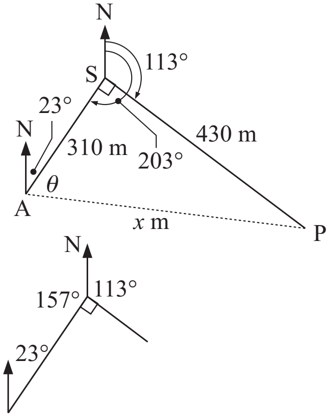
$$\text{Now} \quad \sin 23^o = \frac{h}{x}$$

$$\therefore \quad h = 522.8 \times \sin 23^o$$

$$\therefore \quad h \doteqdot 204$$

So the building is 204 m tall.

7



$$\measuredangle\text{ASP} = 203^o - 113^o = 90^o$$

$$\text{Now} \quad x^2 = 310^2 + 430^2 \quad \{\text{Pythagoras}\}$$

$$\therefore \quad x = \sqrt{310^2 + 430^2}$$

$$\therefore \quad x \doteqdot 530$$

$\therefore$ they are 530 m apart.

$$\text{and} \quad \tan\theta = \tfrac{430}{310}$$

$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{430}{310}\right) \doteqdot 54.2$$

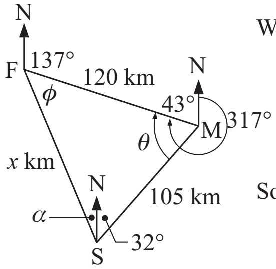
$$\text{and} \quad 23 + \theta = 77.2$$

$\therefore$ bearing of P from A is 077.2^o

8

$$\text{In 45 minutes,} \quad 140 \times \tfrac{3}{4} = 105 \text{ km} \quad \text{is travelled.}$$

$$\text{In 40 minutes,} \quad 180 \times \tfrac{2}{3} = 120 \text{ km} \quad \text{is travelled.}$$



$$\text{We notice that} \quad \theta + 43 + 32 = 180 \quad \{\text{allied angles add to } 180^o\}$$

$$\therefore \quad \theta = 105$$

$$\text{and so,} \quad x = \sqrt{120^2 + 105^2 - 2 \times 120 \times 105 \times \cos 105^o}$$

$$\therefore \quad x \doteqdot 178.74$$

So, is 179 km from the start.

$$\text{Now} \quad \frac{\sin\phi}{105} = \frac{\sin 105^o}{178.74}$$

$$\therefore \quad \sin\phi = \frac{105 \times \sin 105^o}{178.74}$$

$$\therefore \quad \phi \doteqdot 34.6$$

$$\therefore \quad \theta = 180 - 105 - 34.6 - 32 = 8.4 \doteqdot 8$$

So, the bearing is 352^o.

9 If the unknown is an angle, use the cosine rule to avoid the ambiguous case.

10 **a** By the cosine rule,

$$7^2 = 8^2 + x^2 - 2 \times 8 \times x \times \cos 60^o$$

$$\therefore \quad 49 = 64 + x^2 - 16x\left(\tfrac{1}{2}\right)$$

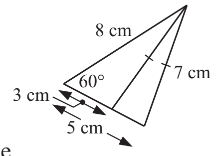
$$\therefore \quad 49 = 64 + x^2 - 8x$$

$$\therefore \quad x^2 - 8x + 15 = 0$$

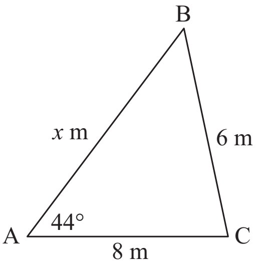
$$\therefore \quad (x-3)(x-5) = 0$$

$$\therefore \quad x = 3 \text{ or } 5$$

b Kady's response should be "Please supply me with additional information as there are two possibilities. Which one do you want?"



11 **a**



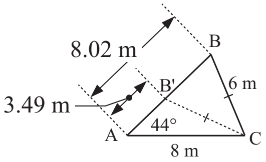
By the cosine rule,

$$\begin{aligned} 6^2 &= x^2 + 8^2 - 2 \times x \times 8 \times \cos 44^o \\ \therefore \quad 36 &= x^2 + 64 - 16x \times \cos 44^o \\ \therefore \quad x^2 - 11.51x + 28 &= 0 \\ \therefore \quad x &= \frac{11.51 \pm \sqrt{11.51^2 - 4(1)(28)}}{2} \\ \therefore \quad x &= \frac{11.51 \pm 4.525}{2} \\ \therefore \quad x &\doteq 8.02 \quad \text{or} \quad 3.49 \end{aligned}$$

Frank needs additional information as there are two possible cases:

(1) when AB $\doteq$ 8.02 m and

(2) when AB $\doteq$ 3.49 m



b Volume

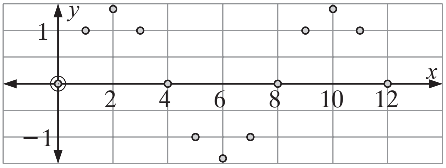
$$\begin{aligned} &= \text{area} \times \text{depth} \\ &= \tfrac{1}{2} \times 8 \times x \times \sin 44^o \times 0.1 \quad \text{and is a max. when} \quad \text{AB} \doteq 8.02 \text{ m} \\ &\doteq 4 \times 8.02 \times \sin 44^o \times 0.1 \\ &\doteq 2.23 \text{ m}^3 \end{aligned}$$

Chapter 13

PERIODIC PHENOMENA

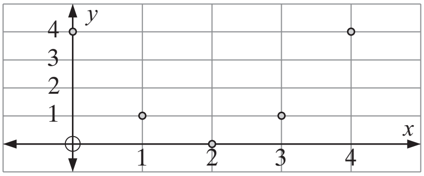
EXERCISE 13A

1 **a**



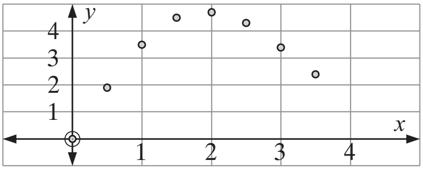
Data exhibits periodic behaviour.

b



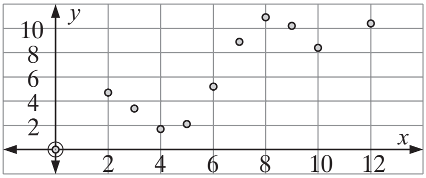
Not enough information to say data is periodic. It may in fact be quadratic.

c



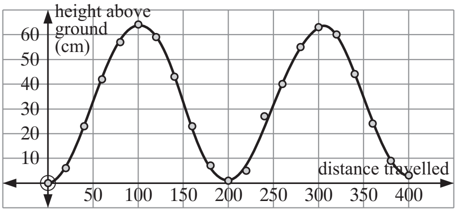
Not enough information to say data is periodic. It may in fact be quadratic.

d



Not enough information to say data is periodic.

2 **a**



b The data is periodic.

i The minimum value from the table is 0 and the maximum value is 64.

So, the principle axis is $y \doteqdot \frac{0+64}{2}$, i.e., $y \doteqdot 32$.

ii The maximum value is 64 cm (approx).

iii The period is $\doteqdot$ 200 cm.

iv The amplitude is $\doteqdot$ 32 cm.

c A curve can be fitted to the data as the distance travelled is continuous.

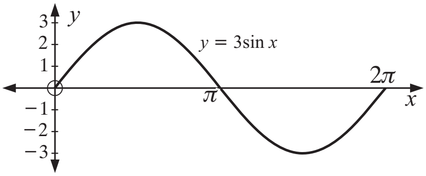
3 **a** periodic **b** periodic **c** periodic **d** not periodic **e** periodic **f** periodic

EXERCISE 13B.1

1 **a** $y = 3\sin x$

has amplitude 3 and period $\frac{2\pi}{1} = 2\pi$

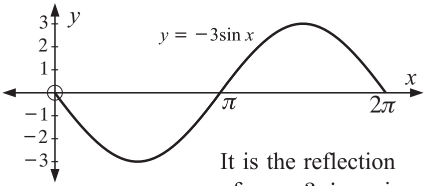
When $x = 0$, $y = 0$.



b $y = -3\sin x$

has amplitude $|-3| = 3$ and period $\frac{2\pi}{1}$ $= 2\pi$.

When $x = 0$, $y = 0$.

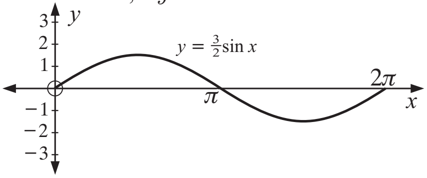


It is the reflection of $y = 3\sin x$ in the x-axis.

c $y = \frac{3}{2}\sin x$

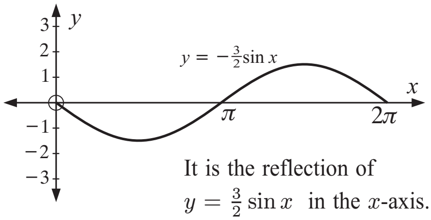
has amplitude $\frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.

When $x = 0$, $y = 0$.



d $y = -\frac{3}{2}\sin x$

has amplitude $\left|-\frac{3}{2}\right| = \frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.

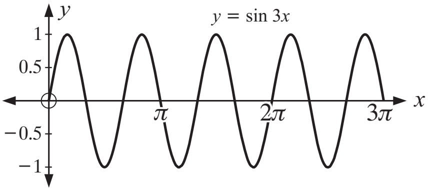


It is the reflection of $y = \frac{3}{2}\sin x$ in the x-axis.

2 **a** $y = \sin 3x$

has amplitude 1 and period $\frac{2\pi}{3}$.

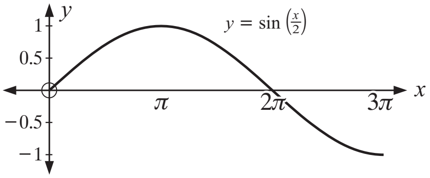
When $x = 0$, $y = 0$.



b $y = \sin\left(\frac{x}{2}\right)$

has amplitude 1 and period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.

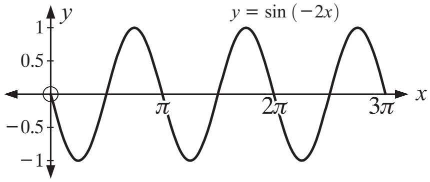
When $x = 0$, $y = 0$.



c $y = \sin(-2x)$

has amplitude 1 and period $\dfrac{2\pi}{|2|} = \pi$.

When $x = 0$, $y = 0$.



It is the reflection of $y = \sin 2x$ in the x-axis.

3 **a** $$\text{period} = \frac{2\pi}{|4|} = \frac{\pi}{2}$$

b $$\text{period} = \frac{2\pi}{|-4|} = \frac{\pi}{2}$$

c $$\text{period} = \frac{2\pi}{\left|\frac{1}{3}\right|} = 6\pi$$

d $$\text{period} = \frac{2\pi}{0.6} = \frac{20\pi}{6} = \frac{10\pi}{3}$$

4 **a** $$\frac{2\pi}{B} = 5\pi \quad \therefore\ B = \frac{2}{5}$$

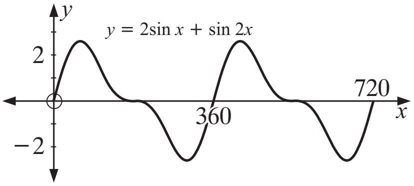
b $$\frac{2\pi}{B} = \frac{2\pi}{3} \quad \therefore\ B = 3$$

c $$\frac{2\pi}{B} = 12\pi \quad \therefore\ B = \frac{1}{6}$$

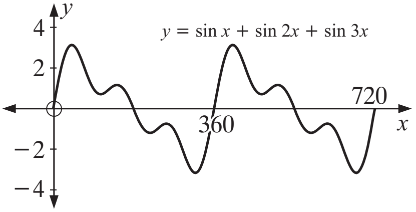
d $$\frac{2\pi}{B} = 4 \quad \therefore\ B = \frac{\pi}{2}$$

e $$\frac{2\pi}{B} = 100 \quad \therefore\ B = \frac{2\pi}{100} = \frac{\pi}{50}$$

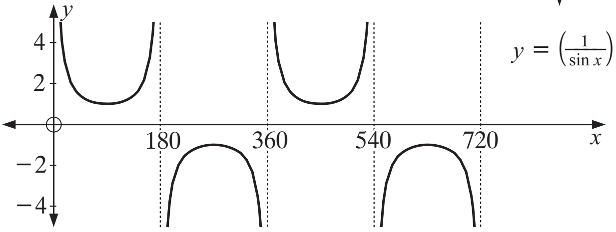
5 **a**



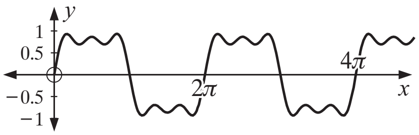
b



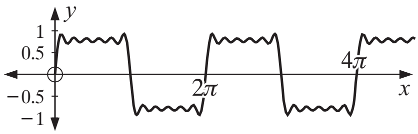
c



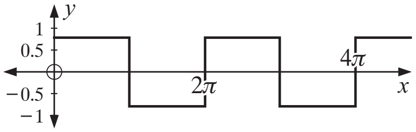
6 **a**



b

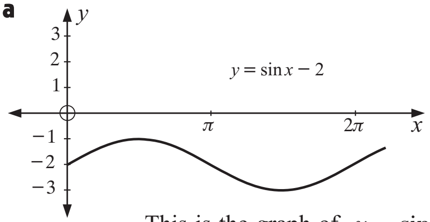


Prediction:



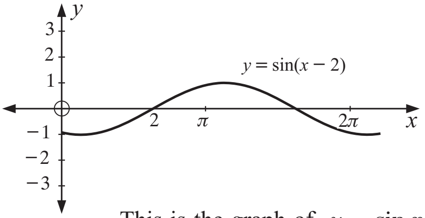
EXERCISE 13B.2

1 **a**



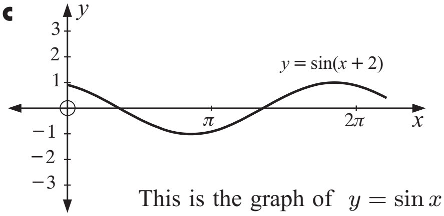
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

b



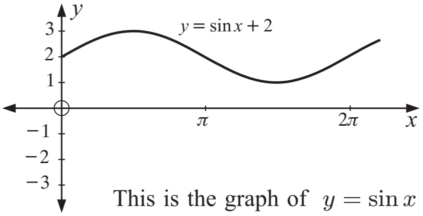
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

c



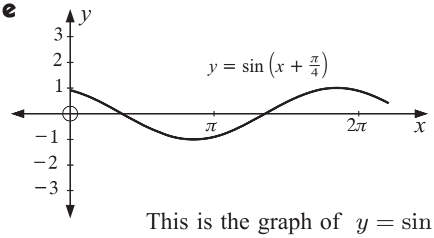
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

d



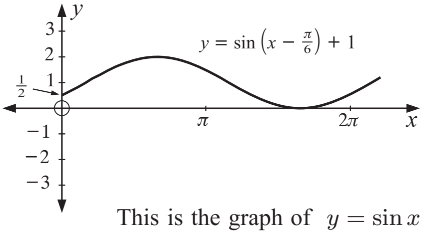
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

e



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} -\frac{\pi}{4} \\ 0 \end{bmatrix}$.

f



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

3 **a** period $= \dfrac{2\pi}{|5|} = \frac{2\pi}{5}$ **b** period $= \dfrac{2\pi}{\left|\frac{1}{4}\right|} = 8\pi$ **c** period $= \dfrac{2\pi}{|-2|} = \pi$

4 **a** $\dfrac{2\pi}{B} = 3\pi$ $\therefore\ B = \frac{2}{3}$

b $\dfrac{2\pi}{B} = \dfrac{\pi}{10}$ $\therefore\ B = 20$

c $\dfrac{2\pi}{B} = 100\pi$ $\therefore\ B = \frac{2}{100} = \frac{1}{50}$

d $\dfrac{2\pi}{B} = 50$ $\therefore\ B = \frac{2\pi}{50} = \frac{\pi}{25}$

5 **a** A translation of $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, i.e., vertically down 1 unit.

b A translation of $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$, i.e., horizontally $\frac{\pi}{4}$ units right.

c A vertical dilation of factor 2. **d** A horizontal dilation of factor $\frac{1}{4}$.

e A vertical dilation of factor $\frac{1}{2}$. **f** A horizontal dilation of factor $\dfrac{1}{\frac{1}{4}} = 4$.

g A reflection in the x-axis. **h** A translation of $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

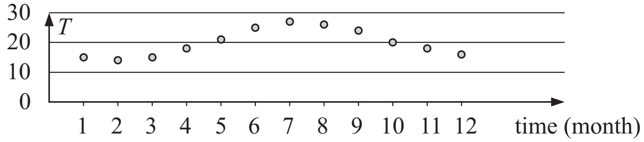
i A vertical dilation of factor 2 followed by a horizontal dilation of factor $\frac{1}{3}$.

j A translation of $\begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$.

EXERCISE 13C

1 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	14	15	18	21	25	27	26	24	20	18	16



The period is 12 months so $\dfrac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6}$ {assuming $B > 0$}.

Amplitude, $A \doteqdot \dfrac{\text{max.} - \text{min.}}{2} \doteqdot \dfrac{27 - 14}{2} \doteqdot 6.5$

As the principal axis is midway between min. and max., then $D \doteqdot \dfrac{27 + 14}{2} \doteqdot 20.5$

When T is 20.5 (midway between min. and max.)

$$C \doteqdot \frac{2 + 7}{2} \doteqdot 4.5 \quad \{\text{average of } t \text{ values}\}$$

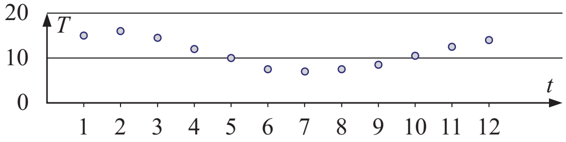
$$\therefore \quad T \doteqdot 6.5 \sin \tfrac{\pi}{6}(t - 4.5) + 20.5 \quad (\textbf{Note:} \quad \tfrac{\pi}{6} \doteqdot 0.524)$$

b Using technology, $T \doteqdot 6.14 \sin(0.575t - 2.70) + 20.4$

i.e., $T \doteqdot 6.14 \sin 0.575(t - 4.70) + 20.4$

2 **a**

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14



The period is $\dfrac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6} \quad \{B > 0\}$

Amplitude, $A \doteqdot \dfrac{\text{max.} - \text{min.}}{2} \doteqdot \dfrac{16 - 7}{2} \doteqdot 4.5$

As the principal axis is midway between min. and max. then $D \doteqdot \dfrac{16 + 7}{2} \doteqdot 11.5$

At min., $t = 7$ and at max., $t = 2 + 12 = 14 \quad \therefore \quad C = \dfrac{7 + 14}{2} = 10.5$

So, $T \doteqdot 4.5 \sin \frac{\pi}{6}(t - 10.5) + 11.5$

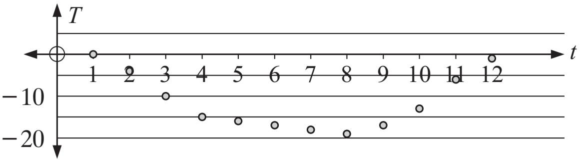
b Using tech., $T \doteqdot 4.29 \sin(0.533t + 0.769) + 11.2$ **Note:** (1) $\frac{\pi}{6} \doteqdot 0.524$ ✓

i.e., $T \doteqdot 4.29 \sin 0.533(t + 1.44) + 11.2$ (2) $\frac{\pi}{6}(1.44 - (-10.5)) \doteqdot 6.25$

$\doteqdot 2\pi$

3

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	0	−4	−10	−15	−16	−17	−18	−19	−17	−13	−6	−1



The period is $\frac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6} \quad \{B > 0\}$

Amplitude, $A \doteqdot \frac{\text{max.} - \text{min.}}{2} \doteqdot \frac{0-(-19)}{2} \doteqdot 9.5$

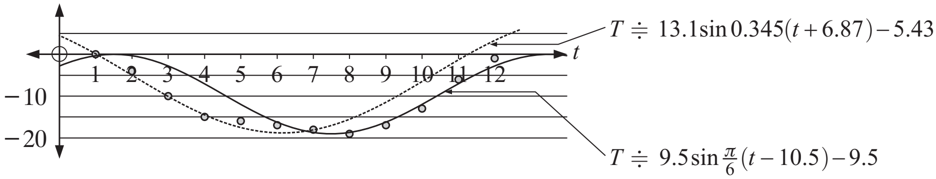
$D \doteqdot \frac{\text{max.} + \text{min.}}{2} \doteqdot \frac{0+(-19)}{2} \doteqdot -9.5$

At min., $t = 8$ and at max., $t = 1 + 12 = 13 \quad \therefore \quad C = \frac{8+13}{2} = 10.5$

So, $T \doteqdot 9.5 \sin \frac{\pi}{6}(t - 10.5) - 9.5$ ...... (1)

From technology, $T \doteqdot 13.1 \sin(0.345t + 2.37) - 5.43$

i.e., $T \doteqdot 13.1 \sin 0.345(t + 6.87) - 5.43$ ...... (2)



The model does not seem appropriate.

4 **a** For the model $H = A \sin B(t - C) + D$

period $= \frac{2\pi}{B} = 12.4$ hours $\quad \therefore \quad B = \frac{2\pi}{12.4} \doteqdot 0.507$

We let the principal axis be 0, i.e., $D = 0$

$\therefore$ A, the amplitude $= 7$, i.e., min. is -7, max. is $+7$

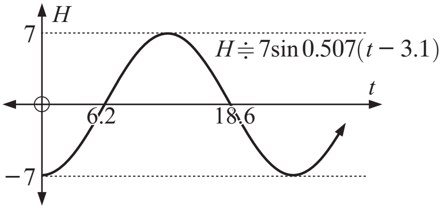
Let $t = 0$ correspond to 'low tide' $\therefore$ $t = 6.2$ corresponds to 'high tide'

$\therefore \quad C = \frac{0+6.2}{2} = 3.1$

So, $H \doteqdot 7 \sin 0.507(t - 3.1) + 0$

i.e., $H \doteqdot 7 \sin 0.507(t - 3.1)$

b



5 Let the model be $H = A \sin B(t - C) + D$ metres

When $t = 0$, $\underset{\substack{\uparrow \\ \text{min.}}}{H = 2}$ and when $t = 50$, $\underset{\substack{\uparrow \\ \text{max.}}}{H = 22}$

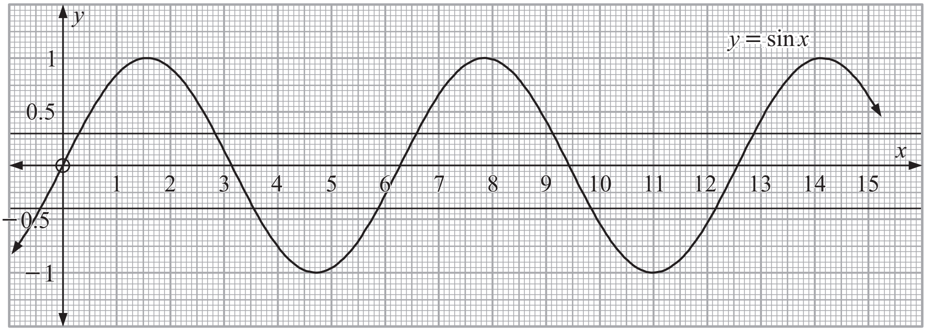
period $= \frac{2\pi}{B} = 100 \quad \therefore \quad B = \frac{2\pi}{100} = \frac{\pi}{50}$

$A = 10$ {from the diagram} $\quad D = \frac{\text{max.} + \text{min.}}{2} = \frac{24}{2} = 12$

$C = \dfrac{0+50}{2} = 25$ {values of t at max. and min.} $\therefore\ H = 10\sin\frac{\pi}{50}(t-25)+12$

EXERCISE 13D.1

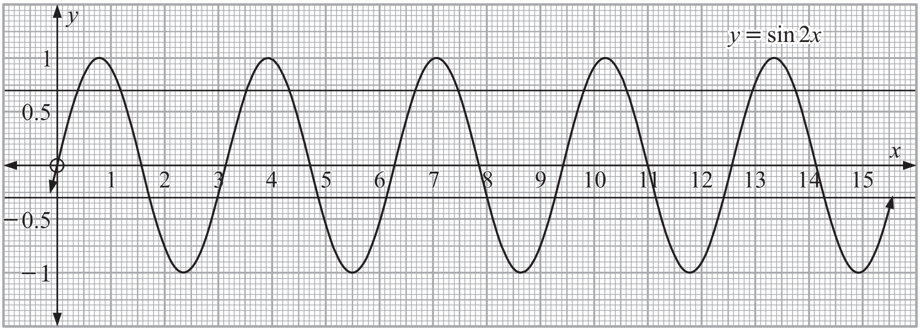
1



a When $\sin x = 0.3,\quad x \doteqdot 0.3,\ 2.8,\ 6.6,\ 9.1,\ 12.9$

b When $\sin x = -0.4,\quad x \doteqdot 5.9,\ 9.8,\ 12.2$

2



a When $\sin(2x) = 0.7,\quad x \doteqdot 0.4,\ 1.2,\ 3.5,\ 4.3,\ 6.7,\ 7.5,\ 9.8,\ 10.6,\ 13.0,\ 13.8$

b When $\sin(2x) = -0.3,\quad x \doteqdot 1.7,\ 3.0,\ 4.8,\ 6.1,\ 8.0,\ 9.3,\ 11.1,\ 12.4,\ 14.3$

EXERCISE 13D.2

1 **a** $\sin x = 0.414$

$\therefore\ x \doteqdot 0.4268,\ 2.715,\ 6.710$

b $\sin x = -0.673$

$\therefore\ x \doteqdot 3.880,\ 5.545$

c $\sin x = 1.289$

There are no solutions as all values of $\sin x$ lie between -1 and $+1$.

d $\sin 2x = 0.162$

$\therefore\ x \doteqdot 0.081\,36,\ 1.489,\ 3.223,\ 4.631,\ 6.365,\ 7.773$

e $\sin\left(\frac{x}{2}\right) = -0.606$

$\therefore\ x \doteqdot 7.585$

f $\sin(x+2) = 0.0652$

$\therefore\ x \doteqdot 1.076,\ 4.348,\ 7.360$

g $\sin(x-1.3) = 0.866$

$\therefore\ x \doteqdot 2.347,\ 3.394$

h $\sin(x-\frac{\pi}{3}) = 0.7063$

$\therefore\ x \doteqdot 1.831,\ 3.405$

i $\sin\left(\frac{2x}{3}\right) = -0.9367 \quad \therefore\ x \doteqdot 6.532,\ 7.605$

EXERCISE 13D.3

1 **a** $x = \frac{\pi}{6} + \frac{k12\pi}{6}$ and $0 \leqslant x \leqslant \frac{36\pi}{6}$

$\therefore\ x = \frac{\pi}{6},\ \frac{13\pi}{6},\ \frac{25\pi}{6}$

b $x = -\frac{\pi}{3} + \frac{k6\pi}{3}$ and $-\frac{6\pi}{3} \leqslant x \leqslant \frac{6\pi}{3}$

$\therefore\ x = -\frac{\pi}{3},\ \frac{5\pi}{3}$

c $x = -\frac{\pi}{2} + \frac{k2\pi}{2}$ and $-\frac{8\pi}{2} \leqslant x \leqslant \frac{8\pi}{2}$

$\therefore\ x = -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2},\ -\frac{3\pi}{2},\ -\frac{5\pi}{2},\ -\frac{7\pi}{2}$

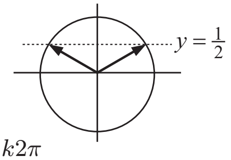
d $x = \frac{5\pi}{6} + \frac{k3\pi}{6}$ and $0 \leqslant x \leqslant \frac{24\pi}{6}$

$\therefore\ x = \frac{2\pi}{6},\ \frac{5\pi}{6},\ \frac{8\pi}{6},\ \frac{11\pi}{6},\ \frac{14\pi}{6},\ \frac{17\pi}{6},\ \frac{20\pi}{6},\ \frac{23\pi}{6}$

i.e., $x = \frac{\pi}{3},\ \frac{5\pi}{6},\ \frac{4\pi}{3},\ \frac{11\pi}{6},\ \frac{7\pi}{3},\ \frac{17\pi}{6},\ \frac{10\pi}{3},\ \frac{23\pi}{6}$

2 **a** $2\sin x = 1, \quad 0 \leqslant x \leqslant 6\pi$

$\therefore \quad \sin x = \frac{1}{2}$

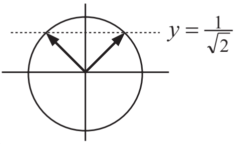


$\therefore \quad x = \left.\begin{array}{l}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{array}\right\} + k2\pi$

$\therefore \quad x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}$

b $\sqrt{2}\sin x = 1, \quad 0 \leqslant x \leqslant 4\pi$

$\therefore \quad \sin x = \frac{1}{\sqrt{2}}$

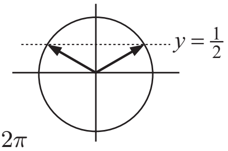


$\therefore \quad x = \left.\begin{array}{l}\frac{\pi}{4}\\ \frac{3\pi}{4}\end{array}\right\} + k2\pi$

$\therefore \quad x = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{4}$

c $2\sin x - 1 = 0, \quad -2\pi \leqslant x \leqslant 2\pi$

$\therefore \quad \sin x = \frac{1}{2}$

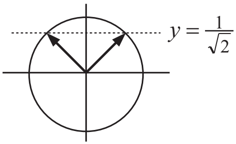


$\therefore \quad x = \left.\begin{array}{l}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{array}\right\} + k2\pi$

$\therefore \quad x = \frac{\pi}{6}, -\frac{11\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}$

d $\sqrt{2}\sin x - 1 = 0, \quad -4\pi \leqslant x \leqslant 0$

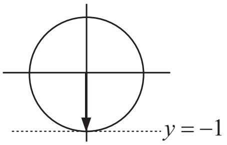
$\therefore \quad \sin x = \frac{1}{\sqrt{2}}$



$\therefore \quad x = \left.\begin{array}{l}\frac{\pi}{4}\\ \frac{3\pi}{4}\end{array}\right\} + k2\pi$

$\therefore \quad x = -\frac{7\pi}{4}, -\frac{15\pi}{4}, -\frac{5\pi}{4}, -\frac{13\pi}{4}$

e $\sin x = -1, \quad 0 \leqslant x \leqslant 6\pi$

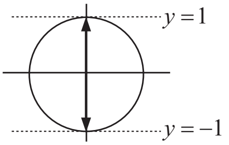


$\therefore \quad x = \frac{3\pi}{2} + k2\pi$

$\therefore \quad x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$

f $\sin^2 x = 1, \quad 0 \leqslant x \leqslant 4\pi$

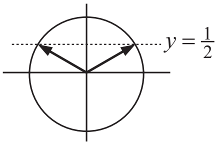
$\therefore \quad \sin x = \pm 1$



$\therefore \quad x = \frac{\pi}{2} + k\pi$

$\therefore \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

g $\sin 2x = \frac{1}{2}, \quad 0 \leqslant x \leqslant 3\pi$

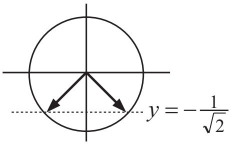


$\therefore \quad 2x = \left.\begin{array}{l}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{array}\right\} + k2\pi \quad \therefore \quad x = \left.\begin{array}{l}\frac{\pi}{12}\\ \frac{5\pi}{12}\end{array}\right\} + k\pi$

$\therefore \quad x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}$

h $\sqrt{2}\sin 3x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin 3x = -\frac{1}{\sqrt{2}}$



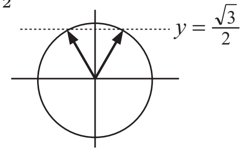
$\therefore \quad 3x = \left.\begin{array}{l}\frac{5\pi}{4}\\ \frac{7\pi}{4}\end{array}\right\} + k2\pi \quad \therefore \quad x = \left.\begin{array}{l}\frac{5\pi}{12}\\ \frac{7\pi}{12}\end{array}\right\} + \frac{k2\pi}{3}$

$\therefore \quad x = \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{21\pi}{12}, \frac{7\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$

i.e., $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

i $2\sin 2x - \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \quad \sin 2x = \frac{\sqrt{3}}{2}$

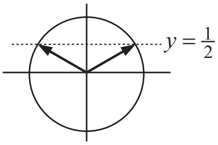


$\therefore \quad 2x = \left.\begin{array}{l}\frac{\pi}{3}\\ \frac{2\pi}{3}\end{array}\right\} + k2\pi \quad \therefore \quad x = \left.\begin{array}{l}\frac{\pi}{6}\\ \frac{\pi}{3}\end{array}\right\} + k\pi$

$\therefore \quad x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$

j $2\sin(x + \frac{\pi}{3}) = 1, \quad -3\pi \leqslant x \leqslant 3\pi$

$\therefore \quad \sin(x + \frac{\pi}{3}) = \frac{1}{2}$



$\therefore \quad x + \frac{\pi}{3} = \left.\begin{array}{l}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{array}\right\} + k2\pi \quad \therefore \quad x = \left.\begin{array}{l}-\frac{\pi}{6}\\ \frac{\pi}{2}\end{array}\right\} + k2\pi$

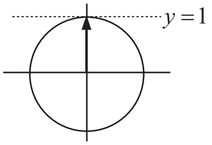
$\therefore \quad x = -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{13\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}$

3 **a**

$$\sin^2 x + \sin x - 2 = 0$$
$$\therefore \quad (\sin x - 1)(\sin x + 2) = 0$$
$$\therefore \quad \sin x = 1 \text{ or } -2$$

But values of sine can only lie between -1 and 1 inclusive.

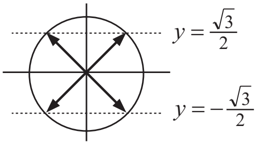
$\therefore \quad \sin x = 1$



$\therefore \quad x = \frac{\pi}{2}, -\frac{3\pi}{2}$

b

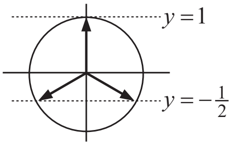
$$4\sin^2 x = 3$$
$$\therefore \quad \sin^2 x = \tfrac{3}{4}$$
$$\therefore \quad \sin x = \pm\tfrac{\sqrt{3}}{2}$$



$\therefore \quad x = \pm\frac{\pi}{3}, \pm\frac{4\pi}{3}, \pm\frac{2\pi}{3}, \pm\frac{5\pi}{3}$

c

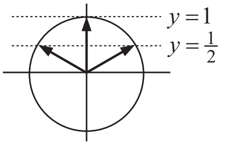
$$2\sin^2 x = \sin x + 1$$
$$\therefore \quad 2\sin^2 x - \sin x - 1 = 0$$
$$\therefore \quad (2\sin x + 1)(\sin x - 1) = 0$$
$$\therefore \quad \sin x = -\tfrac{1}{2} \text{ or } 1$$



$\therefore \quad x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{3\pi}{2}$

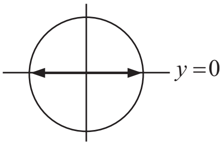
d

$$2\sin^2 x + 1 = 3\sin x$$
$$\therefore \quad 2\sin^2 x - 3\sin x + 1 = 0$$
$$\therefore \quad (2\sin x - 1)(\sin x - 1) = 0$$
$$\therefore \quad \sin x = \tfrac{1}{2} \text{ or } 1$$



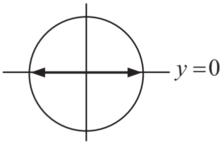
$\therefore \quad x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{3\pi}{2}, -\frac{11\pi}{6}$

4 **a** The zeros of $y = \sin 2x$ are the solutions of $\sin 2x = 0 \quad (0 \leqslant x \leqslant \pi)$



$$\therefore \quad 2x = 0 + k\pi$$
$$\therefore \quad x = 0 + k\,\tfrac{\pi}{2}$$
$$\therefore \quad x = 0, \tfrac{\pi}{2}, \pi$$

b The zeros of $y = \sin(x - \frac{\pi}{4})$ are the solutions of $\sin(x - \frac{\pi}{4}) = 0 \quad (0 \leqslant x \leqslant 3\pi)$



$$\therefore \quad x - \tfrac{\pi}{4} = 0 + k\pi$$
$$\therefore \quad x = \tfrac{\pi}{4} + k\pi$$
$$\therefore \quad x = \tfrac{\pi}{4}, \tfrac{5\pi}{4}, \tfrac{9\pi}{4}$$

EXERCISE 13D.4

1 **a** $P(t) = 7500 + 3000\sin(\frac{\pi t}{8}), \quad 0 \leqslant t \leqslant 12$

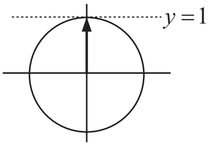
i
$$\begin{aligned} P(0) &= 7500 + 3000\sin 0 \\ &= 7500 + 0 \\ &= 7500 \text{ grass-hoppers} \end{aligned}$$

ii
$$\begin{aligned} P(5) &= 7500 + 3000\sin(\tfrac{5\pi}{8}) \\ &\doteqdot 10\,271.63\,.... \\ &\doteqdot 10\,300 \text{ grass-hoppers} \end{aligned}$$

b The greatest value of $P(t)$ occurs when $\sin\left(\frac{\pi t}{8}\right) = 1$

i.e., is $7500 + 3000 = 10\,500$ grass-hoppers when $\dfrac{\pi t}{8} = \frac{\pi}{2} + k2\pi$

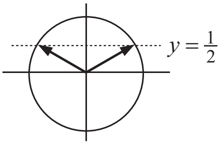
$$\therefore \quad \frac{t}{8} = \tfrac{1}{2} + 2k$$
$$\therefore \quad t = 4 + 16k$$
$$\therefore \quad t = 4 \quad \{\text{as } 0 \leqslant t \leqslant 12\}$$



i.e., after 4 weeks

c **i** When $P(t) = 9000$,

$$7500 + 3000\sin\left(\tfrac{\pi t}{8}\right) = 9000$$
$$\therefore\quad 3000\sin\left(\tfrac{\pi t}{8}\right) = 1500$$
$$\therefore\quad \sin\left(\tfrac{\pi t}{8}\right) = \tfrac{1}{2}$$

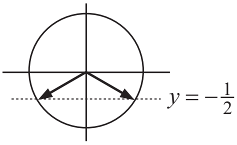


$$\therefore\quad \frac{\pi t}{8} = \left.\begin{matrix}\frac{\pi}{6}\\ \frac{5\pi}{6}\end{matrix}\right\} + k2\pi$$
$$\therefore\quad \frac{t}{8} = \left.\begin{matrix}\frac{1}{6}\\ \frac{5}{6}\end{matrix}\right\} + k\,2$$
$$\therefore\quad t = \left.\begin{matrix}\frac{4}{3}\\ \frac{20}{3}\end{matrix}\right\} + k\,16$$
$$\therefore\quad t = 1\tfrac{1}{3} \text{ or } 6\tfrac{2}{3}$$

i.e., at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks

ii When $P(t) = 6000$,

$$7500 + 3000\sin\left(\tfrac{\pi t}{8}\right) = 6000$$
$$\therefore\quad 3000\sin\left(\tfrac{\pi t}{8}\right) = -1500$$
$$\therefore\quad \sin\left(\tfrac{\pi t}{8}\right) = -\tfrac{1}{2}$$

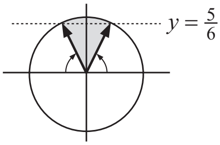


$$\therefore\quad \frac{\pi t}{8} = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi$$
$$\therefore\quad \frac{t}{8} = \left.\begin{matrix}\frac{7}{6}\\ \frac{11}{6}\end{matrix}\right\} + k\,2$$
$$\therefore\quad t = \left.\begin{matrix}\frac{28}{3}\\ \frac{44}{3}\end{matrix}\right\} + k\,16$$
$$\therefore\quad t = 9\tfrac{1}{3}$$

i.e., at $9\frac{1}{3}$ weeks

d If $P(t) > 10\,000$, then

$$7500 + 3000\sin\left(\tfrac{\pi t}{8}\right) > 10\,000$$
$$\therefore\quad 3000\sin\left(\tfrac{\pi t}{8}\right) > 2500$$
$$\therefore\quad \sin\left(\tfrac{\pi t}{8}\right) > \tfrac{5}{6}$$



Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$t \doteqdot 2.51$ or 5.49 So $2.51 \leqslant t \leqslant 5.49$ weeks.

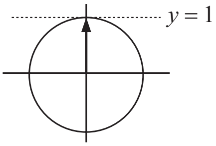
2 $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$

a
$$H(0) = 20 - 19(0) = 20 \text{ m}$$

i.e., 20 m above the ground

b H is smallest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

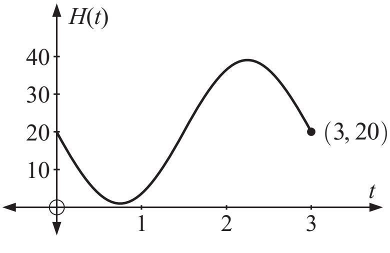
$$\therefore\quad \frac{2\pi t}{3} = \tfrac{\pi}{2} + k2\pi$$
$$\therefore\quad \frac{2t}{3} = \tfrac{1}{2} + k\,2$$
$$\therefore\quad t = \tfrac{3}{4} + k\,3$$
$$\therefore\quad t = \tfrac{3}{4}\text{ min} \quad \{\text{as } k = 0\}$$



c period $= \dfrac{2\pi}{\frac{2\pi}{3}} = 3$ min

$\therefore$ one revolution takes 3 min

d



3 $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$ years

a $P(0) = 400 + 250(0)$

$= 400$ water buffalo

b **i** $P(\frac{1}{2}) = 400 + 250\sin\left(\frac{\pi(\frac{1}{2})}{2}\right)$

$= 400 + 250\sin(\frac{\pi}{4})$

$= 400 + 250 \times \frac{1}{\sqrt{2}}$

$\doteqdot 577$ water buffalo

ii $P(2) = 400 + 250\sin\pi$

$= 400 + 250(0)$

$= 400$ water buffalo

c $P(1) = 400 + 250\sin(\frac{\pi}{2})$

$= 400 + 250 \times 1$

$= 650$ water buffalo

This is the maximum herd size.

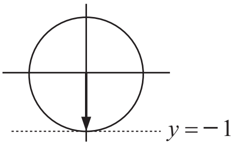
d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$ and is $400 - 250 = 150$ water buffalo.

It occurs when

$$\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$$

$$\therefore \quad \frac{t}{2} = \frac{3}{2} + k\,2$$

$\therefore \quad t = 3 + 4k$ So, the first time is after 3 years.

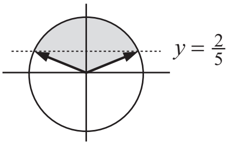


e If $P(t) > 500$ then

$$400 + 250\sin\left(\frac{\pi t}{2}\right) > 500$$

$$\therefore \quad 250\sin\left(\frac{\pi t}{2}\right) > 100$$

$$\therefore \quad \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$$



$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5}$ when $\frac{\pi t}{2} = 0.4115$ or $\pi - 0.4115$

$\therefore \quad t \doteqdot 0.262$ or 1.74

So, for $\sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$, $0.26 < t < 1.74$

4 $C(t) = 9.2\sin\frac{\pi}{7}(t - 4) + 107.8$ cents/L

a **i** 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$

i.e., 98.6 cents/L and 117.0 cents/L

$\uparrow$ min. $\uparrow$ max.

$\therefore$ the statement is true.

ii period $= \dfrac{2\pi}{\frac{\pi}{7}} = 14$ days $\therefore$ true

b $C(7) = 9.2\sin\frac{\pi}{7}(3) + 107.8 \doteqdot 116.8$ cents/L

c When $C(t) = \$1.10$/L then $9.2\sin\frac{\pi}{7}(t - 4) + 107.8 = 110$

$$\therefore \quad \sin\frac{\pi}{7}(t-4) = \frac{2.2}{9.2} \doteqdot 0.239\,13$$

$$\therefore \quad \frac{\pi}{7}(t-4) \doteqdot 0.239\,13 \text{ or } \pi - 0.239\,13$$

$$\therefore \quad t - 4 \doteqdot 0.533 \text{ or } 6.467$$

$$\therefore \quad t \doteqdot 4.53 \text{ or } 10.47$$

i.e., on the 5th and 11th days

d The min. cost/litre is $-9.2+107.8=98.6$ cents/L when $\sin\frac{\pi}{7}(t-4)=-1$

i.e., $\frac{\pi}{7}(t-4)=\frac{3\pi}{2}$

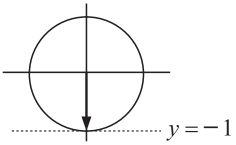
$\therefore\ \frac{t-4}{7}=\frac{3}{2}$

$\therefore\ 2t-8=21$

$\therefore\ 2t=29$

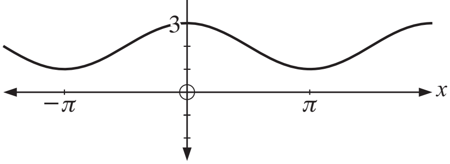
$\therefore\ t=14.5$

i.e., on the 15th day



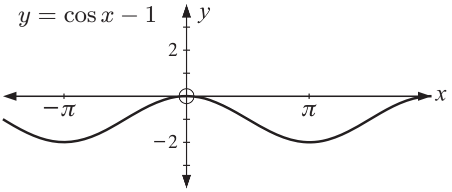
EXERCISE 13E

1 **a** $y=\cos x+2$



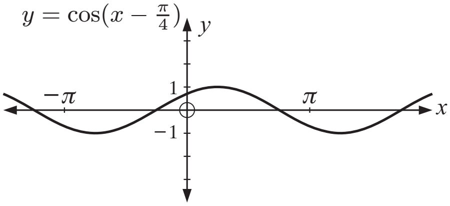
i.e., a vertical translation of $y=\cos x$ through $\begin{bmatrix}0\\2\end{bmatrix}$.

b $y=\cos x-1$



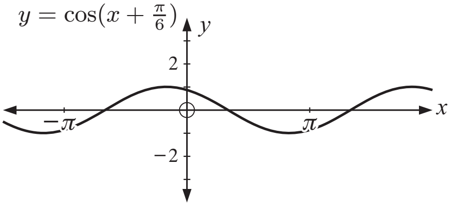
i.e., a vertical translation of $y=\cos x$ through $\begin{bmatrix}0\\-1\end{bmatrix}$.

c $y=\cos(x-\frac{\pi}{4})$



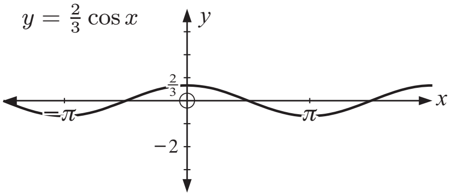
i.e., a horizontal translation of $y=\cos x$ through $\begin{bmatrix}\frac{\pi}{4}\\0\end{bmatrix}$.

d $y=\cos(x+\frac{\pi}{6})$



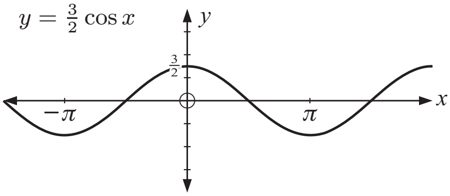
i.e., a horizontal translation of $y=\cos x$ through $\begin{bmatrix}-\frac{\pi}{6}\\0\end{bmatrix}$.

e $y=\frac{2}{3}\cos x$



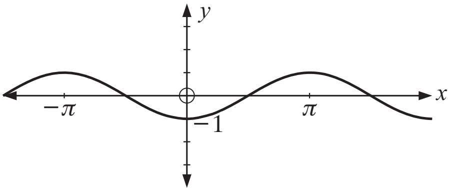
i.e., a vertical dilation of $y=\cos x$ with factor $\frac{2}{3}$.

f $y=\frac{3}{2}\cos x$



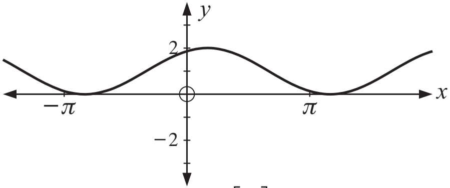
i.e., a vertical dilation of $y=\cos x$ with factor $\frac{3}{2}$.

g $y=-\cos x$



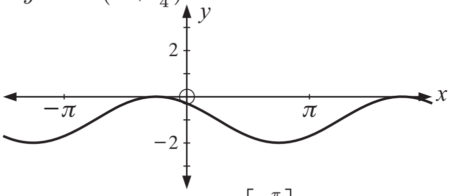
i.e., a reflection of $y=\cos x$ in the x-axis.

h $y=\cos(x-\frac{\pi}{6})+1$



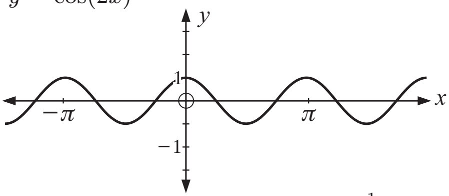
i.e., a translation of $\begin{bmatrix}\frac{\pi}{6}\\1\end{bmatrix}$.

i $y=\cos(x+\frac{\pi}{4})-1$



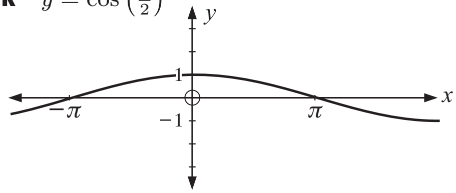
i.e., a translation of $\begin{bmatrix}-\frac{\pi}{4}\\-1\end{bmatrix}$.

j $y=\cos(2x)$



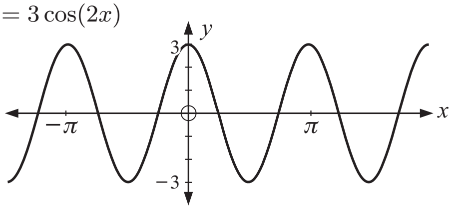
i.e., a horizontal dilation of factor $\frac{1}{2}$.

k $y = \cos\left(\frac{x}{2}\right)$



i.e., a horizontal dilation of factor 2.

l $y = 3\cos(2x)$



i.e., a horizontal dilation of factor $\frac{1}{2}$ followed by a vertical dilation of factor 3.

2 **a** period $= \frac{2\pi}{3}$ **b** period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$ **c** period $= \dfrac{2\pi}{\frac{\pi}{50}} = 100$

3 A controls the amplitude B controls the period, period $= \dfrac{2\pi}{|B|}$

C controls the horizontal translation D controls the vertical translation

4 **a** If $y = A\cos B(x - C) + D$, then $A = 2$, $\pi = \dfrac{2\pi}{B}$ $\therefore$ $B = 2$

C and D are 0 as there is no horizontal or vertical shift. $\therefore$ $y = 2\cos(2x)$

b If $y = A\cos B(x - C) + D$, then $A = 1$, $4\pi = \dfrac{2\pi}{B}$ $\therefore$ $B = \frac{1}{2}$

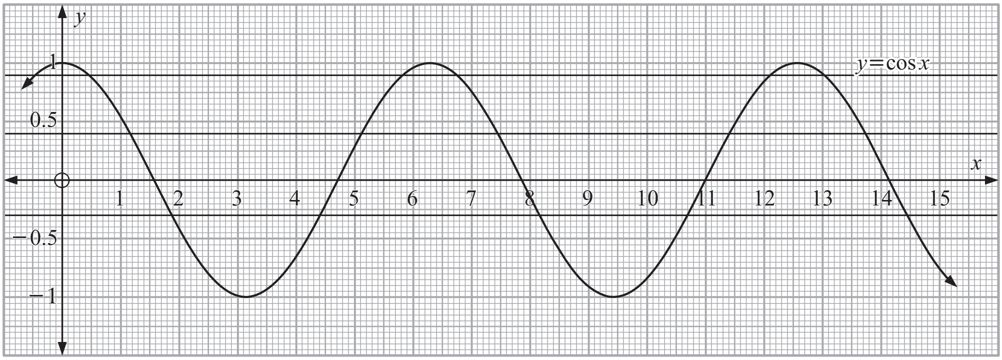
A vertical shift of 2 units, no horizontal shift $\therefore$ $D = 2$, $C = 0$.
So, $y = \cos\frac{1}{2}(x) + 2$ i.e., $y = \cos\left(\frac{x}{2}\right) + 2$.

c If $y = A\cos B(x - C) + D$, then $A = -5$, $6 = \dfrac{2\pi}{B}$ $\therefore$ $B = \frac{\pi}{3}$

$C = D = 0$ {as no translation} $\therefore$ $y = -5\cos\left(\frac{\pi}{3}x\right)$

EXERCISE 13F

1



a $x \doteqdot 1.2, 5.1, 7.4$ **b** $x \doteqdot 4.4, 8.2, 10.7$

2 **a** $\cos x = 0.561$, $0 \leqslant x \leqslant 10$
$\therefore$ $x \doteqdot 0.975, 5.308, 7.258$

b $\cos 2x = 0.782$, $0 \leqslant x \leqslant 6$
$\therefore$ $x \doteqdot 0.336, 2.805, 3.478, 5.947$

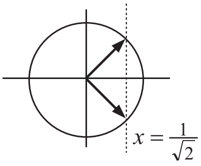
c $\cos(x - 1.3) = -0.609$, $0 \leqslant x \leqslant 12$
$\therefore$ $x \doteqdot 3.526, 5.358, 9.809, 11.641$

d $4\cos 3x + 1 = 0$, $0 \leqslant x \leqslant 5$
$\therefore$ $\cos 3x = -0.25$, $0 \leqslant x \leqslant 5$
$\therefore$ $x \doteqdot 0.608, 1.487, 2.702,$
$3.581, 4.797$

3 **a** $\cos x = \frac{1}{\sqrt{2}}, \quad 0 \leqslant x \leqslant 4\pi$

$$\therefore \quad x = \left.\begin{matrix}\frac{\pi}{4}\\ \frac{7\pi}{4}\end{matrix}\right\} + k2\pi$$

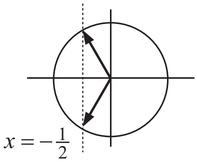
$$\therefore \quad x = \tfrac{\pi}{4}, \tfrac{9\pi}{4}, \tfrac{7\pi}{4}, \tfrac{15\pi}{4}$$



b $\cos x = -\frac{1}{2}, \quad 0 \leqslant x \leqslant 5\pi$

$$\therefore \quad x = \left.\begin{matrix}\frac{2\pi}{3}\\ \frac{4\pi}{3}\end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \tfrac{2\pi}{3}, \tfrac{8\pi}{3}, \tfrac{14\pi}{3}, \tfrac{4\pi}{3}, \tfrac{10\pi}{3}$$

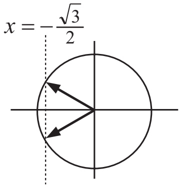


c $2\cos x + \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 3\pi$

$$\therefore \quad \cos x = -\tfrac{\sqrt{3}}{2}$$

$$\therefore \quad x = \left.\begin{matrix}\frac{5\pi}{6}\\ \frac{7\pi}{6}\end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \tfrac{5\pi}{6}, \tfrac{17\pi}{6}, \tfrac{7\pi}{6}$$

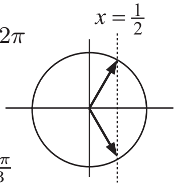


d $\cos(x - \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

$$\therefore \quad x - \tfrac{2\pi}{3} = \left.\begin{matrix}\frac{\pi}{3}\\ \frac{5\pi}{3}\end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \left.\begin{matrix}\pi\\ \frac{7\pi}{3}\end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \pi, -\pi, \tfrac{\pi}{3}, -\tfrac{5\pi}{3}$$



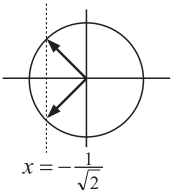
e $\sqrt{2}\cos(x - \frac{\pi}{4}) + 1 = 0, \quad 0 \leqslant x \leqslant 3\pi$

$$\therefore \quad \cos(x - \tfrac{\pi}{4}) = -\tfrac{1}{\sqrt{2}}$$

$$\therefore \quad x - \tfrac{\pi}{4} = \left.\begin{matrix}\frac{3\pi}{4}\\ \frac{5\pi}{4}\end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \left.\begin{matrix}\pi\\ \frac{3\pi}{2}\end{matrix}\right\} + k2\pi$$

$$\therefore \quad x = \pi, 3\pi, \tfrac{3\pi}{2}$$



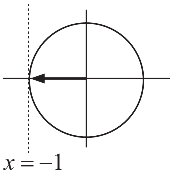
f $\cos 2x + 1 = 0, \quad 0 \leqslant x \leqslant 2\pi$

$$\therefore \quad \cos 2x = -1$$

$$\therefore \quad 2x = \pi + k2\pi$$

$$\therefore \quad x = \tfrac{\pi}{2} + k\pi$$

$$\therefore \quad x = \tfrac{\pi}{2}, \tfrac{3\pi}{2}$$



4 **a** The period is 4 seconds.

$$\therefore \quad \frac{2\pi}{B} = 4$$

$$\therefore \quad B = \tfrac{\pi}{2}$$

Amplitude is 3

$$\therefore \quad A = 3$$

$D = 1 + 3 = 4$

$C = 0$

$$\therefore \quad H(t) = 3\cos\tfrac{\pi}{2}(t - 0) + 4 \text{ metres}$$

i.e., $H(t) = 3\cos(\frac{\pi}{2}t) + 4$ metres

Check: When $t = 0$, $H(0) = 3\cos 0 + 4 = 7$ ✓

b X enters the water when $H(t) = 2$

$$\therefore \quad 3\cos\left(\tfrac{\pi t}{2}\right) + 4 = 2 \qquad \therefore \quad \cos\left(\tfrac{\pi t}{2}\right) = -\tfrac{2}{3}$$

Using technology, $t \doteqdot 1.46$ sec

EXERCISE 13G.1

1 **a** $\sin\theta + \sin\theta$
$= 2\sin\theta$

b $2\cos\theta + \cos\theta$
$= 3\cos\theta$

c $3\sin\theta - \sin\theta$
$= 2\sin\theta$

d $3\sin\theta - 2\sin\theta$
$= \sin\theta$

e $\cos\theta - 3\cos\theta$
$= -2\cos\theta$

f $2\cos\theta - 5\cos\theta$
$= -3\cos\theta$

2 **a** $3\sin^2\theta + 3\cos^2\theta$
$= 3(\sin^2\theta + \cos^2\theta)$
$= 3(1)$
$= 3$

b $-2\sin^2\theta - 2\cos^2\theta$
$= -2(\sin^2\theta + \cos^2\theta)$
$= -2(1)$
$= -2$

c $-\cos^2\theta - \sin^2\theta$
$= -(\cos^2\theta + \sin^2\theta)$
$= -(1)$
$= -1$

d $3 - 3\sin^2\theta$
$= 3(1 - \sin^2\theta)$
$= 3\cos^2\theta$

e $4 - 4\cos^2\theta$
$= 4(1 - \cos^2\theta)$
$= 4\sin^2\theta$

f $\sin^3\theta + \sin\theta\cos^2\theta$
$= \sin\theta(\sin^2\theta + \cos^2\theta)$
$= \sin\theta(1)$
$= \sin\theta$

g
$$\begin{aligned}&\cos^2\theta-1\\=\;&1-\sin^2\theta-1\\=\;&-\sin^2\theta\end{aligned}$$

h
$$\begin{aligned}&\sin^2\theta-1\\=\;&1-\cos^2\theta-1\\=\;&-\cos^2\theta\end{aligned}$$

i
$$\begin{aligned}&2\cos^2\theta-2\\=\;&-2(1-\cos^2\theta)\\=\;&-2\sin^2\theta\end{aligned}$$

j
$$\begin{aligned}&\frac{1-\sin^2\theta}{\cos^2\theta}\\=\;&\frac{\cos^2\theta}{\cos^2\theta}\\=\;&1\end{aligned}$$

k
$$\begin{aligned}&\frac{1-\cos^2\theta}{\sin\theta}\\=\;&\frac{\sin^2\theta}{\sin\theta}\\=\;&\sin\theta\end{aligned}$$

l
$$\begin{aligned}&\frac{\cos^2\theta-1}{-\sin\theta}\\=\;&\frac{1-\sin^2\theta-1}{-\sin\theta}\\=\;&\frac{-\sin^2\theta}{-\sin\theta}\\=\;&\sin\theta\end{aligned}$$

3 **a**
$$\begin{aligned}&(1+\sin\theta)^2\\=\;&1+2\sin\theta+\sin^2\theta\end{aligned}$$

b
$$\begin{aligned}&(\sin\alpha-2)^2\\=\;&\sin^2\alpha-4\sin\alpha+4\end{aligned}$$

c
$$\begin{aligned}&(\cos\alpha-1)^2\\=\;&\cos^2\alpha-2\cos\alpha+1\end{aligned}$$

d
$$\begin{aligned}&(\sin\alpha+\cos\alpha)^2\\=\;&\sin^2\alpha+2\sin\alpha\cos\alpha+\cos^2\alpha\\=\;&1+2\sin\alpha\cos\alpha\end{aligned}$$

e
$$\begin{aligned}&(\sin\beta-\cos\beta)^2\\=\;&\sin^2\beta-2\sin\beta\cos\beta+\cos^2\beta\\=\;&1-2\sin\beta\cos\beta\end{aligned}$$

f
$$\begin{aligned}&-(2-\cos\alpha)^2\\=\;&-[4-4\cos\alpha+\cos^2\alpha]\\=\;&-4+4\cos\alpha-\cos^2\alpha\end{aligned}$$

4 **a**
$$\begin{aligned}&1-\sin^2\theta\\=\;&(1+\sin\theta)(1-\sin\theta)\end{aligned}$$

b
$$\begin{aligned}&\sin^2\alpha-\cos^2\alpha\\=\;&(\sin\alpha+\cos\alpha)(\sin\alpha-\cos\alpha)\end{aligned}$$

c
$$\begin{aligned}&\cos^2\alpha-1\\=\;&(\cos\alpha+1)(\cos\alpha-1)\end{aligned}$$

d
$$\begin{aligned}&2\sin^2\beta-\sin\beta\\=\;&\sin\beta(2\sin\beta-1)\end{aligned}$$

e
$$\begin{aligned}&2\cos\phi+3\cos^2\phi\\=\;&\cos\phi(2+3\cos\phi)\end{aligned}$$

f
$$\begin{aligned}&3\sin^2\theta-6\sin\theta\\=\;&3\sin\theta(\sin\theta-2)\end{aligned}$$

g
$$\begin{aligned}&\sin^2\theta+5\sin\theta+6\\=\;&(\sin\theta+2)(\sin\theta+3)\end{aligned}$$

h
$$\begin{aligned}&2\cos^2\theta+7\cos\theta+3\\=\;&(2\cos\theta+1)(\cos\theta+3)\end{aligned}$$

i
$$\begin{aligned}&6\cos^2\alpha-\cos\alpha-1\\=\;&(3\cos\alpha+1)(2\cos\alpha-1)\end{aligned}$$

5 **a**
$$\begin{aligned}&\frac{1-\sin^2\alpha}{1-\sin\alpha}\\=\;&\frac{(1+\sin\alpha)\cancel{(1-\sin\alpha)}^{1}}{\cancel{1-\sin\alpha}_{1}}\\=\;&1+\sin\alpha\end{aligned}$$

b
$$\begin{aligned}&\frac{\cos^2\beta-1}{\cos\beta+1}\\=\;&\frac{{}^{1}\cancel{(\cos\beta+1)}(\cos\beta-1)}{\cancel{\cos\beta+1}_{1}}\\=\;&\cos\beta-1\end{aligned}$$

c
$$\begin{aligned}&\frac{\cos^2\phi-\sin^2\phi}{\cos\phi+\sin\phi}\\=\;&\frac{{}^{1}\cancel{(\cos\phi+\sin\phi)}(\cos\phi-\sin\phi)}{{}_{1}\cancel{\cos\phi+\sin\phi}}\\=\;&\cos\phi-\sin\phi\end{aligned}$$

d
$$\begin{aligned}&\frac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}\\=\;&\frac{(\cos\phi+\sin\phi)\cancel{(\cos\phi-\sin\phi)}^{1}}{{}_{1}\cancel{\cos\phi-\sin\phi}}\\=\;&\cos\phi+\sin\phi\end{aligned}$$

e
$$\begin{aligned}&\frac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}\\=\;&\frac{{}^{1}\cancel{\sin\alpha+\cos\alpha}}{{}_{1}\cancel{(\sin\alpha+\cos\alpha)}(\sin\alpha-\cos\alpha)}\\=\;&\frac{1}{\sin\alpha-\cos\alpha}\end{aligned}$$

f
$$\begin{aligned}&\frac{3-3\sin^2\theta}{6\cos\theta}\\=\;&\frac{3(1-\sin^2\theta)}{6\cos\theta}\\=\;&\frac{3\cos^2\theta}{6\cos\theta}\\=\;&\frac{\cos\theta}{2}\end{aligned}$$

6 **a**
$$\begin{aligned}&(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2\\ &=\cos^2\theta+\cancel{2\cos\theta\sin\theta}+\sin^2\theta\\ &\quad+\cos^2\theta-\cancel{2\cos\theta\sin\theta}+\sin^2\theta\\ &=2\cos^2\theta+2\sin^2\theta\\ &=2(\cos^2\theta+\sin^2\theta)\\ &=2(1)\\ &=2\end{aligned}$$

b
$$\begin{aligned}&(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2\\ &=4\sin^2\theta+\cancel{12\sin\theta\cos\theta}+9\cos^2\theta\\ &\quad+9\sin^2\theta-\cancel{12\sin\theta\cos\theta}+4\cos^2\theta\\ &=13\sin^2\theta+13\cos^2\theta\\ &=13(\sin^2\theta+\cos^2\theta)\\ &=13(1)\\ &=13\end{aligned}$$

c
$$\begin{aligned}&(1-\cos\theta)\left(1+\frac{1}{\cos\theta}\right)\\ &=1+\frac{1}{\cos\theta}-\cos\theta-1\\ &=\frac{1}{\cos\theta}-\cos\theta\\ &=\frac{1}{\cos\theta}-\cos\theta\left(\frac{\cos\theta}{\cos\theta}\right)\\ &=\frac{1-\cos^2\theta}{\cos\theta}\\ &=\frac{\sin^2\theta}{\cos\theta}\end{aligned}$$

d
$$\begin{aligned}&\left(1+\frac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)\\ &=\sin\theta-\sin^2\theta+1-\sin\theta\\ &=1-\sin^2\theta\\ &=\cos^2\theta\end{aligned}$$

e
$$\begin{aligned}&\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}\\ &=\frac{\sin^2\theta+(1+\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\\ &=\frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}\\ &=\frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}\\ &=\frac{2\cancel{(1+\cos\theta)}^{\,1}}{\sin\theta\cancel{(1+\cos\theta)}_{1}}\\ &=\frac{2}{\sin\theta}\end{aligned}$$

EXERCISE 13G.2

1 **a**
$$\begin{aligned}&\sin\theta+\sin(-\theta)\\ &=\sin\theta-\sin\theta\\ &=0\end{aligned}$$

b
$$\begin{aligned}&\sin(-\theta)-\sin\theta\\ &=-\sin\theta-\sin\theta\\ &=-2\sin\theta\end{aligned}$$

c
$$\begin{aligned}&2\cos\theta+\cos(-\theta)\\ &=2\cos\theta+\cos\theta\\ &=3\cos\theta\end{aligned}$$

d
$$\begin{aligned}&3\sin\theta-\sin(-\theta)\\ &=3\sin\theta--\sin\theta\\ &=3\sin\theta+\sin\theta\\ &=4\sin\theta\end{aligned}$$

e
$$\begin{aligned}&\cos^2(-\alpha)\\ &=\cos(-\alpha)\times\cos(-\alpha)\\ &=\cos\alpha\times\cos\alpha\\ &=\cos^2\alpha\end{aligned}$$

f
$$\begin{aligned}&\sin^2(-\alpha)\\ &=\sin(-\alpha)\times\sin(-\alpha)\\ &=-\sin\alpha\times-\sin\alpha\\ &=\sin^2\alpha\end{aligned}$$

g
$$\begin{aligned}&\cos(-\alpha)\cos\alpha-\sin(-\alpha)\sin\alpha\\ &=\cos\alpha\cos\alpha--\sin\alpha\sin\alpha\\ &=\cos^2\alpha+\sin^2\alpha\\ &=1\end{aligned}$$

2 **a**
$$\begin{aligned}&2\sin\theta-\cos(90^o-\theta)\\ &=2\sin\theta-\sin\theta\\ &=\sin\theta\end{aligned}$$

b
$$\begin{aligned}&\sin(-\theta)-\cos(90^o-\theta)\\ &=-\sin\theta-\sin\theta\\ &=-2\sin\theta\end{aligned}$$

c
$$\begin{aligned}&\sin(90^o-\theta)-\cos\theta\\ &=\cos\theta-\cos\theta\\ &=0\end{aligned}$$

d
$$\begin{aligned}&3\cos(-\theta)-4\sin(\tfrac{\pi}{2}-\theta)\\ &=3\cos\theta-4\cos\theta\\ &=-\cos\theta\end{aligned}$$

e
$$\begin{aligned}&3\cos\theta+\sin(\tfrac{\pi}{2}-\theta)\\ &=3\cos\theta+\cos\theta\\ &=4\cos\theta\end{aligned}$$

f
$$\begin{aligned}&\cos\left(\tfrac{\pi}{2}-\theta\right)+4\sin\theta\\ &=\sin\theta+4\sin\theta\\ &=5\sin\theta\end{aligned}$$

3
$$\begin{aligned}&\sin(\theta-\phi)\\ &=\sin(-[\phi-\theta])\\ &=-\sin(\phi-\theta)\end{aligned}$$

$$\begin{aligned}&\cos(\theta-\phi)\\ &=\cos(-[\phi-\theta])\\ &=\cos(\phi-\theta)\end{aligned}$$

EXERCISE 13H

1 **a** $\sin(M+N)=\sin M\cos N+\cos M\sin N$

b $\cos(T-S)=\cos T\cos S+\sin T\sin S$

c $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

d $\sin(\phi+\theta)=\sin\phi\cos\theta+\cos\phi\sin\theta$

e $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

f $\cos(2\theta-\alpha)=\cos 2\theta\cos\alpha+\sin 2\theta\sin\alpha$

g $\sin(\alpha-2\beta)=\sin\alpha\cos 2\beta-\cos\alpha\sin 2\beta$

h $\cos(3A+B)=\cos 3A\cos B-\sin 3A\sin B$

i $\cos(B-2C)=\cos B\cos 2C+\sin B\sin 2C$

2 **a**
$$\begin{aligned}&\sin(90^o+\theta)\\&=\sin 90^o\cos\theta+\cos 90^o\sin\theta\\&=(1)\cos\theta+(0)\sin\theta\\&=\cos\theta\end{aligned}$$

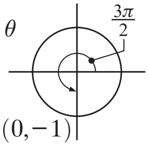
b
$$\begin{aligned}&\cos(90^o+\theta)\\&=\cos 90^o\cos\theta-\sin 90^o\sin\theta\\&=(0)\cos\theta-(1)\sin\theta\\&=-\sin\theta\end{aligned}$$

c
$$\begin{aligned}&\sin(180^o-\alpha)\\&=\sin 180^o\cos\alpha-\cos 180^o\sin\alpha\\&=(0)\cos\alpha-(-1)\sin\alpha\\&=\sin\alpha\end{aligned}$$

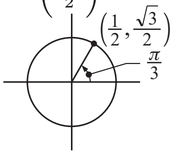
d
$$\begin{aligned}&\cos(\pi+\alpha)\\&=\cos\pi\cos\alpha-\sin\pi\sin\alpha\\&=(-1)\cos\alpha-(0)\sin\alpha\\&=-\cos\alpha\end{aligned}$$

e
$$\begin{aligned}&\sin(2\pi-A)\\&=\sin 2\pi\cos A-\cos 2\pi\sin A\\&=(0)\cos A-(1)\sin A\\&=-\sin A\end{aligned}$$

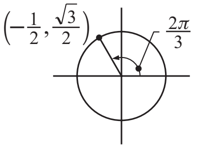
f
$$\begin{aligned}&\cos\left(\tfrac{3\pi}{2}-\theta\right)\\&=\cos\left(\tfrac{3\pi}{2}\right)\cos\theta+\sin\left(\tfrac{3\pi}{2}\right)\sin\theta\\&=(0)\cos\theta+(-1)\sin\theta\\&=-\sin\theta\end{aligned}$$



3 **a**
$$\begin{aligned}&\sin\left(\theta+\tfrac{\pi}{3}\right)\\&=\sin\theta\cos\left(\tfrac{\pi}{3}\right)+\cos\theta\sin\left(\tfrac{\pi}{3}\right)\\&=\sin\theta\times\left(\tfrac{1}{2}\right)+\cos\theta\times\left(\tfrac{\sqrt{3}}{2}\right)\\&=\tfrac{1}{2}\sin\theta+\tfrac{\sqrt{3}}{2}\cos\theta\end{aligned}$$



b
$$\begin{aligned}&\cos\left(\tfrac{2\pi}{3}-\theta\right)\\&=\cos\left(\tfrac{2\pi}{3}\right)\cos\theta+\sin\left(\tfrac{2\pi}{3}\right)\sin\theta\\&=\left(-\tfrac{1}{2}\right)\cos\theta+\left(\tfrac{\sqrt{3}}{2}\right)\sin\theta\\&=-\tfrac{1}{2}\cos\theta+\tfrac{\sqrt{3}}{2}\sin\theta\\&=\tfrac{\sqrt{3}}{2}\sin\theta-\tfrac{1}{2}\cos\theta\end{aligned}$$



c
$$\begin{aligned}&\cos\left(\theta+\tfrac{\pi}{4}\right)\\&=\cos\theta\cos\left(\tfrac{\pi}{4}\right)-\sin\theta\sin\left(\tfrac{\pi}{4}\right)\\&=\tfrac{1}{\sqrt{2}}\cos\theta-\tfrac{1}{\sqrt{2}}\sin\theta\\&=-\tfrac{1}{\sqrt{2}}\sin\theta+\tfrac{1}{\sqrt{2}}\cos\theta\end{aligned}$$

d
$$\begin{aligned}&\sin\left(\tfrac{\pi}{6}-\theta\right)\\&=\sin\left(\tfrac{\pi}{6}\right)\cos\theta-\cos\left(\tfrac{\pi}{6}\right)\sin\theta\\&=\tfrac{1}{2}\cos\theta-\tfrac{\sqrt{3}}{2}\sin\theta\\&=-\tfrac{\sqrt{3}}{2}\sin\theta+\tfrac{1}{2}\cos\theta\end{aligned}$$

4 **a**
$$\begin{aligned}&\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\\&=\cos(2\theta-\theta)\\&=\cos\theta\end{aligned}$$

b
$$\begin{aligned}&\sin 2A\cos A+\cos 2A\sin A\\&=\sin(2A+A)\\&=\sin 3A\end{aligned}$$

c
$$\begin{aligned}&\cos A\sin B-\sin A\cos B\\&=\sin B\cos A-\cos B\sin A\\&=\sin(B-A)\end{aligned}$$

d
$$\begin{aligned}&\sin\alpha\sin\beta+\cos\alpha\cos\beta\\&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\\&=\cos(\alpha-\beta)\end{aligned}$$

e
$$\begin{aligned}&\sin\phi\sin\theta-\cos\phi\cos\theta\\&=-[\cos\phi\cos\theta-\sin\phi\sin\theta]\\&=-\cos(\phi+\theta)\end{aligned}$$

f
$$\begin{aligned}&2\sin\alpha\cos\beta-2\cos\alpha\sin\beta\\&=2\left[\sin\alpha\cos\beta-\cos\alpha\sin\beta\right]\\&=2\sin(\alpha-\beta)\end{aligned}$$

5 **a**
$$\begin{aligned}&\cos(\alpha+\beta)\cos(\alpha-\beta)-\sin(\alpha+\beta)\sin(\alpha-\beta)\\&=\cos\left[(\alpha+\beta)+(\alpha-\beta)\right]\\&=\cos(2\alpha)\end{aligned}$$

b
$$\begin{aligned}&\sin(\theta-2\phi)\cos(\theta+\phi)-\cos(\theta-2\phi)\sin(\theta+\phi)\\&=\sin\left[(\theta-2\phi)-(\theta+\phi)\right]\\&=\sin(-3\phi)\\&=-\sin(3\phi)\end{aligned}$$

c
$$\begin{aligned}&\cos\alpha\cos(\beta-\alpha)-\sin\alpha\sin(\beta-\alpha)\\&=\cos\left[\alpha+(\beta-\alpha)\right]\\&=\cos\beta\end{aligned}$$

6 **a**

$$\begin{aligned} &\cos 75^o \\ &= \cos(45^o + 30^o) \\ &= \cos 45^o \cos 30^o - \sin 45^o \sin 30^o \\ &= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right) \\ &= \tfrac{\sqrt{3}-1}{2\sqrt{2}} \\ &= \left(\tfrac{\sqrt{3}-1}{2\sqrt{2}}\right)\tfrac{\sqrt{2}}{\sqrt{2}} \\ &= \tfrac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

b

$$\begin{aligned} &\sin 105^o \\ &= \sin(60^o + 45^o) \\ &= \sin 60^o \cos 45^o + \cos 60^o \sin 45^o \\ &= \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right) \\ &= \left(\tfrac{\sqrt{3}+1}{2\sqrt{2}}\right)\tfrac{\sqrt{2}}{\sqrt{2}} \\ &= \tfrac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

c

$$\begin{aligned} &\cos\left(\tfrac{13\pi}{12}\right) \\ &= \cos\left(\tfrac{13\times 180^o}{12}\right) \\ &= \cos 195^o \\ &= \cos(150^o + 45^o) \\ &= \cos 150^o \cos 45^o - \sin 150^o \sin 45^o \\ &= \left(-\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right) - \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}\right) \\ &= \left(\tfrac{-\sqrt{3}-1}{2\sqrt{2}}\right)\tfrac{\sqrt{2}}{\sqrt{2}} \\ &= \tfrac{-\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

7 **a**

$$\begin{aligned} &\sqrt{2}\cos\left(\theta + \tfrac{\pi}{4}\right) \\ &= \sqrt{2}\left[\cos\theta\cos\left(\tfrac{\pi}{4}\right) - \sin\theta\sin\left(\tfrac{\pi}{4}\right)\right] \\ &= \sqrt{2}\left[\tfrac{1}{\sqrt{2}}\cos\theta - \tfrac{1}{\sqrt{2}}\sin\theta\right] \\ &= \cos\theta - \sin\theta \end{aligned}$$

b

$$\begin{aligned} &2\cos\left(\theta - \tfrac{\pi}{3}\right) \\ &= 2\left[\cos\theta\cos\left(\tfrac{\pi}{3}\right) + \sin\theta\sin\left(\tfrac{\pi}{3}\right)\right] \\ &= 2\left[\tfrac{1}{2}\cos\theta + \tfrac{\sqrt{3}}{2}\sin\theta\right] \\ &= \cos\theta + \sqrt{3}\sin\theta \end{aligned}$$

c

$$\begin{aligned} &\cos(\alpha+\beta) - \cos(\alpha-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta - [\cos\alpha\cos\beta + \sin\alpha\sin\beta] \\ &= \cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta - \cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta \\ &= -2\sin\alpha\sin\beta \end{aligned}$$

d

$$\begin{aligned} &\cos(\alpha+\beta)\cos(\alpha-\beta) \\ &= [\cos\alpha\cos\beta - \sin\alpha\sin\beta]\,[\cos\alpha\cos\beta + \sin\alpha\sin\beta] \\ &= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta \\ &= \cos^2\alpha\left[1 - \sin^2\beta\right] - \left[1 - \cos^2\alpha\right]\sin^2\beta \\ &= \cos^2\alpha - \cancel{\cos^2\alpha\sin^2\beta} - \sin^2\beta + \cancel{\cos^2\alpha\sin^2\beta} \\ &= \cos^2\alpha - \sin^2\beta \end{aligned}$$

8 **a**

$$\begin{aligned} &\sin(A+B) + \sin(A-B) \\ &= \sin A\cos B + \cancel{\cos A\sin B} + \sin A\cos B - \cancel{\cos A\sin B} \\ &= 2\sin A\cos B \end{aligned}$$

b $\therefore \quad \sin A\cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$

i

$$\begin{aligned} &\sin 3\theta\cos\theta \\ &= \tfrac{1}{2}\sin(3\theta+\theta) + \tfrac{1}{2}\sin(3\theta-\theta) \\ &= \tfrac{1}{2}\sin 4\theta + \tfrac{1}{2}\sin 2\theta \end{aligned}$$

ii

$$\begin{aligned} &\sin 6\alpha\cos\alpha \\ &= \tfrac{1}{2}\sin(6\alpha+\alpha) + \tfrac{1}{2}\sin(6\alpha-\alpha) \\ &= \tfrac{1}{2}\sin 7\alpha + \tfrac{1}{2}\sin 5\alpha \end{aligned}$$

iii

$$\begin{aligned} &2\sin 5\beta\cos\beta \\ &= 2\left[\tfrac{1}{2}\sin(5\beta+\beta) + \tfrac{1}{2}\sin(5\beta-\beta)\right] \\ &= \sin 6\beta + \sin 4\beta \end{aligned}$$

iv

$$\begin{aligned} &4\cos\theta\sin 4\theta \\ &= 4\,[\sin 4\theta\cos\theta] \\ &= 4\left[\tfrac{1}{2}\sin 5\theta + \tfrac{1}{2}\sin 3\theta\right] \\ &= 2\sin 5\theta + 2\sin 3\theta \end{aligned}$$

v

$$\begin{aligned} &6\cos 4\alpha\sin 3\alpha \\ &= 6\sin 3\alpha\cos 4\alpha \\ &= 6\left[\tfrac{1}{2}\sin 7\alpha + \tfrac{1}{2}\sin(-\alpha)\right] \\ &= 3\sin 7\alpha + 3\sin(-\alpha) \\ &= 3\sin 7\alpha - 3\sin\alpha \end{aligned}$$

vi

$$\begin{aligned} &\tfrac{1}{3}\cos 5A\sin 3A \\ &= \tfrac{1}{3}\sin 3A\cos 5A \\ &= \tfrac{1}{3}\left[\tfrac{1}{2}\sin 8A + \tfrac{1}{2}\sin(-2A)\right] \\ &= \tfrac{1}{6}\sin 8A - \tfrac{1}{6}\sin 2A \end{aligned}$$

9 **a**
$$\begin{aligned}&\cos(A+B)+\cos(A-B)\\&=\cos A\cos B-\cancel{\sin A\cos B}+\cos A\cos B+\cancel{\sin A\sin B}\\&=2\cos A\cos B\end{aligned}$$

b $\therefore\quad \cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$

i
$$\begin{aligned}&\cos 4\theta\cos\theta\\&=\tfrac{1}{2}\cos 5\theta+\tfrac{1}{2}\cos 3\theta\end{aligned}$$

ii
$$\begin{aligned}&\cos 7\alpha\cos\alpha\\&=\tfrac{1}{2}\cos 8\alpha+\tfrac{1}{2}\cos 6\alpha\end{aligned}$$

iii
$$\begin{aligned}&2\cos 3\beta\cos\beta\\&=2\left[\tfrac{1}{2}\cos 4\beta+\tfrac{1}{2}\cos 2\beta\right]\\&=\cos 4\beta+\cos 2\beta\end{aligned}$$

iv
$$\begin{aligned}&6\cos x\cos 7x\\&=6\cos 7x\cos x\\&=6\left[\tfrac{1}{2}\cos 8x+\tfrac{1}{2}\cos 6x\right]\\&=3\cos 8x+3\cos 6x\end{aligned}$$

v
$$\begin{aligned}&3\cos P\cos 4P\\&=3\cos 4P\cos P\\&=3\left[\tfrac{1}{2}\cos 5P+\tfrac{1}{2}\cos 3P\right]\\&=\tfrac{3}{2}\cos 5P+\tfrac{3}{2}\cos 3P\end{aligned}$$

vi
$$\begin{aligned}&\tfrac{1}{4}\cos 4x\cos 2x\\&=\tfrac{1}{4}\left[\tfrac{1}{2}\cos 6x+\tfrac{1}{2}\cos 2x\right]\\&=\tfrac{1}{8}\cos 6x+\tfrac{1}{8}\cos 2x\end{aligned}$$

10 **a**
$$\begin{aligned}&\cos(A-B)-\cos(A+B)\\&=\cancel{\cos A\cos B}+\sin A\sin B-[\cancel{\cos A\cos B}-\sin A\sin B]\\&=\sin A\sin B+\sin A\sin B\\&=2\sin A\sin B\end{aligned}$$

b $\therefore\quad \sin A\sin B=\frac{1}{2}\cos(A-B)-\frac{1}{2}\cos(A+B)$

i
$$\begin{aligned}&\sin 3\theta\sin\theta\\&=\tfrac{1}{2}\cos 2\theta-\tfrac{1}{2}\cos 4\theta\end{aligned}$$

ii
$$\begin{aligned}&\sin 6\alpha\sin\alpha\\&=\tfrac{1}{2}\cos 5\alpha-\tfrac{1}{2}\cos 7\alpha\end{aligned}$$

iii
$$\begin{aligned}&2\sin 5\beta\sin\beta\\&=2\left[\tfrac{1}{2}\cos 4\beta-\tfrac{1}{2}\cos 6\beta\right]\\&=\cos 4\beta-\cos 6\beta\end{aligned}$$

iv
$$\begin{aligned}&4\sin\theta\sin 4\theta\\&=4\sin 4\theta\sin\theta\\&=4\left[\tfrac{1}{2}\cos 3\theta-\tfrac{1}{2}\cos 5\theta\right]\\&=2\cos 3\theta-2\cos 5\theta\end{aligned}$$

v
$$\begin{aligned}&10\sin 2A\sin 8A\\&=10\sin 8A\sin 2A\\&=10\left[\tfrac{1}{2}\cos 6A-\tfrac{1}{2}\cos 10A\right]\\&=5\cos 6A-5\cos 10A\end{aligned}$$

vi
$$\begin{aligned}&\tfrac{1}{5}\sin 3M\sin 7M\\&=\tfrac{1}{5}\sin 7M\sin 3M\\&=\tfrac{1}{5}\left[\tfrac{1}{2}\cos 4M-\tfrac{1}{2}\cos 10M\right]\\&=\tfrac{1}{10}\cos 4M-\tfrac{1}{10}\cos 10M\end{aligned}$$

11 (1) becomes $\sin A\cos B=\frac{1}{2}\sin(2A)$

(2) becomes $\cos^2 A=\frac{1}{2}\cos(2A)+\frac{1}{2}\cos 0$, i.e., $\cos^2 A=\frac{1}{2}\cos(2A)+\frac{1}{2}$

(3) becomes $\sin^2 A=\frac{1}{2}\cos 0-\frac{1}{2}\cos(2A)$, i.e., $\sin^2 A=\frac{1}{2}-\frac{1}{2}\cos(2A)$

12 **a** $A+B=S$

$A-B=D \quad \therefore\quad 2A=S+D$ i.e., $A=\frac{S+D}{2}$

and $B=S-A=S-\left(\frac{S+D}{2}\right)=\frac{2S}{2}-\left(\frac{S+D}{2}\right)=\frac{2S-S-D}{2}=\frac{S-D}{2}$

b $\sin A\cos B=\frac{1}{2}\sin(A+B)+\frac{1}{2}\sin(A-B)$ becomes

$\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)=\frac{1}{2}\sin S+\frac{1}{2}\sin D$

i.e., $\sin S+\sin D=2\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$ ...... (4)

c Replacing D by $(-D)$ in (4) gives

$\sin S+\sin(-D)=2\sin\left(\frac{S-D}{2}\right)\cos\left(\frac{S+D}{2}\right)$ ...... (4)

i.e., $\sin S-\sin D=2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

d $\cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$ becomes

$\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)=\frac{1}{2}\cos S+\frac{1}{2}\cos D$

i.e., $\cos S+\cos D=2\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$

e $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$ becomes

$$\sin\left(\tfrac{S+D}{2}\right)\sin\left(\tfrac{S-D}{2}\right) = \tfrac{1}{2}\cos D - \tfrac{1}{2}\cos S$$

i.e., $\cos D - \cos S = 2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

or $\cos S - \cos D = -2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

13 **a**
$$\begin{aligned}&\sin 5x + \sin x\\ &= 2\sin\left(\tfrac{5x+x}{2}\right)\cos\left(\tfrac{5x-x}{2}\right)\\ &= 2\sin 3x\cos 2x\end{aligned}$$

b
$$\begin{aligned}&\cos 8A + \cos 2A\\ &= 2\cos\left(\tfrac{8A+2A}{2}\right)\cos\left(\tfrac{8A-2A}{2}\right)\\ &= 2\cos 5A\cos 3A\end{aligned}$$

c
$$\begin{aligned}&\cos 3\alpha - \cos\alpha\\ &= -2\sin\left(\tfrac{3\alpha+\alpha}{2}\right)\sin\left(\tfrac{3\alpha-\alpha}{2}\right)\\ &= -2\sin 2\alpha\sin\alpha\end{aligned}$$

d
$$\begin{aligned}&\sin 5\theta - \sin 3\theta\\ &= 2\cos\left(\tfrac{5\theta+3\theta}{2}\right)\sin\left(\tfrac{5\theta-3\theta}{2}\right)\\ &= 2\cos 4\theta\sin\theta\end{aligned}$$

e
$$\begin{aligned}&\cos 7\alpha - \cos\alpha\\ &= -2\sin\left(\tfrac{7\alpha+\alpha}{2}\right)\sin\left(\tfrac{7\alpha-\alpha}{2}\right)\\ &= -2\sin 4\alpha\sin 3\alpha\end{aligned}$$

f
$$\begin{aligned}&\sin 3\alpha + \sin 7\alpha\\ &= \sin 7\alpha + \sin 3\alpha\\ &= 2\sin\left(\tfrac{7\alpha+3\alpha}{2}\right)\cos\left(\tfrac{7\alpha-3\alpha}{2}\right)\\ &= 2\sin 5\alpha\cos 2\alpha\end{aligned}$$

g
$$\begin{aligned}&\cos 2B - \cos 4B\\ &= -\left[\cos 4B - \cos 2B\right]\\ &= - - 2\sin\left(\tfrac{4B+2B}{2}\right)\sin\left(\tfrac{4B-2B}{2}\right)\\ &= 2\sin 3B\sin B\end{aligned}$$

h
$$\begin{aligned}&\sin(x+h) - \sin x\\ &= 2\cos\left(\tfrac{x+h+x}{2}\right)\sin\left(\tfrac{x+h-x}{2}\right)\\ &= 2\cos\left(\tfrac{2x+h}{2}\right)\sin\left(\tfrac{h}{2}\right)\\ &= 2\cos\left(x+\tfrac{h}{2}\right)\sin\left(\tfrac{h}{2}\right)\end{aligned}$$

i
$$\begin{aligned}&\cos(x+h) - \cos x\\ &= -2\sin\left(\tfrac{x+h+x}{2}\right)\sin\left(\tfrac{x+h-x}{2}\right)\\ &= -2\sin\left(\tfrac{2x+h}{2}\right)\sin\left(\tfrac{h}{2}\right)\\ &= -2\sin\left(x+\tfrac{h}{2}\right)\sin\left(\tfrac{h}{2}\right)\end{aligned}$$

EXERCISE 13I

1 **a**
$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2(\tfrac{4}{5})(\tfrac{3}{5})\\ &= \tfrac{24}{25}\end{aligned}$$

b
$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \tfrac{9}{25} - \tfrac{16}{25}\\ &= -\tfrac{7}{25}\end{aligned}$$

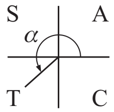
2
$$\begin{aligned}\cos 2A &= 2\cos^2 A - 1\\ &= 2(\tfrac{1}{3})^2 - 1\\ &= 2\times\tfrac{1}{9} - 1\\ &= \tfrac{2}{9} - 1\\ &= -\tfrac{7}{9}\end{aligned}$$

3
$$\begin{aligned}\cos 2\phi &= 1 - 2\sin^2\phi\\ &= 1 - 2(-\tfrac{2}{3})^2\\ &= 1 - 2(\tfrac{4}{9})\\ &= 1 - \tfrac{8}{9}\\ &= \tfrac{1}{9}\end{aligned}$$

4 **a** $\sin\alpha = -\frac{2}{3}$

α is in Q3

$\therefore\ \cos\alpha < 0$



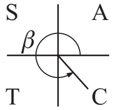
$$\begin{aligned}\cos^2\alpha + \sin^2\alpha &= 1\\ \therefore\ \cos^2\alpha + \tfrac{4}{9} &= 1\\ \therefore\ \cos^2\alpha &= \tfrac{5}{9}\\ \therefore\ \cos\alpha &= -\tfrac{\sqrt{5}}{3}\end{aligned}$$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\left(-\tfrac{2}{3}\right)\left(-\tfrac{\sqrt{5}}{3}\right)\\ &= \tfrac{4\sqrt{5}}{9}\end{aligned}$$

b $\cos\beta = \frac{2}{5}$

β is in Q4

$\therefore\ \sin\beta < 0$



$$\begin{aligned}\cos^2\beta + \sin^2\beta &= 1\\ \therefore\ \tfrac{4}{25} + \sin^2\beta &= 1\\ \therefore\ \sin^2\beta &= \tfrac{21}{25}\\ \therefore\ \sin\beta &= -\tfrac{\sqrt{21}}{5}\end{aligned}$$

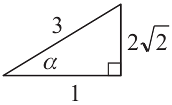
$$\begin{aligned}\sin 2\beta &= 2\sin\beta\cos\beta\\ &= 2\left(-\tfrac{\sqrt{21}}{5}\right)\left(\tfrac{2}{5}\right)\\ &= -\tfrac{4\sqrt{21}}{25}\end{aligned}$$

5 α is acute $\therefore$ $\cos\alpha$ and $\sin\alpha$ are positive

a
$$\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1\\ \therefore\ -\tfrac{7}{9} &= 2\cos^2\alpha - 1\\ \therefore\ 2\cos^2\alpha &= \tfrac{2}{9}\\ \therefore\ \cos^2\alpha &= \tfrac{1}{9}\\ \therefore\ \cos\alpha &= \tfrac{1}{3}\end{aligned}$$

b
$$\begin{aligned}\cos 2\alpha &= 1 - 2\sin^2\alpha\\ \therefore\ -\tfrac{7}{9} &= 1 - 2\sin^2\alpha\\ \therefore\ 2\sin^2\alpha &= 1\tfrac{7}{9} = \tfrac{16}{9}\\ \therefore\ \sin^2\alpha &= \tfrac{8}{9}\\ \therefore\ \sin\alpha &= \tfrac{2\sqrt{2}}{3}\end{aligned}$$

or



$\sin\alpha = \frac{2\sqrt{2}}{3}$

6

a $2\sin\alpha\cos\alpha$
$= \sin 2\alpha$

b $4\cos\alpha\sin\alpha$
$= 2(2\sin\alpha\cos\alpha)$
$= 2\sin 2\alpha$

c $\sin\alpha\cos\alpha$
$= \frac{1}{2}(2\sin\alpha\cos\alpha)$
$= \frac{1}{2}\sin 2\alpha$

d $2\cos^2\beta - 1$
$= \cos 2\beta$

e $1 - 2\cos^2\phi$
$= -(2\cos^2\phi - 1)$
$= -\cos 2\phi$

f $1 - 2\sin^2 N$
$= \cos 2N$

g $2\sin^2 M - 1$
$= -(1 - 2\sin^2 M)$
$= -\cos 2M$

h $\cos^2\alpha - \sin^2\alpha$
$= \cos 2\alpha$

i $\sin^2\alpha - \cos^2\alpha$
$= -(\cos^2\alpha - \sin^2\alpha)$
$= -\cos 2\alpha$

j $2\sin 2A\cos 2A$
$= \sin 2(2A)$
$= \sin 4A$

k $2\cos 3\alpha\sin 3\alpha$
$= \sin 2(3\alpha)$
$= \sin 6\alpha$

l $2\cos^2 4\theta - 1$
$= \cos 2(4\theta)$
$= \cos 8\theta$

m $1 - 2\cos^2 3\beta$
$= -(2\cos^2 3\beta - 1)$
$= -\cos 2(3\beta)$
$= -\cos 6\beta$

n $1 - 2\sin^2 5\alpha$
$= \cos 2(5\alpha)$
$= \cos 10\alpha$

o $2\sin^2 3D - 1$
$= -(1 - 2\sin^2 3D)$
$= -\cos 2(3D)$
$= -\cos 6D$

p $\cos^2 2A - \sin^2 2A$
$= \cos 2(2A)$
$= \cos 4A$

q $\cos^2(\frac{\alpha}{2}) - \sin^2(\frac{\alpha}{2})$
$= \cos 2(\frac{\alpha}{2})$
$= \cos\alpha$

r $2\sin^2 3P - 2\cos^2 3P$
$= -2[\cos^2 3P - \sin^2 3P]$
$= -2\cos 2(3P)$
$= -2\cos 6P$

7

a
$$\begin{aligned}&(\sin\theta + \cos\theta)^2\\ &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= \underbrace{\sin^2\theta + \cos^2\theta} + 2\sin\theta\cos\theta\\ &= 1 + \sin 2\theta\end{aligned}$$

b
$$\begin{aligned}&\cos^4\theta - \sin^4\theta\\ &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)\\ &= 1 \times \cos 2\theta\\ &= \cos 2\theta\end{aligned}$$

EXERCISE 13J.1

1

a $\tan 0^o$
$= 0$

b $\tan 15^o$
$\doteqdot 0.27$

c $\tan 20^o$
$\doteqdot 0.36$

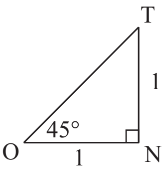
d $\tan 25^o$
$\doteqdot 0.47$

e $\tan 35^o$
$\doteqdot 0.70$

f $\tan 45^o$
$= 1.00$

g $\tan 50^o$
$\doteqdot 1.19$

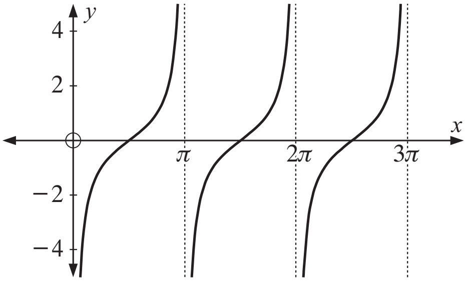
h $\tan 55^o$
$\doteqdot 1.43$



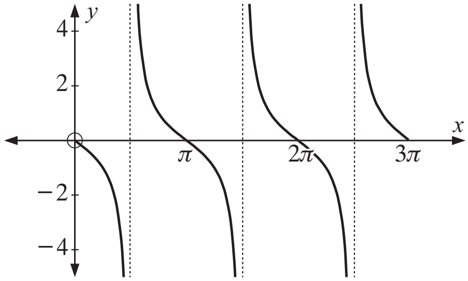
2 In ΔTON, ON = NT = 1 (Δ is isosceles} $\tan 45^o = \frac{\text{NT}}{\text{ON}} = \frac{1}{1} = 1$

EXERCISE 13J.2

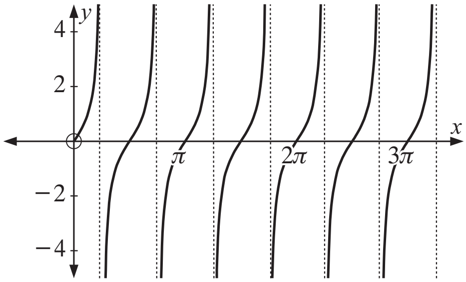
1 **a** **i** $y = \tan(x - \frac{\pi}{2})$ is $y = \tan x$ translated $\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$.



ii $y = -\tan x$ is $y = \tan x$ reflected in the x-axis.

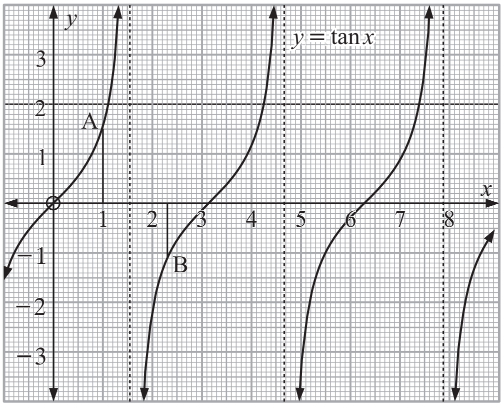


iii $y = \tan 2x$ comes from $y = \tan x$ under a horizontal dilation of factor $\frac{1}{2}$.



2 **a** translation through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **b** reflection in x-axis **c** horizontal dilation, factor $k = 2$

3



a **i** $\tan 1 \doteqdot 1.6$ {point A}

ii $\tan 2.3 \doteqdot -1.1$ {point B}

b **i** $\tan 1 \doteqdot 1.557$

ii $\tan 2.3 \doteqdot -1.119$

c **i** When $\tan x = 2$, $x \doteqdot 1.1, 4.2, 7.4$

ii When $\tan x = -1.4$, $x \doteqdot 2.2, 5.3$

4 **a** period $= \frac{\pi}{1} = \pi$ **b** period $= \frac{\pi}{2}$ **c** period $= \frac{\pi}{n}$

EXERCISE 13K.1

1 $X = \tan^{-1} 2 \quad \therefore \quad X \doteqdot 1.107 + k\pi$

a If $\tan 2x = 2$

$$\begin{aligned} \text{then } 2x &= 1.107 + k\pi \\ \therefore \quad x &= 0.554 + k(\tfrac{\pi}{2}) \end{aligned}$$

b $\tan\left(\frac{x}{3}\right) = 2$

$$\begin{aligned} \therefore \quad \frac{x}{3} &= 1.107 + k\pi \\ \therefore \quad x &= 3.32 + k3\pi \end{aligned}$$

c $\tan(x + 1.2) = 2$

$$\begin{aligned} \therefore \quad x + 1.2 &= 1.107 + k\pi \\ \therefore \quad x &= -0.0929 + k\pi \end{aligned}$$

2 $X = \tan^{-1}(-3) \doteqdot -1.249 + k\pi$

a $\tan(x-2) = -3$

$\therefore\ x - 2 \doteqdot -1.249 + k\pi$

$\therefore\ x \doteqdot 0.751 + k\pi$

b $\tan(3x) = -3$

$\therefore\ 3x = -1.249 + k\pi$

$\therefore\ x \doteqdot -0.416 + \dfrac{k\pi}{3}$

c $\tan\left(\frac{x}{2}\right) = -3$

$\therefore\ \dfrac{x}{2} = -1.249 + k\pi$

$\therefore\ x = -2.50 + k2\pi$

3 $X = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} + k\pi$

a $\tan(x - \frac{\pi}{6}) = \sqrt{3}$

$\therefore\ x - \frac{\pi}{6} = \frac{\pi}{3} + k\pi$

$\therefore\ x = \frac{\pi}{2} + k\pi$

b $\tan 4x = \sqrt{3}$

$\therefore\ 4x = \frac{\pi}{3} + k\pi$

$\therefore\ x = \frac{\pi}{12} + \dfrac{k\pi}{4}$

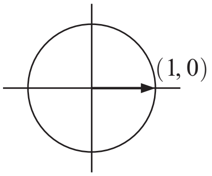
c $\tan^2 x = 3$

$\therefore\ \tan x = \pm\sqrt{3}$

$\therefore\ x = \left.\begin{matrix}\frac{\pi}{3}\\ -\frac{\pi}{3}\end{matrix}\right\} + k\pi$

EXERCISE 13K.2

1 **a**

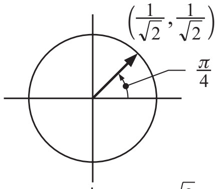


$\tan 0$

$= \frac{0}{1}$

$= 0$

b

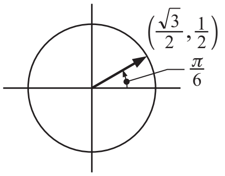


$\tan(\frac{\pi}{4})$

$= \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$= 1$

c

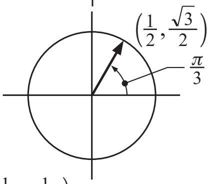


$\tan(\frac{\pi}{6})$

$= \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$= \frac{1}{\sqrt{3}}$

d

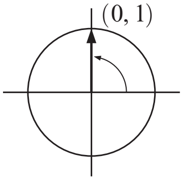


$\tan(\frac{\pi}{3})$

$= \frac{\sqrt{3}}{2} \div \frac{1}{2}$

$= \sqrt{3}$

e

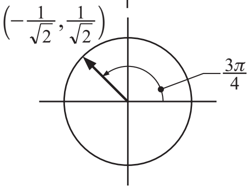


$\tan(\frac{\pi}{2})$

$= \frac{1}{0}$

i.e., undefined

f

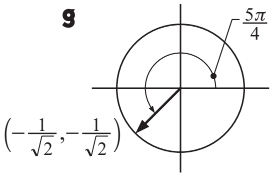


$\tan(\frac{3\pi}{4})$

$= \frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}}$

$= -1$

g

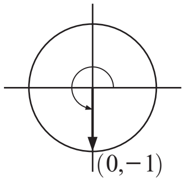


$\tan(\frac{5\pi}{3})$

$= -\frac{\sqrt{3}}{2} \div \frac{1}{2}$

$= -\sqrt{3}$

h

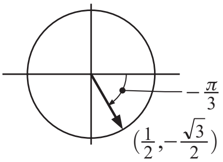


$\tan(\frac{3\pi}{2})$

$= -\frac{1}{0}$

i.e., undefined

i

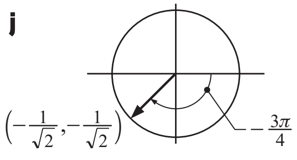


$\tan(-\frac{\pi}{3})$

$= -\frac{\sqrt{3}}{2} \div \frac{1}{2}$

$= -\sqrt{3}$

j



$\tan(-\frac{3\pi}{4})$

$= -\frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}}$

$= 1$

2 **a** $\frac{\pi}{4}, \frac{\pi}{4} + \pi$

i.e., $\frac{\pi}{4}, \frac{5\pi}{4}$

b $\frac{3\pi}{4}, \frac{3\pi}{4} + \pi$

i.e., $\frac{3\pi}{4}, \frac{7\pi}{4}$

c $\frac{\pi}{3}, \frac{\pi}{3} + \pi$

i.e., $\frac{\pi}{3}, \frac{4\pi}{3}$

d $0, 0 + \pi, 0 + 2\pi$

i.e., $0, \pi, 2\pi$

e $\frac{\pi}{6}, \frac{\pi}{6} + \pi$

i.e., $\frac{\pi}{6}, \frac{7\pi}{6}$

f $\frac{5\pi}{3}, \frac{5\pi}{3} - \pi$

i.e., $\frac{5\pi}{3}, \frac{2\pi}{3}$

3 **a** $3\tan x - \tan x$
$= 2\tan x$

b $\tan x - 4\tan x$
$= -3\tan x$

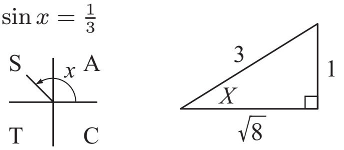
c $\tan x \cos x$
$= \dfrac{\sin x}{\cos x} \times \cos x$
$= \sin x$

d $\dfrac{\sin x}{\tan x}$
$= \sin x \div \dfrac{\sin x}{\cos x}$
$= \sin \times \dfrac{\cos x}{\sin x}$
$= \cos x$

e $3\sin x + 2\cos x \tan x$
$= 3\sin x + 2\cos x \dfrac{\sin x}{\cos x}$
$= 3\sin x + 2\sin x$
$= 5\sin x$

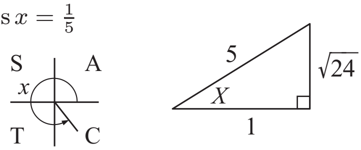
f $\dfrac{2\tan x}{\sin x}$
$= 2\left(\dfrac{\sin x}{\cos x}\right) \div \dfrac{\sin x}{1}$
$= \dfrac{2\sin x}{\cos x} \times \dfrac{1}{\sin x}$
$= \dfrac{2}{\cos x}$

4 **a** $\sin x = \frac{1}{3}$



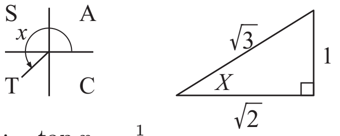
$\therefore \quad \tan x = -\frac{1}{\sqrt{8}} = -\frac{1}{2\sqrt{2}}$

b $\cos x = \frac{1}{5}$



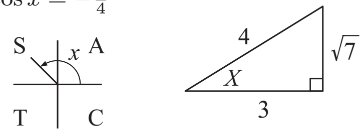
$\therefore \quad \tan x = -\frac{\sqrt{24}}{1} = -2\sqrt{6}$

c $\sin x = -\frac{1}{\sqrt{3}}$



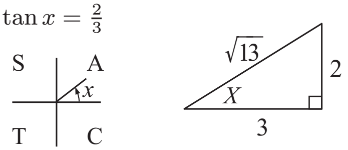
$\therefore \quad \tan x = \frac{1}{\sqrt{2}}$

d $\cos x = -\frac{3}{4}$



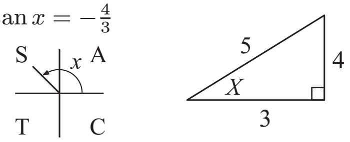
$\therefore \quad \tan x = -\frac{\sqrt{7}}{3}$

5 **a** $\tan x = \frac{2}{3}$



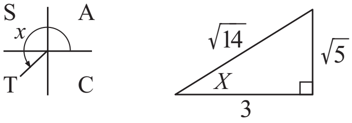
$\therefore \quad \sin x = \frac{2}{\sqrt{13}}, \quad \cos x = \frac{3}{\sqrt{13}}$

b $\tan x = -\frac{4}{3}$



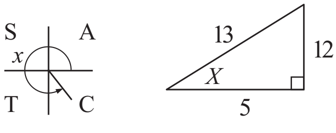
$\therefore \quad \sin x = \frac{4}{5}, \quad \cos x = -\frac{3}{5}$

c $\tan x = \frac{\sqrt{5}}{3}$



$\therefore \quad \sin x = -\frac{\sqrt{5}}{\sqrt{14}}, \quad \cos x = -\frac{3}{\sqrt{14}}$

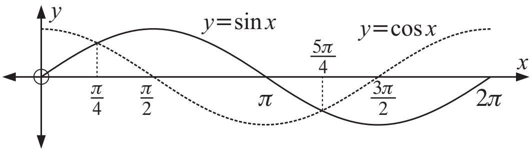
d $\tan x = -\frac{12}{5}$



$\therefore \quad \sin x = -\frac{12}{13}, \quad \cos x = \frac{5}{13}$

EXERCISE 13L

1 **a**



b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

c If $\sin x = \cos x$ then $\dfrac{\sin x}{\cos x} = 1$

$\therefore \quad \tan x = 1$

$\therefore \quad x = \frac{\pi}{4} + k\pi$

$\therefore \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$

2 **a**

$$\begin{aligned}\sin x &= -\cos x\\ \therefore\ \frac{\sin x}{\cos x} &= \frac{-\cos x}{\cos x}\\ \therefore\ \tan x &= -1\\ \therefore\ x &= \tfrac{3\pi}{4} + k\pi\\ \therefore\ x &= \tfrac{3\pi}{4},\ \tfrac{7\pi}{4}\end{aligned}$$

b

$$\begin{aligned}\sin(3x) &= \cos(3x)\\ \therefore\ \frac{\sin(3x)}{\cos(3x)} &= 1\\ \therefore\ \tan(3x) &= 1\\ \therefore\ 3x &= \tfrac{\pi}{4} + k\pi\\ \therefore\ x &= \tfrac{\pi}{12} + \tfrac{k\pi}{3}\\ \therefore\ x &= \tfrac{\pi}{12},\ \tfrac{5\pi}{12},\ \tfrac{9\pi}{12},\ \tfrac{13\pi}{12},\ \tfrac{17\pi}{12},\ \tfrac{21\pi}{12}\\ \therefore\ x &= \tfrac{\pi}{12},\ \tfrac{5\pi}{12},\ \tfrac{3\pi}{4},\ \tfrac{13\pi}{12},\ \tfrac{17\pi}{12},\ \tfrac{7\pi}{4}\end{aligned}$$

c

$$\begin{aligned}\sin(2x) &= \sqrt{3}\cos(2x)\\ \therefore\ \frac{\sin(2x)}{\cos(2x)} &= \sqrt{3}\\ \therefore\ \tan(2x) &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\therefore\ 2x &= \tfrac{\pi}{3} + k\pi\\ \therefore\ x &= \tfrac{\pi}{6} + \tfrac{k\pi}{2}\\ \therefore\ x &= \tfrac{\pi}{6},\ \tfrac{4\pi}{6},\ \tfrac{7\pi}{6},\ \tfrac{10\pi}{6}\\ \therefore\ x &= \tfrac{\pi}{6},\ \tfrac{2\pi}{3},\ \tfrac{7\pi}{6},\ \tfrac{5\pi}{3},\end{aligned}$$

4 **a**

$$\begin{aligned}\sin x &= 5\cos x\\ \therefore\ \tan x &= 5\\ \therefore\ x &= \tan^{-1}(5)\\ \therefore\ x &= 1.373 + k\pi\\ \therefore\ x &\doteqdot 1.37,\ 4.51,\ 7.66\end{aligned}$$

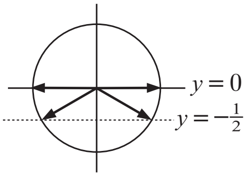
b

$$\begin{aligned}4\sin x + 3\cos x &= 0\\ \therefore\ 4\sin x &= -3\cos x\\ \therefore\ \frac{\sin x}{\cos x} &= -\tfrac{3}{4}\\ \therefore\ \tan x &= -\tfrac{3}{4}\\ \therefore\ x &= \tan^{-1}(-\tfrac{3}{4})\\ \therefore\ x &\doteqdot -1.081 + k\pi\\ \therefore\ x &\doteqdot 2.50,\ 5.64,\ 8.78\end{aligned}$$

EXERCISE 13M

1 **a**

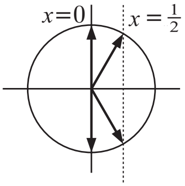
$$\begin{aligned}2\sin^2 x + \sin x &= 0\\ \therefore\ \sin x(2\sin x + 1) &= 0\\ \therefore\ \sin x &= 0 \text{ or } -\tfrac{1}{2}\end{aligned}$$



$$\therefore\ x = 0,\ \pi,\ \tfrac{7\pi}{6},\ \tfrac{11\pi}{6},\ 2\pi$$

b

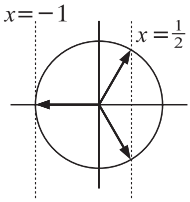
$$\begin{aligned}2\cos^2 x &= \cos x\\ \therefore\ 2\cos^2 x - \cos x &= 0\\ \therefore\ \cos x(2\cos x - 1) &= 0\\ \therefore\ \cos x &= 0 \text{ or } \tfrac{1}{2}\end{aligned}$$



$$\therefore\ x = \tfrac{\pi}{3},\ \tfrac{\pi}{2},\ \tfrac{3\pi}{2},\ \tfrac{5\pi}{3}$$

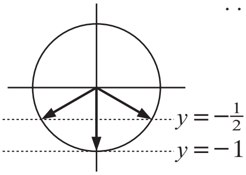
c

$$\begin{aligned}2\cos^2 x + \cos x - 1 &= 0\\ \therefore\ (2\cos x - 1)(\cos x + 1) &= 0\\ \therefore\ \cos x &= \tfrac{1}{2} \text{ or } -1\\ \therefore\ x &= \tfrac{\pi}{3},\ \pi,\ \tfrac{5\pi}{3}\end{aligned}$$



d

$$\begin{aligned}2\sin^2 x + 3\sin x + 1 &= 0\\ \therefore\ (2\sin x + 1)(\sin x + 1) &= 0\\ \therefore\ \sin x &= -\tfrac{1}{2} \text{ or } -1\end{aligned}$$



$$\therefore\ x = \tfrac{7\pi}{6},\ \tfrac{3\pi}{2},\ \tfrac{11\pi}{6}$$

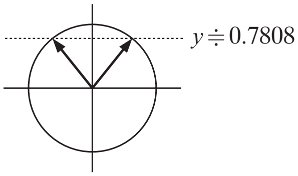
e

$$\begin{aligned}\sin^2 x &= 2 - \cos x\\ \therefore\ 1 - \cos^2 x &= 2 - \cos x\\ \therefore\ \cos^2 x - \cos x + 1 &= 0\\ \text{where}\quad \Delta &= (-1)^2 - 4(1)(1)\\ &= 1 - 4\\ &= -3\end{aligned}$$

$\therefore$ no real solutions exist

f

$$\begin{aligned} 2\cos^2 x &= \sin x \\ \therefore\quad 2(1-\sin^2 x)-\sin x &= 0 \\ \therefore\quad 2-2\sin^2 x-\sin x &= 0 \\ \therefore\quad 2\sin^2 x+\sin x-2 &= 0 \\ \therefore\quad \sin x &= \frac{-1\pm\sqrt{1-4(2)(-2)}}{4} \\ \therefore\quad \sin x &= \frac{-1\pm\sqrt{17}}{4} \\ &\doteqdot 0.7808 \text{ or } 1.281 \end{aligned}$$



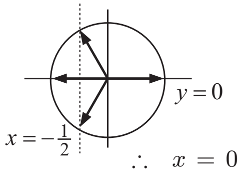
$\therefore\quad \sin x \doteqdot 0.7808$ as $-1 \leqslant \sin x \leqslant 1$

$\therefore\quad x \doteqdot \arcsin(0.7808)$ or $\pi - \arcsin(0.7808)$

$\therefore\quad x \doteqdot 0.896$ or 2.246

2 **a**

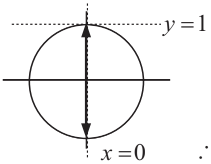
$$\begin{aligned} \sin 2x+\sin x &= 0 \\ \therefore\quad 2\sin x\cos x+\sin x &= 0 \\ \therefore\quad \sin x(2\cos x+1) &= 0 \\ \therefore\quad \sin x = 0 \quad\text{or}\quad \cos x &= -\tfrac{1}{2} \end{aligned}$$



$\therefore\quad x = 0,\ \frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3},\ 2\pi$

b

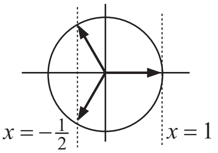
$$\begin{aligned} \sin 2x-2\cos x &= 0 \\ \therefore\quad 2\sin x\cos x-2\cos x &= 0 \\ \therefore\quad 2\cos x(\sin x-1) &= 0 \\ \therefore\quad \cos x = 0 \quad\text{or}\quad \sin x &= 1 \end{aligned}$$



$\therefore\quad x = \frac{\pi}{2},\ \frac{3\pi}{2}$

c

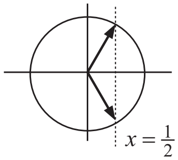
$$\begin{aligned} \cos 2x-\cos x &= 0 \\ \therefore\quad 2\cos^2 x-1-\cos x &= 0 \\ \therefore\quad 2\cos^2 x-\cos x-1 &= 0 \\ \therefore\quad (2\cos x+1)(\cos x-1) &= 0 \\ \therefore\quad \cos x &= -\tfrac{1}{2} \text{ or } 1 \end{aligned}$$



$\therefore\quad x = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ 2\pi$

d

$$\begin{aligned} \cos 2x+3\cos x &= 1 \\ \therefore\quad 2\cos^2 x-1+3\cos x-1 &= 0 \\ \therefore\quad 2\cos^2 x+3\cos x-2 &= 0 \\ \therefore\quad (2\cos x-1)(\cos x+2) &= 0 \\ \therefore\quad \cos x &= \tfrac{1}{2} \text{ or } -2 \\ \therefore\quad \cos x &= \tfrac{1}{2} \quad \{\text{as } -1 \leqslant \cos x \leqslant 1\} \end{aligned}$$



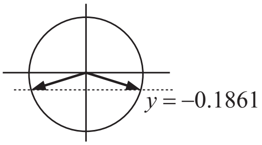
$\therefore\quad x = \frac{\pi}{3},\ \frac{5\pi}{3}$

e

$$\begin{aligned} \cos 2x+5\sin x &= 0 \\ \therefore\quad 1-2\sin^2 x+5\sin x &= 0 \\ \therefore\quad 2\sin^2 x-5\sin x-1 &= 0 \\ \therefore\quad \sin x &= \frac{5\pm\sqrt{25-4(2)(-1)}}{4} \\ &= \frac{5\pm\sqrt{33}}{4} \\ &\doteqdot 2.6861 \text{ or } -0.1861 \end{aligned}$$

$\therefore\quad \sin x = -0.1861$

$\{\text{as } -1 \leqslant \sin x \leqslant 1\}$



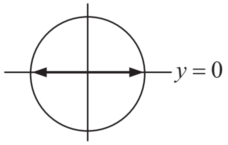
$\therefore\quad x \doteqdot \pi + 0.1872$

or $2\pi - 0.1872$

$\therefore\quad x \doteqdot 3.33$ or 6.10

f

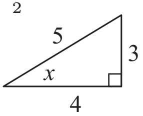
$$\begin{aligned} \sin 2x+3\sin x &= 0 \\ \therefore\quad 2\sin x\cos x+3\sin x &= 0 \\ \therefore\quad \sin x(2\cos x+3) &= 0 \\ \therefore\quad \sin x = 0 \quad\text{or}\quad \cos x &= \underset{\substack{\uparrow \\ \text{impossible}}}{-\tfrac{3}{2}} \end{aligned}$$



$\therefore\quad x = 0,\ \pi,\ 2\pi$

EXERCISE 13N

1 **a** $\sin x = \frac{3}{5}, \quad 0 \leqslant x \leqslant \frac{\pi}{2}$

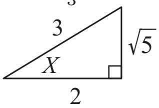


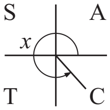
$\therefore \quad \csc x = \dfrac{1}{\sin x} = \frac{5}{3}$

$\sec x = \dfrac{1}{\cos x} = \frac{5}{4}$

$\cot x = \dfrac{1}{\tan x} = \frac{4}{3}$

b $\cos x = \frac{2}{3}$





$\therefore \quad \sin x = -\frac{\sqrt{5}}{3}$ and $\tan x = -\frac{\sqrt{5}}{2}$

$\therefore \quad \csc x = -\frac{3}{\sqrt{5}}$

$\sec x = \frac{3}{2}$

$\cot x = -\frac{2}{\sqrt{5}}$

2 **a** $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$\therefore \quad \csc(\frac{\pi}{3}) = \frac{2}{\sqrt{3}}$

b $\tan(\frac{2\pi}{3}) = -\sqrt{3}$

$\therefore \quad \cot(\frac{2\pi}{3}) = -\frac{1}{\sqrt{3}}$

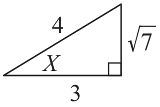
c $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$

$\therefore \quad \sec(\frac{5\pi}{6}) = -\frac{2}{\sqrt{3}}$

d $\tan(\pi) = 0$

$\therefore$ $\cot(\pi)$ is undefined.

3 **a** $\cos x = \frac{3}{4}$



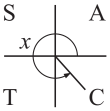
$\therefore \quad \sin x = -\frac{\sqrt{7}}{4}$

$\tan x = -\frac{\sqrt{7}}{3}$

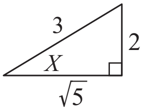
$\csc x = -\frac{4}{\sqrt{7}}$

$\sec x = \frac{4}{3}$

$\cot x = -\frac{3}{\sqrt{7}}$



b $\sin x = -\frac{2}{3}$



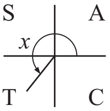
$\therefore \quad \cos x = -\frac{\sqrt{5}}{3}$

$\tan x = \frac{2}{\sqrt{5}}$

$\csc x = -\frac{3}{2}$

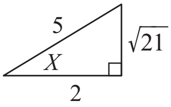
$\sec x = -\frac{3}{\sqrt{5}}$

$\cot x = \frac{\sqrt{5}}{2}$



c $\sec x = \frac{5}{2}$

$\therefore \quad \cos x = \frac{2}{5}$

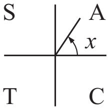


$\therefore \quad \sin x = \frac{\sqrt{21}}{5}$

$\tan x = \frac{\sqrt{21}}{2}$

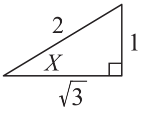
$\csc x = \frac{5}{\sqrt{21}}$

$\cot x = \frac{2}{\sqrt{21}}$



d $\csc x = 2$

$\therefore \quad \sin x = \frac{1}{2}$

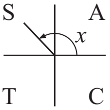


$\therefore \quad \cos x = -\frac{\sqrt{3}}{2}$

$\tan x = -\frac{1}{\sqrt{3}}$

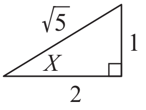
$\sec x = -\frac{2}{\sqrt{3}}$

$\cot x = -\sqrt{3}$



e $\tan x = \frac{1}{2}$

$\therefore \quad \cot x = 2$

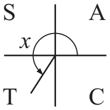


$\sin x = -\frac{1}{\sqrt{5}}$

$\cos x = -\frac{2}{\sqrt{5}}$

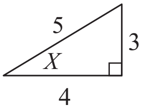
$\csc x = -\sqrt{5}$

$\sec x = -\frac{\sqrt{5}}{2}$



f $\cot x = \frac{4}{3}$

$\therefore \quad \tan x = \frac{3}{4}$

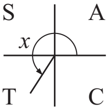


$\therefore \quad \sin x = -\frac{3}{5}$

$\cos x = -\frac{4}{5}$

$\csc x = -\frac{5}{3}$

$\sec x = -\frac{5}{4}$



4 **a**
$$\begin{aligned}&\tan x\cot x\\ &=\frac{\sin x}{\cos x}\times\frac{\cos x}{\sin x}\\ &=1\end{aligned}$$

b
$$\begin{aligned}&\sin x\csc x\\ &=\sin x\times\frac{1}{\sin x}\\ &=1\end{aligned}$$

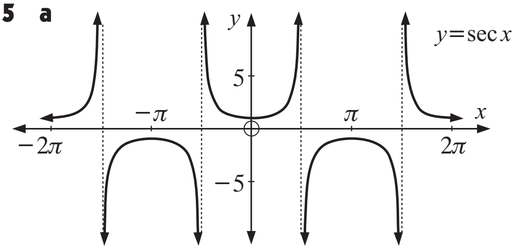
c
$$\begin{aligned}&\csc x\cot x\\ &=\frac{1}{\sin x}\times\frac{\cos x}{\sin x}\\ &=\frac{\cos x}{\sin^2 x}\end{aligned}$$

d
$$\begin{aligned}&\sin x\cot x\\ &=\sin x\times\frac{\cos x}{\sin x}\\ &=\cos x\end{aligned}$$

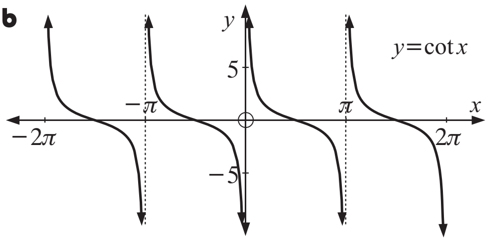
e
$$\begin{aligned}&\frac{\cot x}{\csc x}\\ &=\frac{\cos x}{\sin x}\div\frac{1}{\sin x}\\ &=\frac{\cos x}{\sin x}\times\frac{\sin x}{1}\\ &=\cos x\end{aligned}$$

f
$$\begin{aligned}&\frac{2\sin x\cot x+3\cos x}{\cot x}\\ &=\frac{2\sin x\times\frac{\cos x}{\sin x}+3\cos x}{\frac{\cos x}{\sin x}}\\ &=(2\cos x+3\cos x)\times\frac{\sin x}{\cos x}\\ &=5\cos x\times\frac{\sin x}{\cos x}\\ &=5\sin x\end{aligned}$$

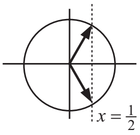
5 **a**



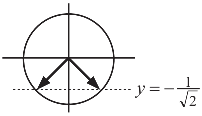
b



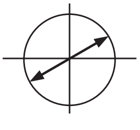
6 **a**
$$\begin{aligned}\sec x&=2\\ \therefore\quad \cos x&=\tfrac{1}{2}\\ \therefore\quad x&=\tfrac{\pi}{3},\ \tfrac{5\pi}{3}\end{aligned}$$



b
$$\begin{aligned}\csc x&=-\sqrt{2}\\ \therefore\quad \sin x&=-\tfrac{1}{\sqrt{2}}\\ \therefore\quad x&=\tfrac{5\pi}{4},\ \tfrac{7\pi}{4}\end{aligned}$$



c
$$\begin{aligned}\cot x&=4\\ \therefore\quad \tan x&=\tfrac{1}{4}\end{aligned}$$



$$\begin{aligned}\therefore\quad x&=\arctan\left(\tfrac{1}{4}\right)+k\pi\\ \therefore\quad x&=0.245\quad\text{or}\quad 3.387\end{aligned}$$

d
$$\begin{aligned}\sec 2x&=\tfrac{1}{3}\\ \therefore\quad \cos 2x&=3\end{aligned}$$
which is impossible as all values of cosine lie between -1 and 1, $\therefore$ no solution exists.

e
$$\begin{aligned}\csc x&=-\tfrac{2}{3}\\ \therefore\quad \sin x&=-\tfrac{3}{2}\end{aligned}$$
which is impossible as $-1\leqslant\sin x\leqslant 1$

$\therefore$ no solutions exist.

f
$$\begin{aligned}\cot\left(2x-\tfrac{\pi}{4}\right)+3&=0\\ \therefore\quad \cot\left(2x-\tfrac{\pi}{4}\right)&=-3\\ \therefore\quad \tan\left(2x-\tfrac{\pi}{4}\right)&=-\tfrac{1}{3}\\ \therefore\quad 2x-\tfrac{\pi}{4}&=\arctan\left(-\tfrac{1}{3}\right)+k\pi\\ \therefore\quad 2x-\tfrac{\pi}{4}&\doteqdot -0.3218+k\pi\\ \therefore\quad 2x&\doteqdot 0.4636+k\pi\\ \therefore\quad x&\doteqdot 0.232+k\tfrac{\pi}{2}\\ \therefore\quad x&=0.232,\ 1.803,\ 3.373,\ 4.944\end{aligned}$$

7 **a**
$$\begin{aligned}\sin^2 x+\cos^2 x&=1\\ \therefore\quad \frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}&=\frac{1}{\cos^2 x}\\ \therefore\quad \tan^2 x+1&=\sec^2 x\end{aligned}$$

b
$$\begin{aligned}\sin^2 x+\cos^2 x&=1\\ \therefore\quad \frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}&=\frac{1}{\sin^2 x}\\ \therefore\quad 1+\cot^2 x&=\csc^2 x\end{aligned}$$

8 **a**

$$\begin{aligned}&\sin^2 x+\cot^2 x\sin^2 x\\&=\sin^2 x+\frac{\cos^2 x}{\sin^2 x}\sin^2 x\\&=\sin^2 x+\cos^2 x\\&=1\end{aligned}$$

b

$$\begin{aligned}&\tan x+\cot x\\&=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\\&=\frac{\sin x}{\cos x}\left(\frac{\sin x}{\sin x}\right)+\frac{\cos x}{\sin x}\left(\frac{\cos x}{\cos x}\right)\\&=\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}\\&=\frac{1}{\cos x\sin x}\\&=\frac{1}{\sin x}\times\frac{1}{\cos x}\\&=\csc x\sec x\end{aligned}$$

c

$$\begin{aligned}&\sec x-\tan x\sin x\\&=\frac{1}{\cos x}-\left(\frac{\sin x}{\cos x}\right)\sin x\\&=\frac{1-\sin^2 x}{\cos x}\\&=\frac{\cos^2 x}{\cos x}\\&=\cos x\end{aligned}$$

d

$$\begin{aligned}&\frac{\sin\theta}{1-\cos\theta}-\frac{\sin\theta}{1+\cos\theta}\\&=\frac{\sin\theta}{1-\cos\theta}\left(\frac{1+\cos\theta}{1+\cos\theta}\right)-\frac{\sin\theta}{1+\cos\theta}\left(\frac{1-\cos\theta}{1-\cos\theta}\right)\\&=\frac{\cancel{\sin\theta}+\sin\theta\cos\theta-\cancel{\sin\theta}+\sin\theta\cos\theta}{(1-\cos\theta)(1+\cos\theta)}\\&=\frac{2\sin\theta\cos\theta}{1-\cos^2\theta}\\&=\frac{2\sin\theta\cos\theta}{\sin^2\theta}\\&=\frac{2\cos\theta}{\sin\theta}\\&=2\cot\theta\end{aligned}$$

e

$$\begin{aligned}&\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\\&=\frac{1}{1-\sin\theta}\left(\frac{1+\sin\theta}{1+\sin\theta}\right)+\frac{1}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right)\\&=\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}\\&=\frac{2}{1-\sin^2\theta}\\&=\frac{2}{\cos^2\theta}\\&=2\sec^2\theta\end{aligned}$$

f

$$\begin{aligned}&\frac{(1-\cot\theta)^2}{\csc^2\theta}+\sin 2\theta\\&=\frac{1-2\cot\theta+\cot^2\theta}{\dfrac{1}{\sin^2\theta}}+\sin 2\theta\\&=\left(1-\frac{2\cos\theta}{\sin\theta}+\frac{\cos^2\theta}{\sin^2\theta}\right)\sin^2\theta+2\sin\theta\cos\theta\\&=\sin^2\theta-\cancel{2\sin\theta\cos\theta}+\cos^2\theta+\cancel{2\sin\theta\cos\theta}\\&=1\end{aligned}$$

EXERCISE 13O

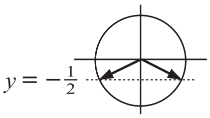
1 **a** $1+\sin x+\sin^2 x+\sin^3 x+\ldots\ldots+\sin^{n-1} x$ is a geometric series with $u_1=1,\quad r=\sin x$

$$\begin{aligned}\therefore\quad \text{sum}&=\frac{u_1(1-r^n)}{1-r}\\&=\frac{1(1-\sin^n x)}{1-\sin x}\\&=\frac{1-\sin^n x}{1-\sin x}\end{aligned}$$

b $S_\infty=\dfrac{u_1}{1-r}=\dfrac{1}{1-\sin x}$

as $-1\leqslant\sin x\leqslant 1\ \Rightarrow\ -1\leqslant r\leqslant 1$

c If $S_\infty=\frac{2}{3}$, $\dfrac{1}{1-\sin x}=\frac{2}{3}$



$$\begin{aligned}\therefore\quad 3&=2-2\sin x\\\therefore\quad 2\sin x&=-1\\\therefore\quad \sin x&=-\tfrac{1}{2}\\\therefore\quad x&=\tfrac{7\pi}{6}\ \text{ or }\ \tfrac{11\pi}{6}\end{aligned}$$

2 **a** **i**
$$\begin{aligned}&2\sin x(\cos x+\cos 3x)\\ &=2\sin x\cos x+2\sin x\cos 3x\\ &=\sin 2x+\sin 4x+\sin(-2x)\\ &=\cancel{\sin 2x}+\sin 4x-\cancel{\sin 2x}\\ &=\sin 4x\end{aligned}$$

ii
$$\begin{aligned}&2\sin x(\cos x+\cos 3x+\cos 5x)\\ &=2\sin x(\cos x+\cos 3x)+2\sin x\cos 5x\\ &=\sin 4x+\sin 6x+\sin(-4x)\quad\{\text{from } \textbf{i}\}\\ &=\cancel{\sin 4x}+\sin 6x-\cancel{\sin 4x}\\ &=\sin 6x\end{aligned}$$

$\{2\sin A\cos B=\sin(A+B)+\sin(A-B)$ was used$\}$

b **i** $2\sin x(\cos x+\cos 3x+\cos 5x+.....+\cos 7x)=\sin 8x$

ii $2\sin x(\cos x+\cos 3x+\cos 5x+......+\cos 19x)=\sin 20x$

$$\therefore\quad \cos x+\cos 3x+\cos 5x+......+\cos 19x=\frac{\sin 20x}{2\sin x}$$

c In general, $\cos x+\cos 3x+\cos 5x+......+\cos(2n-1)x=\dfrac{\sin 2nx}{2\sin x}$

3 **a** **i**
$$\begin{aligned}&\sin x\cos x\cos 2x\\ &=\tfrac{1}{2}(2\sin x\cos x)\cos 2x\\ &=\tfrac{1}{2}\sin 2x\cos 2x\\ &=\tfrac{1}{4}2\sin 2x\cos 2x\\ &=\tfrac{1}{4}\sin 4x\quad ......\ (1)\\ &=\frac{\sin(2^2x)}{2^2}\end{aligned}$$

ii
$$\begin{aligned}&(\sin x\cos x\cos 2x)\cos 4x\\ &=\tfrac{1}{4}\sin 4x\cos 4x\quad\{\text{from (1)}\}\\ &=\tfrac{1}{8}(2\sin 4x\cos 4x)\\ &=\frac{\sin 8x}{8}\\ &=\frac{\sin(2^3x)}{2^3}\end{aligned}$$

b **i** $\dfrac{\sin(2^4x)}{2^4}$ **ii** $\dfrac{\sin(2^6x)}{2^6}$

c $\sin x\cos x\cos 2x\cos 4x......\cos(2^nx)=\dfrac{\sin(2^{n+1}x)}{2^{n+1}}$

or $\cos x\cos 2x\cos 4x......\cos(2^nx)=\dfrac{\sin(2^{n+1}x)}{2^{n+1}\sin x}$

4 **a** P_n is "$\cos\theta+\cos 3\theta+\cos 5\theta+......+\cos(2n-1)\theta=\dfrac{\sin 2n\theta}{2\sin\theta},\quad n\in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n=1$, LHS $=\cos\theta$ and RHS $=\dfrac{\sin 2\theta}{2\sin\theta}=\dfrac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta$ $\therefore$ P_1 is true.

(2) If P_k is true, then

$$\cos\theta+\cos 3\theta+\cos 5\theta+......+\cos(2k-1)\theta=\frac{\sin 2k\theta}{2\sin\theta}$$

$$\begin{aligned}\therefore\quad &\cos\theta+\cos 3\theta+\cos 5\theta+......+\cos(2k-1)\theta+\cos(2k+1)\theta\\ &=\frac{\sin 2k\theta}{2\sin\theta}+\cos(2k+1)\theta\\ &=\frac{\sin 2k\theta+2\sin\theta\cos(2k+1)\theta}{2\sin\theta}\\ &=\frac{\sin 2k\theta+\sin[\theta+(2k+1)\theta]+\sin[\theta-(2k+1)\theta]}{2\sin\theta}\\ &=\frac{\sin 2k\theta+\sin(\theta+2k\theta+\theta)+\sin(\theta-2k\theta-\theta)}{2\sin\theta}\\ &=\frac{\sin 2k\theta+\sin(2k\theta+2\theta)+\sin(-2k\theta)}{2\sin\theta}\\ &=\frac{\cancel{\sin 2k\theta}+\sin 2(k+1)\theta-\cancel{\sin 2k\theta}}{2\sin\theta}\\ &=\frac{\sin 2(k+1)\theta}{2\sin\theta}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true {Principle of Mathematical Induction}

b $\cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos 31\theta = \dfrac{\sin 32\theta}{2\sin\theta}$ $\{n = 16\}$

5 P_n is "$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2n-1)\theta = \dfrac{1-\cos 2n\theta}{2\sin\theta}$, $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS $= \sin\theta$ and RHS $= \dfrac{1-\cos 2\theta}{\sin\theta} = \dfrac{1-(1-2\sin^2\theta)}{\sin\theta} = \dfrac{2\sin^2\theta}{2\sin\theta} = \sin\theta$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$$\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2k-1)\theta = \frac{1-\cos 2k\theta}{2\sin\theta}$$

$\therefore$ $\sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2k-1)\theta + \sin(2k+1)\theta$

$$= \frac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta$$

$$= \frac{1-\cos 2k\theta + 2\sin(2k+1)\theta\sin\theta}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta + \cos[(2k+1)\theta-\theta] - \cos[(2k+1)\theta+\theta]}{2\sin\theta}$$

$$= \frac{1-\cancel{\cos 2k\theta} + \cancel{\cos 2k\theta} - \cos[(2k+2)\theta]}{2\sin\theta}$$

$$= \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true {Principle of Mathematical Induction}

Thus $\sin\frac{\pi}{7} + \sin\frac{3\pi}{7} + \sin\frac{5\pi}{7} + ... + \sin\frac{13\pi}{7}$ has $2n-1 = 13$ and $\theta = \frac{\pi}{7}$

i.e., $n = 7$ and $\theta = \frac{\pi}{7}$

$\therefore$ the sum is $\dfrac{1-\cos(2\times 7\times\frac{\pi}{7})}{2\sin\frac{\pi}{7}} = \dfrac{1-\cos 2\pi}{2\sin\frac{\pi}{7}} = \dfrac{1-1}{2\sin\frac{\pi}{7}} = 0$

6 P_n is "$\cos x \times \cos 2x \times \cos 4x \times \cos 8x \times ...... \times \cos(2^{n-1}x) = \dfrac{\sin(2^n x)}{2^n \times \sin x}$, $n \in Z^+$"

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS $= \cos x$, RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x\cos x}{2\sin x} = \cos x$ $\therefore$ P_1 is true.

(2) If P_k is true, then

$$\cos x \times \cos 2x \times \cos 4x \times ...... \times \cos(2^{k-1}x) = \frac{\sin(2^k x)}{2^k \sin x}$$

$\therefore$ $\cos x \times \cos 2x \times \cos 4x \times ...... \times \cos(2^{k-1}x) \times \cos(2^k x)$

$$= \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x)$$

$$= \frac{2\sin(2^k x)\cos(2^k x)}{2 \times 2^k \sin x}$$

$$= \frac{\sin(2\times 2^k x)}{2^{k+1}\sin x} \quad \{2\sin\theta\cos\theta = \sin 2\theta\}$$

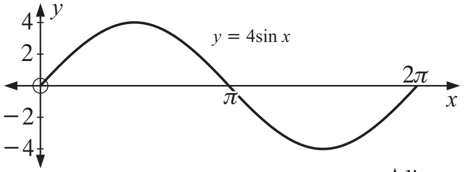
$$= \frac{\sin(2^{k+1}x)}{2^{k+1}\sin x}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true {Principle of Mathematical Induction}

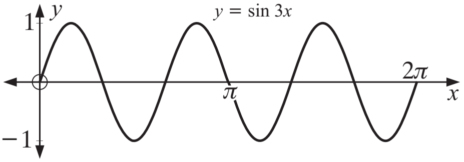
REVIEW SET 13A

1 This is the graph of $y = \sin x$ under a vertical stretch of factor 4. The amplitude is 4.



2 This is the graph of $y = \sin x$ under a horizontal stretch of factor $\frac{1}{3}$.

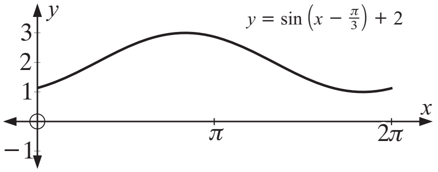
The period is $\frac{2\pi}{2}$.



3 **a** period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

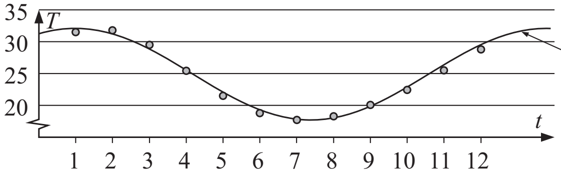
b period $= \frac{2\pi}{4} = \frac{\pi}{2}$

4



5

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8



$T = A\sin B(t - C) + D$ period $= \dfrac{2\pi}{B} = 12$, $\therefore$ $B = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$

max. $= 31.8$ $\therefore$ $A = \dfrac{\text{max.} - \text{min.}}{2} \doteq \dfrac{31.8 - 17.7}{2} \doteq 7.05$

min. $= 17.7$

$D = \dfrac{\text{max.} + \text{min.}}{2} \doteq \dfrac{31.8 + 17.7}{2} \doteq 24.75$

$C = \dfrac{2 + 7}{2} = 4.5$ {values of t at max. and min.}

So, $T \doteq 7.05\sin\frac{\pi}{6}(t - 10.5) + 24.75$

From technology, $T \doteq 7.21\sin(0.488t + 1.082) + 24.75$

$\doteq 7.21\sin 0.488(t + 2.22) + 24.75$ **Note:** $0.488(2.22 - -10.5) \doteq 6.21$

$\doteq 2\pi$

6 **a** $\sin x = 0.382$

$\therefore$ $x \doteq 0.392, 2.750, 6.675$

b $\sin\left(\frac{x}{2}\right) = -0.458$

$\therefore$ $x \doteq 7.235$

7 **a** $\sin(x - 2.4) = 0.754$

$\therefore$ $x \doteq 3.254, 4.687$

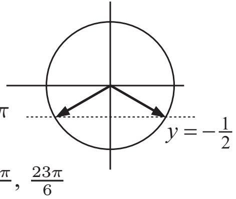
b $\sin\left(x + \frac{\pi}{3}\right) = 0.6049$

$\therefore$ $x \doteq 1.445, 5.89, 7.73$

8 **a** $2\sin x = -1$, $0 \leqslant x \leqslant 4\pi$

$\therefore$ $\sin x = -\frac{1}{2}$

$\therefore$ $x = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi$

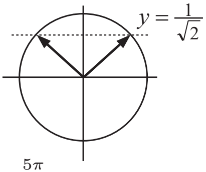


$\therefore$ $x = \frac{7\pi}{6}, \frac{19\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}$

b $\sqrt{2}\sin x - 1 = 0$, $-2\pi \leqslant x \leqslant 2\pi$

$\therefore$ $\sin x = \frac{1}{\sqrt{2}}$

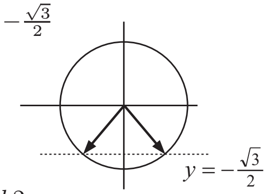
$\therefore$ $x = \left.\begin{matrix}\frac{\pi}{4}\\ \frac{3\pi}{4}\end{matrix}\right\} + k2\pi$



$\therefore$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{7\pi}{4}, -\frac{5\pi}{4}$

9 **a** $2\sin 3x + \sqrt{3} = 0, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore \quad \sin 3x = -\frac{\sqrt{3}}{2}$



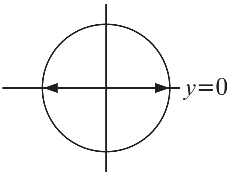
$\therefore \quad 3x = \left.\begin{matrix} \frac{4\pi}{3} \\ \frac{5\pi}{3} \end{matrix}\right\} + k2\pi$

$\therefore \quad x = \left.\begin{matrix} \frac{4\pi}{9} \\ \frac{5\pi}{9} \end{matrix}\right\} + k\,\frac{2\pi}{3}$

$\therefore \quad x = \frac{4\pi}{9}, \frac{10\pi}{9}, \frac{16\pi}{9}, \frac{5\pi}{9}, \frac{11\pi}{9}, \frac{17\pi}{9}$

b $\sqrt{2}\sin(x + \frac{\pi}{4}) = 0, \quad 0 \leqslant x \leqslant 3\pi$

$\therefore \quad \sin(x + \frac{\pi}{4}) = 0$



$\therefore \quad x + \frac{\pi}{4} = 0 + k\pi$

$\therefore \quad x = -\frac{\pi}{4} + k\pi$

$\therefore \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

10 $P(t) = 5 + 2\sin\left(\frac{\pi t}{3}\right), \quad 0 \leqslant t \leqslant 8$

a $P(0) = 5 + 2\sin 0$
$= 5$

i.e., 5000 water beetles

b Smallest $P = 5 + 2(-1) = 3$
Largest $P = 5 + 2(1) = 7$

$\therefore$ smallest is 3000 water beetles
largest is 7000 water beetles

c If population is > 6000,
then $P(t) > 6$

$\therefore \quad 5 + 2\sin\left(\frac{\pi t}{3}\right) > 6$

$\therefore \quad 2\sin\left(\frac{\pi t}{3}\right) > 1$

$\therefore \quad \sin\left(\frac{\pi t}{3}\right) > \frac{1}{2}$

Using technology,
$0.5 < t < 2.5$ and $6.5 < t < 8$

REVIEW SET 13B

1 **a** $\sin^2 x - \sin x - 2 = 0$

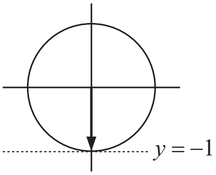
$\therefore \quad (\sin x - 2)(\sin x + 1) = 0$

$\therefore \quad \sin x = 2 \text{ or } -1$

But $\sin x$ values lie between -1 and 1 inclusive

$\therefore \quad \sin x = -1$

$\therefore \quad x = \frac{3\pi}{2} + k2\pi$

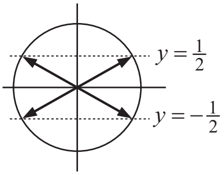


b $4\sin^2 x = 1$

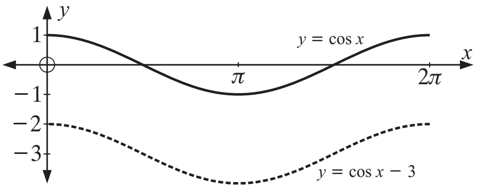
$\therefore \quad \sin^2 x = \frac{1}{4}$

$\therefore \quad \sin x = \pm\frac{1}{2}$

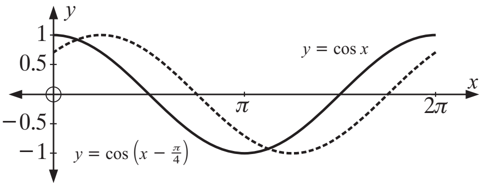
$\therefore \quad x = \left.\begin{matrix} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{matrix}\right\} + k\pi$



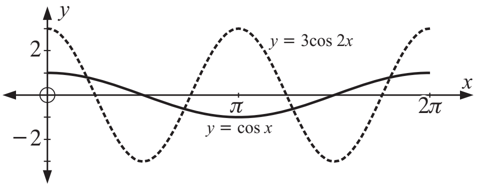
2 **a**



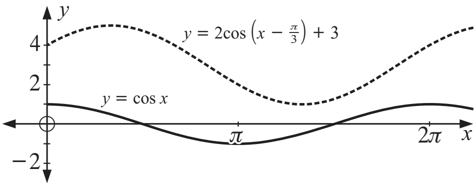
b



c



d



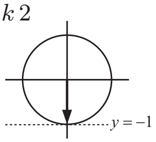
3 $P(t) = 40+12\sin\frac{2\pi}{7}\left(t-\frac{37}{12}\right)$ m^3

a $P(t)$ is a minimum of $\begin{aligned} &40+12(-1) \\ &= 28 \text{ mg/m}^3\end{aligned}$

b when

$$\begin{aligned} \sin\tfrac{2\pi}{7}(t-\tfrac{37}{12}) &= -1 \\ \therefore\ \tfrac{2\pi}{7}(t-\tfrac{37}{12}) &= \tfrac{3\pi}{2}+k2\pi \\ \therefore\ \tfrac{2}{7}(t-\tfrac{37}{12}) &= \tfrac{3}{2}+k\,2 \end{aligned}$$

$$\begin{aligned} \text{So,}\quad t-\tfrac{37}{12} &= \tfrac{21}{4}+k\,7 \\ \therefore\ t &= 8\tfrac{1}{3}+k\,7 \\ \therefore\ t &= 1\tfrac{1}{3},\ 8\tfrac{1}{3},\ 15\tfrac{1}{3}, \text{ etc.} \end{aligned}$$



∴ on Mondays at 8.00 am

{$1\frac{1}{3}$ days after midnight Sat.}

4 **a** If $y = A\cos B(t-C)+D$

$$\text{then}\quad A = -4,\qquad \frac{2\pi}{B} = \pi \qquad \therefore\ B = 2$$

$$\begin{aligned} C = D &= 0 \\ \therefore\ y &= -4\cos 2x \end{aligned}$$

b If $y = A\cos B(x-C)+D$

$$\text{then}\quad A = 1,\quad \frac{2\pi}{B} = 8 \quad \therefore\ B = \tfrac{\pi}{4}$$

$$\begin{aligned} D &= \frac{\text{max.} + \text{min.}}{2} = \frac{3+1}{2} = 2 \\ C &= 0 \\ \text{So,}\quad y &= \cos\left(\tfrac{\pi}{4}x\right)+2 \end{aligned}$$

5 **a** $\cos x = 0.4379, \quad 0 \leqslant x \leqslant 10$

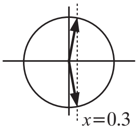
$\therefore\ x \doteqdot 1.12,\ 5.17,\ 7.40$

b $\cos(x-2.4) = -0.6014, \quad 0 \leqslant x \leqslant 6$

$\therefore\ x \doteqdot 0.184,\ 4.62$

6 **a** $\cos 4x = 0.3$, for all x

$\therefore\ 4x = \cos^{-1}(0.3)$



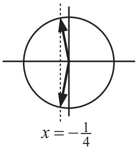
$$\therefore\ 4x = \left.\begin{matrix} 1.2661 \\ 2\pi-1.2661 \end{matrix}\right\} + k2\pi$$

$$\therefore\ x = \left.\begin{matrix} 0.317 \\ 1.254 \end{matrix}\right\} + k(\tfrac{\pi}{2})$$

b $4\cos 2x + 1 = 0, \quad 0 \leqslant x \leqslant 5$

$\therefore\ \cos 2x = -\frac{1}{4}$

$$\therefore\ 2x = \left.\begin{matrix} \pi-1.318 \\ \pi+1.318 \end{matrix}\right\} + k2\pi$$

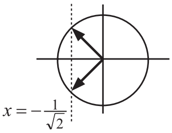


$$\therefore\ x = \left.\begin{matrix} 0.912 \\ 2.230 \end{matrix}\right\} + k\pi$$

$\therefore\ x = 0.912,\ 4.05,\ 2.23$

7 **a** $\cos x = -\frac{1}{\sqrt{2}}, \quad 0 \leqslant x \leqslant 4\pi$

$$x = \left.\begin{matrix} \frac{3\pi}{4} \\ \frac{5\pi}{4} \end{matrix}\right\} + k2\pi$$

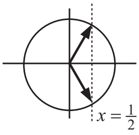


$\therefore\ x = \frac{3\pi}{4},\ \frac{11\pi}{4},\ \frac{5\pi}{4},\ \frac{13\pi}{4}$

b $\cos\left(x+\frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

$$x+\tfrac{2\pi}{3} = \left.\begin{matrix} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{matrix}\right\} + k2\pi$$

$$\therefore\ x = \left.\begin{matrix} -\frac{\pi}{3} \\ \pi \end{matrix}\right\} + k2\pi$$



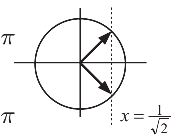
$\therefore\ x = -\frac{\pi}{3},\ \frac{5\pi}{3},\ \pi,\ -\pi$

8 **a** $\sqrt{2}\cos(x+\frac{\pi}{4})-1 = 0, \quad 0 \leqslant x \leqslant 4\pi$

$\therefore\ \cos(x+\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$\therefore\ x+\tfrac{\pi}{4} = \left.\begin{matrix} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{matrix}\right\} + k2\pi$$

$$\therefore\ x = \left.\begin{matrix} 0 \\ \frac{3\pi}{2} \end{matrix}\right\} + k2\pi$$



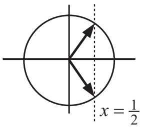
$\therefore\ x = 0,\ 2\pi,\ 4\pi,\ \frac{3\pi}{2},\ \frac{7\pi}{2}$

b $2\cos 2x - 1 = 0$, for all x

$\therefore\ \cos 2x = \frac{1}{2}$

$$\therefore\ 2x = \left.\begin{matrix} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{matrix}\right\} + k2\pi$$

$$\therefore\ x = \left.\begin{matrix} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{matrix}\right\} + k\pi$$



9 **a**
$$\begin{aligned}&\cos^3\theta+\sin^2\theta\cos\theta\\&=\cos\theta(\cos^2\theta+\sin^2\theta)\\&=\cos\theta(1)\\&=\cos\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\cos^2\theta-1}{\sin\theta}\\&=\frac{-(1-\cos^2\theta)}{\sin\theta}\\&=-\frac{\sin^2\theta}{\sin\theta}\\&=-\sin\theta\end{aligned}$$

c
$$\begin{aligned}&3\cos\theta-\cos\theta\\&=2\cos\theta\end{aligned}$$

d
$$\begin{aligned}&5-5\sin^2\theta\\&=5(1-\sin^2\theta)\\&=5\cos^2\theta\end{aligned}$$

e
$$\begin{aligned}&\frac{\sin^2\theta-1}{\cos\theta}\\&=-\frac{(1-\sin^2\theta)}{\cos\theta}\\&=-\frac{\cos^2\theta}{\cos\theta}\\&=-\cos\theta\end{aligned}$$

10 **a**
$$\begin{aligned}&(2\sin\alpha-1)^2\\&=4\sin^2\alpha-4\sin\alpha+1\end{aligned}$$

b
$$\begin{aligned}&(\cos\alpha-\sin\alpha)^2\\&=\cos^2\alpha-2\sin\alpha\cos\alpha+\sin^2\alpha\\&=\cos^2\alpha+\sin^2\alpha-2\sin\alpha\cos\alpha\\&=1-\sin2\alpha\end{aligned}$$

REVIEW SET 13C

1 **a**
$$\begin{aligned}&\frac{1-\cos^2\theta}{1+\cos\theta}\\&=\frac{{}^{1}\cancel{(1+\cos\theta)}(1-\cos\theta)}{\cancel{1+\cos\theta}_{\,1}}\\&=1-\cos\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\sin\alpha-\cos\alpha}{\sin^2\alpha-\cos^2\alpha}\\&=\frac{\cancel{\sin\alpha-\cos\alpha}\ ^{1}}{(\sin\alpha+\cos\alpha)\cancel{(\sin\alpha-\cos\alpha)}\ ^{1}}\\&=\frac{1}{\sin\alpha+\cos\alpha}\end{aligned}$$

c
$$\begin{aligned}&\frac{4\sin^2\alpha-4}{8\cos\alpha}\\&=\frac{-4(1-\sin^2\alpha)}{8\cos\alpha}\\&=\frac{-4\cos^2\alpha}{8\cos\alpha}\\&=-\tfrac{1}{2}\cos\alpha\end{aligned}$$

2 **a**
$$\begin{aligned}&\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}\\&=\frac{\cos^2\theta+(1+\sin\theta)^2}{(1+\sin\theta)\cos\theta}\\&=\frac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{(1+\sin\theta)\cos\theta}\\&=\frac{2+2\sin\theta}{(1+\sin\theta)\cos\theta}\qquad\{\cos^2\theta+\sin^2\theta=1\}\\&=\frac{2\cancel{(1+\sin\theta)}^{\,1}}{{}_{1}\cancel{(1+\sin\theta)}\cos\theta}\\&=\frac{2}{\cos\theta}\quad\text{or}\quad 2\sec\theta\end{aligned}$$

b
$$\begin{aligned}&\left(1+\frac{1}{\cos\theta}\right)\left(\cos\theta-\cos^2\theta\right)\\&=\cancel{\cos\theta}-\cos^2\theta+1-\cancel{\cos\theta}\\&=1-\cos^2\theta\\&=\sin^2\theta\end{aligned}$$

3 **a**
$$\begin{aligned}\sin2A&=2\sin A\cos A\\&=2(\tfrac{5}{13})(\tfrac{12}{13})\\&=\tfrac{120}{169}\end{aligned}$$

b
$$\begin{aligned}\cos2A&=\cos^2A-\sin^2A\\&=(\tfrac{12}{13})^2-(\tfrac{5}{13})^2\\&=\tfrac{144-25}{169}\\&=\tfrac{119}{169}\end{aligned}$$

4

$$\begin{aligned}\cos^2\alpha+\sin^2\alpha &= 1\\ \therefore\ \cos^2\alpha+\tfrac{9}{16} &= 1\\ \therefore\ \cos^2\alpha &= \tfrac{7}{16}\\ \therefore\ \cos\alpha &= \pm\tfrac{\sqrt{7}}{4}\\ \text{But in Q3,}\quad \cos\alpha &< 0\\ \therefore\ \cos\alpha &= -\tfrac{\sqrt{7}}{4}\end{aligned}$$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2(-\tfrac{3}{4})(-\tfrac{\sqrt{7}}{4})\\ &= \tfrac{3\sqrt{7}}{8}\end{aligned}$$

5

$$\begin{aligned}\cos 2A &= 1-2\sin^2 A\\ \therefore\ \cos x &= 1-2\sin^2\left(\tfrac{x}{2}\right) \quad \{\text{letting } 2A=x,\ A=\tfrac{x}{2}\}\\ \therefore\ -\tfrac{3}{4} &= 1-2\sin^2\left(\tfrac{x}{2}\right)\\ \therefore\ 2\sin^2\left(\tfrac{x}{2}\right) &= \tfrac{7}{4}\\ \therefore\ \sin^2\left(\tfrac{x}{2}\right) &= \tfrac{7}{8}\\ \therefore\ \sin\left(\tfrac{x}{2}\right) &= \pm\tfrac{\sqrt{7}}{2\sqrt{2}}\end{aligned}$$

But $\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}$ (in Q2) $\quad\therefore\ \sin\left(\frac{x}{2}\right)=\frac{\sqrt{7}}{2\sqrt{2}}$

6 **a** **i**
$$\begin{aligned}\tan x &= 4\\ \therefore\ x &\doteqdot 1.326+k\pi\\ \text{i.e.,}\quad x &\doteqdot 1.33+k\pi\end{aligned}$$

ii
$$\begin{aligned}\tan\left(\tfrac{x}{4}\right) &= 4\\ \therefore\ \tfrac{x}{4} &\doteqdot 1.326+k\pi\\ \therefore\ x &\doteqdot 5.30+k4\pi\end{aligned}$$

iii
$$\begin{aligned}\tan(x-1.5) &= 4\\ \therefore\ x-1.5 &\doteqdot 1.326+k2\pi\\ \therefore\ x &\doteqdot 2.83+k\pi\end{aligned}$$

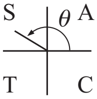
b **i**
$$\begin{aligned}\tan(x+\tfrac{\pi}{6}) &= -\sqrt{3}\\ \therefore\ x+\tfrac{\pi}{6} &= \tfrac{2\pi}{3}+k\pi\\ \therefore\ x &= \tfrac{\pi}{2}+k\pi\end{aligned}$$

ii
$$\begin{aligned}\tan 2x &= -\sqrt{3}\\ \therefore\ 2x &= \tfrac{2\pi}{3}+k\pi\\ \therefore\ x &= \tfrac{\pi}{3}+\tfrac{k\pi}{2}\end{aligned}$$

iii
$$\begin{aligned}\tan^2 x-3 &= 0\\ \therefore\ \tan x &= \pm\sqrt{3}\\ \therefore\ x &= \left.\begin{matrix}\tfrac{\pi}{3}\\ \tfrac{2\pi}{3}\end{matrix}\right\}+k\pi\end{aligned}$$

c
$$\begin{aligned}3\tan(x-1.2) &= -2\\ \therefore\ \tan(x-1.2) &= -\tfrac{2}{3}\\ \therefore\ x-1.2 &\doteqdot -0.588+k\pi\\ \therefore\ x &\doteqdot 0.612+k\pi\end{aligned}$$

7
$$\begin{aligned}\tan\theta &= -\tfrac{2}{3},\quad \tfrac{\pi}{2}<\theta<\pi\\ \therefore\ \frac{\sin\theta}{\cos\theta} &= -\frac{2}{3}\\ \therefore\ \sin\theta &= -2k,\quad \cos\theta = 3k\\ \text{but}\quad \cos^2\theta+\sin^2\theta &= 1\\ \therefore\ 9k^2+4k^2 &= 1\\ \therefore\ 13k^2 &= 1\\ \therefore\ k &= \pm\tfrac{1}{\sqrt{13}}\end{aligned}$$

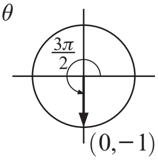


But in Q2,
$$\begin{aligned}\sin\theta &> 0,\ \cos\theta<0\\ \therefore\ k &= -\tfrac{1}{\sqrt{13}}\\ \therefore\ \sin\theta &= \tfrac{2}{\sqrt{13}},\quad \cos\theta=-\tfrac{3}{\sqrt{13}}\end{aligned}$$

8
$$\begin{aligned}\frac{\sin 2\alpha-\sin\alpha}{\cos 2\alpha-\cos\alpha+1} &= \frac{2\sin\alpha\cos\alpha-\sin\alpha}{2\cos^2\alpha-1-\cos\alpha+1}\\ &= \frac{\sin\alpha\cancel{(2\cos\alpha-1)}^{\,1}}{\cos\alpha\cancel{(2\cos\alpha-1)}_{\,1}}\\ &= \frac{\sin\alpha}{\cos\alpha}\\ &= \tan\alpha\end{aligned}$$

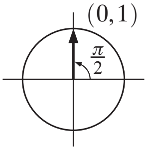
9 **a**

$$\cos\left(\tfrac{3\pi}{2}-\theta\right)$$
$$= \cos\left(\tfrac{3\pi}{2}\right)\cos\theta + \sin\tfrac{3\pi}{2}\sin\theta$$
$$= (0)\cos\theta + (-1)\sin\theta$$
$$= -\sin\theta$$



b

$$\sin\left(\theta+\tfrac{\pi}{2}\right)$$
$$= \sin\theta\cos\left(\tfrac{\pi}{2}\right) + \cos\theta\sin\left(\tfrac{\pi}{2}\right)$$
$$= \sin\theta(0) + \cos\theta(1)$$
$$= \cos\theta$$



10 **a**

$$\tan(\alpha+\beta)$$
$$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$
$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

b In **a** let $\beta=\alpha$

$\therefore \quad \tan(\alpha+\alpha)$

$$= \frac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}$$

i.e., $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

c If $\tan 2\alpha = 3$, then

$$\frac{2\tan\alpha}{1-\tan^2\alpha} = 3$$
$$\therefore \quad 2\tan\alpha = 3-3\tan^2\alpha$$
$$\therefore \quad 3\tan^2\alpha+2\tan\alpha-3=0$$
$$\therefore \quad \tan\alpha = \frac{-2\pm\sqrt{40}}{6}$$
$$\therefore \quad \tan\alpha = \frac{-1\pm\sqrt{10}}{3}$$

REVIEW SET 13D

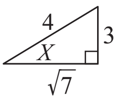
1 **a**

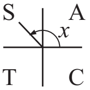
$$\sqrt{2}\cos\left(\theta+\tfrac{\pi}{4}\right)$$
$$= \sqrt{2}\left[\cos\theta\cos\left(\tfrac{\pi}{4}\right)-\sin\theta\sin\left(\tfrac{\pi}{4}\right)\right]$$
$$= \sqrt{2}\left[\cos\theta\times\tfrac{1}{\sqrt{2}}-\sin\theta\times\tfrac{1}{\sqrt{2}}\right]$$
$$= \cos\theta-\sin\theta$$

b

$$\cos\alpha\cos(\beta-\alpha)-\sin\alpha\sin(\beta-\alpha)$$
$$= \cos\left[\alpha+(\beta-\alpha)\right]$$
$$= \cos\left[\alpha+\beta-\alpha\right]$$
$$= \cos\beta$$

2 **a** $\sin x = \frac{3}{4}$





$\therefore \quad \cos x = -\frac{\sqrt{7}}{4}$

b

$$\sin 2x$$
$$= 2\sin x\cos x$$
$$= 2\left(\tfrac{3}{4}\right)\left(-\tfrac{\sqrt{7}}{4}\right)$$
$$= \tfrac{-3\sqrt{7}}{8}$$

c

$$\cos 2x$$
$$= 1-2\sin^2 x$$
$$= 1-2\left(\tfrac{9}{16}\right)$$
$$= 1-\tfrac{9}{8}$$
$$= -\tfrac{1}{8}$$

d

$$\tan 2x$$
$$= \frac{\sin 2x}{\cos 2x}$$
$$= -\tfrac{3\sqrt{7}}{8} \div -\tfrac{1}{8}$$
$$= 3\sqrt{7}$$

3

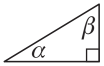
$$\cos 2\theta = 1-2\sin^2\theta$$
$$\therefore \quad \cos\left(\tfrac{\pi}{4}\right) = 1-2\sin^2\left(\tfrac{\pi}{8}\right) \quad \{\text{letting } \theta=\tfrac{\pi}{8}\}$$
$$\therefore \quad \tfrac{1}{\sqrt{2}} = 1-2\sin^2\left(\tfrac{\pi}{8}\right)$$
$$\therefore \quad 2\sin^2\left(\tfrac{\pi}{8}\right) = 1-\tfrac{1}{\sqrt{2}} = \tfrac{\sqrt{2}-1}{\sqrt{2}}$$
$$\therefore \quad \sin^2\left(\tfrac{\pi}{8}\right) = \tfrac{\sqrt{2}-1}{2\sqrt{2}}$$
$$\therefore \quad \sin^2\left(\tfrac{\pi}{8}\right) = \left(\tfrac{\sqrt{2}-1}{2\sqrt{2}}\right)\times\tfrac{\sqrt{2}}{\sqrt{2}}$$
$$\therefore \quad \sin^2\left(\tfrac{\pi}{8}\right) = \tfrac{2-\sqrt{2}}{4}$$
$$\therefore \quad \sin^2\left(\tfrac{\pi}{8}\right) = \pm\tfrac{\sqrt{2-\sqrt{2}}}{2}$$

But $\sin\left(\frac{\pi}{8}\right)$ is positive as $\frac{\pi}{8}$ is in quad 1.

$$\therefore \quad \sin\left(\tfrac{\pi}{8}\right) = \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

4

$$\alpha+\beta = \tfrac{\pi}{2} \quad \{\text{angles of a } \Delta\}$$
$$\therefore \quad \beta = \tfrac{\pi}{2}-\alpha$$



So, $\sin 2\beta = \sin(\pi-2\alpha)$

$$= \sin\pi\cos 2\alpha-\cos\pi\sin 2\alpha$$
$$= (0)\cos 2\alpha-(-1)\sin 2\alpha$$
$$= \sin 2\alpha$$

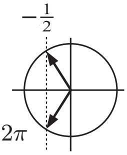
5 **a**
$$\begin{aligned}&(\sin\theta+\cos\theta)^2\\ &=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &=\left[\sin^2\theta+\cos^2\theta\right]+2\sin\theta\cos\theta\\ &=1+\sin 2\theta\end{aligned}$$

b
$$\begin{aligned}\csc(2x)+\cot(2x)&=\frac{1}{\sin 2x}+\frac{\cos 2x}{\sin 2x}\\ &=\frac{1+\cos 2x}{\sin 2x}\\ &=\frac{\cancel{1}+2\cos^2 x\cancel{-1}}{2\sin x\cos x}\\ &=\frac{\cancel{2}\cos x\cancel{\cos x}}{\cancel{2}\sin x\cancel{\cos x}}\\ &=\cot x\end{aligned}$$

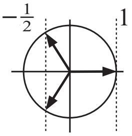
6 **a**
$$\begin{aligned}&\sin 3\theta\\ &=\sin(2\theta+\theta)\\ &=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta\\ &=2\sin\theta\cos\theta\times\cos\theta+\left[1-2\sin^2\theta\right]\sin\theta\\ &=2\sin\theta\cos^2\theta+\sin\theta-2\sin^3\theta\\ &=2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta\\ &=2\sin\theta-2\sin^3\theta+\sin\theta-2\sin^3\theta\\ &=3\sin\theta-4\sin^3\theta\end{aligned}$$

b
$$\begin{aligned}&\cos 3\theta\\ &=\cos(2\theta+\theta)\\ &=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta\\ &=\left[2\cos^2\theta-1\right]\cos\theta-2\sin\theta\cos\theta\sin\theta\\ &=[2\cos^2\theta-1]\cos\theta-2\cos\theta\sin^2\theta\\ &=2\cos^3\theta-\cos\theta-2\cos\theta\left[1-\cos^2\theta\right]\\ &=2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta\\ &=4\cos^3\theta-3\cos\theta\end{aligned}$$
$\therefore\quad a=4$ and $b=-3$

7 **a**
$$\begin{aligned}2\cos(2x)+1&=0\\ \therefore\quad\cos(2x)&=-\tfrac{1}{2}\\ \therefore\quad 2x&=\left.\begin{matrix}\frac{2\pi}{3}\\ \frac{4\pi}{3}\end{matrix}\right\}+k2\pi\\ \therefore\quad x&=\left.\begin{matrix}\frac{\pi}{3}\\ \frac{2\pi}{3}\end{matrix}\right\}+k\pi\\ \therefore\quad x&=\tfrac{\pi}{3},\ \tfrac{4\pi}{3},\ \tfrac{2\pi}{3},\ \tfrac{5\pi}{3}\end{aligned}$$



b
$$\begin{aligned}\cos 2x&=\cos x\\ \therefore\quad 2\cos^2 x-1&=\cos x\\ \therefore\quad 2\cos^2 x-\cos x-1&=0\\ \therefore\quad(2\cos x+1)(\cos x-1)&=0\\ \therefore\quad\cos x&=-\tfrac{1}{2}\text{ or }1\\ \therefore\quad x&=0,\ \tfrac{2\pi}{3},\ \tfrac{4\pi}{3},\ 2\pi\end{aligned}$$



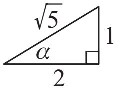
8 **a**
$$\begin{aligned}\sin 2\theta&=2\sin\theta\cos\theta\\ &=2\left(\frac{b}{c}\right)\left(\frac{a}{c}\right)\\ &=\frac{2ab}{c^2}\end{aligned}$$

b
$$\begin{aligned}\cos 2\theta&=\cos^2\theta-\sin^2\theta\\ &=\left(\frac{a}{c}\right)^2-\left(\frac{b}{c}\right)^2\\ &=\frac{a^2-b^2}{c^2}\end{aligned}$$

9
$$\begin{aligned}\tan 2\alpha&=\tfrac{4}{3},\quad 0<\alpha<\tfrac{\pi}{2}\\ \therefore\quad\frac{2\tan\alpha}{1-\tan^2\alpha}&=\tfrac{4}{3}\\ \therefore\quad 6\tan\alpha&=4-4\tan^2\alpha\\ \therefore\quad 4\tan^2\alpha+6\tan\alpha-4&=0\\ \therefore\quad 2\tan^2\alpha+3\tan\alpha-2&=0\\ \therefore\quad(2\tan\alpha-1)(\tan\alpha+2)&=0\\ \therefore\quad\tan\alpha&=\tfrac{1}{2}\text{ or }-2\end{aligned}$$

But α is in quad 1.

$\therefore\quad\tan\alpha=\frac{1}{2}$



$\therefore\quad\sin\alpha=\frac{1}{\sqrt{5}}$

10 **a** By the sine rule $\dfrac{\sin 2\alpha}{5}=\dfrac{\sin\alpha}{3}$
$$\begin{aligned}\therefore\quad\frac{2\sin\alpha\cos\alpha}{\sin\alpha}&=\tfrac{5}{3}\\ \therefore\quad 2\cos\alpha&=\tfrac{5}{3}\quad\text{p.v. }\sin\alpha\neq 0\text{ and this is so.}\\ \therefore\quad\cos\alpha&=\tfrac{5}{6}\end{aligned}$$

b Using the cosine rule
$$\begin{aligned}3^2&=x^2+5^2-2\times x\times 5\times\cos\alpha\\ \therefore\quad 9&=x^2+25-10x\left(\tfrac{5}{6}\right)\\ \therefore\quad x^2-\tfrac{25}{3}x+16&=0\\ \therefore\quad 3x^2-25x+48&=0\end{aligned}$$

c
$$\begin{aligned}(3x-16)(x-3)&=0\\ \therefore\quad x&=\tfrac{16}{3}\text{ or }3\end{aligned}$$

Chapter 14

MATRICES

EXERCISE 14A

1 **a** 1 row and 4 columns $\therefore$ 1×4 **b** 2 rows and 1 column $\therefore$ 2×1

c 2 rows and 2 columns $\therefore$ 2×2 **d** 3 rows and 3 columns $\therefore$ 3×3

2 $\begin{bmatrix} 2 & 1 & 6 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1.95 \\ 2.35 \\ 0.15 \\ 0.95 \end{bmatrix}$ **c** $(2 \times 1.95) + (1 \times 2.35)$ $+ (6 \times 0.15) + (1 \times 0.95)$ represents the total cost of the groceries.

3

$$\begin{array}{cc} \begin{array}{ccc} 200\text{ g} & 300\text{ g} & 500\text{ g} \end{array} & \\ \begin{bmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{bmatrix} & \begin{array}{l} \text{week 1} \\ \text{week 2} \\ \text{week 3} \\ \text{week 4} \end{array} \end{array}$$

4

$$\begin{array}{cc} \begin{array}{cccc} \text{pies} & \text{pasties} & \text{rolls} & \text{buns} \end{array} & \\ \begin{bmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{bmatrix} & \begin{array}{c} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \\ \text{Monday} \end{array} \end{array}$$

EXERCISE 14B

1 **a** $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}$$

b $\mathbf{A} + \mathbf{B} + \mathbf{C}$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ -1 & 1 \end{bmatrix}$$

c $\mathbf{B} + \mathbf{C}$

$$= \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ -6 & -1 \end{bmatrix}$$

d $\mathbf{C} + \mathbf{B} - \mathbf{A}$

$$= \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -11 & -3 \end{bmatrix}$$

2 **a** $\mathbf{P} + \mathbf{Q}$

$$= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{bmatrix}$$

b $\mathbf{P} - \mathbf{Q}$

$$= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} - \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{bmatrix}$$

c $$\mathbf{Q} - \mathbf{P} = \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{bmatrix}$$

3 **a**

$$\begin{array}{cc} \text{Friday} & \text{Saturday} \\ \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} & \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} \end{array}$$

b Total for Friday and Saturday $= \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} + \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} = \begin{bmatrix} 187 \\ 229 \\ 101 \end{bmatrix}$

4 **a** **i** Cost price

$$\begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix}$$

ii Selling price

$$\begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix}$$

b In order to find David's profit/loss matrix we subtract the cost price matrix from the selling price matrix.

c Profit/Loss matrix $= \begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix} - \begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.90 \\ 0.41 \\ -0.28 \\ -0.05 \end{bmatrix}$

5 **a**

$$\begin{array}{cc} \text{Lou} & \text{Rose} \\ \end{array}$$
$$\begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix}$$

b

$$\begin{array}{cc} \text{Lou} & \text{Rose} \\ \end{array}$$
$$\begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix}$$

c Total sales for November and December

$$= \begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix} + \begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix} = \begin{bmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{bmatrix}$$

6 **a** $\begin{bmatrix} x & x^2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} y & 4 \\ 3 & y+1 \end{bmatrix}$

Equating corresponding elements:

$x = y,\ \ x^2 = 4$ and $-1 = y+1$

$\therefore\ \ y = -2$ and $x = \pm 2$

But $x = y \ \ \therefore\ \ x = y = -2$

b $\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} -y & x \\ x & -y \end{bmatrix}$

Equating corresponding elements:

$$\left.\begin{array}{l} x = -y \\ y = x \end{array}\right\} \ \therefore\ \ y = 0,\ x = 0$$

7 **a**

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2+(-1) & 1+2 \\ 3+2 & -1+3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{B} + \mathbf{A} &= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1+2 & 2+1 \\ 2+3 & 3+(-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \end{aligned}$$

b $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ because addition of numbers is commutative.

8 **a**

$$\begin{aligned} (\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \left(\begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \right) + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A} + (\mathbf{B} + \mathbf{C}) &= \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \left(\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

b Let $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\begin{aligned} \therefore \quad & (\mathbf{A}+\mathbf{B})+\mathbf{C} \\ &= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right)+\begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}+\begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & \mathbf{A}+(\mathbf{B}+\mathbf{C}) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}+\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}+\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} \\ &= \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix} \\ &= (\mathbf{A}+\mathbf{B})+\mathbf{C} \end{aligned}$$

EXERCISE 14C

1 **a** $2\mathbf{B}$

$$= 2\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 48 & 12 \end{bmatrix}$$

b $\frac{1}{3}\mathbf{B}$

$$= \tfrac{1}{3}\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix}$$

c $\frac{1}{12}\mathbf{B}$

$$= \tfrac{1}{12}\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$$

d $-\frac{1}{2}\mathbf{B}$

$$= -\tfrac{1}{2}\begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -12 & -3 \end{bmatrix}$$

2 **a** $\mathbf{A}+\mathbf{B}$

$$\begin{aligned} &= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2+1 & 3+2 & 5+1 \\ 1+1 & 6+2 & 4+3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{bmatrix} \end{aligned}$$

b $\mathbf{A}-\mathbf{B}$

$$\begin{aligned} &= \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix}-\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2-1 & 3-2 & 5-1 \\ 1-1 & 6-2 & 4-3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix} \end{aligned}$$

c $2\mathbf{A}+\mathbf{B}$

$$\begin{aligned} &= 2\begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 & 10 \\ 2 & 12 & 8 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+1 & 6+2 & 10+1 \\ 2+1 & 12+2 & 8+3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{bmatrix} \end{aligned}$$

d $3\mathbf{A}-\mathbf{B}$

$$\begin{aligned} &= 3\begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix}-\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 9 & 15 \\ 3 & 18 & 12 \end{bmatrix}-\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6-1 & 9-2 & 15-1 \\ 3-1 & 18-2 & 12-3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{bmatrix} \end{aligned}$$

3 **a**

Increase of 15% $= 1.15\begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{bmatrix}$ rounded to the nearest whole number.

b

Decrease of 15% $= 0.85\begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{bmatrix}$ rounded to the nearest whole number.

4 **a**

$$\begin{array}{ccl} \text{Weekdays} & \text{Weekends} & \\ \begin{bmatrix}75\\27\\102\end{bmatrix} & \begin{bmatrix}136\\43\\129\end{bmatrix} & \begin{array}{l}\text{VHS}\\ \text{DVD}\\ \text{games}\end{array}\end{array}$$

b

$$\begin{bmatrix}75\\27\\102\end{bmatrix}+\begin{bmatrix}136\\43\\129\end{bmatrix}=\begin{bmatrix}211\\70\\231\end{bmatrix}$$

c The sum matrix of **b** represents total weekly average hirings.

5 The matrix is $12\mathbf{F} = 12\begin{bmatrix}1\\4\\2\\1\end{bmatrix}=\begin{bmatrix}12\\48\\24\\12\end{bmatrix}$

EXERCISE 14D

1 **a** $\mathbf{A}+2\mathbf{A}=3\mathbf{A}$ **b** $3\mathbf{B}-3\mathbf{B}=\mathbf{O}$ **c** $\mathbf{C}-2\mathbf{C}=-\mathbf{C}$

d $-\mathbf{B}+\mathbf{B}=\mathbf{O}$ **e** $2(\mathbf{A}+\mathbf{B})=2\mathbf{A}+2\mathbf{B}$ **f** $-(\mathbf{A}+\mathbf{B})=-\mathbf{A}-\mathbf{B}$

g
$$\begin{aligned}&-(2\mathbf{A}-\mathbf{C})\\&=-2\mathbf{A}+\mathbf{C}\end{aligned}$$

h
$$\begin{aligned}&3\mathbf{A}-(\mathbf{B}-\mathbf{A})\\&=3\mathbf{A}-\mathbf{B}+\mathbf{A}\\&=4\mathbf{A}-\mathbf{B}\end{aligned}$$

i
$$\begin{aligned}&\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})\\&=\mathbf{A}+2\mathbf{B}-\mathbf{A}+\mathbf{B}\\&=3\mathbf{B}\end{aligned}$$

2 **a**
$$\begin{aligned}\text{if}\quad \mathbf{X}+\mathbf{B}&=\mathbf{A}\\ \text{then}\quad \mathbf{X}+\mathbf{B}+(-\mathbf{B})&=\mathbf{A}+(-\mathbf{B})\\ \therefore\quad \mathbf{X}+\mathbf{O}&=\mathbf{A}-\mathbf{B}\\ \therefore\quad \mathbf{X}&=\mathbf{A}-\mathbf{B}\end{aligned}$$

b
$$\begin{aligned}\text{if}\quad \mathbf{B}+\mathbf{X}&=\mathbf{C}\\ \text{then}\quad \mathbf{B}+\mathbf{X}+(-\mathbf{B})&=\mathbf{C}+(-\mathbf{B})\\ \therefore\quad \mathbf{O}+\mathbf{X}&=\mathbf{C}-\mathbf{B}\\ \therefore\quad \mathbf{X}&=\mathbf{C}-\mathbf{B}\end{aligned}$$

c
$$\begin{aligned}\text{if}\quad 4\mathbf{B}+\mathbf{X}&=2\mathbf{C}\\ \text{then}\quad 4\mathbf{B}+\mathbf{X}+(-4\mathbf{B})&=2\mathbf{C}+(-4\mathbf{B})\\ \therefore\quad \mathbf{O}+\mathbf{X}&=2\mathbf{C}-4\mathbf{B}\\ \therefore\quad \mathbf{X}&=2\mathbf{C}-4\mathbf{B}\end{aligned}$$

d
$$\begin{aligned}\text{if}\quad 2\mathbf{X}&=\mathbf{A}\\ \text{then}\quad \tfrac{1}{2}(2\mathbf{X})&=\tfrac{1}{2}\mathbf{A}\\ \therefore\quad 1\mathbf{X}&=\tfrac{1}{2}\mathbf{A}\quad\text{i.e.,}\quad \mathbf{X}=\tfrac{1}{2}\mathbf{A}\end{aligned}$$

e
$$\begin{aligned}\text{if}\quad 3\mathbf{X}&=\mathbf{B}\\ \text{then}\quad \tfrac{1}{3}(3\mathbf{X})&=\tfrac{1}{3}\mathbf{B}\\ \therefore\quad 1\mathbf{X}&=\tfrac{1}{3}\mathbf{B}\quad\text{i.e.,}\quad \mathbf{X}=\tfrac{1}{3}\mathbf{B}\end{aligned}$$

f
$$\begin{aligned}\text{if}\quad \mathbf{A}-\mathbf{X}&=\mathbf{B}\\ \text{then}\quad \mathbf{A}-\mathbf{X}+\mathbf{X}&=\mathbf{B}+\mathbf{X}\\ \therefore\quad \mathbf{A}+\mathbf{O}&=\mathbf{B}+\mathbf{X}\\ \therefore\quad \mathbf{A}&=\mathbf{B}+\mathbf{X}\\ \text{and}\quad \mathbf{A}+(-\mathbf{B})&=\mathbf{B}+\mathbf{X}+(-\mathbf{B})\\ \therefore\quad \mathbf{A}-\mathbf{B}&=\mathbf{X}+\mathbf{O}\\ \therefore\quad \mathbf{A}-\mathbf{B}&=\mathbf{X}\\ \therefore\quad \mathbf{X}&=\mathbf{A}-\mathbf{B}\end{aligned}$$

g
$$\begin{aligned}\text{if}\quad \tfrac{1}{2}\mathbf{X}&=\mathbf{C}\\ \text{then}\quad 2(\tfrac{1}{2}\mathbf{X})&=2\mathbf{C}\\ \therefore\quad 1\mathbf{X}&=2\mathbf{C}\quad\text{i.e.,}\quad \mathbf{X}=2\mathbf{C}\end{aligned}$$

h
$$\begin{aligned}\text{if}\quad 2(\mathbf{X}+\mathbf{A})&=\mathbf{B}\\ \text{then}\quad \tfrac{1}{2}\left[2(\mathbf{X}-\mathbf{A})\right]&=\tfrac{1}{2}\mathbf{B}\\ \therefore\quad 1(\mathbf{X}+\mathbf{A})&=\tfrac{1}{2}\mathbf{B}\\ \therefore\quad \mathbf{X}+\mathbf{A}&=\tfrac{1}{2}\mathbf{B}\\ \therefore\quad \mathbf{X}+\mathbf{A}+(-\mathbf{A})&=\tfrac{1}{2}\mathbf{B}+(-\mathbf{A})\\ \therefore\quad \mathbf{X}+\mathbf{O}&=\tfrac{1}{2}\mathbf{B}-\mathbf{A}\\ \therefore\quad \mathbf{X}&=\tfrac{1}{2}\mathbf{B}-\mathbf{A}\end{aligned}$$

i
$$\begin{aligned}\text{if}\quad \mathbf{A}-4\mathbf{X}&=\mathbf{C}\\ \text{then}\quad \mathbf{A}-4\mathbf{X}+4\mathbf{X}&=\mathbf{C}+4\mathbf{X}\\ \therefore\quad \mathbf{A}+\mathbf{O}&=\mathbf{C}+4\mathbf{X}\\ \therefore\quad \mathbf{A}&=\mathbf{C}+4\mathbf{X}\\ \text{and}\quad \mathbf{A}-\mathbf{C}&=4\mathbf{X}\\ \therefore\quad \tfrac{1}{4}(\mathbf{A}-\mathbf{C})&=\tfrac{1}{4}(4\mathbf{X})\\ \therefore\quad \mathbf{X}&=\tfrac{1}{4}(\mathbf{A}-\mathbf{C})\end{aligned}$$

3 **a**
$$\begin{aligned}\text{if}\quad \tfrac{1}{3}\mathbf{X}&=\mathbf{M}\\ \text{then}\quad 3\left(\tfrac{1}{3}\mathbf{X}\right)&=3\mathbf{M}\\ \therefore\quad \mathbf{X}=3\mathbf{M}&=3\begin{bmatrix}1&2\\3&6\end{bmatrix}\\ &=\begin{bmatrix}3&6\\9&18\end{bmatrix}\end{aligned}$$

b
$$\begin{aligned}&\text{if}\quad 4\mathbf{X}=\mathbf{N}\quad\text{then}\quad \tfrac{1}{4}(4\mathbf{X})=\tfrac{1}{4}\mathbf{N}\\ &\therefore\quad \mathbf{X}=\tfrac{1}{4}\mathbf{N}\\ &\therefore\quad \mathbf{X}=\tfrac{1}{4}\begin{bmatrix}2&-1\\3&5\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{4}\\ \frac{3}{4}&\frac{5}{4}\end{bmatrix}\end{aligned}$$

c

$$\begin{aligned}\text{if}\quad \mathbf{A}-2\mathbf{X} &= 3\mathbf{B}\\ \text{then}\quad \mathbf{A}-2\mathbf{X}+2\mathbf{X} &= 3\mathbf{B}+2\mathbf{X}\\ \therefore\quad \mathbf{A} &= 3\mathbf{B}-2\mathbf{X}\\ \therefore\quad \mathbf{A}+(-3\mathbf{B}) &= 3\mathbf{B}-2\mathbf{X}+(-3\mathbf{B})\\ \therefore\quad \mathbf{A}-3\mathbf{B} &= 2\mathbf{X}\\ \therefore\quad \tfrac{1}{2}(\mathbf{A}-3\mathbf{B}) &= \tfrac{1}{2}(2\mathbf{X})\\ \therefore\quad \tfrac{1}{2}(\mathbf{A}-3\mathbf{B}) &= 1\mathbf{X}\end{aligned}$$

$$\begin{aligned}\therefore\quad \mathbf{X} &= \tfrac{1}{2}(\mathbf{A}-3\mathbf{B})\\ &= \tfrac{1}{2}\left(\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}-3\begin{bmatrix}1 & 4\\ -1 & 1\end{bmatrix}\right)\\ &= \tfrac{1}{2}\begin{bmatrix}-2 & -12\\ 2 & -1\end{bmatrix}\\ &= \begin{bmatrix}-1 & -6\\ 1 & -\frac{1}{2}\end{bmatrix}\end{aligned}$$

EXERCISE 14E.1

1 **a**

$$\begin{aligned}&\begin{bmatrix}3 & -1\end{bmatrix}\begin{bmatrix}5\\ 4\end{bmatrix}\\ &= [3\times 5+(-1)\times 4]\\ &= [15-4]\\ &= [11]\end{aligned}$$

b

$$\begin{aligned}&\begin{bmatrix}1 & 3 & 2\end{bmatrix}\begin{bmatrix}5\\ 1\\ 7\end{bmatrix}\\ &= [1\times 5+3\times 1+2\times 7]\\ &= [5+3+14]\\ &= [22]\end{aligned}$$

c

$$\begin{aligned}\begin{bmatrix}6 & -1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\\ 4\end{bmatrix} &= [6\times 1+-1\times 0+2\times -1+3\times 4]\\ &= [6+0-2+12]\\ &= [16]\end{aligned}$$

2

$$\begin{bmatrix}w & x & y & z\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}w+x+y+z\end{bmatrix}$$

$\therefore$ $\frac{1}{4}(w+x+y+z)$, which is the average of w, x, y and z, can be represented as $\begin{bmatrix}w & x & y & z\end{bmatrix}\begin{bmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}$

3 **a** $\mathbf{Q}=\begin{bmatrix}4\\ 3\\ 2\end{bmatrix}$ $\quad\mathbf{P}=\begin{bmatrix}27 & 35 & 39\end{bmatrix}$

b

$$\begin{aligned}\text{total cost} &= \mathbf{PQ}=\begin{bmatrix}27 & 35 & 39\end{bmatrix}\begin{bmatrix}4\\ 3\\ 2\end{bmatrix}\\ &= [27\times 4+35\times 3+39\times 2]\\ &= [291]\quad\therefore\ \text{total cost is \$291}\end{aligned}$$

4 **a** $\mathbf{P}=\begin{bmatrix}10 & 6 & 3 & 1\end{bmatrix}$ $\quad\mathbf{N}=\begin{bmatrix}3\\ 2\\ 4\\ 2\end{bmatrix}$

b

$$\begin{aligned}\text{total points} &= \mathbf{PN}=\begin{bmatrix}10 & 6 & 3 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\\ 4\\ 2\end{bmatrix}\\ &= [10\times 3+6\times 2+3\times 4+1\times 2]\\ &= [30+12+12+2]\\ &= [56]\end{aligned}$$

So, the number of points awarded is 56.

EXERCISE 14E.2

1 $\mathbf{A}=\begin{bmatrix}4 & 2 & 1\end{bmatrix}$ which is 1 row × 3 columns

$\mathbf{B}=\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 0\end{bmatrix}$ which is 2 rows × 3 columns

AB cannot be found because the number of columns in **A** does not equal the number of rows in **B**.

2 $\mathbf{A}$ is $2\times n$ and $\mathbf{B}$ is $m\times 3$.

a We can find $\mathbf{AB}$ if the number of columns in $\mathbf{A}$ equals the number of rows in $\mathbf{B}$, i.e., $n=m$.

b If $\mathbf{AB}$ can be found its order is 2×3.

c $\mathbf{BA}$ cannot be found because the number of columns in $\mathbf{B}$ does not equal the number of rows in $\mathbf{A}$.

3 **a** $\mathbf{B}$ is $1\times \underbracket{2 \text{ and } \mathbf{A} \text{ is } 2}\times 2$

$\therefore$ $\mathbf{BA}$ is 1×2

$$\begin{aligned}\mathbf{BA} &= \begin{bmatrix} 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\\ &= \begin{bmatrix} 5\times 2+6\times 3 & 5\times 1+6\times 4 \end{bmatrix}\\ &= \begin{bmatrix} 10+18 & 5+24 \end{bmatrix}\\ &= \begin{bmatrix} 28 & 29 \end{bmatrix}\end{aligned}$$

b **i** $\mathbf{A}$ is $1\times \underbracket{3 \text{ and } \mathbf{B} \text{ is } 3}\times 1$, $\therefore$ $\mathbf{AB}$ is 1×1 and

$$\mathbf{AB} = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = [2\times 1+0\times 4+3\times 2] = [2+0+6] = [8]$$

ii $\mathbf{B}$ is $3\times \underbracket{1 \text{ and } \mathbf{A} \text{ is } 1}\times 3$, $\therefore$ $\mathbf{BA}$ is 3×3 and

$$\mathbf{BA} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1\times 2 & 1\times 0 & 1\times 3 \\ 4\times 2 & 4\times 0 & 4\times 3 \\ 2\times 2 & 2\times 0 & 2\times 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{bmatrix}$$

4 **a**

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{is } 1\times \underbracket{3 \text{ by } 3}\times 3 \quad \therefore \quad \text{resultant matrix is } 1\times 3$$

$$\begin{aligned}&= \begin{bmatrix} 1\times 2+2\times 0+1\times 1 & 1\times 3+2\times 1+1\times 0 & 1\times 1+2\times 0+1\times 2 \end{bmatrix}\\ &= \begin{bmatrix} 2+0+1 & 3+2+0 & 1+0+2 \end{bmatrix}\\ &= \begin{bmatrix} 3 & 5 & 3 \end{bmatrix}\end{aligned}$$

b

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{is } 3\times \underbracket{3 \text{ by } 3}\times 1 \quad \therefore \quad \text{resultant matrix is } 3\times 1$$

$$= \begin{bmatrix} 1\times 2+0\times 3+(-1)\times 4 \\ (-1)\times 2+1\times 3+0\times 4 \\ 0\times 2+(-1)\times 3+1\times 4 \end{bmatrix} = \begin{bmatrix} 2+0-4 \\ -2+3+0 \\ 0-3+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

5 **a**

$$\mathbf{C} = \begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix} \qquad \mathbf{N} = \begin{array}{cc} \text{adults} & \text{children} \\ \left[\begin{matrix} 2375 \\ 2502 \end{matrix}\right. & \left.\begin{matrix} 5156 \\ 3612 \end{matrix}\right] \end{array} \begin{matrix} \\ \text{first day} \\ \text{second day} \end{matrix}$$

b $\mathbf{N}$ is 2×2 and $\mathbf{C}$ is 2×1 $\therefore$ $\mathbf{NC}$ is 2×1

$$\begin{aligned}\mathbf{NC} &= \begin{bmatrix} 2375 & 5156 \\ 2502 & 3612 \end{bmatrix}\begin{bmatrix} 12.5 \\ 9.5 \end{bmatrix} = \begin{bmatrix} 2375\times 12.5+5156\times 9.5 \\ 2502\times 12.5+3612\times 9.5 \end{bmatrix} = \begin{bmatrix} 29\,687.5+48\,982 \\ 31\,275+34\,314 \end{bmatrix}\\ &= \begin{bmatrix} 78\,669.5 \\ 65\,589 \end{bmatrix} \begin{matrix} \text{income from adults' rides} \\ \text{income from children's rides} \end{matrix}\end{aligned}$$

c Total income $= \$78\,669.50 + \$65\,589 = \$144\,258.50$

6 **a**

$$\begin{array}{cc} & \text{me} \quad \text{friend} & \\ \mathbf{R} = & \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} & \begin{array}{c}\text{hammers}\\ \text{screwdrivers}\\ \text{cans of paint}\end{array}\end{array}$$

b

$$\mathbf{P} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \begin{array}{l}\text{store A}\\ \text{store B}\end{array}$$

c $\mathbf{P}$ is 2×3 and $\mathbf{R}$ is 3×2, $\therefore$ $\mathbf{PR}$ is 2×2

$$\mathbf{PR} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7\times1+3\times1+19\times2 & 7\times1+3\times2+19\times3 \\ 6\times1+2\times1+22\times2 & 6\times1+2\times2+22\times3 \end{bmatrix}$$

$$\therefore \quad \mathbf{PR} = \begin{bmatrix} 7+3+38 & 7+6+57 \\ 6+2+44 & 6+4+66 \end{bmatrix} = \begin{bmatrix} 48 & 70 \\ 52 & 76 \end{bmatrix}$$

d My costs at Store A are \$48; my friend's costs at Store B are \$76.

e My costs at Store B are \$52. Therefore I should shop at Store A, which is cheaper.

EXERCISE 14F

1 **a** $\begin{bmatrix} 16 & 18 & 15 \\ 13 & 21 & 16 \\ 10 & 22 & 24 \end{bmatrix}$

b $\begin{bmatrix} 10 & 6 & -7 \\ 9 & 3 & 0 \\ 4 & -4 & -10 \end{bmatrix}$

c $\begin{bmatrix} 22 & 0 & 132 & 176 & 198 \\ 44 & 154 & 88 & 110 & 0 \\ 176 & 44 & 88 & 88 & 132 \end{bmatrix}$

d $\begin{bmatrix} 115 \\ 136 \\ 46 \\ 106 \end{bmatrix}$

2 **a**

$$\text{Numbers matrix } \mathbf{N} = \begin{array}{ccc} \text{nights} & \text{breakfasts} & \text{dinners} \\ \downarrow & \downarrow & \downarrow \\ \end{array}\begin{bmatrix} 3 & 3 & 2 \end{bmatrix}$$

b

$$\text{Prices matrix} \quad \mathbf{P} = \begin{array}{c} \begin{array}{ccc}\text{Bay View} & \text{Terrace} & \text{Staunton Star}\end{array} \\ \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} \end{array} \begin{array}{l}\text{room}\\ \text{breakfast}\\ \text{dinner}\end{array}$$

c Total prices for each venue = numbers matrix × prices matrix = $\mathbf{NP}$

$\mathbf{N}$ is 1×3 and $\mathbf{P}$ is 3×3, $\therefore$ $\mathbf{NP}$ is 1×3

$$\mathbf{NP} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ \$657 for Bay View, \$730 for Terrace, \$670 for Staunton Star.

d

$$\text{Total prices} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 369 & 420 & 385 \end{bmatrix}, \quad \{\text{using technology}\}$$

i.e., \$369 for Bay View, \$420 for Terrace, \$385 for Staunton Star.

e To include both scenarios we calculate

$$\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \\ 369 & 420 & 385 \end{bmatrix}, \quad \text{using technology}$$

3 Prices matrix $= \begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix}$ To find total income we calculate:

prices matrix × numbers matrix.

$$= \begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix} \begin{bmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{bmatrix} = \begin{bmatrix} 51\,145 \\ 60\,655 \\ 61\,575 \\ 51\,285 \end{bmatrix} \text{ using tech.}$$

$\therefore$ total income $= \$51\,145 + \$60\,655 + \$61\,575 + \$51\,285 = \$224\,660$

4 **a**

$$\text{Income matrix } \mathbf{I} = \begin{bmatrix} 125 & 195 & 225 \end{bmatrix}$$

$$\text{Cost matrix } \mathbf{C} = \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$$

$$\text{Numbers (bookings) matrix } \mathbf{N} = \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{profit per day} &= (\text{income from room}) \times (\text{bookings per day}) \\ &\quad - (\text{maintenance cost per room}) \times (\text{bookings per day}) \\ &= \mathbf{IN} - \mathbf{CN} \\ &= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1185 & 800 & 1350 & 970 & 845 & 1130 & 845 \end{bmatrix}, \text{ using technology} \end{aligned}$$

$$\begin{aligned} \therefore \text{ profit for the week} &= \$1185 + \$800 + \$1350 + \$970 + \$845 + \$1130 + \$845 \\ &= \$7125 \end{aligned}$$

b If the hotel maintained every room every day we would need to calculate (income from room) × (bookings per day) − (maintenance costs per room) × (number of rooms)

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \times \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \times \begin{bmatrix} 20 & 20 & 20 & 20 & 20 & 20 & 20 \\ 15 & 15 & 15 & 15 & 15 & 15 & 15 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$\therefore$ using technology, profit per day $= \begin{bmatrix} -820 & -1840 & -455 & -1485 & -1725 & -920 & -1785 \end{bmatrix}$

$\therefore$ the profit per week would be $(-\$820) + ... + (-\$1785) = -\$9030$, i.e., a loss of \$9030.

c Profit per room matrix

$$\begin{aligned} &= \text{income per room matrix} - \text{cost per room matrix} \\ &= \mathbf{I} - \mathbf{C} \\ &= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \\ &= \begin{bmatrix} 40 & 75 & 95 \end{bmatrix} \end{aligned}$$

$\therefore$ the result in **a** can be obtained if we calculate:

$$\begin{bmatrix} 40 & 75 & 95 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

This checks using technology.

EXERCISE 14G

1 $$\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1+0 & 1+0 \\ -1+0 & 1+6 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 7 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 0+2 \\ 0+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 6 \end{bmatrix}$$

$\mathbf{AB} \neq \mathbf{BA}$ $\therefore$ in the general case **AB** does not necessarily equal **BA**.

2 $$\mathbf{AO} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$$

$$\mathbf{OA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O} \quad \therefore \quad \mathbf{AO} = \mathbf{OA} = \mathbf{O}$$

3 **a** Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, say

$$\begin{aligned} &\mathbf{A}(\mathbf{B}+\mathbf{C}) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3+4 & 4+4 \\ 9+8 & 12+8 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &\mathbf{AB}+\mathbf{AC} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 1+2 \\ 3+4 & 3+4 \end{bmatrix} + \begin{bmatrix} 2+2 & 3+2 \\ 6+4 & 9+4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 10 & 13 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix} \\ &= \mathbf{A}(\mathbf{B}+\mathbf{C}) \end{aligned}$$

b $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} = \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB}+\mathbf{AC} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} + \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} \\ &= \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix} = \mathbf{A}(\mathbf{B}+\mathbf{C}) \end{aligned}$$

c Using the matrices in **a**,

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6+3 & 9+3 \\ 14+7 & 21+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3+6 & 4+8 \\ 9+12 & 12+16 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} = (\mathbf{AB})\mathbf{C} \end{aligned}$$

d Using the matrices in **b**,

$$\begin{aligned}\mathbf{(AB)C} &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A(BC)} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left[\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right] \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{bmatrix} \\ &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix} = \mathbf{(AB)C}\end{aligned}$$

4 **a** If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore$ equating coefficients of corresponding elements: $w=1$, $y=0$, $x=0$, $z=1$

which checks with the coefficients in the second line. $\therefore$ $w=z=1$, $x=y=0$ i.e., $\mathbf{X}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b In **a** we showed that $\mathbf{AX}=\mathbf{A}$ where $\mathbf{X}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore$ $\mathbf{AI}=\mathbf{IA}=\mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ where $\mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5 **a**

$$\begin{aligned}\mathbf{A}^2 &= \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 4+3 & 2+(-2) \\ 6+(-6) & 3+4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\end{aligned}$$

b

$$\begin{aligned}\mathbf{A}^3 &= \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 25+(-2) & -5+(-4) \\ 10+8 & -2+16 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 23 & -9 \\ 18 & 14 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 115+(-18) & -23+(-36) \\ 90+28 & -18+56 \end{bmatrix} \\ &= \begin{bmatrix} 97 & -59 \\ 118 & 38 \end{bmatrix}\end{aligned}$$

6 **a** $\mathbf{A}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\therefore$ $\mathbf{A}^2$ is 3×2 by 3×2

$2\neq 3$ so $\mathbf{A}^2$ does not exist.

b We can square a matrix when the number of columns equals the number of rows, i.e., if it is a square matrix.

7 $\mathbf{I}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$ $\therefore$ $\mathbf{I}^3 = \mathbf{II}^2 = \mathbf{II} = \mathbf{I}^2 = \mathbf{I}$

8 **a**

$$\begin{aligned}&\mathbf{A(A+I)} \\ &= \mathbf{A}^2+\mathbf{AI} \\ &= \mathbf{A}^2+\mathbf{A}\end{aligned}$$

b

$$\begin{aligned}&\mathbf{(B+2I)B} \\ &= \mathbf{B}^2+2\mathbf{IB} \\ &= \mathbf{B}^2+2\mathbf{B}\end{aligned}$$

c

$$\begin{aligned}&\mathbf{A}(\mathbf{A}^2-2\mathbf{A}+\mathbf{I}) \\ &= \mathbf{A}^3-2\mathbf{A}^2+\mathbf{AI} \\ &= \mathbf{A}^3-2\mathbf{A}^2+\mathbf{A}\end{aligned}$$

d
$$\begin{aligned}&\mathbf{A}(\mathbf{A}^2+\mathbf{A}-2\mathbf{I})\\&=\mathbf{A}^3+\mathbf{A}^2-2\mathbf{AI}\\&=\mathbf{A}^3+\mathbf{A}^2-2\mathbf{A}\end{aligned}$$

e
$$\begin{aligned}&(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})\\&=(\mathbf{A}+\mathbf{B})\mathbf{C}+(\mathbf{A}+\mathbf{B})\mathbf{D}\\&=\mathbf{AC}+\mathbf{BC}+\mathbf{AD}+\mathbf{BD}\end{aligned}$$

f
$$\begin{aligned}&(\mathbf{A}+\mathbf{B})^2\\&=(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})\\&=(\mathbf{A}+\mathbf{B})\mathbf{A}+(\mathbf{A}+\mathbf{B})\mathbf{B}\\&=\mathbf{A}^2+\mathbf{BA}+\mathbf{AB}+\mathbf{B}^2\end{aligned}$$

g
$$\begin{aligned}&(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})\\&=(\mathbf{A}+\mathbf{B})\mathbf{A}-(\mathbf{A}+\mathbf{B})\mathbf{B}\\&=\mathbf{A}^2+\mathbf{BA}-\mathbf{AB}-\mathbf{B}^2\end{aligned}$$

h
$$\begin{aligned}&(\mathbf{A}+\mathbf{I})^2\\&=(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I})\\&=(\mathbf{A}+\mathbf{I})\mathbf{A}+(\mathbf{A}+\mathbf{I})\mathbf{I}\\&=\mathbf{A}^2+\mathbf{IA}+\mathbf{AI}+\mathbf{I}^2\\&=\mathbf{A}^2+\mathbf{A}+\mathbf{A}+\mathbf{I}\\&=\mathbf{A}^2+2\mathbf{A}+\mathbf{I}\end{aligned}$$

i
$$\begin{aligned}&(3\mathbf{I}-\mathbf{B})^2\\&=(3\mathbf{I}-\mathbf{B})(3\mathbf{I}-\mathbf{B})\\&=(3\mathbf{I}-\mathbf{B})3\mathbf{I}-(3\mathbf{I}-\mathbf{B})\mathbf{B}\\&=9\mathbf{I}^2-3\mathbf{BI}-3\mathbf{IB}+\mathbf{B}^2\\&=9\mathbf{I}-3\mathbf{B}-3\mathbf{B}+\mathbf{B}^2\\&=9\mathbf{I}-6\mathbf{B}+\mathbf{B}^2\end{aligned}$$

9 **a** $\mathbf{A}^2=2\mathbf{A}-\mathbf{I}$

$$\begin{aligned}\therefore\ \mathbf{A}^3&=\mathbf{A}\times\mathbf{A}^2\\&=\mathbf{A}(2\mathbf{A}-\mathbf{I})\\&=2\mathbf{A}^2-\mathbf{AI}\\&=2(2\mathbf{A}-\mathbf{I})-\mathbf{A}\\&=4\mathbf{A}-2\mathbf{I}-\mathbf{A}\\&=3\mathbf{A}-2\mathbf{I}\end{aligned}$$

and
$$\begin{aligned}\mathbf{A}^4&=\mathbf{A}\times\mathbf{A}^3\\&=\mathbf{A}(3\mathbf{A}-2\mathbf{I})\\&=3\mathbf{A}^2-2\mathbf{AI}\\&=3(2\mathbf{A}-\mathbf{I})-2\mathbf{A}\\&=6\mathbf{A}-3\mathbf{I}-2\mathbf{A}\\&=4\mathbf{A}-3\mathbf{I}\end{aligned}$$

b $\mathbf{B}^2=2\mathbf{I}-\mathbf{B}$

$$\begin{aligned}\therefore\ \mathbf{B}^3&=\mathbf{B}\times\mathbf{B}^2\\&=\mathbf{B}(2\mathbf{I}-\mathbf{B})\\&=2\mathbf{BI}-\mathbf{B}^2\\&=2\mathbf{B}-(2\mathbf{I}-\mathbf{B})\\&=2\mathbf{B}-2\mathbf{I}+\mathbf{B}\\&=3\mathbf{B}-2\mathbf{I}\end{aligned}$$

and
$$\begin{aligned}\mathbf{B}^4&=\mathbf{B}\times\mathbf{B}^3\\&=\mathbf{B}(3\mathbf{B}-2\mathbf{I})\\&=3\mathbf{B}^2-2\mathbf{BI}\\&=3(2\mathbf{I}-\mathbf{B})-2\mathbf{B}\\&=6\mathbf{I}-3\mathbf{B}-2\mathbf{B}\\&=6\mathbf{I}-5\mathbf{B}\end{aligned}$$

and
$$\begin{aligned}\mathbf{B}^5&=\mathbf{B}\times\mathbf{B}^4\\&=\mathbf{B}(6\mathbf{I}-5\mathbf{B})\\&=6\mathbf{BI}-5\mathbf{B}^2\\&=6\mathbf{B}-5(2\mathbf{I}-\mathbf{B})\\&=6\mathbf{B}-10\mathbf{I}+5\mathbf{B}\\&=11\mathbf{B}-10\mathbf{I}\end{aligned}$$

c $\mathbf{C}^2=4\mathbf{C}-3\mathbf{I}$

$$\begin{aligned}\mathbf{C}^3&=\mathbf{C}\times\mathbf{C}^2\\&=\mathbf{C}(4\mathbf{C}-3\mathbf{I})\\&=4\mathbf{C}^2-3\mathbf{CI}\\&=4(4\mathbf{C}-3\mathbf{I})-3\mathbf{C}\\&=16\mathbf{C}-12\mathbf{I}-3\mathbf{C}\\&=13\mathbf{C}-12\mathbf{I}\end{aligned}$$

$$\begin{aligned}\mathbf{C}^5&=\mathbf{C}^2\times\mathbf{C}^3\\&=(4\mathbf{C}-3\mathbf{I})(13\mathbf{C}-12\mathbf{I})\\&=(4\mathbf{C}-3\mathbf{I})13\mathbf{C}-(4\mathbf{C}-3\mathbf{I})12\mathbf{I}\\&=52\mathbf{C}^2-39\mathbf{IC}-48\mathbf{CI}+36\mathbf{I}^2\\&=52(4\mathbf{C}-3\mathbf{I})-39\mathbf{C}-48\mathbf{C}+36\mathbf{I}\\&=208\mathbf{C}-156\mathbf{I}-87\mathbf{C}+36\mathbf{I}\\&=121\mathbf{C}-120\mathbf{I}\end{aligned}$$

10 **a** If $\mathbf{A}^2=\mathbf{I}$:

i
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+2\mathbf{I})\\&=\mathbf{A}^2+2\mathbf{AI}\\&=\mathbf{I}+2\mathbf{A}\end{aligned}$$

ii
$$\begin{aligned}&(\mathbf{A}-\mathbf{I})^2\\&=(\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{I})\\&=(\mathbf{A}-\mathbf{I})\mathbf{A}-(\mathbf{A}-\mathbf{I})\mathbf{I}\\&=\mathbf{A}^2-\mathbf{IA}-\mathbf{AI}+\mathbf{I}^2\\&=\mathbf{I}-\mathbf{A}-\mathbf{A}+\mathbf{I}\\&=2\mathbf{I}-2\mathbf{A}\end{aligned}$$

iii
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+3\mathbf{I})^2\\&=\mathbf{A}(\mathbf{A}+3\mathbf{I})(\mathbf{A}+3\mathbf{I})\\&=\mathbf{A}[(\mathbf{A}+3\mathbf{I})\mathbf{A}+(\mathbf{A}+3\mathbf{I})3\mathbf{I}]\\&=\mathbf{A}\left[\mathbf{A}^2+3\mathbf{IA}+3\mathbf{AI}+9\mathbf{I}^2\right]\\&=\mathbf{A}[\mathbf{I}+3\mathbf{A}+3\mathbf{A}+9\mathbf{I}]\\&=\mathbf{A}[10\mathbf{I}+6\mathbf{A}]\\&=10\mathbf{AI}+6\mathbf{A}^2\\&=10\mathbf{A}+6\mathbf{I}\end{aligned}$$

b If $\mathbf{A}^3 = \mathbf{I}$,
$$\begin{aligned}\mathbf{A}^2(\mathbf{A}+\mathbf{I})^2 &= \mathbf{A}^2(\mathbf{A}^2+2\mathbf{A}+\mathbf{I})\\ &= \mathbf{A}^4+2\mathbf{A}^3+\mathbf{A}^2\mathbf{I}\\ &= \mathbf{A}(\mathbf{A}^3)+2\mathbf{A}^3+\mathbf{A}^2\mathbf{I}\\ &= \mathbf{A}\mathbf{I}+2\mathbf{I}+\mathbf{A}^2\\ &= \mathbf{A}^2+\mathbf{A}+2\mathbf{I}\end{aligned}$$

c If $\mathbf{A}^2 = \mathbf{O}$:

i
$$\begin{aligned}&\mathbf{A}(2\mathbf{A}-3\mathbf{I})\\ &= 2\mathbf{A}^2-3\mathbf{A}\mathbf{I}\\ &= 2\mathbf{O}-3\mathbf{A}\\ &= -3\mathbf{A}\end{aligned}$$

ii
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+2\mathbf{I})(\mathbf{A}-\mathbf{I})\\ &= \mathbf{A}[(\mathbf{A}+2\mathbf{I})\mathbf{A}-(\mathbf{A}+2\mathbf{I})\mathbf{I}]\\ &= \mathbf{A}(\mathbf{A}^2+2\mathbf{I}\mathbf{A}-\mathbf{A}\mathbf{I}-2\mathbf{I}^2)\\ &= \mathbf{A}(\mathbf{O}+\mathbf{A}-2\mathbf{I})\\ &= \mathbf{A}^2-2\mathbf{A}\mathbf{I}\\ &= \mathbf{O}-2\mathbf{A}\\ &= -2\mathbf{A}\end{aligned}$$

iii
$$\begin{aligned}&\mathbf{A}(\mathbf{A}+\mathbf{I})^3\\ &= \mathbf{A}(\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I})^2\\ &= \mathbf{A}(\mathbf{A}+\mathbf{I})(\mathbf{A}^2+2\mathbf{A}+\mathbf{I})\\ &= (\mathbf{A}^2+\mathbf{A}\mathbf{I})(\mathbf{O}+2\mathbf{A}+\mathbf{I})\\ &= (\mathbf{O}+\mathbf{A})(2\mathbf{A}+\mathbf{I})\\ &= 2\mathbf{A}^2+\mathbf{A}\mathbf{I}\\ &= 2\mathbf{O}+\mathbf{A}\\ &= \mathbf{A}\end{aligned}$$

11 **a**
$$\mathbf{AB} = \begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0+0&0+0\\0+0&0+0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} = \mathbf{O}$$

b
$$\mathbf{A}^2 = \begin{bmatrix}\frac12&\frac12\\\frac12&\frac12\end{bmatrix}\begin{bmatrix}\frac12&\frac12\\\frac12&\frac12\end{bmatrix} = \begin{bmatrix}\frac14+\frac14&\frac14+\frac14\\\frac14+\frac14&\frac14+\frac14\end{bmatrix} = \begin{bmatrix}\frac12&\frac12\\\frac12&\frac12\end{bmatrix} = \mathbf{A}$$

c
$$\begin{aligned}\mathbf{A}^2 &= \mathbf{A}\\ \therefore\ \mathbf{A}^2-\mathbf{A} &= \mathbf{O}\\ \therefore\ \mathbf{A}(\mathbf{A}-\mathbf{I}) &= \mathbf{O}\\ \therefore\ \mathbf{A} &= \mathbf{O} \text{ or } \mathbf{A}-\mathbf{I}=\mathbf{O}\\ \therefore\ \mathbf{A} &= \mathbf{O} \text{ or } \mathbf{I}\end{aligned}$$

The argument contains a false step. As the example in **a** illustrates, $\mathbf{AB}=\mathbf{O}$ does not imply that $\mathbf{A}=\mathbf{O}$ or $\mathbf{B}=\mathbf{O}$.

This is a property of real numbers that does not hold for matrices. Therefore it is false to say that if $\mathbf{A}(\mathbf{A}-\mathbf{I})=\mathbf{O}$, then $\mathbf{A}=\mathbf{O}$ or $\mathbf{A}-\mathbf{I}$ equals $\mathbf{O}$.

d Let $\mathbf{A} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $\mathbf{A}^2=\mathbf{A}$,
$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$$
$$\begin{bmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$$

Equating corresponding elements:
$$\begin{aligned}a^2+bc &= a &\therefore\quad bc &= a(1-a) &\text{...... (1)}\\ ab+bd &= b &\therefore\quad b(a+d-1) &= 0 &\text{...... (2)}\\ ac+cd &= c &\therefore\quad c(a+d-1) &= 0 &\text{...... (3)}\\ bc+d^2 &= d &\therefore\quad bc &= d(1-d) &\text{...... (4)}\end{aligned}$$

<u>If $a+d-1\neq 0$</u> then from (2) and (3), $b=c=0$.

$\therefore$ from (1) and (4), $a=0$ or 1 and $d=0$ or 1

$\therefore$ $a=0,\ d=0$ or $a=0,\ d=1$ or $a=1,\ d=0$ or $a=1,\ d=1$

where the last two cases are not possible as $a+d\neq 1$.

So, if $a=0,\ d=0$ then $\mathbf{A}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ and if $a=1,\ d=1$ then $\mathbf{A}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

<u>If $a+d-1=0$</u> then $d=1-a$ and $c=\dfrac{a-a^2}{b}$

So $\mathbf{A}$ is $\begin{bmatrix}a&b\\\dfrac{a-a^2}{b}&1-a\end{bmatrix}$ provided $b\neq 0$.

12 Choose $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$,

then $\mathbf{A}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+(-1) & -1+1 \\ 1+(-1) & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $\mathbf{A}^2 = \mathbf{O}$, but $\mathbf{A} \neq \mathbf{O}$ so "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a false statement.

13 **a** Since $\mathbf{A}^2 = a\mathbf{A}+b\mathbf{I}$, $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1+(-2) & 2+4 \\ -1+(-2) & -2+4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ -a & 2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 6 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} a+b & 2a \\ -a & 2a+b \end{bmatrix}$$

$\therefore$ $a+b = -1$ and $2a = 6$

i.e., $a = 3$ and $b = -4$

Checking for consistency: $-a = -3$, $2a+b = 6+(-4) = 2$ ✓

$\therefore$ $\mathbf{A}^2 = 3\mathbf{A} - 4\mathbf{I}$

b Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} = a\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 9+2 & 3+(-2) \\ 6+(-4) & 2+4 \end{bmatrix} = \begin{bmatrix} 3a & a \\ 2a & -2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} 11 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3a+b & a \\ 2a & -2a+b \end{bmatrix}$$

$\therefore$ $3a+b = 11$ and $a = 1$

$\therefore$ $a = 1$ and $b = 8$

Checking for consistency $2a = 2(1) = 2$, $-2a+b = -2(1)+8 = 6$ ✓

$\therefore$ $\mathbf{A}^2 = \mathbf{A} + 8\mathbf{I}$

14 If $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = p\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+(-2) & 2+(-6) \\ -1+3 & -2+9 \end{bmatrix} = \begin{bmatrix} p & 2p \\ -p & -3p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} p+q & 2p \\ -p & -3p+q \end{bmatrix}$$

$\therefore$ $p+q = -1$ and $2p = -4$

$\therefore$ $p = -2$ and $q = 1$

Checking for consistency $-p = -(-2) = 2$, $-3p+q = -3(-2)+1 = 7$ ✓

$\therefore$ $\mathbf{A}^2 = -2\mathbf{A} + \mathbf{I}$

a
$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 \\ &= \mathbf{A}(-2\mathbf{A} + \mathbf{I}) \\ &= -2\mathbf{A}^2 + \mathbf{AI} \\ &= -2(-2\mathbf{A} + \mathbf{I}) + \mathbf{A} \\ &= 4\mathbf{A} - 2\mathbf{I} + \mathbf{A} \\ &= 5\mathbf{A} - 2\mathbf{I} \end{aligned}$$

b
$$\begin{aligned} \mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\ &= \mathbf{A}(5\mathbf{A} - 2\mathbf{I}) \\ &= 5\mathbf{A}^2 - 2\mathbf{AI} \\ &= 5(-2\mathbf{A} + \mathbf{I}) - 2\mathbf{A} \\ &= -10\mathbf{A} + 5\mathbf{I} - 2\mathbf{A} \\ &= -12\mathbf{A} + 5\mathbf{I} \end{aligned}$$

EXERCISE 14H

1 **a**
$$\begin{bmatrix}5&6\\2&3\end{bmatrix}\begin{bmatrix}3&-6\\-2&5\end{bmatrix}$$
$$=\begin{bmatrix}3&0\\0&3\end{bmatrix}$$
$$=3\mathbf{I}$$
$$\therefore\ \begin{bmatrix}5&6\\2&3\end{bmatrix}\times\tfrac{1}{3}\begin{bmatrix}3&-6\\-2&5\end{bmatrix}=\mathbf{I}$$
$$\therefore\ \begin{bmatrix}5&6\\2&3\end{bmatrix}^{-1}=\tfrac{1}{3}\begin{bmatrix}3&-6\\-2&5\end{bmatrix}$$
$$=\begin{bmatrix}1&-2\\-\frac{2}{3}&\frac{5}{3}\end{bmatrix}$$

b
$$\begin{bmatrix}3&-4\\1&2\end{bmatrix}\begin{bmatrix}2&4\\-1&3\end{bmatrix}$$
$$=\begin{bmatrix}10&0\\0&10\end{bmatrix}$$
$$=10\mathbf{I}$$
$$\therefore\ \begin{bmatrix}3&-4\\1&2\end{bmatrix}\times\tfrac{1}{10}\begin{bmatrix}2&4\\-1&3\end{bmatrix}=\mathbf{I}$$
$$\therefore\ \begin{bmatrix}3&-4\\1&2\end{bmatrix}^{-1}=\tfrac{1}{10}\begin{bmatrix}2&4\\-1&3\end{bmatrix}$$
$$=\begin{bmatrix}0.2&0.4\\-0.1&0.3\end{bmatrix}$$

c
$$\begin{bmatrix}2&0&3\\1&5&2\\1&-3&1\end{bmatrix}\begin{bmatrix}-11&9&15\\-1&1&1\\8&-6&-10\end{bmatrix}=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}=2\mathbf{I}$$
$$\therefore\ \begin{bmatrix}2&0&3\\1&5&2\\1&-3&1\end{bmatrix}\times\tfrac{1}{2}\begin{bmatrix}-11&9&15\\-1&1&1\\8&-6&-10\end{bmatrix}=\mathbf{I}$$
and so
$$\begin{bmatrix}2&0&3\\1&5&2\\1&-3&1\end{bmatrix}^{-1}=\begin{bmatrix}-\frac{11}{2}&\frac{9}{2}&\frac{15}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\4&-3&-5\end{bmatrix}$$

2 **a** $|\mathbf{A}|=12-14$
$=-2$

b $|\mathbf{A}|=2-3$
$=-1$

c $|\mathbf{A}|=0-0$
$=0$

d $|\mathbf{A}|=1-0$
$=1$

3 **a** $\det\mathbf{B}$
$=12--14$
$=26$

b $\det\mathbf{B}=6-0$
$=6$

c $\det\mathbf{B}=0-1$
$=-1$

d $\det\mathbf{B}=a^2--a$
$=a^2+a$

4
$$\mathbf{A}=\begin{bmatrix}2&-1\\-1&-1\end{bmatrix}$$

a $|\mathbf{A}|=2(-1)-(-1)(-1)=-3$

b
$$\mathbf{A}^2=\begin{bmatrix}2&-1\\-1&-1\end{bmatrix}\begin{bmatrix}2&-1\\-1&-1\end{bmatrix}$$
$$=\begin{bmatrix}4+1&-2+1\\-2+1&1+1\end{bmatrix}$$
$$=\begin{bmatrix}5&-1\\-1&2\end{bmatrix}$$
$$\therefore\ \left|\mathbf{A}^2\right|=5(2)-(-1)(-1)=9$$

c
$$2\mathbf{A}=2\begin{bmatrix}2&-1\\-1&-1\end{bmatrix}$$
$$=\begin{bmatrix}4&-2\\-2&-2\end{bmatrix}$$
$$\therefore\ |2\mathbf{A}|=4(-2)-(-2)(-2)=-12$$

5 Let $\mathbf{A}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ then $k\mathbf{A}=\begin{bmatrix}ka&kb\\kc&kd\end{bmatrix}$ and
$$|k\mathbf{A}|=ka(kd)-kb(kc)$$
$$=k^2(ad-bc)$$
$$=k^2\,|\mathbf{A}|$$

6 $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

a $|\mathbf{A}| = ad - bc$
and $|\mathbf{B}| = wz - xy$

b
$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} \end{aligned}$$
$$\therefore \quad |\mathbf{AB}| = (aw+by)(cx+dz) - (ax+bz)(cw+dy)$$

c Expanding brackets,
$$\begin{aligned} |\mathbf{AB}| &= awcx + awdz + bycx + bydz - axcw - axdy - bzcw - bzdy \\ &= wz(ad-bc) - xy(ad-bc) \\ &= (ad-bc)(wz-xy) \\ &= |\mathbf{A}|\,|\mathbf{B}| \end{aligned}$$

7 $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$

a **i**
$$\begin{aligned} |\mathbf{A}| &= 1(4) - 2(3) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$

ii
$$\begin{aligned} |2\mathbf{A}| &= 2^2\,|\mathbf{A}| \\ &= 4(-2) \\ &= -8 \end{aligned}$$

iii
$$\begin{aligned} |-\mathbf{A}| &= (-1)^2\,|\mathbf{A}| \\ &= 1(-2) \\ &= -2 \end{aligned}$$

iv
$$\begin{aligned} |\mathbf{B}| &= (-1)(1) - 2(0) \\ &= -1 \end{aligned}$$
$$\begin{aligned} \therefore \quad |-3\mathbf{B}| &= (-3)^2\,|\mathbf{B}| \\ &= 9(-1) \\ &= -9 \end{aligned}$$

v
$$\begin{aligned} |\mathbf{AB}| &= |\mathbf{A}|\,|\mathbf{B}| \\ &= (-2)(-1) \\ &= 2 \end{aligned}$$

b *Checking:*

ii $2\mathbf{A} = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

$|2\mathbf{A}| = 2(8) - 4(6) = -8$ ✓

iii $|-\mathbf{A}| = \begin{vmatrix} -1 & -2 \\ -3 & -4 \end{vmatrix}$

$= (-1)(-4) - (-2)(-3) = -2$ ✓

iv
$$\begin{aligned} -3\mathbf{B} &= -3\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 \\ 0 & -3 \end{bmatrix} \end{aligned}$$
$$\begin{aligned} |-3\mathbf{B}| &= (3)(-3) - (-6)(0) \\ &= -9 \ \checkmark \end{aligned}$$

v
$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1+0 & 2+2 \\ -3+0 & 6+4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -3 & 10 \end{bmatrix} \end{aligned}$$
$|\mathbf{AB}| = (-1)(10) - 4(-3) = 2$ ✓

8 **a** $$\begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{2(5) - 4(-1)}\begin{bmatrix} 5 & -4 \\ -(-1) & 2 \end{bmatrix} = \tfrac{1}{14}\begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix}$$

b $$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{1(-1) - 0(1)}\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = -\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

c $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}^{-1}$ does not exist, since $ad - bc = 2(2) - 4(1) = 0$

d $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{1(1) - 0(0)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

e $\begin{bmatrix} 3 & 5 \\ -6 & -10 \end{bmatrix}^{-1}$ does not exist, since $ad - bc = 3(-10) - 5(-6) = 0$

f $$\begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix}^{-1} = \frac{1}{(-1)(7)-2(4)}\begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix} = -\tfrac{1}{15}\begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix}$$

g $$\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{3(2)-(-1)(4)}\begin{bmatrix} 2 & -4 \\ -(-1) & 3 \end{bmatrix} = \tfrac{1}{10}\begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}$$

h $$\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{(-1)3-(-1)2}\begin{bmatrix} 3 & -(-1) \\ -2 & -1 \end{bmatrix} = -\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

EXERCISE 14I

1 **a** $$\left.\begin{aligned} 3x-y &= 8 \\ 2x+3y &= 6 \end{aligned}\right\} \text{ can be written as } \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

b $$\left.\begin{aligned} 4x-3y &= 11 \\ 3x+2y &= -5 \end{aligned}\right\} \text{ can be written as } \begin{bmatrix} 4 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

c $$\left.\begin{aligned} 3a-b &= 6 \\ 2a+7b &= -4 \end{aligned}\right\} \text{ can be written as } \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

2 **a** $2x-y=6$, $x+3y=14$ In matrix form, the system is: $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 14 \end{bmatrix} = \tfrac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \tfrac{1}{7}\begin{bmatrix} 18+14 \\ -6+28 \end{bmatrix} = \begin{bmatrix} \frac{32}{7} \\ \frac{22}{7} \end{bmatrix} \quad \text{and so} \quad x = \tfrac{32}{7},\ y = \tfrac{22}{7}$$

b $5x-4y=5$, $2x+3y=-13$ In matrix form, the system is: $\begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -13 \end{bmatrix} = \tfrac{1}{23}\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ -13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \tfrac{1}{23}\begin{bmatrix} 15+(-52) \\ -10+(-65) \end{bmatrix} = \begin{bmatrix} -\frac{37}{23} \\ -\frac{75}{23} \end{bmatrix} \quad \text{and so} \quad x = -\tfrac{37}{23},\ y = -\tfrac{75}{23}$$

c $x-2y=7$, $5x+3y=-2$ In matrix form, the system is: $\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ -2 \end{bmatrix} = \tfrac{1}{13}\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \tfrac{1}{13}\begin{bmatrix} 21+(-4) \\ -35+(-2) \end{bmatrix} = \begin{bmatrix} \frac{17}{13} \\ -\frac{37}{13} \end{bmatrix} \quad \text{and so} \quad x = \tfrac{17}{13},\ y = -\tfrac{37}{13}$$

d $3x+5y=4$, $2x-y=11$ In matrix form, the system is: $\begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 11 \end{bmatrix} = \tfrac{1}{-13}\begin{bmatrix} -1 & -5 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\tfrac{1}{13}\begin{bmatrix} -4+(-55) \\ -8+33 \end{bmatrix} = \begin{bmatrix} \frac{59}{13} \\ \frac{-25}{13} \end{bmatrix} \quad \text{and so} \quad x = \tfrac{59}{13},\ y = -\tfrac{25}{13}$$

e $4x-7y=8$ $\quad$ In matrix form, the system is: $\begin{bmatrix}4&-7\\3&-5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\0\end{bmatrix}$
$3x-5y=0$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4&-7\\3&-5\end{bmatrix}^{-1}\begin{bmatrix}8\\0\end{bmatrix}=1\begin{bmatrix}-5&7\\-3&4\end{bmatrix}\begin{bmatrix}8\\0\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-40+0\\-24+0\end{bmatrix}\quad\text{and so}\quad x=-40,\quad y=-24$$

f $7x+11y=18$ $\quad$ In matrix form, the system is: $\begin{bmatrix}7&11\\11&-7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}18\\-11\end{bmatrix}$
$11x-7y=-11$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}7&11\\11&-7\end{bmatrix}^{-1}\begin{bmatrix}18\\-11\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\tfrac{1}{-170}\begin{bmatrix}-7&-11\\-11&7\end{bmatrix}\begin{bmatrix}18\\-11\end{bmatrix}=-\tfrac{1}{170}\begin{bmatrix}-126+121\\-198-77\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=-\tfrac{1}{170}\begin{bmatrix}-5\\-275\end{bmatrix}=\begin{bmatrix}\frac{1}{34}\\\frac{55}{34}\end{bmatrix}\quad\text{and so}\quad x=\tfrac{1}{34},\quad y=\tfrac{55}{34}$$

3 **a** If $\mathbf{AX}=\mathbf{B}$
then $\mathbf{A}^{-1}\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$ {pre-mult by $\mathbf{A}^{-1}$}
$\therefore\ \mathbf{IX}=\mathbf{A}^{-1}\mathbf{B}$
$\therefore\ \mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$

If $\mathbf{XA}=\mathbf{B}$
then $\mathbf{XAA}^{-1}=\mathbf{BA}^{-1}$ {post-mult by $\mathbf{A}^{-1}$}
$\therefore\ \mathbf{XI}=\mathbf{BA}^{-1}$
$\therefore\ \mathbf{X}=\mathbf{BA}^{-1}$

b **i** $\mathbf{X}\begin{bmatrix}1&2\\5&-1\end{bmatrix}=\begin{bmatrix}14&-5\\22&0\end{bmatrix}$

$$\therefore\ \mathbf{X}=\begin{bmatrix}14&-5\\22&0\end{bmatrix}\begin{bmatrix}1&2\\5&-1\end{bmatrix}^{-1}$$
$$=\begin{bmatrix}14&-5\\22&0\end{bmatrix}\tfrac{1}{-11}\begin{bmatrix}-1&-2\\-5&1\end{bmatrix}$$
$$=-\tfrac{1}{11}\begin{bmatrix}-14+25&-28-5\\-22+0&-44+0\end{bmatrix}$$
$$=\begin{bmatrix}-1&3\\2&4\end{bmatrix}$$

ii $\begin{bmatrix}1&3\\2&-1\end{bmatrix}\mathbf{X}=\begin{bmatrix}1&-3\\4&2\end{bmatrix}$

$$\therefore\ \mathbf{X}=\begin{bmatrix}1&3\\2&-1\end{bmatrix}^{-1}\begin{bmatrix}1&-3\\4&2\end{bmatrix}$$
$$=\tfrac{1}{-7}\begin{bmatrix}-1&-3\\-2&1\end{bmatrix}\begin{bmatrix}1&-3\\4&2\end{bmatrix}$$
$$=-\tfrac{1}{7}\begin{bmatrix}-1+(-12)&3+(-6)\\-2+4&6+2\end{bmatrix}$$
$$=\begin{bmatrix}\frac{13}{7}&\frac{3}{7}\\-\frac{2}{7}&-\frac{8}{7}\end{bmatrix}$$

4 **a** **i** $|\mathbf{A}|=0$ when $2k-(-6)=0$ i.e., $k=-3$ i.e., singular when $k=-3$

ii $\mathbf{A}=\begin{bmatrix}k&1\\-6&2\end{bmatrix}$ $\quad\therefore\ \mathbf{A}^{-1}=\dfrac{1}{2k+6}\begin{bmatrix}2&-1\\6&k\end{bmatrix}$, provided that $k\neq-3$

b **i** $|\mathbf{A}|=0$ when $3k-0=0$ i.e., $k=0$ i.e., singular when $k=0$

ii $\mathbf{A}=\begin{bmatrix}3&-1\\0&k\end{bmatrix}$ $\quad\therefore\ \mathbf{A}^{-1}=\dfrac{1}{3k}\begin{bmatrix}k&1\\0&3\end{bmatrix}$, provided that $k\neq0$

c **i** $|\mathbf{A}|=0$ when $k(k+1)-2=0$
i.e., $k^2+k-2=0$
$(k-1)(k+2)=0$
$\therefore\ k=1$ or -2 i.e., singular when $k=1$ or -2

ii
$$\mathbf{A} = \begin{bmatrix} k+1 & 2 \\ 1 & k \end{bmatrix} \quad \therefore \quad \mathbf{A}^{-1} = \frac{1}{k(k+1)-2}\begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix},$$
$$= \frac{1}{(k+2)(k-1)}\begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix}, \quad \text{provided that } k \neq -2 \text{ or } 1$$

5 **a** $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×2 $\quad\therefore\quad$ $\mathbf{AB}$ is 2×2

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+0+2 & 2+0+(-2) \\ 1+(-4)+3 & -2+6+(-3) \end{bmatrix}$$
$$\therefore \quad \mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

b $\mathbf{B}$ is 3×2 and $\mathbf{A}$ is 2×3 $\;\therefore\;$ $\mathbf{BA}$ is 3×3 whereas $\mathbf{AB}$ is 2×2. Hence $\mathbf{BA} \neq \mathbf{AB}$, and $\mathbf{A}$ and $\mathbf{B}$ are not inverses. The inverse of a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1}$. This requires that $\mathbf{A}$ has the same number of rows as columns, i.e., that $\mathbf{A}$ is a square matrix.

6 $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$

Since $\mathbf{AXB} = \mathbf{C}$,

then $\mathbf{A}^{-1}\mathbf{AXB} = \mathbf{A}^{-1}\mathbf{C}$ {premult. by $\mathbf{A}^{-1}$}

$\therefore\;\mathbf{IXB} = \mathbf{A}^{-1}\mathbf{C}$

$\therefore\;\mathbf{XB} = \mathbf{A}^{-1}\mathbf{C}$

and

$\mathbf{XBB}^{-1} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$ {postmult. by $\mathbf{B}^{-1}$}

$\therefore\;\mathbf{XI} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$

$\therefore\;\mathbf{X} = \mathbf{A}^{-1}\mathbf{CB}^{-1}$

$$\therefore\quad \mathbf{X} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}\tfrac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 0+(-1) & 3+(-2) \\ 0+2 & 0+4 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$
$$\therefore\quad \mathbf{X} = \tfrac{1}{4}\begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 0+1 & 2+1 \\ 0+4 & -4+4 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 1 & 0 \end{bmatrix}$$

7 **a** **i**
$$\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$
and $|\mathbf{A}| = -2 - -12$
$= -2 + 12$
$= 10$

ii As $|\mathbf{A}| \neq 0$, the system has a unique solution
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$$
$$= \tfrac{1}{10}\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$$
$$= \tfrac{1}{10}\begin{bmatrix} 25 \\ -10 \end{bmatrix}$$
$$= \begin{bmatrix} 2\frac{1}{2} \\ -1 \end{bmatrix}$$
$\therefore\quad x = 2\frac{1}{2},\quad y = -1$

b **i**
$$\begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$
and $|\mathbf{A}| = -2 - 4k$

ii The system has a unique solution if
$-2-4k \neq 0$ i.e., $k \neq -\frac{1}{2}$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$$
$$= \frac{1}{-2-4k}\begin{bmatrix} -1 & -k \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$$
$$= \frac{1}{-2-4k}\begin{bmatrix} -8-11k \\ -10 \end{bmatrix}$$
$$\therefore\quad x = \frac{8+11k}{2+4k},\quad y = \frac{5}{1+2k},\quad k \neq -\tfrac{1}{2}$$
is the unique solution

iii When $k = -\frac{1}{2}$, the equations are
$$\begin{cases} 2x - \frac{1}{2}y = 8 \\ 4x - y = 11 \end{cases} \quad \text{i.e.,} \quad \begin{aligned} 4x - y &= 16 \\ 4x - y &= 11 \end{aligned}$$
So, we have no solutions (as the lines are parallel and so do not meet).

8 **a** If $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{AA} = \mathbf{AA}^{-1} = \mathbf{I}$

b If $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is its own inverse, then

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{lllll} \therefore & a^2+b^2 = 1 & & \text{then } a = 0 & \text{or } b = 0 \\ & \text{and } 2ab = 0 & & \text{and } b^2 = 1 & \text{and } a^2 = 1 \\ & \text{If } 2ab = 0, & \therefore & a = 0 \text{ and } b = \pm 1 & \therefore\ a = \pm 1 \text{ and } b = 0 \end{array}$$

This gives four possible combinations: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

9 **a**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \therefore\ \mathbf{A}^{-1} = \tfrac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\therefore\ (\mathbf{A}^{-1})^{-1} = \frac{1}{\frac{1}{2}}\begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \mathbf{A}$$

b If $\mathbf{A}^{-1} = \mathbf{B}$

then $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1}) = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$

and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1} = \mathbf{BB}^{-1} = \mathbf{I}$

c We can deduce from **b** that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ and $(\mathbf{A}^{-1})^{-1}$ is the inverse of $\mathbf{A}^{-1}$.

10 **a** $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$

i $\mathbf{A}^{-1} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

ii $\mathbf{B}^{-1} = \frac{1}{-2}\begin{bmatrix} -3 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

iii $(\mathbf{AB})^{-1}$

$$= \left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0+2 & 1+(-3) \\ 0+(-2) & 2+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}^{-1}$$

$$= \tfrac{1}{6}\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{i.e.,} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

iv $(\mathbf{BA})^{-1}$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0+2 & 0+(-1) \\ 2+(-6) & 2+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}^{-1}$$

$$= \tfrac{1}{6}\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \text{i.e.,} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

v $\mathbf{A}^{-1}\mathbf{B}^{-1}$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{6}+\frac{1}{3} & \frac{1}{6}+0 \\ \frac{6}{6}+\left(-\frac{1}{3}\right) & \frac{2}{6}+0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

vi $\mathbf{B}^{-1}\mathbf{A}^{-1}$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{6}+\frac{2}{6} & \frac{3}{6}+\left(-\frac{1}{6}\right) \\ \frac{1}{3}+0 & \frac{1}{3}+0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

b Choose appropriate vectors and repeat question **a**.

c The results of **a** and **b** suggest that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

d $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{AB})(\mathbf{AB})^{-1}$ {from **c**} $= \mathbf{I}$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = (\mathbf{AB})^{-1}(\mathbf{AB})$ {from **c**} $= \mathbf{I}$

$\therefore$ $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$ i.e., $\mathbf{AB}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ are inverses.

11 $(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = k \times \frac{1}{k}(\mathbf{AA}^{-1}) = \mathbf{I}$ also $\left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \frac{1}{k} \times k(\mathbf{A}^{-1}\mathbf{A}) = \mathbf{I}$

$\therefore$ $(k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \mathbf{I}$ i.e., $k\mathbf{A}$ and $\frac{1}{k}\mathbf{A}^{-1}$ are inverses.

12 $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$

a $\mathbf{X} = \mathbf{AY} = \mathbf{A}(\mathbf{BZ}) = \mathbf{ABZ}$

b $(\mathbf{AB})^{-1}\mathbf{X} = (\mathbf{AB})^{-1}\mathbf{ABZ}$ {premult. by $(\mathbf{AB})^{-1}$}

$(\mathbf{AB})^{-1}\mathbf{X} = \mathbf{IZ}$

$\therefore$ $\mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X}$ {as $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$}

13 In each example we premultiply by $\mathbf{A}^{-1}$.

a
$$\begin{aligned} \mathbf{A}^2 &= 4\mathbf{A} - \mathbf{I} \\ \therefore\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(4\mathbf{A} - \mathbf{I}) \\ \therefore\ \mathbf{A}^{-1}\mathbf{AA} &= 4\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \\ \therefore\ \mathbf{IA} &= 4\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\ \mathbf{A} - 4\mathbf{I} &= -\mathbf{A}^{-1} \\ \therefore\ \mathbf{A}^{-1} &= 4\mathbf{I} - \mathbf{A} \end{aligned}$$

b
$$\begin{aligned} 5\mathbf{A} &= \mathbf{I} - \mathbf{A}^2 \\ \therefore\ \mathbf{A}^{-1}5\mathbf{A} &= \mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}^2) \\ \therefore\ 5\mathbf{A}^{-1}\mathbf{A} &= \mathbf{A}^{-1}\mathbf{I} - \mathbf{A}^{-1}\mathbf{AA} \\ \therefore\ 5\mathbf{I} &= \mathbf{A}^{-1} - \mathbf{IA} \\ \therefore\ 5\mathbf{I} &= \mathbf{A}^{-1} - \mathbf{A} \\ \therefore\ \mathbf{A}^{-1} &= 5\mathbf{I} + \mathbf{A} \end{aligned}$$

c
$$\begin{aligned} 2\mathbf{I} &= 3\mathbf{A}^2 - 4\mathbf{A} \\ \therefore\ \mathbf{A}^{-1}2\mathbf{I} &= \mathbf{A}^{-1}3\mathbf{A}^2 - \mathbf{A}^{-1}4\mathbf{A} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{A}^{-1}\mathbf{AA} - 4\mathbf{A}^{-1}\mathbf{A} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{IA} - 4\mathbf{I} \\ \therefore\ 2\mathbf{A}^{-1} &= 3\mathbf{A} - 4\mathbf{I} \\ \therefore\ \mathbf{A}^{-1} &= \tfrac{3}{2}\mathbf{A} - 2\mathbf{I} \end{aligned}$$

14 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$, let $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\therefore \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} = p\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} + q\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 9+(-4) & 6+(-2) \\ -6+2 & -4+1 \end{bmatrix} = \begin{bmatrix} 3p & 2p \\ -2p & -p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 3p+q & 2p \\ -2p & -p+q \end{bmatrix}$$

$\therefore$ $5 = 3p + q$ and $4 = 2p$

$\therefore$ $p = 2$ and $q = -1$ and $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$

Checking for consistency: $-2p = -2(2) = -4$ ✓

$-p + q = -2 + (-1) = -3$ ✓

$$\begin{aligned} \text{Now}\quad \mathbf{A}^2 &= 2\mathbf{A} - \mathbf{I} \\ \therefore\quad \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}2\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \qquad \{\text{premultiplying by } \mathbf{A}^{-1}\} \\ \therefore\quad \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 2\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1} \\ \therefore\quad \mathbf{I}\mathbf{A} &= 2\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\quad \mathbf{A} &= 2\mathbf{I} - \mathbf{A}^{-1} \\ \therefore\quad \mathbf{A}^{-1} &= 2\mathbf{I} - \mathbf{A} \end{aligned}$$

15 If $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$,

$$\begin{aligned} \text{then}\quad \mathbf{A}^2 &= \mathbf{AA} \\ &= (\mathbf{AB})\mathbf{A} \\ &= \mathbf{A}(\mathbf{BA}) \qquad \{\text{associative rule}\} \\ &= \mathbf{AB} \\ \therefore\quad \mathbf{A}^2 &= \mathbf{A} \end{aligned}$$

$ab = ac$ implies that $b = c$ for real numbers, but this property does not hold for matrices.

Thus from $\mathbf{AB} = \mathbf{AI} = \mathbf{A}$ it does not follow that $\mathbf{B} = \mathbf{I}$.

16

$$\begin{aligned} \text{If}\quad \mathbf{AB} &= \mathbf{AC} \\ \text{then}\quad \mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\mathbf{AC} \qquad \{\text{premultiplying by } \mathbf{A}^{-1}\} \\ \therefore\quad \mathbf{IB} &= \mathbf{IC} \\ \therefore\quad \mathbf{B} &= \mathbf{C} \end{aligned}$$

i.e., if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$ if $\mathbf{A}^{-1}$ exists, i.e., if $ad - bc \neq 0$.

17 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$

$$\begin{aligned} \text{then}\quad \mathbf{X}^3 &= (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) \\ &= (\mathbf{P}^{-1}\mathbf{A})(\mathbf{PP}^{-1})\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} \qquad \{\text{associative rule}\} \\ &= \mathbf{P}^{-1}\mathbf{AIAIAP} \\ &= \mathbf{P}^{-1}\mathbf{AAAP} \\ &= \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P} \\ &= \mathbf{P}^{-1}\mathbf{IP} \\ &= \mathbf{P}^{-1}\mathbf{P} \qquad \text{and so}\quad \mathbf{X}^3 = \mathbf{I} \end{aligned}$$

18

$$\begin{aligned} \text{If}\quad a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} &= \mathbf{O} \quad \text{and} \quad \mathbf{X} = \mathbf{P}^{-1}\mathbf{AP} \\ \text{then}\quad a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} &= a(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I} \\ &= a\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I} \\ &= a\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I} \\ &= \mathbf{P}^{-1}(a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{P} \\ &= \mathbf{P}^{-1}\mathbf{OP} \\ &= \mathbf{O} \end{aligned}$$

EXERCISE 14J

1 **a**

$$\begin{aligned} \begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix} &= 2\begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} \\ &= 2(10 - 0) + 3(2 - -5) + 0 \\ &= 41 \end{aligned}$$

b

$$\begin{aligned} \begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix} &= -1\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} \\ &= -1(0 - 0) + 2(0 - 1) - 3(2 - 0) \\ &= -8 \end{aligned}$$

c $$\begin{aligned}\begin{vmatrix}2&1&3\\-1&1&2\\2&1&3\end{vmatrix} &= 2\begin{vmatrix}1&2\\1&3\end{vmatrix}+1\begin{vmatrix}2&-1\\3&2\end{vmatrix}+3\begin{vmatrix}-1&1\\2&1\end{vmatrix}\\ &= 2(3-2)+1(4--3)+3(-1-2)\\ &= 0\end{aligned}$$

d $$\begin{vmatrix}1&0&0\\0&2&0\\0&0&3\end{vmatrix} = 1\begin{vmatrix}2&0\\0&3\end{vmatrix}+0\begin{vmatrix}0&0\\3&0\end{vmatrix}+0\begin{vmatrix}0&2\\0&0\end{vmatrix} = 1(6-0)=6$$

e $$\begin{vmatrix}0&0&2\\0&1&0\\3&0&0\end{vmatrix} = 0\begin{vmatrix}1&0\\0&0\end{vmatrix}+0\begin{vmatrix}0&0\\0&3\end{vmatrix}+2\begin{vmatrix}0&1\\3&0\end{vmatrix} = 2(0-3)=-6$$

f $$\begin{aligned}\begin{vmatrix}4&1&3\\-1&0&2\\-1&1&1\end{vmatrix} &= 4\begin{vmatrix}0&2\\1&1\end{vmatrix}+1\begin{vmatrix}2&-1\\1&-1\end{vmatrix}+3\begin{vmatrix}-1&0\\-1&1\end{vmatrix}\\ &= 4(0-2)+1(-2--1)+3(-1-0)\\ &= -12\end{aligned}$$

2 **a** $$\begin{aligned}\begin{vmatrix}x&2&9\\3&1&2\\-1&0&x\end{vmatrix} &= x\begin{vmatrix}1&2\\0&x\end{vmatrix}+2\begin{vmatrix}2&3\\x&-1\end{vmatrix}+9\begin{vmatrix}3&1\\-1&0\end{vmatrix}\\ &= x(x)+2(-2-3x)+9(1)\\ &= x^2-4-6x+9\\ &= x^2-6x+5\\ &= (x-5)(x-1)\end{aligned}$$

The matrix is singular if its determinant is 0
i.e., when $x=5$ or $x=1$

b This means that the matrix has an inverse for all x in R, $x \neq 1$ or 5.

3 **a** $$\begin{vmatrix}a&0&0\\0&b&0\\0&0&c\end{vmatrix} = a\begin{vmatrix}b&0\\0&c\end{vmatrix}+0\begin{vmatrix}0&0\\c&0\end{vmatrix}+0\begin{vmatrix}0&b\\0&0\end{vmatrix} = a(bc-0)=abc$$

b $$\begin{aligned}\begin{vmatrix}0&x&y\\-x&0&z\\-y&-z&0\end{vmatrix} = 0\begin{vmatrix}0&z\\-z&0\end{vmatrix}+x\begin{vmatrix}z&-x\\0&-y\end{vmatrix}+y\begin{vmatrix}-x&0\\-y&-z\end{vmatrix} &= x(-zy)+y(xz)\\ &= -xyz+xyz\\ &= 0\end{aligned}$$

c $$\begin{aligned}\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix} &= a\begin{vmatrix}c&a\\a&b\end{vmatrix}+b\begin{vmatrix}a&b\\b&c\end{vmatrix}+c\begin{vmatrix}b&c\\c&a\end{vmatrix}\\ &= a(cb-a^2)+b(ac-b^2)+c(ba-c^2)\\ &= abc-a^3+abc-b^3+abc-c^3\\ &= 3abc-a^3-b^3-c^3\end{aligned}$$

4 $\left\{\begin{aligned}x+2y-3z &= 5\\ 2x-y-z &= 8\\ kx+y+2z &= 14\end{aligned}\right.$ has matrix equation $\underset{\mathbf{A}}{\begin{bmatrix}1&2&-3\\2&-1&-1\\k&1&2\end{bmatrix}}\underset{\mathbf{X}}{\begin{bmatrix}x\\y\\z\end{bmatrix}} = \underset{\mathbf{B}}{\begin{bmatrix}5\\8\\14\end{bmatrix}}$

$\therefore$ $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$ has a unique solution if $|\mathbf{A}| \neq 0$.

Now $$\begin{aligned}\begin{vmatrix}1&2&-3\\2&-1&-1\\k&1&2\end{vmatrix} &= 1\begin{vmatrix}-1&-1\\1&2\end{vmatrix}+2\begin{vmatrix}-1&2\\2&k\end{vmatrix}+(-3)\begin{vmatrix}2&-1\\k&1\end{vmatrix}\\ &= 1(-2--1)+2(-k-4)-3(2--k)\\ &= -1-2k-8-6-3k\\ &= -15-5k\end{aligned}$$

i.e., a unique solution for all k provided $k \neq -3$.

5 $\left\{\begin{aligned} 2x - y - 4z &= 8 \\ 3x - ky + z &= 1 \\ 5x - y + kz &= -2 \end{aligned}\right.$ has matrix equation $\underset{\mathbf{A}}{\begin{bmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{bmatrix}} \underset{\mathbf{X}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\mathbf{B}}{\begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}}$

This has a unique solution if $|\mathbf{A}| \neq 0$.

Now
$$\begin{vmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{vmatrix} = 2\begin{vmatrix} -k & 1 \\ -1 & k \end{vmatrix} + (-1)\begin{vmatrix} 1 & 3 \\ k & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -k \\ 5 & -1 \end{vmatrix}$$
$$= 2(-k^2 - -1) - 1(5 - 3k) - 4(-3 - -5k)$$
$$= -2k^2 + 2 - 5 + 3k + 12 - 20k$$
$$= -2k^2 - 17k + 9$$
$$= -(2k^2 + 17k - 9)$$
$$= -(2k - 1)(k + 9)$$

$\therefore$ there is a unique solution for all k provided $k \neq \frac{1}{2}$ and $k \neq -9$.

6 **a** $\begin{vmatrix} 1 & k & 3 \\ k & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 7$ $\quad \therefore \quad 1\begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} + k\begin{vmatrix} -1 & k \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} k & 1 \\ 3 & 4 \end{vmatrix} = 7$

$$\therefore \quad 1(2 - -4) + k(-3 - 2k) + 3(4k - 3) = 7$$
$$\therefore \quad 6 - 3k - 2k^2 + 12k - 9 = 7$$
$$\therefore \quad 2k^2 - 9k + 10 = 0$$
$$\therefore \quad (2k - 5)(k - 2) = 0$$
and so $k = \frac{5}{2}$ or 2

b
$$\begin{vmatrix} k & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & k \end{vmatrix} = 0$$
$$\therefore \quad k\begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ k & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = 0$$
$$\therefore \quad k(k^2 - 4) + 2(2 - 2k) + (4 - k) = 0$$
$$\therefore \quad k^3 - 4k + 4 - 4k + 4 - k = 0$$
$$\therefore \quad k^3 - 9k + 8 = 0$$

Using technology there is one rational zero, $k = 1$

$$\therefore \quad (k - 1)(k^2 + k - 8) = 0$$
$$\therefore \quad k = 1 \quad \text{or} \quad k = \frac{-1 \pm \sqrt{1 - 4(1)(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2}$$

7 Using technology,

a
$$\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{vmatrix} = 16$$

$$\text{inverse} = \begin{bmatrix} -\frac{21}{16} & -\frac{17}{16} & \frac{5}{4} & \frac{11}{16} \\ -\frac{17}{16} & -\frac{29}{16} & \frac{5}{4} & \frac{15}{16} \\ \frac{5}{4} & \frac{5}{4} & -1 & -\frac{3}{4} \\ \frac{11}{16} & \frac{15}{16} & -\frac{3}{4} & \frac{-5}{16} \end{bmatrix}$$

b
$$\begin{vmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{vmatrix} = -34$$

$$\text{inverse} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -1 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{15}{34} & \frac{1}{2} & -\frac{4}{17} & \frac{29}{34} & -\frac{23}{34} \\ -\frac{29}{34} & \frac{3}{2} & -\frac{61}{17} & \frac{149}{34} & -\frac{83}{34} \\ \frac{39}{34} & -\frac{3}{2} & \frac{58}{17} & -\frac{157}{34} & \frac{87}{34} \\ \frac{1}{17} & 0 & -\frac{4}{17} & \frac{6}{17} & -\frac{3}{17} \end{bmatrix}$$

8 **a** Let o, a, p, c, l represent (respectively) the cost (in dollars) of each orange, apple, pear, cabbage and lettuce. The system, in matrix form, becomes:

$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 5 & 2 & 2 \end{bmatrix}} \underset{\mathbf{X}}{\begin{bmatrix} o \\ a \\ p \\ c \\ l \end{bmatrix}} = \underset{\mathbf{B}}{\begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 10.9 \end{bmatrix}}$$

b $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

Using technology, $|\mathbf{A}| = 0$, $\therefore$ $\mathbf{A}^{-1}$ does not exist and $\mathbf{X}$ cannot be found using this information.

c If the last line is amended, the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \det\mathbf{A} = 6$$

$$\therefore \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 9.2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.7 \\ 2 \\ 1.5 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ oranges cost 50 cents, apples cost 80 cents, pears cost 70 cents, cabbages cost \$2.00, and lettuces cost \$1.50

EXERCISE 14K

1 **a** $\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}$

2 **a**

$$\mathbf{B}^{-1} = \begin{bmatrix} 0.050\,23 & -0.011\,48 & -0.066\,34 \\ 4.212 \times 10^{-4} & 0.013\,53 & 0.027\,75 \\ -0.029\,90 & 0.039\,33 & 0.030\,06 \end{bmatrix} \doteqdot \begin{bmatrix} 0.050 & -0.011 & -0.066 \\ 0.000 & 0.014 & 0.028 \\ -0.030 & 0.039 & 0.030 \end{bmatrix}$$

b

$$\mathbf{B}^{-1} = \begin{bmatrix} 1.596 & -0.9964 & -0.1686 \\ -3.224 & 1.925 & 0.6291 \\ 2.000 & -1.086 & -0.3958 \end{bmatrix} \doteqdot \begin{bmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{bmatrix}$$

EXERCISE 14L

1 **a**

$$\left.\begin{aligned} x - y - z &= 2 \\ x + y + 3z &= 7 \\ 9x - y - 3z &= -1 \end{aligned}\right\} \text{ has matrix equation } \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$$

b

$$\left.\begin{aligned} 2x + y - z &= 3 \\ y + 2z &= 6 \\ x - y + z &= 13 \end{aligned}\right\} \text{ has matrix equation } \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix}$$

c

$$\left.\begin{aligned} a + b - c &= 7 \\ a - b + c &= 6 \\ 2a + b - 3c &= -2 \end{aligned}\right\} \text{ has matrix equation } \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -2 \end{bmatrix}$$

2 $$\mathbf{AB} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}\begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 8-1-6 & 14-2-12 & -6+1+5 \\ -4-2+6 & -7-4+12 & 3+2-5 \\ 0-6+6 & 0-12+12 & 0+6-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \therefore \quad \mathbf{AB} = \mathbf{I} \quad \text{and so} \quad \mathbf{A} = \mathbf{B}^{-1}$$

$$\left.\begin{aligned} 4a+7b-3c &= -8 \\ -a-2b+c &= 3 \\ 6a+12b-5c &= -15 \end{aligned}\right\} \text{ has matrix equation } \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}^{-1}\begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -16+3+15 \\ 8+6-15 \\ 0+18-15 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \qquad \therefore \quad a = 2, \quad b = -1, \quad c = 3$$

3 $$\mathbf{MN} = \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15+3-14 & 10-3-7 & 15+6-21 \\ -3-3+6 & -2+3+3 & -3-6+9 \\ -9-1+10 & -6+1+5 & -9-2+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \therefore \quad \mathbf{MN} = 4\mathbf{I} \quad \therefore \quad \left(\tfrac{1}{4}\mathbf{M}\right)\mathbf{N} = \mathbf{I} \quad \text{and so} \quad \tfrac{1}{4}\mathbf{M} = \mathbf{N}^{-1}$$

Now $$\left.\begin{aligned} 3u+2v+3w &= 18 \\ u-v+2w &= 6 \\ 2u+v+3w &= 16 \end{aligned}\right\} \text{ has matrix equation } \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix}\begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 90+18-112 \\ -18-18+48 \\ -54-6+80 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 12 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \qquad \therefore \quad u = -1, \quad v = 3, \quad w = 5$$

4 **a** $$\begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix}$$

Using technology, $x = \frac{23}{10}$, $y = \frac{13}{10}$, $z = -\frac{9}{2}$

b $\begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$$

Using technology,

$x = -\frac{1}{3}$,

$y = -\frac{95}{21}$,

$z = \frac{2}{21}$

c $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}$$

Using technology,

$x = 2$,

$y = 4$,

$z = -1$

5 **a** $x = 2, \quad y = -1, \quad z = 5$

b $x = 4, \quad y = -2, \quad z = 1$

c $x = 4, \quad y = -3, \quad z = 2$

d $x = 4, \quad y = 6, \quad z = -7$

e $x = 3, \quad y = 11, \quad z = -7$

f $x = 0.326, \quad y = 7.652, \quad z = 4.156$

6 **a** Let x be the cost of a football in dollars,
y be the cost of baseball in dollars, and
z be the cost of a basketball in dollars.

Using technology, $x = 14, \quad y = 11, \quad z = 17.$

b Cost of 4 footballs and 5 baseballs is $4x + 5y = 4(14) + 5(11) = 111$, i.e., \$111

$\therefore$ amount left for basketballs is \$315 – \$111 = \$204

Number of basketballs bought $= \dfrac{204}{17} = 12$ i.e., 12 basketballs are bought.

7 **a** System of equations is:

$$\begin{aligned} 2x + 3y + 8z &= 352 \\ x + 5y + 4z &= 274 \\ x + 2y + 11z &= 351 \end{aligned}$$

b Using technology, $x = 42, \quad y = 28, \quad z = 23,$
i.e., the salaries are: manager \$42 000, clerk \$28 000 and labourer \$23 000.

c Salary bill is

$$\begin{aligned} & 3x + 8y + 37z \\ &= 3(42) + 8(28) + 37(23) \\ &= 1201 \quad \text{(thousands of dollars)} \end{aligned}$$

i.e., \$1 201 000

8 Let x be the cost in dollars of 1 kg of cashews,
y be the cost in dollars of 1 kg of macadamias, and
z be the cost in dollars of 1 kg of Brazil nuts.

The cost of 1 kg of mix A is $0.5x + 0.3y + 0.2z = 12.5$,
the cost of 1 kg of mix B is $0.2x + 0.4y + 0.4z = 12.4$,
the cost of 1 kg of mix C is $0.6x + 0.1y + 0.3z = 11.7$.

Using technology, $x = 12, \quad y = 15$ and $z = 10$
i.e., the cost of 1 kg of cashews is \$12, the cost of 1 kg of macadamias is \$15 and the cost of 1 kg of Brazil nuts is \$10.

Cost per kg of 400 g cashews, 200 g macadamias and 400 g Brazil nuts
$= 0.4 \times 12 + 0.2 \times 15 + 0.4 \times 10$ dollars
$=$ \$11.80

9 **a** Number of students who study Chemistry is $\frac{1}{3}p + \frac{1}{3}q + \frac{2}{5}r = 27$ ..... (1)

number of students who study Maths is $\frac{1}{2}p + \frac{2}{3}q + \frac{1}{5}r = 35$ ..... (2)

number of students who study Geography is $\frac{1}{4}p + \frac{1}{3}q + \frac{3}{5}r = 30$ ..... (3)

The required system of equations is

$$\begin{aligned} 5p + 5q + 6r &= 405 & \{(1) \times 15\} \\ 15p + 20q + 6r &= 1050 & \{(2) \times 30\} \\ 15p + 20q + 36r &= 1800 & \{(3) \times 60\} \end{aligned}$$

b Using technology, $p = 24$, $q = 27$, $r = 25$.

10 **a** As t is the number of years after 2000, then

profit in year 2000 is $P(0) = b + \dfrac{c}{4} = 160\,000$

profit in year 2001 is $P(1) = a + b + \dfrac{c}{5} = 198\,000$

profit in year 2002 is $P(2) = 2a + b + \dfrac{c}{6} = 240\,000$

Using technology, $a = 50\,000$, $b = 100\,000$ and $c = 240\,000$.

b Using the model given, the profit in 1999 would be

$$P(-1) = -a + b + \frac{c}{3} - 50\,000 + 100\,000 + 80\,000 = 130\,000$$

i.e., the profit would be \$130 000, which fits the model.

c Predicted profit in 2003 is $P(3) = 3a + b + \dfrac{c}{7} = 3(50\,000) + 100\,000 + \dfrac{240\,000}{7}$

$\doteqdot \$284\,000$

Predicted profit in 2005 is $P(5) = 5a + b + \dfrac{c}{9} = 5(50\,000) + 100\,000 + \dfrac{240\,000}{9}$

$\doteqdot \$377\,000$

EXERCISE 14M.1

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 4 & 1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 9 & -27 \end{array}\right] \quad \leftarrow \text{Replace } R_2 \text{ with } R_2 - 4R_1$$

$$\begin{array}{rrr} 4 & 1 & 5 \\ -4 & 8 & -32 \\ \hline 0 & 9 & -27 \end{array}$$

From R_2, $9y = -27$

i.e., $y = -3$

Now $x - 2y = 8$

$\therefore\ x - 2(-3) = 8$

$\therefore\ x = 2$

So the solution is $x = 2$, $y = -3$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 4 & 5 & 21 \\ 5 & -3 & -20 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 4 & 5 & 21 \\ 0 & -37 & -185 \end{array}\right] \quad \leftarrow \text{Replace } R_2 \text{ with } 4R_2 - 5R_1$$

$$\begin{array}{rrr} 20 & -12 & -80 \\ -20 & -25 & -105 \\ \hline 0 & -37 & -185 \end{array}$$

From R_2, $-37y = -185$

i.e., $y = 5$

Now $4x + 5y = 21$

$\therefore\ 4x + 25 = 21$

$\therefore\ 4x = -4$

$\therefore\ x = -1$

So the solution is $x = -1$, $y = 5$.

c In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & 1 & -10 \\ 2 & 5 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & 1 & -10 \\ 0 & 13 & -52 \end{array}\right] \leftarrow \text{Replace } R_2 \text{ with } 3R_2 - 2R_1 \qquad \begin{array}{rrr} 6 & 15 & -72 \\ -6 & -2 & 20 \\ \hline 0 & 13 & -52 \end{array}$$

From R_2, $$\begin{aligned} 13y &= -52 \\ \text{i.e., } y &= -4 \end{aligned}$$

$$\begin{aligned} \text{Now } 3x + y &= -10 \\ \therefore\ 3x + (-4) &= -10 \\ \therefore\ 3x &= -6 \\ \therefore\ x &= -2 \end{aligned}$$

So the solution is $x = -2$, $y = -4$.

2 **a** One equation is not a multiple of the other and their gradients are not the same, so the lines are intersecting.

b $x + y = 7$ can be written as $3x + 3y = 21$ and the other line is $3x + 3y = 1$, $\therefore$ the lines are parallel.

c The lines intersect at $(2\frac{1}{2}, 2)$.

d $x - 2y = 4$ can be written as $2x - 4y = 8$, so the lines are coincident.

e The lines are intersecting.

f $3x - 4y = 5$ can be written as $-3x + 4y = -5$ and the other line is $-3x + 4y = 2$, $\therefore$ the lines are parallel.

3 **a** $x + 2y = 3$ can be written as $2x + 4y = 6$, $\therefore$ the equations represent coincident lines, i.e., there are an infinite number of solutions (all the points on the line).

b As the second equation is an exact multiple of the first, it will give the same solutions as the first so it can be ignored.

c **i** If $x = t$, $$\begin{aligned} t + 2y &= 3 \\ \therefore\ 2y &= 3 - t \\ \therefore\ y &= \frac{3-t}{2}, \end{aligned}$$ i.e., the solutions are $x = t$, $y = \frac{3-t}{2}$, $t \in \mathcal{R}$.

ii If $y = s$, $$\begin{aligned} x + 2s &= 3 \\ \therefore\ x &= 3 - 2s, \end{aligned}$$ i.e., the solutions are $x = 3 - 2s$, $y = s$, $s \in \mathcal{R}$.

4 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 2 & 3 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right] \leftarrow \text{Replace } R_2 \text{ with } R_2 - R_1 \qquad \begin{array}{rrr} 2 & 3 & 11 \\ -2 & -3 & -5 \\ \hline 0 & 0 & 6 \end{array}$$

R_2 shows $0x + 0y = 6$ i.e., there are no solutions {as the lines are parallel}

b In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & 6 & 10 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{Replace } R_2 \text{ with } R_2 - 2R_1 \qquad \begin{array}{rrr} 4 & 6 & 10 \\ -4 & -6 & -10 \\ \hline 0 & 0 & 0 \end{array}$$

R_2 shows $0x + 0y = 0$, which is true for all x and y

i.e., there are infinitely many solutions {as the lines are coincident}

5 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ 6 & -2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{Replace } R_2 \text{ by } R_2 - 2R_1 \qquad \begin{array}{ccc} 6 & -2 & 4 \\ -6 & 2 & -4 \\ \hline 0 & 0 & 0 \end{array}$$

R_2 shows $0x + 0y = 0$, which is true for all x and y

i.e., there are infinitely many solutions {as the lines are coincident}

Substitute $x = t$ in the first equation

$$\begin{aligned} 3x - y &= 2 \\ \therefore \quad 3t - y &= 2 \\ y &= 3t - 2 \end{aligned}$$

i.e., the solutions have form $x = t$, $y = 3t - 2$, t in $\mathcal{R}$.

b

$$\begin{aligned} 3x - y &= 2 \quad \text{...... (1)} \\ 6x - 2y &= k \quad \text{...... (2)} \end{aligned}$$

If $k = 4$ then $6x - 2y = 4$, which is an exact multiple (×2) of equation (1), $\therefore$ the lines are coincident and there are an infinite number of solutions of the form $x = t$, $y = 3t - 2$.

If $k \neq 4$ then the equations represent parallel lines $\therefore$ there are no solutions.

6 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 3 & -1 & 8 \\ 6 & -2 & k \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 3 & -1 & 8 \\ 0 & 0 & k-16 \end{array}\right] \leftarrow \text{Replace } R_2 \text{ by } R_2 - 2R_1 \qquad \begin{array}{ccc} 6 & -2 & k \\ -6 & 2 & -16 \\ \hline 0 & 0 & k-16 \end{array}$$

b If $k - 16 = 0$, i.e., $k = 16$, there are infinitely many solutions.

c Substitute $x = t$ in $3x - y = 8$, then $3t - y = 8$ i.e., $y = 3t - 8$.
The solutions are $x = t$, $y = 3t - 8$, t in $\mathcal{R}$.

d The system has no solutions when $k - 16 \neq 0$, i.e., $k \neq 16$.

7 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 4 & 8 & 1 \\ 2 & -a & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 4 & 8 & 1 \\ 0 & -2a-8 & 21 \end{array}\right] \leftarrow \text{Replace } R_2 \text{ by } 2R_2 - R_1 \qquad \begin{array}{ccc} 4 & -2a & 22 \\ -4 & -8 & -1 \\ \hline 0 & -2a-8 & 21 \end{array}$$

b A unique solution exists provided $-2a - 8 \neq 0$ i.e., provided $a \neq -4$.

c From R_2,

$$\begin{aligned} (-2a-8)y &= 21 \\ \therefore \quad y &= \frac{-21}{2a+8} \\ \text{and} \quad 4x + 8y &= 1 \\ \therefore \quad 4x + 8\left(\frac{-21}{2a+8}\right) &= 1 \end{aligned}$$

$$\begin{aligned} \therefore \quad 4x(2a+8) - 168 &= 2a + 8 \\ \therefore \quad 2x(2a+8) - 84 &= a + 4 \\ \therefore \quad 2x(2a+8) &= a + 88 \\ \therefore \quad x &= \frac{a+88}{4a+16} \end{aligned}$$

The solution is $x = \dfrac{a+88}{4a+16}$, $y = \dfrac{-21}{2a+8}$, $a \neq -4$.

d When $a = -4$ there are no solutions as the lines are parallel.

8 In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} m & 2 & 6 \\ 2 & m & 6 \end{array}\right]$$

$$\begin{array}{ccc} 2m & m^2 & 6m \\ -2m & -4 & -12 \\ \hline 0 & m^2-4 & 6m-12 \end{array}$$

$$\sim \left[\begin{array}{cc|c} m & 2 & 6 \\ 0 & m^2-4 & 6m-12 \end{array}\right] \quad \leftarrow \text{Replace } R_2 \text{ by} \quad mR_2 - 2R_1$$

A unique solution exists provided $m^2 - 4 \neq 0$

i.e., there is a unique solution for all m except $m = \pm 2$.

a In R_2, $(m^2-4)y = 6m - 12$

$$\therefore \quad y = \frac{6(m-2)}{(m-2)(m+2)}$$

$$\therefore \quad y = \frac{6}{m+2} \quad \text{provided} \quad m \neq \pm 2$$

Substituting in $mx + 2y = 6$

gives $mx + 2\left(\dfrac{6}{m+2}\right) = 6$

$$\begin{aligned} \therefore \quad m(m+2)x + 12 &= 6(m+2) \\ \therefore \quad m(m+2)x &= 6m + 12 - 12 \\ \therefore \quad m(m+2)x &= 6m \\ \therefore \quad x &= \frac{6}{m+2} \end{aligned}$$

i.e., the unique solution is $x = \dfrac{6}{m+2}$, $y = \dfrac{6}{m+2}$ when $x \neq \pm 2$.

b When $m = 2$, the equations are $2x + 2y = 6$ and $2x + 2y = 6$, i.e., the lines are coincident so there are an infinite number of solutions of the form $x = t$, $y = \dfrac{6-2t}{2} = 3 - t$ for all t in $\mathcal{R}$.

When $m = -2$, the equations are $-2x + 2y = 6$ and $2x - 2y = 6$

i.e., $-2x + 2y = -6$

$\therefore$ the lines are parallel and there are no solutions.

EXERCISE 14M.2

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & 5 \\ 2 & 3 & 1 & 6 \end{array}\right]$$

$$\begin{array}{cccc} 2 & 4 & 1 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & 2 & -1 & -7 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 1 & -1 & -6 \end{array}\right] \quad \begin{array}{l} \leftarrow R_2 \to R_2 - 2R_1 \\ \leftarrow R_3 \to R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 3 & 1 & 6 \\ -2 & -2 & -2 & -12 \\ \hline 0 & 1 & -1 & -6 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -1 & -5 \end{array}\right] \quad \leftarrow R_3 \to 2R_3 - R_2$$

$$\begin{array}{cccc} 0 & 2 & -2 & -12 \\ 0 & -2 & 1 & 7 \\ \hline 0 & 0 & -1 & -5 \end{array}$$

The last row gives $-z = -5$ i.e., $z = 5$

$$\begin{aligned} \therefore \quad \text{in row 2,} \quad 2y - z &= -7 \\ \therefore \quad 2y - 5 &= -7 \\ \therefore \quad 2y &= -2 \\ \therefore \quad y &= -1 \end{aligned}$$

and in row 1,

$$\begin{aligned} x + y + z &= 6 \\ \therefore \quad x - 1 + 5 &= 6 \\ \therefore \quad x + 4 &= 6 \\ \therefore \quad x &= 2 \end{aligned}$$

Thus we have a unique solution $x = 2$, $y = -1$, $z = 5$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 1 & 6 & 17 & 9 \\ 1 & 4 & 8 & 4 \end{array}\right] \qquad \begin{array}{rrrr} 1 & 6 & 17 & 9 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 2 & 6 & 2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -3 & -3 \end{array}\right] \begin{array}{l} \leftarrow R_2 \to R_2 - R_1 \\ \leftarrow R_3 \to R_3 - R_1 \end{array} \qquad \begin{array}{rrrr} 1 & 4 & 8 & 4 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 0 & -3 & -3 \end{array}$$

The last row gives $-3z = -3 \qquad \therefore \quad z = 1$

$\therefore$ in row 2, $2y + 6z = 2$

$\therefore \quad 2y + 6 = 2$

$\therefore \quad y = -2$

and in row 1, $x + 4y + 11z = 7$

$\therefore \quad x + 4(-2) + 11(1) = 7$

$\therefore \quad x + 3 = 7$

$\therefore \quad x = 4$

Thus we have a unique solution $x = 4, \quad y = -2, \quad z = 1$

c In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 2 & -2 & -5 & 4 \\ 3 & 2 & 2 & 10 \end{array}\right] \qquad \begin{array}{rrrr} 2 & -2 & -5 & 4 \\ -2 & 1 & -3 & -17 \\ \hline 0 & -1 & -8 & -13 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 7 & -5 & -31 \end{array}\right] \begin{array}{l} \leftarrow R_2 \to R_2 - R_1 \\ \leftarrow R_3 \to 2R_3 - 3R_1 \end{array} \qquad \begin{array}{rrrr} 6 & 4 & 4 & 20 \\ -6 & 3 & -9 & -51 \\ \hline 0 & 7 & -5 & -31 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 0 & -61 & -122 \end{array}\right] \leftarrow R_3 \to R_3 + 7R_2 \qquad \begin{array}{rrrr} 0 & 7 & -5 & -31 \\ 0 & -7 & -56 & -91 \\ \hline 0 & 0 & -61 & -122 \end{array}$$

The last row gives $-61z = -122$

$\therefore \quad z = 2$

$\therefore$ in row 2, $-y - 8z = -13$

$\therefore \quad -y - 16 = -13$

$\therefore \quad y = -3$

and in row 1, $2x - y + 3z = 17$

$\therefore \quad 2x + 3 + 6 = 17$

$\therefore \quad 2x = 8$

$\therefore \quad x = 4$

Thus we have a unique solution $x = 4, \quad y = -3, \quad z = 2.$

EXERCISE 14M.3

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -4 & 8 & 2 \\ -3 & 6 & 7 & -3 \end{array}\right] \qquad \begin{array}{rrrr} 2 & -4 & 8 & 2 \\ -2 & 4 & -10 & -2 \\ \hline 0 & 0 & -2 & 0 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 22 & 0 \end{array}\right] \begin{array}{l} \leftarrow R_2 \to R_2 - 2R_1 \\ \leftarrow R_3 \to R_3 + 3R_1 \end{array} \qquad \begin{array}{rrrr} -3 & 6 & 7 & -3 \\ 3 & -6 & 15 & 3 \\ \hline 0 & 0 & 22 & 0 \end{array}$$

Rows 2 and 3 show $-2z = 0$ and $22z = 0$ i.e., $z = 0$

Row 1 becomes $x - 2y + 5\,(0) = 1$

let $y = t$, then $x - 2t = 1$

i.e., $x = 1 + 2t$

$\therefore$ there are infinitely many solutions of the form $x = 1 + 2t, \; y = t, \; z = 0 \quad$ (t is in $\mathcal{R}$)

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4\\ 3 & 2 & 1 & 7\\ 5 & 2 & 3 & 11\end{array}\right]$$

$$\begin{array}{rrrr} 3 & 2 & 1 & 7\\ -3 & -6 & 3 & -12\\ \hline 0 & -4 & 4 & -5\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4\\ 0 & -4 & 4 & -5\\ 0 & -8 & 8 & -9\end{array}\right] \begin{array}{l} \leftarrow R_2 \to R_2 - 3R_1\\ \leftarrow R_3 \to R_3 - 5R_1\end{array}$$

$$\begin{array}{rrrr} 5 & 2 & 3 & 11\\ -5 & -10 & 5 & -20\\ \hline 0 & -8 & 8 & -9\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4\\ 0 & -4 & 4 & -5\\ 0 & 0 & 0 & 1\end{array}\right] \leftarrow R_3 \to R_3 - 2R_2$$

$$\begin{array}{rrrr} 0 & -8 & 8 & -9\\ 0 & 8 & -8 & 10\\ \hline 0 & 0 & 0 & 1\end{array}$$

In row 3, $0z = 1$ which has no solution. $\therefore$ the system has no real solutions.

c In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 1\\ 3 & 5 & 0 & 1\\ 5 & 13 & 7 & 4\end{array}\right]$$

$$\begin{array}{rrrr} 6 & 10 & 0 & 2\\ -6 & -12 & -3 & -3\\ \hline 0 & -2 & -3 & -1\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & 1\\ 0 & -2 & -3 & -1\\ 0 & 6 & 9 & 3\end{array}\right] \begin{array}{l} R_2 \to 2R_2 - 3R_1\\ R_3 \to 2R_3 - 5R_1\end{array}$$

$$\begin{array}{rrrr} 10 & 26 & 14 & 8\\ -10 & -20 & -5 & -5\\ \hline 0 & 6 & 9 & 3\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & 1\\ 0 & -2 & -3 & -1\\ 0 & 0 & 0 & 0\end{array}\right] R_3 \to R_3 + 3R_2$$

$$\begin{array}{rrrr} 0 & 6 & 9 & 3\\ 0 & -6 & -9 & -3\\ \hline 0 & 0 & 0 & 0\end{array}$$

The row of zeros indicates infinitely many solutions.

If we let $z = t$ in row 2, $-2y - 3t = -1$

$$\therefore \quad 2y = 1 - 3t$$

$$\therefore \quad y = \frac{1-3t}{2}$$

Thus in equation 1, $2x + 4\left(\dfrac{1-3t}{2}\right) + t = 1$

$$\therefore \quad 2x + 2(1-3t) + t = 1$$

$$\therefore \quad 2x + 2 - 6t + t = 1$$

$$\therefore \quad 2x = 5t - 1 \quad \text{and so} \quad x = \frac{5t-1}{2}$$

$\therefore$ the solutions have form $x = \dfrac{5t-1}{2}$, $y = \dfrac{1-3t}{2}$, $z = t$, for all t in $\mathcal{R}$.

d In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1\\ 5 & 6 & 7 & 2\\ 8 & 9 & 10 & 4\end{array}\right]$$

$$\begin{array}{rrrr} 10 & 12 & 14 & 4\\ -10 & -15 & -20 & -5\\ \hline 0 & -3 & -6 & -1\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1\\ 0 & -3 & -6 & -1\\ 0 & -3 & -6 & 0\end{array}\right] \begin{array}{l} R_2 \to 2R_2 - 5R_1\\ R_3 \to R_3 - 4R_1\end{array}$$

$$\begin{array}{rrrr} 8 & 9 & 10 & 4\\ -8 & -12 & -16 & -4\\ \hline 0 & -3 & -6 & 0\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1\\ 0 & -3 & -6 & -1\\ 0 & 0 & 0 & 1\end{array}\right] R_3 \to R_3 - R_2$$

In row 3, $0z = 1$ which has no solution. $\therefore$ the system has no real solutions.

2 In augmented matrix form, the system is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 4 & 1 \\ 1 & 7 & -1 & k \end{array}\right]$

a $$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 4 & 1 \\ 1 & 7 & -1 & k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 5 & -2 & k-3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 2 & -1 & 4 & 1 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -5 & 2 & -5 \end{array} \qquad \begin{array}{rrrr} 1 & 7 & -1 & k \\ -1 & -2 & -1 & -3 \\ \hline 0 & 5 & -2 & k-3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 0 & k-8 \end{array}\right] \quad R_3 \to R_3 + R_2$$

b If $k - 8 \neq 0$, we have $0z \neq 0$, so the system is inconsistent and there are no solutions.

If $k - 8 = 0$, we have $0z = 0$, which is true for all z, so we have infinitely many solutions.

Let $z = t$ in row 2, then $-5y + 2t = -5 \quad \therefore \quad 5y = 2t + 5$

$$\therefore \quad y = \frac{2t+5}{5}$$

and substituting in row 1, $x + 2\left(\dfrac{2t+5}{5}\right) + t = 3$

$$\therefore \quad x + \tfrac{4}{5}t + 2 + t = 3$$

$$\therefore \quad x = 1 - \tfrac{9}{5}t = \frac{5-9t}{5}$$

$\therefore$ the solutions have form $x = \dfrac{5-9t}{5}$, $y = \dfrac{2t+5}{5}$, $z = t$, t in $\mathcal{R}$.

c There is no unique solution because the system reduces to 2 equations in 3 unknowns when $k = 8$.

3 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 1 & -1 & 3 & -1 \\ 1 & -7 & k & -k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & -9 & k+2 & -k-5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 1 & -1 & 3 & -1 \\ -1 & -2 & 2 & -5 \\ \hline 0 & -3 & 5 & -6 \end{array} \qquad \begin{array}{rrrr} 1 & -7 & k & -k \\ -1 & -2 & 2 & -5 \\ \hline 0 & -9 & k+2 & -k-5 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & 0 & k-13 & 13-k \end{array}\right] \quad R_3 \to R_3 + 3R_2$$

$$\begin{array}{rrrr} 0 & -9 & k+2 & -k-5 \\ 0 & 9 & -15 & 18 \\ \hline 0 & 0 & k-13 & 13-k \end{array}$$

b If $k = 13$, row 3 is a row of zeros, so there are infinitely many solutions.

Let $z = t$ in row 2, then $-3y + 5t = -6 \quad \therefore \quad 3y = 6 + 5t$

$$\therefore \quad y = \frac{6+5t}{3}$$

and substituting in row 1 gives $x + 2\left(\dfrac{6+5t}{3}\right) - 2t = 5$

$$\therefore \quad x + 4 + \tfrac{10}{3}t - 2t = 5$$

$$\therefore \quad x = 1 - \tfrac{4}{3}t = \frac{3-4t}{3}$$

i.e., there are infinitely many solutions of the form $x = \dfrac{3-4t}{3}$, $y = \dfrac{6+5t}{3}$, $z = t$, t in $\mathcal{R}$.

c If $k \neq 13$, then $(k-13)z = 13-k \quad \therefore \quad z = \dfrac{13-k}{k-13} = -1$

From row 2, $-3y + 5(-1) = -6 \quad \therefore \quad -3y - 5 = -6 \quad \therefore \quad y = \frac{1}{3}$

and from row 1, $x + 2y - 2z = x + \frac{2}{3} + 2 = 5 \quad \therefore \quad x = 2\frac{1}{3}$

i.e., the unique solution is $x = \frac{7}{3}, \quad y = \frac{1}{3}, \quad z = -1.$

4 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array}\right] \qquad \begin{array}{cccc} 2 & -1 & 1 & 7 \\ -2 & -6 & -6 & -2a+2 \\ \hline 0 & -7 & -5 & 9-2a \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 3 & -5 & a & 16 \\ -3 & -9 & -9 & -3a+3 \\ \hline 0 & -14 & a-9 & 19-3a \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array} \qquad \begin{array}{cccc} 0 & -14 & a-9 & 19-3a \\ 0 & 14 & 10 & 4a-18 \\ \hline 0 & 0 & a+1 & a+1 \end{array}$$

b If $a = -1$, row 3 is a row of zeros, so there are infinitely many solutions, as we have 2 equations in 3 unknowns.

Let $z = t$ in row 2, then $-7y - 5t = 9 - 2(-1) \quad \therefore \quad -7y - 5t = 11$

$$\therefore \quad y = \frac{-5t-11}{7}$$

and substituting in row 1 gives $x + 3\left(\dfrac{-5t-11}{7}\right) + 3t = (-1) - 1$

$$\therefore \quad x - \frac{15t}{7} - \frac{33}{7} + 3t = -2 \text{ and so, } x = \frac{19}{7} - \frac{6t}{7}$$

i.e., there are infinitely many solutions of form $x = \dfrac{19-6t}{7}$, $y = \dfrac{-5t-11}{7}$, $z = t$, t in $\mathcal{R}$.

c If $a \neq -1$, then $(a+1)z = a+1 \quad \therefore \quad z = 1$

From row 2, $-7y - 5(1) = 9 - 2a \quad \therefore \quad -7y = -2a + 14$

$$y = \frac{2a-14}{7}$$

and substituting in row 1 gives $x + 3\left(\dfrac{2a-14}{7}\right) + 3 = a - 1$

$$\therefore \quad x + \frac{6a}{7} - 6 + 3 = a - 1 \quad \therefore \quad x = \frac{a}{7} + 2 = \frac{a+14}{7}$$

i.e., the unique solution is $x = \dfrac{a+14}{7}, \quad y = \dfrac{2a-14}{7}, \quad z = 1.$

5 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ m & -2 & 1 & 1 \\ 1 & 2 & m & -1 \end{array}\right] \qquad \begin{array}{cccc} 2m & -4 & 2 & 2 \\ -2m & -m & m & -3m \\ \hline 0 & -m-4 & m+2 & -3m+2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -m-4 & m+2 & -3m+2 \\ 0 & 3 & 2m+1 & -5 \end{array}\right] \begin{array}{l} \\ R_2 \to 2R_2 - mR_1 \\ R_3 \to 2R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 2 & 4 & 2m & -2 \\ -2 & -1 & 1 & -3 \\ \hline 0 & 3 & 2m+1 & -5 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -m-4 & m+2 & -3m+2 \\ 0 & 0 & 2(m+1)(m+5) & -14(m+1) \end{array}\right] \begin{array}{l} \\ \\ R_3 \to (m+4)R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 3m+12 & (m+4)(2m+1) & -5(m+4) \\ 0 & -3m-12 & 3(m+2) & -9m+6 \\ \hline 0 & 0 & 2m^2+12m+10 & -14m-14 \\ & & =2(m^2+6m+5) & =-14(m+1) \\ & & =2(m+1)(m+5) & \end{array}$$

a If $m=-5$, row 3 becomes $0x+0y+0z=-56$, i.e., the system is inconsistent and there are no solutions.

b If $m=-1$, row 3 is a row of zeros, so we have 2 equations in 3 unknowns and $\therefore$ there are infinitely many solutions.

c If $m \neq -1$ and $m \neq -5$, then row 3 becomes

$$2(m+5)(m+1)z = -14(m+1)$$

$$\therefore \quad z = \frac{-7}{m+5} \qquad \text{and substituting in row 2 gives}$$

$$\begin{aligned} -(m+4)y+(m+2)\left(\frac{-7}{m+5}\right) &= -3m+2 \\ \therefore \quad -(m+4)(m+5)y-7(m+2) &= (-3m+2)(m+5) \\ \therefore \quad -(m+4)(m+5)y &= -3m^2-13m+10+7m+14 \\ \therefore \quad -(m+4)(m+5)y &= -3m^2-6m+24 \\ \therefore \quad -(m+4)(m+5)y &= -3(m^2+2m-8) \\ \therefore \quad (m+4)(m+5)y &= 3(m-2)(m+4) \\ \therefore \quad y &= \frac{3(m-2)}{m+5} \end{aligned}$$

and substituting in row 1 gives

$$\begin{aligned} 2x+\frac{3(m-2)}{m+5}-\frac{-7}{m+5} &= 3 \\ \therefore \quad 2x(m+5)+3(m-2)+7 &= 3(m+5) \\ \therefore \quad 2x(m+5)+3m-6+7 &= 3m+15 \\ \therefore \quad 2x(m+5) &= 14 \\ \therefore \quad x &= \frac{7}{m+5} \end{aligned}$$

$\therefore$ the system has a unique solution for all m except $m=-5$ and $m=-1$, and the solution is $x=\dfrac{7}{m+5}$, $y=\dfrac{3(m-2)}{m+5}$, $z=\dfrac{-7}{m+5}$.

6 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ k & -2 & 3 & k \\ 4 & -3 & 10 & 5 \end{array}\right] \qquad \begin{array}{cccc} k & -2 & 3 & k \\ -k & -3k & -k^2 & -2k \\ \hline 0 & -2-3k & 3-k^2 & -k \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2-3k & 3-k^2 & -k \\ 0 & -15 & 10-4k & -3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2-kR_1 \\ R_3 \to R_3-4R_1 \end{array} \qquad \begin{array}{cccc} 4 & -3 & 10 & 5 \\ -4 & -12 & -4k & -8 \\ \hline 0 & -15 & 10-4k & -3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2-3k & 3-k^2 & -k \\ 0 & 0 & (3k+25)(k-1) & 6(k-1) \end{array}\right] \begin{array}{l} \\ \\ R_3 \to (3k+2)R_3-15R_2 \end{array}$$

$$\begin{array}{cccc}
0 & -15(3k+2) & (3k+2)(10-4k) & -3(3k+2) \\
0 & 30+45k & 15k^2-45 & 15k \\ \hline
0 & 0 & -12k^2+22k+20+15k^2-45 & 6k-6 \\
 & & =3k^2+22k-25 & =6(k-1) \\
 & & =(3k+25)(k-1) &
\end{array}$$

b If $k=1$, row 3 is a row of zeros, so there are infinitely many solutions.

The system becomes $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -5 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Let $z=t$ in row 2, then

$$-5y+2t=-1$$
$$\therefore \quad -5y=-1-2t$$
$$\therefore \quad y=\frac{1+2t}{5}$$

and substituting in row 1 gives $x+3\left(\dfrac{1+2t}{5}\right)+t=2$

$$\therefore \quad x+\tfrac{3}{5}+\tfrac{6}{5}t+t=2$$
$$\therefore \quad x=\tfrac{7}{5}-\tfrac{11}{5}t=\frac{7-11t}{5}$$

$\therefore$ the solutions are of the form $x=\dfrac{7-11t}{5}$, $y=\dfrac{1+2t}{5}$, $z=t$, for all t in $\mathcal{R}$.

c If $k=-\frac{25}{3}$ the system is inconsistent $(0z=-56)$ $\therefore$ there are no real solutions.

d The system has a unique solution for all $k\neq-1$ or $-\frac{25}{3}$.

EXERCISE 14M.4

1 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -3 & 1 & 1 \end{array}\right] \quad R_2\to 2R_2-R_1 \qquad \begin{array}{cccc} 2 & -2 & 2 & 6 \\ -2 & -1 & -1 & -5 \\ \hline 0 & -3 & 1 & 1 \end{array}$$

Let $z=t$ in row 2, then $-3y+t=1$ $\therefore$ $-3y=1-t$ $\therefore$ $y=\dfrac{t-1}{3}$

and substituting in equation 1 gives $2x+\left(\dfrac{t-1}{3}\right)+t=5$

$$\therefore \quad 2x+\tfrac{1}{3}t-\tfrac{1}{3}+t=5$$
$$\therefore \quad 2x=\tfrac{16}{3}-\tfrac{4}{3}t$$
$$\therefore \quad x=\tfrac{8}{3}-\tfrac{2}{3}t=\frac{8-2t}{3}$$

$\therefore$ the solutions are $x=\dfrac{8-2t}{3}$, $y=\dfrac{t-1}{3}$, $z=t$, t in $\mathcal{R}$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 10 \\ 1 & -2 & 1 & -4 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 3 & 1 & 2 & 10 \\ 0 & -7 & 1 & -22 \end{array}\right] \quad R_2\to 3R_2-R_1 \qquad \begin{array}{cccc} 3 & -6 & 3 & -12 \\ -3 & -1 & -2 & -10 \\ \hline 0 & -7 & 1 & -22 \end{array}$$

Let $z=t$ in row 2, then $-7y+t=-22$ $\therefore$ $-7y=-t-22$

$$\therefore \quad y=\frac{t+22}{7}$$

and substituting in equation 1 gives $3x + \left(\frac{t+22}{7}\right) + 2t = 10$

$$\therefore \quad 3x + \tfrac{1}{7}t + \tfrac{22}{7} + 2t = 10$$

$$\therefore \quad 3x = \tfrac{48}{7} - \tfrac{15}{7}t$$

$$\therefore \quad x = \tfrac{16}{7} - \tfrac{5}{7}t = \frac{16-5t}{7}$$

$\therefore$ the solutions are $x = \frac{16-5t}{7}$, $y = \frac{t+22}{7}$, $z = t$, t in $\mathcal{R}$.

c In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 4 & 2 & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 0 & 0 & 6 \end{array}\right] \quad R_2 \to R_2 - 2R_1 \qquad \begin{array}{cccc} 2 & 4 & 2 & 16 \\ -2 & -4 & -2 & -10 \\ \hline 0 & 0 & 0 & 6 \end{array}$$

Row 2 shows $0z = 6$ which has no solution

$\therefore$ the system is inconsistent and has no real solutions.

2 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 7 & -4 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1 \qquad \begin{array}{cccc} 2 & 1 & -2 & 0 \\ -2 & 6 & -2 & 0 \\ \hline 0 & 7 & -4 & 0 \end{array}$$

Let $z = t$ in row 2, then $7y - 4t = 0$ $\quad\therefore\quad y = \frac{4}{7}t$

and substituting in equation 1 gives $x - 3(\frac{4}{7}t) + t = 0$ $\quad\therefore\quad x = -t + \frac{12}{7}t = \frac{5}{7}t$

$\therefore$ the solution is $x = \frac{5}{7}t$, $y = \frac{4}{7}t$, $z = t$ for t in $\mathcal{R}$

(or $x = 5s$, $y = 4s$, $z = 7s$ for s in $\mathcal{R}$)

To solve the new system, substitute the solution of the first two equations into the third equation.

$$\therefore \quad 3(\tfrac{5}{7}t) - (\tfrac{4}{7}t) + t = 18 \qquad \therefore \quad 15t - 4t + 7t = 126$$

$$\therefore \quad 18t = 126$$

$$\therefore \quad t = 7$$

$\therefore$ the solution is $x = 5$, $y = 4$, $z = 7$

3 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & -5 & 3 & 0 \end{array}\right] \quad R_2 \to 2R_2 - R_1 \qquad \begin{array}{cccc} 2 & -2 & 4 & 0 \\ -2 & -3 & -1 & 0 \\ \hline 0 & -5 & 3 & 0 \end{array}$$

Let $z = t$ in row 2, then $-5y + 3t = 0$

$$\therefore \quad y = \tfrac{3}{5}t$$

and substituting in equation 1 gives $2x + 3(\frac{3}{5}t) + t = 0$ $\quad\therefore\quad 2x = -\frac{14}{5}t$

$$\therefore \quad x = -\tfrac{7}{5}t$$

$\therefore$ the solution is $x = -\frac{7}{5}t$, $y = \frac{3}{5}t$, $z = t$ for t in $\mathcal{R}$

(or $x = -7s$, $y = 3s$, $z = 5s$ for s in $\mathcal{R}$)

To solve the new system, substitute the solution of the first two equations into the third equation.

$$\therefore \quad a(-\tfrac{7}{5}t) + (\tfrac{3}{5}t) - t = 0 \qquad \therefore \quad -7at + 3t - 5t = 0$$

$$-7at - 2t = 0$$

$$\therefore \quad t(7a+2) = 0$$

$$\therefore \quad t = 0 \quad \text{or} \quad a = -\tfrac{2}{7}$$

If $t = 0$, then the solution is $x = 0$, $y = 0$, $z = 0$.

If $a = -\frac{2}{7}$, then the solution is $x = -\frac{7}{5}t$, $y = \frac{3}{5}t$, $z = t$ for t in $\mathcal{R}$

(or $x = -7s$, $y = 3s$, $z = 5s$ for s in $\mathcal{R}$)

4 **a** $P(x) = ax^2 + bx + c$ in thousands of dollars where x is in thousands.

Profit is $8 \times \$1000$ when 1×1000 items are produced.

$\therefore \quad P(1) = a + b + c = 8$

and profit is $17 \times \$1000$ when 4×1000 items are produced

$\therefore \quad P(4) = 16a + 4b + c = 17$

i.e., $a + b + c = 8$ and $16a + 4b + c = 17$

b If $a = t$, $b = 3 - 5t$, $c = 5 + 4t$,

then $a + b + c = t + 3 - 5t + 5 + 4t = 8$

and $16a + 4b + c = 16t + 4(3-5t) + (5+4t) = 16t + 12 - 20t + 5 + 4t = 17$

$\therefore \quad a = t$, $b = 3 - 5t$, $c = 5 + 4t$ are the possible solutions for the system.

c
$$\begin{aligned} \text{Now} \quad P(x) &= tx^2 + (3-5t)x + (5+4t) \\ \text{and} \quad P(2.5) &= t(2.5)^2 + (3-5t)(2.5) + 5 + 4t \\ &= 6.25t + 7.5 - 12.5t + 5 + 4t \\ &= -2.25t + 12.5 \end{aligned}$$

$$\text{But} \quad P(2.5) = 19.75 \quad \therefore \quad -2.25t + 12.5 = 19.75$$

$$\therefore \quad -2.25t = 7.25$$

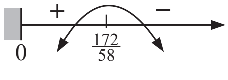
$$\therefore \quad t = -\frac{7.25}{2.25} = -\tfrac{29}{9}$$

$$\therefore \quad a = -\tfrac{29}{9}, \quad b = 3 - 5(-\tfrac{29}{9}) = \tfrac{172}{9} \quad \text{and} \quad c = 5 + 4(-\tfrac{29}{9}) = -\tfrac{71}{9}$$

$$\therefore \quad P(x) = -\tfrac{29}{9}x^2 + \tfrac{172}{9}x - \tfrac{71}{9}$$

d Now $P'(x) = -\frac{58}{9}x + \frac{172}{9} = \frac{1}{9}(-58x + 172)$

Sign diagram for $P'(x)$ is:



$\therefore \quad P(x)$ is a maximum when $x \doteqdot 2.966$. Now $P(2.966) \doteqdot 20.448$

$\therefore$ maximum profit is approximately $20 448 when 2966 items are produced.

EXERCISE 14N

1 **a**
$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{M}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{M}^3 = \mathbf{M}^2\mathbf{M} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{M}^4 = \mathbf{M}^3\mathbf{M} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$$

b P_n is: "If $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{M}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ for $n \in Z^+$."

c **Proof:** (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{M}^1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \mathbf{M}$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\mathbf{M}^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned}\therefore\ \mathbf{M}^{k+1} &= \mathbf{M}^k\mathbf{M} \\ &= \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2+2k \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2(k+1) \\ 0 & 1 \end{bmatrix}\end{aligned}$$

Thus, P_{k+1} is true whenever P_k is true and P_1 is true.

$\Rightarrow$ P_n is true {Principle of Math. Induction}

2 **a** $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$$\begin{aligned}\mathbf{A}^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} \\ &= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{A}^3\mathbf{A} \\ &= \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 80 \\ 0 & 81 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{A}^5 &= \mathbf{A}^4\mathbf{A} \\ &= \begin{bmatrix} 1 & 80 \\ 0 & 81 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 242 \\ 0 & 243 \end{bmatrix}\end{aligned}$$

b P_n is "If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then $\mathbf{A}^n = \begin{bmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{bmatrix}$, $n \in Z^+$."

c **Proof:** (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{A}^1 = \begin{bmatrix} 1 & 3-1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\therefore$ P_1 is true.

(2) If P_k is true, then $\mathbf{A}^k = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}$

$$\begin{aligned}\therefore\ \mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} &= \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2+3(3^k-1) \\ 0 & 3^k \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2+3^{k+1}-3 \\ 0 & 3^{k+1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3^{k+1}-1 \\ 0 & 3^{k+1} \end{bmatrix}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\Rightarrow$ P_n is true {Princ. of Math. Induct.}

d If $n = -1$, $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3^{-1}-1 \\ 0 & 3^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

But $\mathbf{A}^{-1} = \dfrac{1}{3-0}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

So, the result is true when $n = -1$.

3 **a** $\mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \mathbf{P}^2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$ $\quad \mathbf{P}^3 = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$

$\mathbf{P}^4 = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$

b P_n is "If $\mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ then $\mathbf{P}^n = \begin{bmatrix} n+1 & n \\ -n & 1-n \end{bmatrix}$, $n \in Z^+$."

c **Proof:** (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{P}^1 = \mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \therefore$ P_1 is true.

(2) If P_k is true then $\mathbf{P}^k = \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix}$

$$\begin{aligned} \therefore\ \mathbf{P}^{k+1} &= \mathbf{P}^k\mathbf{P} \\ &= \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2k+2-k & k+1+0 \\ -2k-1+k & -k+0 \end{bmatrix} \\ &= \begin{bmatrix} k+2 & k+1 \\ -k-1 & -k \end{bmatrix} \\ &= \begin{bmatrix} (k+1)+1 & (k+1) \\ -(k+1) & 1-(k+1) \end{bmatrix} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true {Princ. of Math.Induc.}

4 P_n is "If $u_1 = u_2 = 1$ and $u_{n+2} = u_n + u_{n+1}$ and $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then $\mathbf{A}^{n+1} = \begin{bmatrix} u_{n+2} & u_{n+1} \\ u_{n+1} & u_1 \end{bmatrix}$ for $n \in Z^+$."

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{A}^2 = \begin{bmatrix} u_3 & u_2 \\ u_2 & u_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{A}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \therefore$ P_1 is true.

(2) If P_k is true, then $\mathbf{A}^{k+1} = \begin{bmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{bmatrix}$

$\therefore$ $\mathbf{A}^{k+2} = \mathbf{A}^{k+1}\mathbf{A} = \begin{bmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u_{k+2}+u_{k+1} & u_{k+2}+0 \\ u_{k+1}+u_k & u_{k+1}+0 \end{bmatrix}$

$$= \begin{bmatrix} u_{k+3} & u_{k+2} \\ u_{k+2} & u_{k+1} \end{bmatrix}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.
∴ P_n is true {Principle of Mathematical Induction}

REVIEW SET 14A

1 **a** $\mathbf{A}+\mathbf{B}$

$$\begin{aligned} &= \begin{bmatrix}3&2\\0&-1\end{bmatrix}+\begin{bmatrix}1&0\\-2&4\end{bmatrix} \\ &= \begin{bmatrix}4&2\\-2&3\end{bmatrix}\end{aligned}$$

b $3\mathbf{A}$

$$\begin{aligned} &= 3\begin{bmatrix}3&2\\0&-1\end{bmatrix} \\ &= \begin{bmatrix}9&6\\0&-3\end{bmatrix}\end{aligned}$$

c $-2\mathbf{B}$

$$\begin{aligned} &= -2\begin{bmatrix}1&0\\-2&4\end{bmatrix} \\ &= \begin{bmatrix}-2&0\\4&-8\end{bmatrix}\end{aligned}$$

d $\mathbf{A}-\mathbf{B}$

$$\begin{aligned} &= \begin{bmatrix}3&2\\0&-1\end{bmatrix}-\begin{bmatrix}1&0\\-2&4\end{bmatrix} \\ &= \begin{bmatrix}2&2\\2&-5\end{bmatrix}\end{aligned}$$

e $\mathbf{B}-2\mathbf{A}$

$$\begin{aligned} &= \begin{bmatrix}1&0\\-2&4\end{bmatrix}-\begin{bmatrix}6&4\\0&-2\end{bmatrix} \\ &= \begin{bmatrix}-5&-4\\-2&6\end{bmatrix}\end{aligned}$$

f $3\mathbf{A}-2\mathbf{B}$

$$\begin{aligned} &= 3\begin{bmatrix}3&2\\0&-1\end{bmatrix}-2\begin{bmatrix}1&0\\-2&4\end{bmatrix} \\ &= \begin{bmatrix}7&6\\4&-11\end{bmatrix}\end{aligned}$$

g $\mathbf{AB}$

$$\begin{aligned} &= \begin{bmatrix}3&2\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\-2&4\end{bmatrix} \\ &= \begin{bmatrix}-1&8\\2&-4\end{bmatrix}\end{aligned}$$

h $\mathbf{BA}$

$$\begin{aligned} &= \begin{bmatrix}1&0\\-2&4\end{bmatrix}\begin{bmatrix}3&2\\0&-1\end{bmatrix} \\ &= \begin{bmatrix}3&2\\-6&-8\end{bmatrix}\end{aligned}$$

i $\mathbf{A}^{-1}$

$$\begin{aligned} &= \frac{1}{-3}\begin{bmatrix}-1&-2\\0&3\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\0&-1\end{bmatrix}\end{aligned}$$

j $\mathbf{A}^2$

$$\begin{aligned} &= \begin{bmatrix}3&2\\0&-1\end{bmatrix}\begin{bmatrix}3&2\\0&-1\end{bmatrix} \\ &= \begin{bmatrix}9&4\\0&1\end{bmatrix}\end{aligned}$$

k $\mathbf{ABA}$

$$\begin{aligned} &= (\mathbf{AB})\mathbf{A} \\ &= \begin{bmatrix}-1&8\\2&-4\end{bmatrix}\begin{bmatrix}3&2\\0&-1\end{bmatrix} \\ &= \begin{bmatrix}-3&-10\\6&8\end{bmatrix}\end{aligned}$$

l $(\mathbf{AB})^{-1}$

$$\begin{aligned} &= \begin{bmatrix}-1&8\\2&-4\end{bmatrix}^{-1} \\ &= \frac{1}{4-16}\begin{bmatrix}-4&-8\\-2&-1\end{bmatrix} \\ &= \tfrac{1}{-12}\begin{bmatrix}-4&-8\\-2&-1\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\ \frac{1}{6}&\frac{1}{12}\end{bmatrix}\end{aligned}$$

2 **a** Equating corresponding elements,

$$\begin{aligned} a &= -a \\ b-2 &= 3 \\ c &= 2-c \\ d &= -4 \end{aligned} \qquad \therefore \quad \begin{aligned} &a=0, \quad b=5 \\ &c=1, \quad d=-4 \end{aligned}$$

b Equating corresponding elements,

$$\begin{aligned} 3+b &= a \\ 2a-a &= 2 \\ b+c &= 2 \\ -2+d &= 6 \end{aligned} \qquad \therefore \quad \begin{aligned} a&=2 \\ b&=-1 \\ c&=3 \\ d&=8 \end{aligned}$$

3 **a**

$$\begin{aligned} \mathbf{B}-\mathbf{Y} &= \mathbf{A} \\ \therefore\ -\mathbf{Y} &= \mathbf{A}-\mathbf{B} \\ \therefore\ \mathbf{Y} &= -(\mathbf{A}-\mathbf{B}) \\ \therefore\ \mathbf{Y} &= \mathbf{B}-\mathbf{A} \end{aligned}$$

b

$$\begin{aligned} 2\mathbf{Y}+\mathbf{C} &= \mathbf{D} \\ \therefore\ 2\mathbf{Y} &= \mathbf{D}-\mathbf{C} \\ \therefore\ \mathbf{Y} &= \tfrac{1}{2}(\mathbf{D}-\mathbf{C}) \end{aligned}$$

c

$$\begin{aligned} \mathbf{AY} &= \mathbf{B} \\ \therefore\ \mathbf{A}^{-1}\mathbf{AY} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{IY} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{Y} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

d

$$\begin{aligned} \mathbf{YB} &= \mathbf{C} \\ \therefore\ \mathbf{YBB}^{-1} &= \mathbf{CB}^{-1} \\ \therefore\ \mathbf{YI} &= \mathbf{CB}^{-1} \\ \therefore\ \mathbf{Y} &= \mathbf{CB}^{-1} \end{aligned}$$

e

$$\begin{aligned} \mathbf{C}-\mathbf{AY} &= \mathbf{B} \\ \therefore\ -\mathbf{AY} &= \mathbf{B}-\mathbf{C} \\ \therefore\ \mathbf{AY} &= \mathbf{C}-\mathbf{B} \\ \therefore\ \mathbf{A}^{-1}\mathbf{AY} &= \mathbf{A}^{-1}(\mathbf{C}-\mathbf{B}) \\ \therefore\ \mathbf{Y} &= \mathbf{A}^{-1}(\mathbf{C}-\mathbf{B}) \end{aligned}$$

f

$$\begin{aligned} \mathbf{AY}^{-1} &= \mathbf{B} \\ \therefore\ \mathbf{A}^{-1}\mathbf{AY}^{-1} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ \mathbf{Y}^{-1} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\ (\mathbf{Y}^{-1})^{-1} &= (\mathbf{A}^{-1}\mathbf{B})^{-1} \\ \therefore\ \mathbf{Y} &= \mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1} \\ \therefore\ \mathbf{Y} &= \mathbf{B}^{-1}\mathbf{A} \end{aligned}$$

4 **a**

$$\begin{aligned} 3x-4y &= 2 \\ 5x+2y &= -1 \end{aligned}$$

$$\begin{aligned}
\therefore \quad \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \tfrac{1}{26}\begin{bmatrix} 2 & 4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \tfrac{1}{26}\begin{bmatrix} 0 \\ -13 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

$$\therefore \quad x = 0, \quad y = -\tfrac{1}{2}$$

b

$$\begin{aligned} 4x-y &= 5 \\ 2x+3y &= 9 \end{aligned}$$

$$\begin{aligned}
\therefore \quad \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 5 \\ 9 \end{bmatrix} \\
\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix} \\
&= \tfrac{1}{14}\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} \\
&= \tfrac{1}{14}\begin{bmatrix} 24 \\ 26 \end{bmatrix} \\
&= \begin{bmatrix} \frac{12}{7} \\ \frac{13}{7} \end{bmatrix}
\end{aligned}$$

$$\therefore \quad x = \tfrac{12}{7}, \quad y = \tfrac{13}{7}$$

c

$$\begin{aligned}
\mathbf{X}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}\tfrac{1}{-1}\begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} -1 & 8 \\ -2 & 6 \end{bmatrix}
\end{aligned}$$

d

$$\begin{aligned}
\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{X} &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \tfrac{1}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}
\end{aligned}$$

e

$$\begin{aligned}
\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{X} &= \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \tfrac{1}{-3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \tfrac{1}{-3}\begin{bmatrix} -14 \\ -1 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} \frac{14}{3} \\ \frac{1}{3} \end{bmatrix}
\end{aligned}$$

f

$$\begin{aligned}
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{X}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \\
\therefore \quad \mathbf{X} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\
\therefore \quad \mathbf{X} &= \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}\tfrac{1}{-3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \\
&= -\tfrac{1}{6}\begin{bmatrix} 5 & -2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \\
&= -\tfrac{1}{6}\begin{bmatrix} -3 & -9 \\ -9 & 3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

5 **a** $2\mathbf{B}$

$$= 2\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{bmatrix}$$

b $\frac{1}{2}\mathbf{B}$

$$= \tfrac{1}{2}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$$

c $\mathbf{AB}$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$1 \times \textcircled{3} \longleftrightarrow \textcircled{3} \times 2$$

$$= \begin{bmatrix} 11 & 12 \end{bmatrix}$$

d $\mathbf{B}$ is $3 \times \textcircled{2}$ and $\mathbf{A}$ is $1 \times \textcircled{3}$. not equal

$\therefore$ $\mathbf{BA}$ does not exist.

6 **a** $\mathbf{P} + \mathbf{Q}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{bmatrix}$$

b $\mathbf{Q} - \mathbf{P}$

$$= \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{bmatrix}$$

c $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

$$= \begin{bmatrix} \frac{3}{2} & 3 \\ \frac{3}{2} & 0 \\ 3 & \frac{9}{2} \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{bmatrix}$$

7 In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ k & 3 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 3-4k & -6-2k \end{array}\right] \quad R_2 \to R_2 - kR_1$$

$$\begin{array}{ccc} k & 3 & -6 \\ -k & -4k & -2k \\ \hline 0 & 3-4k & -6-2k \end{array}$$

Row 2 gives $(3-4k)y = -6-2k$

i.e., $y = \dfrac{-2(3+k)}{3-4k}$ which is not defined if $k = \frac{3}{4}$

So the system has a unique solution for all $k \neq \frac{3}{4}$, k in $\mathcal{R}$.

If $k = \frac{3}{4}$, equation 2 becomes $\frac{3}{4}x + 3y = -6$

$$\therefore \quad 3x + 12y = -24$$

$$\text{i.e.,} \quad x + 4y = -8$$

which is the equation of a line parallel to $x + 4y = 2$ $\therefore$ there are no solutions.

8 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 2 & 3 & -1 & -3 \\ 1 & -2 & 3 & 2 \end{array}\right]$$

$$\begin{array}{cccc} 6 & 9 & -3 & -9 \\ -6 & 2 & -4 & -6 \\ \hline 0 & 11 & -7 & -15 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 11 & -7 & -15 \\ 0 & -5 & 7 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to 3R_2 - 2R_1 \\ R_3 \to 3R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -6 & 9 & 6 \\ -3 & 1 & -2 & -3 \\ \hline 0 & -5 & 7 & 3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 11 & -7 & -15 \\ 0 & 0 & 42 & -42 \end{array}\right] \quad R_3 \to 11R_3 + 5R_2$$

$$\begin{array}{cccc} 0 & -55 & 77 & 33 \\ 0 & 55 & -35 & -75 \\ \hline 0 & 0 & 42 & -42 \end{array}$$

From row 3, $42z = -42$ $\therefore$ $z = -1$

Substituting in row 2 gives

$$\begin{aligned} 11y - 7(-1) &= -15 \\ \therefore 11y + 7 &= -15 \\ \therefore 11y &= -22 \\ \therefore \quad y &= -2 \end{aligned}$$

and from row 1,

$$\begin{aligned} 3x - (-2) + 2(-1) &= 3 \\ \therefore \quad 3x + 2 - 2 &= 3 \\ \therefore \quad x &= 1 \end{aligned}$$

i.e., $x = 1$, $y = -2$, $z = -1$

9 $x^2+y^2+ax+by+c=0$

a $(-2,\ 4)$ lies on the circle, $\therefore\ 4+16-2a+4b+c=0$, i.e., $2a-4b-c=20$

$(1,\ 3)$ lies on the circle, $\therefore\ 1+9+a+3b+c=0$, i.e., $a+3b+c=-10$

In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & 20 \\ 1 & 3 & 1 & -10 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 2 & -4 & -1 & 20 \\ 0 & 10 & 3 & -40 \end{array}\right] \quad R_2 \to 2R_2 - R_1 \qquad \begin{array}{cccc} 2 & 6 & 2 & -20 \\ -2 & 4 & 1 & -20 \\ \hline 0 & 10 & 3 & -40 \end{array}$$

Let $c=t$ in row 3, then $10b+3t=-40$ $\quad 10b=-3t-40$ $\quad b=\dfrac{-3t-40}{10}$

and substituting in row 1 gives $\quad 2a-4\left(\dfrac{-3t-40}{10}\right)-t=20$

$$\therefore\ 20a+12t+160-10t=200$$

$$\therefore\ 20a=40-2t \quad \therefore\ a=\frac{40-2t}{20}$$

i.e., $a=2-\dfrac{t}{10}$, $\ b=-4-\dfrac{3t}{10}$, $\ c=t$ for all t in $\mathcal{R}$.

b There are infinitely many solutions as we have 2 equations in 3 unknowns.

c If $(2,\ 2)$ is on the circle then $(2,\ 2)$ satisfies the equation of the circle.

$$\therefore\ 2^2+2^2+\left(\frac{20-t}{10}\right)(2)+\left(\frac{-3t-40}{10}\right)(2)+t=0$$

$$\therefore\ 40+40+40-2t-6t-80+10t=0 \quad \therefore\ 2t=-40$$

$$\therefore\ t=-20$$

and when $t=-20$, $\ a=2-(\frac{-20}{10})=4$, $\ b=-4-(\frac{-60}{10})=2$, $\ c=-20$

i.e., the equation of the circle is $\ x^2+y^2+4x+2y-20=0$

10 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 13 \\ 1 & -1 & 3 & -1 \\ 3 & 7 & -11 & k \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 2 & 3 & -4 & 13 \\ 3 & 7 & -11 & k \end{array}\right] \quad \text{interchanging } R_1 \text{ and } R_2 \qquad \begin{array}{cccc} 2 & 3 & -4 & 13 \\ -2 & 2 & -6 & 2 \\ \hline 0 & 5 & -10 & 15 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 5 & -10 & 15 \\ 0 & 10 & -20 & k+3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 3 & 7 & -11 & k \\ -3 & 3 & -9 & 3 \\ \hline 0 & 10 & -20 & k+3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 5 & -10 & 15 \\ 0 & 0 & 0 & k-27 \end{array}\right] \quad R_3 \to R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & 10 & -20 & k+3 \\ 0 & -10 & 20 & -30 \\ \hline 0 & 0 & 0 & k-27 \end{array}$$

If $k-27=0$, i.e., $k=27$ we have a row of zeros.

$\therefore$ we have 2 equations in 3 unknowns which gives an infinite number of solutions.

Let $z=t$ in row 2. then $5y-10t=15$ $\quad$ and $\quad x-(3+2t)+3t=-1$

$\therefore\ y=3+2t$ $\quad \therefore\ x-3-2t+3t=-1$

$\therefore\ x=2-t$

i.e., if $k=27$, we have solutions of the form $x=2-t$, $\ y=3+2t$, $\ z=t$ where t is in $\mathcal{R}$.

If $k \neq 27$, the system is inconsistent and there are no solutions.

11 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 2 & 7 & 0 \end{array}\right] \quad R_2 \to 3R_2 - R_1 \qquad \begin{array}{rrrr} 3 & 3 & 6 & 0 \\ -3 & -1 & 1 & 0 \\ \hline 0 & 2 & 7 & 0 \end{array}$$

Let $z = t$ in row 2.

then $2y + 7t = 0$ i.e., $y = -\frac{7}{2}t$ and $3x + (-\frac{7}{2}t) - t = 0$

$$\therefore \quad 3x = \tfrac{9}{2}t$$

$$\therefore \quad x = \tfrac{3}{2}t$$

$\therefore$ we have infinitely many solutions of the form $x = \frac{3}{2}t$, $y = -\frac{7}{2}t$, $z = t$ where t is in $\mathcal{R}$ (or $x = 3t$, $y = -7t$, $z = 2t$ for t in $\mathcal{R}$).

REVIEW SET 14B

1
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \therefore\ a = 1, \quad b = 0 \qquad \text{i.e., } a = 1,\ b = 0,\ c = 0,\ d = 1$$

$$\therefore \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{array}{l} a + c = 1, \\ b + d = 1 \end{array} \qquad \therefore \text{ matrix is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

2 **a** **AB**

$$= \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= [10]$$

b **BA**

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\begin{bmatrix} 4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

c **AC**

$$= \begin{bmatrix} 4 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 18 & 21 \end{bmatrix}$$

d **CA** does not exist as **C** is 3×3 and **A** is 1×3

e
$$\mathbf{CB} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$$

3 **a**
$$\mathbf{A}^{-1} = \frac{1}{42 - 40}\begin{bmatrix} 7 & -8 \\ -5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{bmatrix}$$

b $\mathbf{A}^{-1}$ does not exist as $|\mathbf{A}|$

$= -24 - -24$

$= 0$

c
$$\mathbf{A}^{-1} = \tfrac{1}{-3}\begin{bmatrix} -3 & -5 \\ 6 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{bmatrix}$$

4 $\mathbf{A} = 2\mathbf{A}^{-1}$

a
$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \times \mathbf{A} \\ &= \mathbf{A}(2\mathbf{A}^{-1}) \\ &= 2\mathbf{A}\mathbf{A}^{-1} \\ &= 2\mathbf{I} \end{aligned}$$

b
$$\begin{aligned} (\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I}) &= (\mathbf{A} - \mathbf{I})\mathbf{A} + (\mathbf{A} - \mathbf{I})3\mathbf{I} \\ &= \mathbf{A}^2 - \mathbf{I}\mathbf{A} + 3\mathbf{A}\mathbf{I} - 3\mathbf{I}^2 \\ &= 2\mathbf{I} - \mathbf{A} + 3\mathbf{A} - 3\mathbf{I} \\ &= 2\mathbf{A} - \mathbf{I} \end{aligned}$$

5 Sales matrix is
$$\begin{bmatrix} 42 - 27 & 54 - 31 \\ 36 - 28 & 27 - 15 \\ 34 - 28 & 30 - 22 \end{bmatrix} = \begin{bmatrix} 15 & 23 \\ 8 & 12 \\ 6 & 8 \end{bmatrix}$$
Totals matrix is
$$\begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix}$$

$$\therefore \text{ total profit} = \begin{bmatrix} 0.75 & 0.55 & 1.20 \end{bmatrix}\begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix} = [56.3] \text{ dollars} \quad \text{i.e., } \$56.30$$

6

$$\mathbf{AB} = \begin{bmatrix}1&2&3\\2&5&7\\-2&-4&-5\end{bmatrix}\begin{bmatrix}3&-2&-1\\-4&1&-1\\2&0&1\end{bmatrix} = \begin{bmatrix}3+-8+6 & -2+2+0 & -1-2+3\\ 6-20+14 & -4+5+0 & -2-5+7\\ -6+16-10 & 4-4+0 & 2+4-5\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix}3&-2&-1\\-4&1&-1\\2&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\2&5&7\\-2&-4&-5\end{bmatrix} = \begin{bmatrix}3-4+2 & 6-10+4 & 9-14+5\\ -4+2+2 & -8+5+4 & -12+7+5\\ 2+0-2 & 4+0-4 & 6+0-5\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

i.e., $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ $\quad\therefore\quad \mathbf{A}^{-1} = \mathbf{B}$

7 Using technology, the solution of

$$\begin{aligned}2x+y+z &= 8\\ 4x-7y+3z &= 10\\ 3x-2y+z &= 1\end{aligned}$$

is $x=2$, $y=1$, $z=3$

8 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c}1&2&-3&3\\6&3&2&4\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&2&-3&3\\0&-9&20&-14\end{array}\right] \quad R_2 \to R_2 - 6R_1 \qquad \begin{array}{cccc}6&3&2&4\\-6&-12&18&-18\\ \hline 0&-9&20&-14\end{array}$$

Let $z=t$ in row 2, then

$$\begin{aligned}-9y+20t &= -14\\ \therefore -9y &= -14-20t\\ \text{i.e., } y &= \frac{14+20t}{9}\end{aligned}$$

and in row 1,

$$\begin{aligned}x+2\left(\frac{14+20t}{9}\right)-3t &= 3\\ \therefore\quad 9x+28+40t-27t &= 27\\ \therefore\quad 9x &= -1-13t\\ \therefore\quad x &= \frac{-1-13t}{9}\end{aligned}$$

$\therefore$ there are infinitely many solutions of the form $x = \dfrac{-1-13t}{9}$, $y = \dfrac{14+20t}{9}$, $z = t$ where t is in $\mathcal{R}$.

9 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c}2&-3&9\\m&-7&n\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c}2&-3&9\\3m-14&0&3n-63\end{array}\right] \quad R_2 \to 3R_2 - 7R_1 \qquad \begin{array}{ccc}3m&-21&3n\\-14&21&-63\\ \hline 3m-14&0&3n-63\end{array}$$

$$\sim \left[\begin{array}{cc|c}2&-3&9\\14-3m&0&63-3n\end{array}\right] \quad R_2 \to -R_2$$

b When $14-3m \neq 0$, i.e., when $m \neq \frac{14}{3}$, the system has a unique solution.

{If $m = \frac{14}{3}$, the second row becomes all zeros so there are infinitely many solutions.}

REVIEW SET 14C

1 In matrix form $\begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ has a unique solution if $\begin{vmatrix} k & 3 \\ 1 & k+2 \end{vmatrix} \neq 0$

$$\begin{aligned} \text{i.e.,} \quad k^2+2k-3 &\neq 0 \\ (k-1)(k+3) &\neq 0 \\ k &\neq 1 \text{ or } -3 \end{aligned}$$

and if $k \neq 1$ or -3,
$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix}^{-1}\begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &= \frac{1}{(k-1)(k+3)}\begin{bmatrix} k+2 & -3 \\ -1 & k \end{bmatrix}\begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &= \frac{1}{(k-1)(k+3)}\begin{bmatrix} -6k-18 \\ 6+2k \end{bmatrix} \\ &= \frac{1}{(k-1)(k+3)}(k+3)\begin{bmatrix} -6 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{-6}{k-1} \\ \dfrac{2}{k-1} \end{bmatrix} \end{aligned}$$

So the unique solution is $x = \dfrac{-6}{k-1}$, $y = \dfrac{2}{k-1}$

If $k = 1$, the equations are: $\left.\begin{aligned} x+3y &= -6 \\ x+3y &= 2 \end{aligned}\right\}$ parallel lines $\therefore$ no solutions exist

If $k = -3$, the equations are: $\left.\begin{aligned} -3x+3y &= -6 \\ x-y &= 2 \end{aligned}\right\}$ coincident lines $\therefore$ infinitely many solutions

2
$$\begin{aligned} \begin{vmatrix} x & 2 & 0 \\ 2 & x+1 & -2 \\ 0 & -2 & x+2 \end{vmatrix} &= x\begin{vmatrix} x+1 & -2 \\ -2 & x+2 \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ x+2 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & x+1 \\ 0 & -2 \end{vmatrix} \\ &= x\,[(x+1)(x+2)-4] + 2\,[0-2(x+2)] \\ &= x(x^2+3x+2-4) - 4(x+2) \\ &= x(x^2+3x-2) - 4x - 8 \\ &= x^3+3x^2-2x-4x-8 \\ &= x^3+3x^2-6x-8 \\ &= (x+4)(x+1)(x-2) \qquad \{\text{using technology}\} \end{aligned}$$

But $(x+4)(x+1)(x-2) = 0$ {given}

$\therefore \quad x = -4, \ -1$ or 2

3 $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}$

a $2\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} -4 & 6 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} -14 & 18 \\ 18 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -12 \\ -10 & 4 \end{bmatrix}$

b **A** is 2×2 and **C** is 2×3 $\therefore$ **AC** is 2×3

$\mathbf{AC} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+6 & -6+3 \\ -4+0 & 0-2 & 12-1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{bmatrix}$

c **C** is 2×3 and **B** is 2×2 $\therefore$ **CB** is not possible.

d

$$\begin{aligned} \mathbf{DA} &= \mathbf{B} \\ \therefore\ \mathbf{DAA}^{-1} &= \mathbf{BA}^{-1} \qquad \{\text{post multiplying by } \mathbf{A}^{-1}\} \\ \therefore\ \mathbf{D} &= \mathbf{BA}^{-1} \\ \therefore\ \mathbf{D} &= \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix} = \frac{1}{-2(-1)-3(4)} \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = -\tfrac{1}{10} \begin{bmatrix} 7-36 & 21-18 \\ -9+12 & -27+6 \end{bmatrix} \\ \therefore\ \mathbf{D} &= \begin{bmatrix} \frac{29}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{21}{10} \end{bmatrix} \end{aligned}$$

4

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\mathbf{X} &= \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix} \\ \therefore\ \mathbf{X} &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix} \\ &= \tfrac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix} \\ &= \tfrac{1}{11}\begin{bmatrix} 11 & 33 & 22 \\ -11 & 11 & 33 \end{bmatrix} \\ \therefore\ \mathbf{X} &= \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 3 \end{bmatrix} \end{aligned}$$

5

$$\begin{aligned} 5\mathbf{A}^2 - 6\mathbf{B} &= 3\mathbf{I} \\ \therefore\ \mathbf{A}(5\mathbf{A} - 6\mathbf{I}) &= 3\mathbf{I} \\ \therefore\ \mathbf{A} \times \tfrac{1}{3}(5\mathbf{A} - 6\mathbf{I}) &= \mathbf{I} \\ \therefore\ \mathbf{A}^{-1} &= \tfrac{5}{3}\mathbf{A} - 2\mathbf{I} \end{aligned}$$

6 **a** **i** If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{ABB}^{-1} = \mathbf{BB}^{-1}$ provided $\mathbf{B}^{-1}$ exists.

$\therefore$ $\mathbf{A} = \mathbf{I}$ provided $\mathbf{B}^{-1}$ exists, i.e., provided that $|\mathbf{B}| \neq 0$.

ii

$$\begin{aligned} (\mathbf{A}+\mathbf{B})^2 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 \\ &= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 \quad \text{provided that } \mathbf{AB} = \mathbf{BA}. \end{aligned}$$

b $\mathbf{M} = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix} \begin{bmatrix} k-1 & -2 \\ -3 & -k \end{bmatrix}$ and $\mathbf{M}^{-1}$ exists provided that $|\mathbf{M}| \neq 0$.

$$\begin{aligned} \text{Now } |\mathbf{M}| = \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} \begin{vmatrix} k-1 & -2 \\ -3 & k \end{vmatrix} &= (k^2-4)(k^2-k-6) \\ &= (k+2)(k-2)(k+2)(k-3) \end{aligned}$$

$\therefore$ $\mathbf{M}^{-1}$ exists provided that $k \neq 3$ or ± 2

7 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ t & 1 & -1 & 3t \\ 1 & 3 & t & 13 \end{array}\right]$$

$$\begin{array}{rrrr} t & 1 & -1 & 3t \\ -t & t & 2t & 3t \\ \hline 0 & 1+t & -1+2t & 6t \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1+t & -1+2t & 6t \\ 0 & 4 & t+2 & 16 \end{array}\right] \quad \begin{array}{l} \\ R_2 \to R_2 - tR_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 1 & 3 & t & 13 \\ -1 & 1 & 2 & 3 \\ \hline 0 & 4 & t+2 & 16 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1+t & -1+2t & 6t \\ 0 & 0 & t^2-5t+6 & 16-8t \end{array}\right] \quad R_3 \to (1+t)R_3 - 4R_2$$

$$\begin{array}{rrrr} 0 & 4+4t & t^2+3t+2 & 16+16t \\ 0 & -4-4t & 4-8t & -24t \\ \hline 0 & 0 & t^2-5t+6 & 16-8t \end{array}$$

Row 3 gives $(t-3)(t-2)z = -8(t-2)$ $\therefore$ $z = \dfrac{-8(t-2)}{(t-3)(t-2)}$

If $t = 3$, there are no solutions, $\therefore$ there is no unique solution.

If $t = 2$, row 3 becomes a row of zeros so there are infinitely many solutions, i.e., there is no unique solution.

The reduced matrix becomes $\left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Row 2 gives $3y + 3z = 12$ i.e., $y + z = 4$. Let $z = s$, then $y = 4 - s$

and substituting in row 1 gives

$$\begin{aligned} x - (4 - s) - 2s &= -3 \\ \therefore\ x - 4 + s - 2s &= -3 \\ \text{i.e., } x &= 1 + s \end{aligned}$$

i.e., when $t = 2$, the solutions are of the form $x = 1 + s$, $y = 4 - s$, $z = s$ where s is in $\mathcal{R}$.

If $t \neq 2$ and $t \neq 3$, then $z = \dfrac{-8}{t-3}$.

Substituting in row 2 gives

$$\begin{aligned} (1+t)y - \frac{8(-1+2t)}{t-3} &= 6t \\ \therefore\ (t-3)(1+t)y + 8 - 16t &= 6t(t-3) \\ \therefore\ (t-3)(1+t)y &= 6t^2 - 18t + 16t - 8 \\ \therefore\ y &= \frac{2(3t^2 - t - 4)}{(t-3)(t+1)} \\ &= \frac{2(3t-4)(t+1)}{(t-3)(t+1)} \\ &= \frac{2(3t-4)}{t-3} \end{aligned}$$

and

$$\begin{aligned} x - \frac{2(3t-4)}{t-3} + \frac{2 \times 8}{t-3} &= -3 \\ \therefore\ (t-3)x - 6t + 8 + 16 &= -3t + 9 \\ \therefore\ (t-3)x &= 3t - 15 \\ \therefore\ x &= \frac{3(t-5)}{t-3} \end{aligned}$$

So if $t \neq 2$ or 3, then the unique solution is:

$$x = \frac{3(t-5)}{t-3}, \quad y = \frac{2(3t-4)}{t-3}, \quad z = \frac{-8}{t-3} \quad \text{for } t \text{ in } \mathcal{R}$$

8 **a** $s = at^2 + bt + c$

At $t = 1$, $s(1) = 63$ $\therefore$ $a + b + c = 63$

At $t = 2$, $s(2) = 72$ $\therefore$ $4a + 2b + c = 72$

At $t = 7$, $s(7) = 27$ $\therefore$ $49a + 7b + c = 27$

Using technology, $a = -3$, $b = 18$, $c = 48$ and $\therefore$ $s(t) = -3t^2 + 18t + 48$

b $s(0) = 48$ $\therefore$ the height of the cliff is 48 m

c The rock reaches sea level when $s(t) = 0$

$$\begin{aligned} \therefore\ -3(t^2 - 6t - 16) &= 0 \\ \therefore\ -3(t-8)(t+2) &= 0 \\ \therefore\ t &= 8 \text{ or } -2 \end{aligned}$$

but $t \geqslant 0$, i.e., it reaches sea level after 8 seconds.

REVIEW SET 14D

1 This system in matrix form is $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \text{So,} \quad x = 1, \quad y = -1, \quad z = 2. \quad \{\text{using technology}\}$$

2
$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = (a+b)\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} + c\begin{vmatrix} a & a \\ c+a & b \end{vmatrix} + c\begin{vmatrix} a & b+c \\ b & b \end{vmatrix}$$

$$\begin{aligned} &= (a+b)\,[(b+c)(c+a) - ab] + c\,[ab - a(c+a)] + c\,[ab - (b+c)b] \\ &= (a+b)(bc + ab + c^2 + ac - ab) + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= abc + b^2c + ac^2 + bc^2 + a^2c + abc + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= 4abc \end{aligned}$$

3 If $\mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I}$,

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A}(5\mathbf{A} + 2\mathbf{I}) \\ &= 5\mathbf{A}^2 + 2\mathbf{A} \\ &= 5(5\mathbf{A} + 2\mathbf{I}) + 2\mathbf{A} \\ &= 25\mathbf{A} + 10\mathbf{I} + 2\mathbf{A} \\ &= 27\mathbf{A} + 10\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^4 &= \mathbf{A}(27\mathbf{A} + 10\mathbf{I}) \\ &= 27\mathbf{A}^2 + 10\mathbf{AI} \\ &= 27(5\mathbf{A} + 2\mathbf{I}) + 10\mathbf{A} \\ &= 135\mathbf{A} + 54\mathbf{I} + 10\mathbf{A} \\ &= 145\mathbf{A} + 54\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^5 &= \mathbf{A}(145\mathbf{A} + 54\mathbf{I}) \\ &= 145\mathbf{A}^2 + 54\mathbf{AI} \\ &= 145(5\mathbf{A} + 2\mathbf{I}) + 54\mathbf{A} \\ &= 725\mathbf{A} + 290\mathbf{I} + 54\mathbf{A} \\ &= 779\mathbf{A} + 290\mathbf{I} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{A}(779\mathbf{A} + 290\mathbf{I}) \\ &= 779\mathbf{A}^2 + 290\mathbf{AI} \\ &= 779(5\mathbf{A} + 2\mathbf{I}) + 290\mathbf{AI} \\ &= 4185\mathbf{A} + 1558\mathbf{I} \end{aligned}$$

4 **a** $C(0) = 80 \quad \therefore \quad a(0) + b(0) + c(0) + d = 80 \quad \therefore \quad d = 80$

b
$$\begin{aligned} C(1) &= 100 & \therefore \quad a + b + c + 80 &= 100 \\ C(2) &= 148 & 8a + 4b + 2c + 80 &= 148 \\ C(4) &= 376 & 64a + 16b + 4c + 80 &= 376 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

{technology}

$\therefore \quad a = 2, \quad b = 8, \quad c = 10$

5
$$\begin{aligned} \mathbf{AXB} &= \mathbf{C} \\ \therefore \quad \mathbf{A}^{-1}\mathbf{AXBB}^{-1} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \\ \therefore \quad \mathbf{IXI} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \\ \therefore \quad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \end{aligned}$$

$$\text{So} \quad \mathbf{X} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -12 & -11 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}^{-1}$$

$$\therefore \quad \mathbf{X} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \qquad \{\text{using technology}\}$$

6 **a** $$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \quad x = 6, \quad y = -2, \quad z = 1$$

b $$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{7}{6} \\ -\frac{7}{6} \end{bmatrix}$$

$$\therefore \quad x = \tfrac{3}{2}, \quad y = -\tfrac{7}{6}, \quad z = -\tfrac{7}{6}$$

7 In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} k & 2 & 1 \\ 2 & k & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} k & 2 & 1 \\ 0 & (k+2)(k-2) & -2(k+1) \end{array}\right] \quad R_2 \to kR_2 - 2R_1 \qquad \begin{array}{ccc} 2k & k^2 & -2k \\ -2k & -4 & -2 \\ \hline 0 & k^2-4 & -2k-2 \end{array}$$

Row 2 gives $(k+2)(k-2)y = -2(k+1)$ i.e., $y = \dfrac{-2(k+1)}{(k+2)(k-2)}$

If $k \neq \pm 2$, then substituting in row 1 gives

$$kx + 2\left(\frac{-2k-2}{(k+2)(k-2)}\right) = 1$$

$$\therefore \quad k(k+2)(k-2)x - 4k - 4 = k^2 - 4$$

$$\therefore \quad x = \frac{k^2+4k}{k(k+2)(k-2)}$$

$$\therefore \quad = \frac{k+4}{(k+2)(k-2)}$$

$\therefore$ if $k \neq \pm 2$, then the unique solution is $x = \dfrac{k+4}{(k+2)(k-2)}$, $y = \dfrac{-2(k+1)}{(k+2)(k-2)}$

If $k = \pm 2$, then row 2 becomes $0z = -6$ or $0z = 2$ which has no solution, i.e., the system has no solutions.

8 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 9 \\ 3 & 2 & 5 & 19 \\ 1 & 1 & -3 & 1 \end{array}\right]$$

Using technology, $x = 6$, $y = -2$, $z = 1$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & -3 & 0 & 5 \end{array}\right]$$

Using technology, $x = \frac{3}{2}$, $y = -\frac{7}{6}$, $z = -\frac{7}{6}$.

9 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 10 \\ 4 & -6 & k & m \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 1 & 10 \\ 0 & 0 & k-2 & m-20 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\begin{array}{cccc} 4 & -6 & k & m \\ -4 & 6 & -2 & -20 \\ \hline 0 & 0 & k-2 & m-20 \end{array}$$

If $k = 2$ and $m = 20$, the last row is all zeros so there are an infinite number of solutions.

Let $y = s$ and $z = t$ in equation 1.

The solutions are of the form $x = \dfrac{10 + 3s - t}{2}$, $y = s$, $z = t$ for all s, t in $\mathcal{R}$.

If $k = 2$ and $m \neq 20$, the system is inconsistent.

If $k \neq 2$, then from row 2, $z = \dfrac{m-20}{k-2}$.

Let $y = t$ in equation 1, then

$$\begin{aligned} 2x - 3t + \frac{m-20}{k-2} &= 10 \\ \therefore\quad 2(k-2)x - 3(k-2)t + m - 20 &= 10k - 20 \\ \therefore\quad 2(k-2)x &= 10k - m + 3(k-2)t \\ \therefore\quad x &= \frac{10k - m + 3(k-2)t}{2(k-2)} \end{aligned}$$

i.e., there are an infinite number of solutions of the form

$$x = \frac{10k - m + 3(k-2)t}{2(k-2)}, \quad y = t, \quad z = \frac{m-20}{k-2}, \quad t \text{ is in } \mathcal{R}.$$

10 **a** In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ k & 1 & -1 & 3 \\ 5 & -k & 3 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & 1-5k & 6k-1 & 3-2k \\ 0 & -k-25 & 33 & -3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - kR_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & 1-5k & 6k-1 & 3-2k \\ 0 & 0 & 2(k-2)(3k-2) & -2(k-2)(k+18) \end{array}\right] \quad R_3 \to (1-5k)R_3 + (k+25)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & 1-5k & 6k-1 & 3-2k \\ 0 & 0 & (k-2)(3k-2) & -(k-2)(k+18) \end{array}\right] \quad R_3 \to R_3 \div 2 \qquad \text{as required}$$

$$\begin{array}{cccc} k & 1 & -1 & 3 \\ -k & -5k & 6k & -2k \\ \hline 0 & 1-5k & 6k-1 & 3-2k \end{array} \qquad \begin{array}{cccc} 5 & -k & 3 & 7 \\ -5 & -25 & 30 & -10 \\ \hline 0 & -k-25 & 33 & -3 \end{array}$$

$$\begin{array}{cccc} 0 & 5k^2 + 124k - 25 & 33 - 165k & -3 + 15k \\ 0 & -5k^2 - 124k + 25 & 6k^2 + 149k - 25 & -2k^2 - 47k + 75 \\ \hline 0 & 0 & 6k^2 - 16k + 8 & -2k^2 - 32k + 72 \\ & & = 2(3k^2 - 8k + 4) & = -2(k^2 + 16k - 36) \end{array}$$

b k has a unique solution provided $k \neq 2$ or $\frac{2}{3}$

c If $k = 2$, then row 3 becomes a row of zeros so there are infinitely many solutions.

The system becomes $\left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & -9 & 11 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Let $z = t$ in row 2, then $-9y + 11t = -1$ i.e., $y = \dfrac{1+11t}{9}$

Substituting in row 1 gives $x + 5\left(\dfrac{1+11t}{9}\right) - 6t = 2$

$\therefore \quad 9x + 5 + 55t - 54t = 18$

$\therefore \quad x = \dfrac{13-t}{9}$

i.e., the solutions are of the form $x = \dfrac{13-t}{9}$, $y = \dfrac{1+11t}{9}$, $z = t$, where t is in $\mathcal{R}$.

d If $k = \frac{2}{3}$, then the system is inconsistent and there are no solutions.

REVIEW SET 14E

1 **a** Let $\$x$ be the cost of an opera ticket
$\$y$ be the cost of a play ticket
$\$z$ be the cost of a concert ticket

$$\begin{aligned} 3x + 2y + 5z &= 267 \\ 2x + 3y + z &= 145 \\ x + 5y + 4z &= 230 \end{aligned}$$

b So, $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix}$ $\quad\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 18 \\ 27 \end{bmatrix}$ using technology

$\therefore$ cost of each ticket is \$32 for opera, \$18 for play, \$27 for concert.

c Total cost $= 4 \times \$32 + 1 \times \$18 + 2 \times \$27$
$= \$128 + \$18 + \$54 = \$200$

2 $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix}$ $\quad\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$\therefore \quad x = 2, \quad y = 1, \quad z = 3$ {using technology}

3 **a** $3\mathbf{A} = 3\begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 6 & 6 \\ 3 & -3 & 0 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ 5 & 3 \end{bmatrix}$

c $\mathbf{BA}$
$= \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$
$= \begin{bmatrix} -2 & 0 & 4 \\ 10 & -7 & -6 \\ -1 & 0 & 2 \end{bmatrix}$

d **A** is 2×3 and **C** is 2×2
$\therefore$ **AC** does not exist.

e $\mathbf{BC} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$
$= \begin{bmatrix} 0 & 22 \\ 7 & -12 \\ 0 & 11 \end{bmatrix}$

4 $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix}$ $\quad\mathbf{B} = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1+4+3 & 2-2+4 & -3+8+1 \\ -2+8+18 & 4-4+24 & -6+16+6 \\ -3+2+6 & 6-1+8 & -9+4+2 \end{bmatrix}$$

$$\therefore \quad \mathbf{AB} = \begin{bmatrix} 6 & 4 & 6 \\ 24 & 24 & 16 \\ 5 & 13 & -3 \end{bmatrix}$$

$$\text{and} \quad \det(\mathbf{AB}) = 6\begin{vmatrix} 24 & 16 \\ 13 & -3 \end{vmatrix} + 4\begin{vmatrix} 16 & 24 \\ -3 & 5 \end{vmatrix} + 6\begin{vmatrix} 24 & 24 \\ 5 & 13 \end{vmatrix}$$

$$= 6(-72-208) + 4(80+72) + 6(312-120)$$

$$= 80$$

$$\det\mathbf{A} = 1\begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 1(8-6) + 2(18-4) + 1(2-12)$$

$$= 2 + 28 - 10$$

$$= 20 \qquad \text{and} \quad \det\mathbf{B} = -1\begin{vmatrix} -1 & 4 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= -1(-1-16) + 2(12-2) - 3(8 - -3)$$

$$= 4 \quad \therefore \quad \det\mathbf{A} \times \det\mathbf{B} = 20 \times 4 = 80 = \det(\mathbf{AB})$$

5 If $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$ then

$$\mathbf{A}^3 = \mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{A}^2 - \mathbf{AI} = \mathbf{A} - \mathbf{I} - \mathbf{A} = -\mathbf{I}$$
$$\mathbf{A}^4 = \mathbf{AA}^3 = \mathbf{A}(-\mathbf{I}) = -\mathbf{A}$$
$$\mathbf{A}^5 = \mathbf{AA}^4 = \mathbf{A}(-\mathbf{A}) = -\,\mathbf{A}^2 = -(\mathbf{A} - \mathbf{I}) = \mathbf{I} - \mathbf{A}$$
$$\mathbf{A}^6 = \mathbf{AA}^5 = \mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{AI} - \mathbf{A}^2 = \mathbf{A} - (\mathbf{A} - \mathbf{I}) = \mathbf{I}$$
$$\mathbf{A}^7 = \mathbf{AA}^6 = \mathbf{AI} = \mathbf{A}$$
$$\mathbf{A}^8 = \mathbf{AA}^7 = \mathbf{AA} = \mathbf{A}^2 = \mathbf{A} - \mathbf{I}$$

a $\mathbf{A}^{6n+3} = -\mathbf{I}$

$\mathbf{A}^{6n+5} = \mathbf{I} - \mathbf{A}$

b Now $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$

$\therefore \quad \mathbf{A}^{-1}\mathbf{AA} = \mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I}$ {premultiplying by $\mathbf{A}^{-1}$}

$\therefore \quad \mathbf{IA} = \mathbf{I} - \mathbf{A}^{-1}$

$\therefore \quad \mathbf{A}^{-1} = \mathbf{I} - \mathbf{A}$

c P_n is "$\mathbf{A}^{6n+5} = \mathbf{I} - \mathbf{A}$ for all n in Z^+".

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{A}^{11} = \mathbf{A}^5\mathbf{A}^6$

$= (\mathbf{I} - \mathbf{A})\mathbf{I}$

$= \mathbf{I} - \mathbf{A} \quad \therefore \quad P_1$ is true

(2) If P_k is true, then $\mathbf{A}^{6k+5} = \mathbf{I} - \mathbf{A}$

$\therefore \quad \mathbf{A}^{6(k+1)+5} = \mathbf{A}^{6k+6+5}$

$= \mathbf{A}^{6k+5}\mathbf{A}^6$

$= (\mathbf{I} - \mathbf{A})\mathbf{I}$

$= \mathbf{I} - \mathbf{A}$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

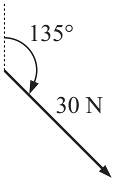
$\Rightarrow$ P_n is true {Principle of Mathematical Induction)

Chapter 15

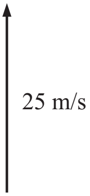
VECTORS IN 2 AND 3 DIMENSIONS

EXERCISE 15A.1

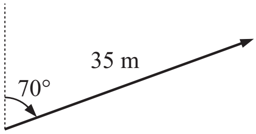
1 **a**



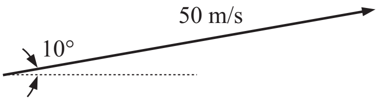
b



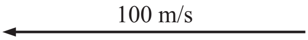
c



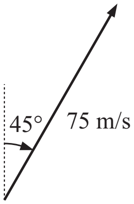
d



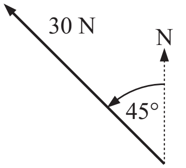
2 **a**



b

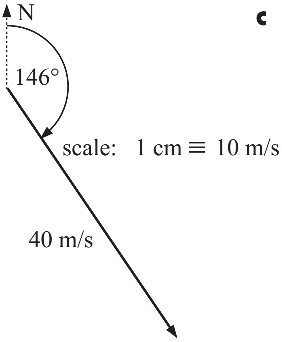


3 **a**

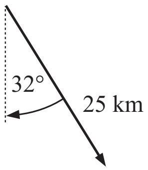


scale: 1 cm $\equiv$ 10 N

b

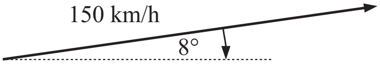


c



Scale: 1 cm $\equiv$ 10 km

d



Scale: 1 cm $\equiv$ 30 km/h

EXERCISE 15A.2

1 **a** If they are equal in magnitude, they have the same length. These are **p**, **q**, **s** and **t**.

b Those parallel are **p**, **q**, **r** and **t**.

c Those in the same direction are: **p** and **r**, **q** and **t**.

d To be equal they must have the same direction and be equal in length $\therefore$ $\mathbf{q} = \mathbf{t}$.

e **p** and **q** are negatives (equal length, but opposite direction). Likewise **p** and **t** are negatives. We write $\mathbf{p} = -\mathbf{q}$ and $\mathbf{p} = -\mathbf{t}$.

2 **a** True, as they have the same length and are parallel.

b True, as they are sides of an equilateral triangle.

c False, as they do not have the same direction.

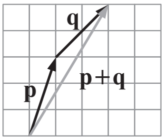
d False, as they have opposite directions.

e True, as they have the same length and direction.

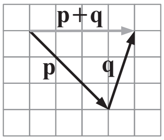
f False, as they do not have the same direction.

EXERCISE 15B.1

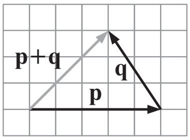
1 **a**



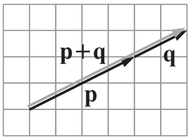
b



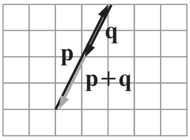
c



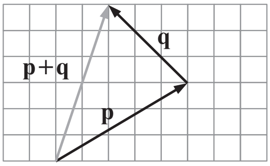
d



e



f



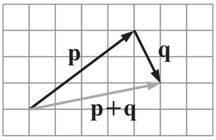
2 **a** $\overrightarrow{AB}+\overrightarrow{BC}$
$=\overrightarrow{AC}$

b $\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{BD}$

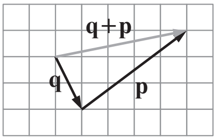
c $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{AC}+\overrightarrow{CD}$
$=\overrightarrow{AD}$

d $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}$
$=\overrightarrow{AB}+\overrightarrow{BD}$
$=\overrightarrow{AD}$

3 **a** **i**



ii



b yes

4 $\overrightarrow{PS}=\overrightarrow{PR}+\overrightarrow{RS}$
$=(\mathbf{a}+\mathbf{b})+\mathbf{c}$

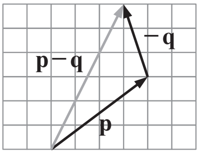
But $\overrightarrow{PS}=\overrightarrow{PQ}+\overrightarrow{QS}$
$=\mathbf{a}+(\mathbf{b}+\mathbf{c})$

$\therefore\ (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ {as both are equal to $\overrightarrow{PS}$}

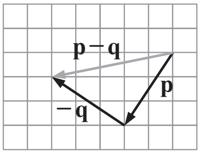
Note: $\overrightarrow{PS}=\mathbf{a}+\mathbf{b}+\mathbf{c}$ also

EXERCISE 15B.2

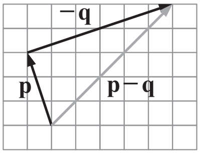
1 **a**



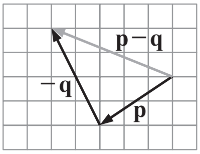
b



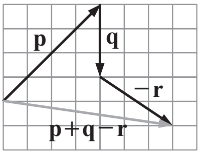
c



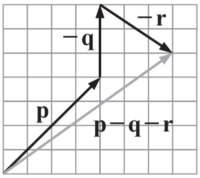
d



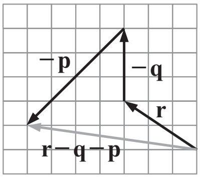
2 **a**



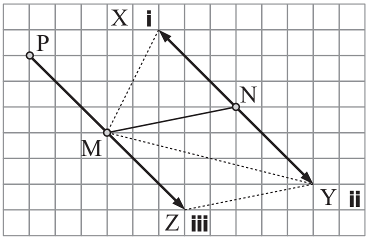
b



c



3 **a**



b a parallelogram

4 **a** $\overrightarrow{AC}+\overrightarrow{CB}$
$=\overrightarrow{AB}$

b $\overrightarrow{AD}-\overrightarrow{BD}$
$=\overrightarrow{AD}+\overrightarrow{DB}$
$=\overrightarrow{AB}$

c $\overrightarrow{AC}+\overrightarrow{CA}$
$=\overrightarrow{AA}$
$=\mathbf{0}$

d $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{AC}+\overrightarrow{CD}$
$=\overrightarrow{AD}$

e $\overrightarrow{BA}-\overrightarrow{CA}+\overrightarrow{CB}$
$=\overrightarrow{BA}+\overrightarrow{AC}+\overrightarrow{CB}$
$=\overrightarrow{BC}+\overrightarrow{CB}$
$=\overrightarrow{BB}$
$=\mathbf{0}$

f $\overrightarrow{AB}-\overrightarrow{CB}-\overrightarrow{DC}$
$=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
$=\overrightarrow{AC}+\overrightarrow{CD}$
$=\overrightarrow{AD}$

5 **a** $\mathbf{t}=\mathbf{r}+\mathbf{s}$ **b** $\mathbf{r}=-\mathbf{s}-\mathbf{t}$ **c** $\mathbf{r}=-\mathbf{p}-\mathbf{q}-\mathbf{s}$ **d** $\mathbf{r}=\mathbf{q}-\mathbf{p}+\mathbf{s}$
e $\mathbf{p}=\mathbf{t}+\mathbf{s}+\mathbf{r}-\mathbf{q}$ **f** $\mathbf{p}=-\mathbf{u}+\mathbf{t}+\mathbf{s}-\mathbf{r}-\mathbf{q}$

6 **a** **i** $\overrightarrow{OB}$
$=\overrightarrow{OA}+\overrightarrow{AB}$
$=\mathbf{r}+\mathbf{s}$

ii $\overrightarrow{CA}$
$=\overrightarrow{CB}+\overrightarrow{BA}$
$=-\overrightarrow{BC}-\overrightarrow{AB}$
$=-\mathbf{t}-\mathbf{s}$

iii $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}$
$=\mathbf{r}+\mathbf{s}+\mathbf{t}$

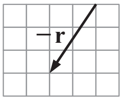
b **i** $\overrightarrow{AD}$
$=\overrightarrow{AB}+\overrightarrow{BD}$
$=\mathbf{p}+\mathbf{q}$

ii $\overrightarrow{BC}$
$=\overrightarrow{BD}+\overrightarrow{DC}$
$=\mathbf{q}+\mathbf{r}$

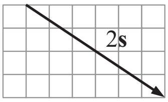
iii $\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$
$=\mathbf{p}+\mathbf{q}+\mathbf{r}$

EXERCISE 15B.3

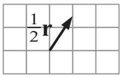
1 **a**



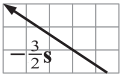
b



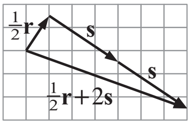
c



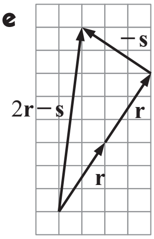
d



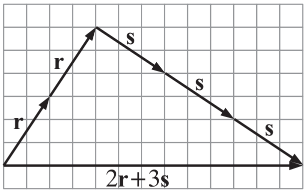
g



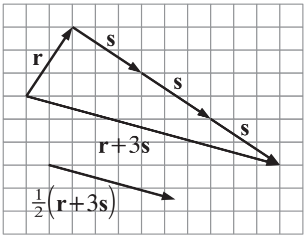
e



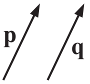
f



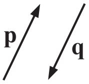
h



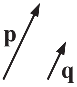
2 **a**



b



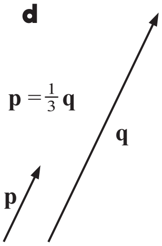
c



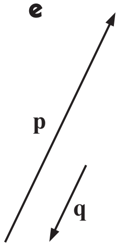
$\mathbf{p}=2\mathbf{q}$

d

$\mathbf{p}=\frac{1}{3}\mathbf{q}$



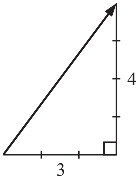
e



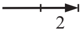
$\mathbf{p}=-3\mathbf{q}$

EXERCISE 15C.1

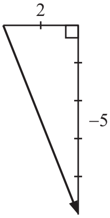
1 **a**



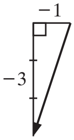
b



c



d



2 **a** $\begin{bmatrix}7\\3\end{bmatrix}$ **b** $\begin{bmatrix}-6\\0\end{bmatrix}$ **c** $\begin{bmatrix}2\\-5\end{bmatrix}$ **d** $\begin{bmatrix}0\\6\end{bmatrix}$ **e** $\begin{bmatrix}-6\\3\end{bmatrix}$ **f** $\begin{bmatrix}-5\\-5\end{bmatrix}$

EXERCISE 15C.2

1 **a**
$$\begin{aligned}&\mathbf{a}+\mathbf{b}\\&=\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}1\\4\end{bmatrix}\\&=\begin{bmatrix}-2\\6\end{bmatrix}\end{aligned}$$

b
$$\begin{aligned}&\mathbf{b}+\mathbf{a}\\&=\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix}\\&=\begin{bmatrix}-2\\6\end{bmatrix}\end{aligned}$$

c
$$\begin{aligned}&\mathbf{b}+\mathbf{c}\\&=\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix}\\&=\begin{bmatrix}-1\\-1\end{bmatrix}\end{aligned}$$

d
$$\begin{aligned}&\mathbf{c}+\mathbf{b}\\&=\begin{bmatrix}-2\\-5\end{bmatrix}+\begin{bmatrix}1\\4\end{bmatrix}\\&=\begin{bmatrix}-1\\-1\end{bmatrix}\end{aligned}$$

e
$$\begin{aligned}&\mathbf{a}+\mathbf{c}\\&=\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix}\\&=\begin{bmatrix}-5\\-3\end{bmatrix}\end{aligned}$$

f
$$\begin{aligned}&\mathbf{c}+\mathbf{a}\\&=\begin{bmatrix}-2\\-5\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix}\\&=\begin{bmatrix}-5\\-3\end{bmatrix}\end{aligned}$$

g
$$\begin{aligned}&\mathbf{a}+\mathbf{a}\\&=\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix}\\&=\begin{bmatrix}-6\\4\end{bmatrix}\end{aligned}$$

h
$$\begin{aligned}&\mathbf{b}+\mathbf{a}+\mathbf{c}\\&=\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix}\\&=\begin{bmatrix}-2\\6\end{bmatrix}+\begin{bmatrix}-2\\-5\end{bmatrix}\\&=\begin{bmatrix}-4\\1\end{bmatrix}\end{aligned}$$

2 **a**
$$\begin{aligned}&\mathbf{p}-\mathbf{q}\\&=\begin{bmatrix}-4\\2\end{bmatrix}-\begin{bmatrix}-1\\-5\end{bmatrix}\\&=\begin{bmatrix}-3\\7\end{bmatrix}\end{aligned}$$

b
$$\begin{aligned}&\mathbf{q}-\mathbf{r}\\&=\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}\\&=\begin{bmatrix}-4\\-3\end{bmatrix}\end{aligned}$$

c
$$\begin{aligned}&\mathbf{p}+\mathbf{q}-\mathbf{r}\\&=\begin{bmatrix}-4\\2\end{bmatrix}+\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}\\&=\begin{bmatrix}-8\\-1\end{bmatrix}\end{aligned}$$

d
$$\begin{aligned}&\mathbf{p}-\mathbf{q}-\mathbf{r}\\&=\begin{bmatrix}-4\\2\end{bmatrix}-\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}\\&=\begin{bmatrix}-6\\9\end{bmatrix}\end{aligned}$$

e
$$\begin{aligned}&\mathbf{q}-\mathbf{r}-\mathbf{p}\\&=\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}-\begin{bmatrix}-4\\2\end{bmatrix}\\&=\begin{bmatrix}0\\-5\end{bmatrix}\end{aligned}$$

f
$$\begin{aligned}&\mathbf{r}+\mathbf{q}-\mathbf{p}\\&=\begin{bmatrix}3\\-2\end{bmatrix}+\begin{bmatrix}-1\\-5\end{bmatrix}-\begin{bmatrix}-4\\2\end{bmatrix}\\&=\begin{bmatrix}6\\-9\end{bmatrix}\end{aligned}$$

3 **a**
$$\begin{aligned}&\overrightarrow{\mathrm{AC}}\\&=\overrightarrow{\mathrm{AB}}+\overrightarrow{\mathrm{BC}}\\&=-\overrightarrow{\mathrm{BA}}+\overrightarrow{\mathrm{BC}}\\&=-\begin{bmatrix}2\\-3\end{bmatrix}+\begin{bmatrix}-3\\1\end{bmatrix}\\&=\begin{bmatrix}-5\\4\end{bmatrix}\end{aligned}$$

b
$$\begin{aligned}&\overrightarrow{\mathrm{CB}}\\&=\overrightarrow{\mathrm{CA}}+\overrightarrow{\mathrm{AB}}\\&=\begin{bmatrix}2\\-1\end{bmatrix}+\begin{bmatrix}-1\\3\end{bmatrix}\\&=\begin{bmatrix}1\\2\end{bmatrix}\end{aligned}$$

c
$$\begin{aligned}&\overrightarrow{\mathrm{SP}}\\&=\overrightarrow{\mathrm{SR}}+\overrightarrow{\mathrm{RQ}}+\overrightarrow{\mathrm{QP}}\\&=-\overrightarrow{\mathrm{RS}}+\overrightarrow{\mathrm{RQ}}-\overrightarrow{\mathrm{PQ}}\\&=-\begin{bmatrix}-3\\2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}-1\\4\end{bmatrix}\\&=\begin{bmatrix}6\\-5\end{bmatrix}\end{aligned}$$

EXERCISE 15C.3

1 **a**
$$\begin{aligned}&-3\mathbf{p}\\&=-3\begin{bmatrix}1\\5\end{bmatrix}\\&=\begin{bmatrix}-3\\-15\end{bmatrix}\end{aligned}$$

b
$$\begin{aligned}&\tfrac{1}{2}\mathbf{q}\\&=\tfrac{1}{2}\begin{bmatrix}-2\\4\end{bmatrix}\\&=\begin{bmatrix}-1\\2\end{bmatrix}\end{aligned}$$

c
$$\begin{aligned}&2\mathbf{p}+\mathbf{q}\\&=2\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix}\\&=\begin{bmatrix}2\\10\end{bmatrix}+\begin{bmatrix}-2\\4\end{bmatrix}\\&=\begin{bmatrix}0\\14\end{bmatrix}\end{aligned}$$

d
$$\begin{aligned}&\mathbf{p}-2\mathbf{q}\\&=\begin{bmatrix}1\\5\end{bmatrix}-2\begin{bmatrix}-2\\4\end{bmatrix}\\&=\begin{bmatrix}1\\5\end{bmatrix}-\begin{bmatrix}-4\\8\end{bmatrix}\\&=\begin{bmatrix}5\\-3\end{bmatrix}\end{aligned}$$

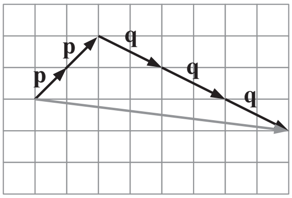
e
$$\begin{aligned}&\mathbf{p}-\tfrac{1}{2}\mathbf{r}\\&=\begin{bmatrix}1\\5\end{bmatrix}-\begin{bmatrix}-\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}\\&=\begin{bmatrix}2\frac{1}{2}\\5\frac{1}{2}\end{bmatrix}\end{aligned}$$

f
$$\begin{aligned}&2\mathbf{p}+3\mathbf{r}\\&=2\begin{bmatrix}1\\5\end{bmatrix}+3\begin{bmatrix}-3\\-1\end{bmatrix}\\&=\begin{bmatrix}2\\10\end{bmatrix}+\begin{bmatrix}-9\\-3\end{bmatrix}\\&=\begin{bmatrix}-7\\7\end{bmatrix}\end{aligned}$$

g
$$\begin{aligned}&2\mathbf{q}-3\mathbf{r}\\&=2\begin{bmatrix}-2\\4\end{bmatrix}-3\begin{bmatrix}-3\\-1\end{bmatrix}\\&=\begin{bmatrix}-4\\8\end{bmatrix}-\begin{bmatrix}-9\\-3\end{bmatrix}\\&=\begin{bmatrix}5\\11\end{bmatrix}\end{aligned}$$

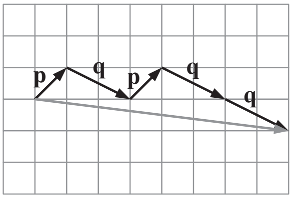
h
$$\begin{aligned}&2\mathbf{p}-\mathbf{q}+\tfrac{1}{3}\mathbf{r}\\&=\begin{bmatrix}2\\10\end{bmatrix}-\begin{bmatrix}-2\\4\end{bmatrix}+\begin{bmatrix}-1\\-\frac{1}{3}\end{bmatrix}\\&=\begin{bmatrix}3\\5\frac{2}{3}\end{bmatrix}\end{aligned}$$

2 **a**



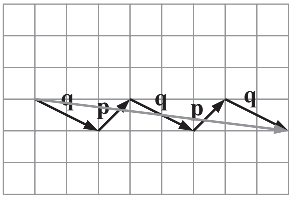
$= \begin{bmatrix}8\\-1\end{bmatrix}$

b



$= \begin{bmatrix}8\\-1\end{bmatrix}$

c



$= \begin{bmatrix}8\\-1\end{bmatrix}$

All vector expressions are equal {each consists of 2**p**s and 3**q**s}.
Each expression is equal to $2\mathbf{p} + 3\mathbf{q}$.

EXERCISE 15C.4

1 **a** $|\mathbf{r}|$
$= \sqrt{2^2+3^2}$
$= \sqrt{13}$ units

b $|\mathbf{s}|$
$= \sqrt{(-1)^2+4^2}$
$= \sqrt{17}$ units

c $\mathbf{r}+\mathbf{s}$
$= \begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}-1\\4\end{bmatrix}$
$= \begin{bmatrix}1\\7\end{bmatrix}$
$\therefore\ |\mathbf{r}+\mathbf{s}|$
$= \sqrt{1^2+7^2}$
$= \sqrt{50}$ units

d $\mathbf{r}-\mathbf{s}$
$= \begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}-1\\4\end{bmatrix}$
$= \begin{bmatrix}3\\-1\end{bmatrix}$
$\therefore\ |\mathbf{r}-\mathbf{s}|$
$= \sqrt{3^2+(-1)^2}$
$= \sqrt{10}$ units

e $\mathbf{s}-2\mathbf{r}$
$= \begin{bmatrix}-1\\4\end{bmatrix} - \begin{bmatrix}4\\6\end{bmatrix}$
$= \begin{bmatrix}-5\\-2\end{bmatrix}$
$\therefore\ |\mathbf{s}-2\mathbf{r}|$
$= \sqrt{(-5)^2+(-2)^2}$
$= \sqrt{29}$ units

2 **a** $|\mathbf{p}|$
$= \sqrt{1^2+3^2}$
$= \sqrt{10}$ units

b $2\mathbf{p} = \begin{bmatrix}2\\6\end{bmatrix}$
$\therefore\ |2\mathbf{p}| = \sqrt{2^2+6^2}$
$= \sqrt{40}$
$= 2\sqrt{10}$ units

c $-2\mathbf{p} = \begin{bmatrix}-2\\-6\end{bmatrix}$
$\therefore\ |-2\mathbf{p}| = \sqrt{(-2)^2+(-6)^2}$
$= \sqrt{4+36}$
$= \sqrt{40}$
$= 2\sqrt{10}$ units

d $3\mathbf{p} = \begin{bmatrix}3\\9\end{bmatrix}$
$\therefore\ |3\mathbf{p}| = \sqrt{3^2+9^2}$
$= \sqrt{9+81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

e $-3\mathbf{p} = \begin{bmatrix}-3\\-9\end{bmatrix}$
$\therefore\ |-3\mathbf{p}| = \sqrt{(-3)^2+(-9)^2}$
$= \sqrt{9+81}$
$= \sqrt{90}$
$= 3\sqrt{10}$ units

f $|\mathbf{q}| = \sqrt{(-2)^2+4^2}$
$= \sqrt{4+16}$
$= \sqrt{20}$
$= 2\sqrt{5}$ units

g $4\mathbf{q} = \begin{bmatrix}-8\\16\end{bmatrix}$
$\therefore\ |4\mathbf{q}| = \sqrt{(-8)^2+16^2}$
$= \sqrt{64+256}$
$= \sqrt{320}$
$= 8\sqrt{5}$ units

h $-4\mathbf{q} = \begin{bmatrix}8\\-16\end{bmatrix}$
$\therefore\ |-4\mathbf{q}| = \sqrt{8^2+(-16)^2}$
$= \sqrt{64+256}$
$= \sqrt{320}$
$= 8\sqrt{5}$ units

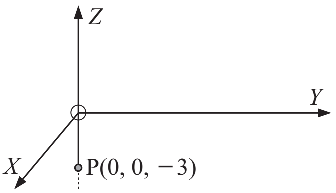
i $\frac{1}{2}\mathbf{q} = \begin{bmatrix}-1\\2\end{bmatrix}$
$\therefore\ \left|\frac{1}{2}\mathbf{q}\right| = \sqrt{(-1)^2+2^2}$
$= \sqrt{5}$ units

j $-\frac{1}{2}\mathbf{q} = \begin{bmatrix}1\\-2\end{bmatrix}$
$\therefore\ \left|-\frac{1}{2}\mathbf{q}\right| = \sqrt{1^2+(-2)^2}$
$= \sqrt{5}$ units

3 $k\mathbf{a} = \begin{bmatrix}ka_1\\ka_2\end{bmatrix}$ $\therefore\ |k\mathbf{a}| = \sqrt{(ka_1)^2+(ka_2)^2}$
$= \sqrt{k^2a_1^2+k^2a_2^2}$
$= \sqrt{k^2(a_1^2+a_2^2)}$
$= \sqrt{k^2}\sqrt{a_1^2+a_2^2}$
$= |k|\sqrt{a_1^2+a_2^2}$
$= |k|\,|\mathbf{a}|$

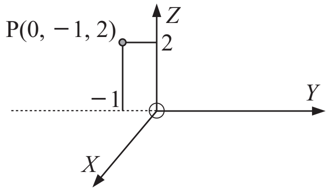
EXERCISE 15D

1 **a**



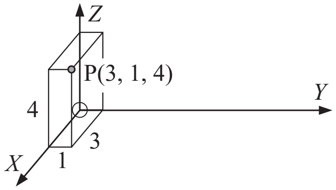
$$OP = \sqrt{0^2 + 0^2 + (-3)^2} = 3 \text{ units}$$

b



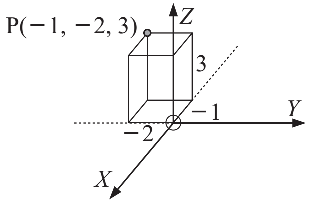
$$OP = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5} \text{ units}$$

c



$$\begin{aligned} OP &= \sqrt{3^2 + 1^2 + 4^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

d



$$\begin{aligned} OP &= \sqrt{(-1)^2 + (-2)^2 + 3^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

2 **a** **i** AB

$$\begin{aligned} &= \sqrt{(0 - -1)^2 + (-1 - 2)^2 + (1 - 3)^2} \\ &= \sqrt{1 + 9 + 4} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{-1+0}{2}, \frac{2-1}{2}, \frac{3+1}{2}\right)$

i.e., $\left(-\frac{1}{2}, \frac{1}{2}, 2\right)$

b **i** AB

$$\begin{aligned} &= \sqrt{(2 - 0)^2 + (-1 - 0)^2 + (3 - 0)^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{0+2}{2}, \frac{0-1}{2}, \frac{0+3}{2}\right)$

i.e., $\left(1, -\frac{1}{2}, \frac{3}{2}\right)$

c **i** AB

$$\begin{aligned} &= \sqrt{(-1 - 3)^2 + (0 - -1)^2 + (1 - -1)^2} \\ &= \sqrt{16 + 1 + 4} \\ &= \sqrt{21} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{3-1}{2}, \frac{-1+0}{2}, \frac{-1+1}{2}\right)$

i.e., $\left(1, -\frac{1}{2}, 0\right)$

d **i** AB

$$\begin{aligned} &= \sqrt{(0 - 2)^2 + (1 - 0)^2 + (0 - -3)^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

ii Midpoint is at $\left(\frac{2+0}{2}, \frac{0+1}{2}, \frac{-3+0}{2}\right)$

i.e., $\left(1, \frac{1}{2}, -\frac{3}{2}\right)$

3 P(0, 4, 4) Q(2, 6, 5) R(1, 4, 3)

$$\begin{aligned} PQ &= \sqrt{(2-0)^2 + (6-4)^2 + (5-4)^2} \\ &= \sqrt{4 + 4 + 1} \\ &= 3 \end{aligned}$$

$$\begin{aligned} PR &= \sqrt{(1-0)^2 + (4-4)^2 + (3-4)^2} \\ &= \sqrt{1 + 0 + 1} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(1-2)^2 + (4-6)^2 + (3-5)^2} \\ &= \sqrt{1 + 4 + 4} \\ &= 3 \end{aligned}$$

$\therefore$ PQ = QR and so Δ PQR is isosceles.

4 **a** A(2, −1, 7) B(3, 1, 4) C(5, 4, 5)

$$\begin{aligned} AB &= \sqrt{(3-2)^2+(1--1)^2+(4-7)^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(5-2)^2+(4--1)^2+(5-7)^2} \\ &= \sqrt{9+25+4} \\ &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(5-3)^2+(4-1)^2+(5-4)^2} \\ &= \sqrt{4+9+1} \\ &= \sqrt{14} \end{aligned}$$

$\therefore$ as AB = BC, Δ ABC is isosceles.

b A(0, 0, 3) B(2, 8, 1) C(−9, 6, 18)

$$\begin{aligned} AB &= \sqrt{(2-0)^2+(8-0)^2+(1-3)^2} \\ &= \sqrt{4+64+4} \\ &= \sqrt{72} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(-9-0)^2+(6-0)^2+(18-3)^2} \\ &= \sqrt{81+36+225} \\ &= \sqrt{342} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-9-2)^2+(6-8)^2+(18-1)^2} \\ &= \sqrt{121+4+289} \\ &= \sqrt{414} \end{aligned}$$

Since $BC^2 = AB^2 + AC^2$, Δ ABC is right-angled.

c A(5, 6, −2) B(6, 12, 9) C(2, 4, 2)

$$\begin{aligned} AB &= \sqrt{(6-5)^2+(12-6)^2+(9--2)^2} \\ &= \sqrt{1+36+121} \\ &= \sqrt{158} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(2-5)^2+(4-6)^2+(2--2)^2} \\ &= \sqrt{9+4+16} \\ &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(2-6)^2+(4-12)^2+(2-9)^2} \\ &= \sqrt{16+64+49} \\ &= \sqrt{129} \end{aligned}$$

Since $AB^2 = AC^2 + BC^2$, Δ ABC is right-angled.

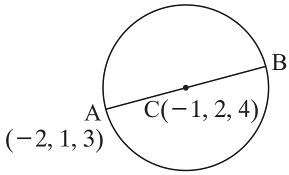
d A(1, 0, −3) B(2, 2, 0) C(4, 6, 6)

$$\begin{aligned} AB &= \sqrt{(2-1)^2+(2-0)^2+(0--3)^2} \\ &= \sqrt{1^2+2^2+3^2} \\ &= \sqrt{14} \end{aligned} \qquad \begin{aligned} AC &= \sqrt{(4-1)^2+(6-0)^2+(6--3)^2} \\ &= \sqrt{3^2+6^2+9^2} \\ &= \sqrt{126} \\ &= 3\sqrt{14} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(4-2)^2+(6-2)^2+(6-0)^2} \\ &= \sqrt{2^2+4^2+6^2} \\ &= \sqrt{56} \\ &= 2\sqrt{14} \end{aligned}$$

Since AB + BC = AC, the points A, B and C lie on a straight line, i.e., do not form a triangle.

5



If B is (a, b, c) then $\dfrac{a-2}{2} = -1, \quad \dfrac{b+1}{2} = 2, \quad \dfrac{c+3}{2} = 4$

$\therefore \quad a = 0, \quad b = 3, \quad c = 5$

$\therefore$ B is (0, 3, 5)

$$\begin{aligned} r = AC &= \sqrt{(-1--2)^2+(2-1)^2+(4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

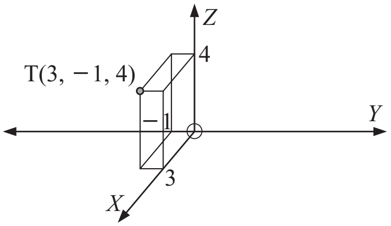
6 **a** $(0,\ y,\ 0)$ for any y

b Distance between $(0,\ y,\ 0)$ and $B(-1,\ -1,\ 2)$ is $\sqrt{(-1)^2+(-1-y)^2+2^2}$

$$\begin{aligned}\therefore\quad \sqrt{1+(y+1)^2+4} &= \sqrt{14}\\ \therefore\quad (y+1)^2 &= 9\\ \therefore\quad y+1 &= \pm 3\\ \therefore\quad y &= -1\pm 3\\ \therefore\quad y &= -4 \text{ or } 2 \qquad \therefore \text{ the two points are } (0,\ -4,\ 0) \text{ and } (0,\ 2,\ 0)\end{aligned}$$

EXERCISE 15E.1

1 **a**



b $\overrightarrow{OT}=\begin{bmatrix}3\\-1\\4\end{bmatrix}$

c $$\begin{aligned}OT &= \sqrt{(3-0)^2+(-1-0)^2+(4-0)^2}\\ &= \sqrt{9+1+16}\\ &= \sqrt{26}\text{ units}\end{aligned}$$

2 **a** $\overrightarrow{AB}=\begin{bmatrix}1-(-3)\\0-1\\-1-2\end{bmatrix}=\begin{bmatrix}4\\-1\\-3\end{bmatrix}$

$\overrightarrow{BA}=\begin{bmatrix}-3-1\\1-0\\2-(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\\3\end{bmatrix}$

b $$\begin{aligned}|\,\overrightarrow{AB}\,| &= \sqrt{4^2+(-1)^2+(-3)^2}\\ &= \sqrt{26}\text{ units}\end{aligned}$$

$$\begin{aligned}|\,\overrightarrow{BA}\,| &= \sqrt{(-4)^2+1^2+3^2}\\ &= \sqrt{26}\text{ units}\end{aligned}$$

3 $\overrightarrow{OA}=\begin{bmatrix}3\\1\\0\end{bmatrix}$ $\quad\overrightarrow{OB}=\begin{bmatrix}-1\\1\\2\end{bmatrix}$ $\quad\overrightarrow{AB}=\begin{bmatrix}-1-3\\1-1\\2-0\end{bmatrix}=\begin{bmatrix}-4\\0\\2\end{bmatrix}$

4 **a** The position vector of M from N

$=\overrightarrow{NM}$

$=\begin{bmatrix}4-(-1)\\-2-2\\-1-0\end{bmatrix}=\begin{bmatrix}5\\-4\\-1\end{bmatrix}$

b The position vector of N from M

$=\overrightarrow{MN}$

$=\begin{bmatrix}-1-4\\2-(-2)\\0-(-1)\end{bmatrix}=\begin{bmatrix}-5\\4\\1\end{bmatrix}$

c $|\,\overrightarrow{MN}\,| = \sqrt{(-5)^2+4^2+1^2}=\sqrt{25+16+1}=\sqrt{42}$ units

5 **a** The position vector of A from O

$=\overrightarrow{OA}=\begin{bmatrix}-1\\2\\5\end{bmatrix}$

$$\begin{aligned}|\,\overrightarrow{OA}\,| &= \sqrt{(-1)^2+2^2+5^2}\\ &= \sqrt{1+4+25}\\ &= \sqrt{30}\text{ units}\end{aligned}$$

b The position vector of C from A

$=\overrightarrow{AC}=\begin{bmatrix}-3-(-1)\\1-2\\0-5\end{bmatrix}=\begin{bmatrix}-2\\-1\\-5\end{bmatrix}$

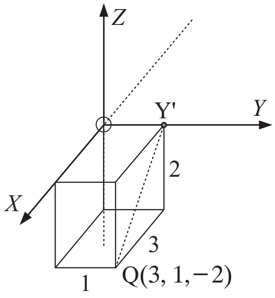
$$\begin{aligned}|\,\overrightarrow{AC}\,| &= \sqrt{(-2)^2+(-1)^2+(-5)^2}\\ &= \sqrt{4+1+25}\\ &= \sqrt{30}\text{ units}\end{aligned}$$

c The position vector of B from C

$=\overrightarrow{CB}=\begin{bmatrix}5\\-1\\3\end{bmatrix}$

and $$\begin{aligned}|\,\overrightarrow{CB}\,| &= \sqrt{5^2+(-1)^2+3^2}\\ &= \sqrt{25+1+9}\\ &= \sqrt{35}\text{ units}\end{aligned}$$

6



a The distance from Q to the Y-axis is the distance from Q to Y$'$(0, 1, 0)

$$\therefore \quad QY' = \sqrt{(3-0)^2+(1-1)^2+(-2-0)^2} = \sqrt{9+4} = \sqrt{13} \text{ units}$$

b The distance from Q to the origin is

$$QO = \sqrt{(3-0)^2+(1-0)^2+(-2-0)^2} = \sqrt{9+1+4} = \sqrt{14} \text{ units}$$

c The distance from Q to the ZOY plane is the distance from Q to (0, 1, −2), i.e., 3 units.

EXERCISE 15E.2

1 **a** $\begin{bmatrix} a-4 \\ b-3 \\ c+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$

$\therefore \quad a-4 = 1,$
$b-3 = 3,$
$c+2 = -4$
$\therefore \quad a=5, \quad b=6, \quad c=-6$

b $\begin{bmatrix} a-5 \\ b-2 \\ c+3 \end{bmatrix} = \begin{bmatrix} 3-a \\ 2-b \\ 5-c \end{bmatrix}$

$\therefore \quad a-5 = 3-a,$
$b-2 = 2-b,$
$c+3 = 5-c,$
$\therefore \quad 2a = 8, \quad 2b=4, \quad 2c=2$
$\therefore \quad a=4, \quad b=2, \quad c=1$

2 **a** $2\begin{bmatrix} 1 \\ 0 \\ 3a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \\ 6a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$

$\therefore \quad 6a=2, \quad b=2, \quad c-1=0 \quad \therefore \quad a=\frac{1}{3}, \quad b=2, \quad c=1$

b $\begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} = \begin{bmatrix} b \\ a^2 \\ a+b \end{bmatrix}$

$\therefore \quad b=2, \quad a^2=a, \quad a+b=3$
i.e., $a=1, \quad b=2$

c $a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

$\therefore \quad a+2b=-1$ ...... (1), $\quad a+c=3$ ...... (2) and $-b+c = 3$ ...... (3)

(1) − (2) gives: $2b-c = -4$ ...... (4)
Adding (3) and (4) gives $b = -1$
$\therefore$ using (3), $c = 2$
and using (2), $a = 1$
$\therefore \quad a = 1, \quad b=-1, \quad c=2$

3 A(−1, 3, 4) B(2, 5, −1) C(−1, 2, −2) D(r, s, t)

a If $\overrightarrow{AC} = \overrightarrow{BD}$ then $\begin{bmatrix} -1-(-1) \\ 2-3 \\ -2-4 \end{bmatrix} = \begin{bmatrix} r-2 \\ s-5 \\ t+1 \end{bmatrix}$

$\therefore \quad r-2=0, \quad s-5=-1$ and $t+1=-6 \quad \therefore \quad r=2, \quad s=4$ and $t=-7$

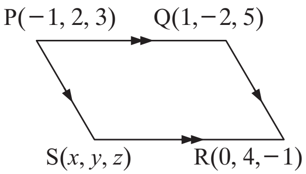
b If $\overrightarrow{AB} = \overrightarrow{DC}$ then $\begin{bmatrix} 2-(-1) \\ 5-3 \\ -1-4 \end{bmatrix} = \begin{bmatrix} -1-r \\ 2-s \\ -2-t \end{bmatrix}$

$\therefore \quad -1-r=3, \quad 2-s=2$ and $-2-t=-5 \quad \therefore \quad r=-4, \quad s=0$ and $t=3$

4 **a** $\overrightarrow{AB} = \begin{bmatrix} 3-1 \\ -3-2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\overrightarrow{DC} = \begin{bmatrix} 7-5 \\ -4-1 \\ 5-6 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

5 **a** Suppose S is at (x, y, z) $\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}



$$\therefore \begin{bmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 0-x \\ 4-y \\ -1-z \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -x \\ 4-y \\ -1-z \end{bmatrix}$$

$\therefore\ -x = 2$ $\qquad 4-y = -4$ $\qquad -1-z = 2$

$\therefore\ x = -2$ $\qquad y = 8$ $\qquad z = -3$

$\therefore$ S is at (−2, 8, −3)

b The midpoint of PR is $\left(\dfrac{-1+0}{2}, \dfrac{2+4}{2}, \dfrac{3+(-1)}{2}\right)$ i.e., $(-\frac{1}{2}, 3, 1)$

The midpoint of QS is $\left(\dfrac{1+(-2)}{2}, \dfrac{-2+8}{2}, \dfrac{5+(-3)}{2}\right)$ i.e., $(-\frac{1}{2}, 3, 1)$

i.e., PR and QS have the same midpoint.

EXERCISE 15F

1 **a** $2\mathbf{x} = \mathbf{q}$

$\therefore\ \frac{1}{2}(2\mathbf{x}) = \frac{1}{2}\mathbf{q}$

$\therefore\ \mathbf{x} = \frac{1}{2}\mathbf{q}$

b $\frac{1}{2}\mathbf{x} = \mathbf{n}$

$\therefore\ 2(\frac{1}{2}\mathbf{x}) = 2\mathbf{n}$

$\therefore\ \mathbf{x} = 2\mathbf{n}$

c $-3\mathbf{x} = \mathbf{p}$

$\therefore\ 3\mathbf{x} = -\mathbf{p}$

$\therefore\ \frac{1}{3}(3\mathbf{x}) = -\frac{1}{3}\mathbf{p}$

$\therefore\ \mathbf{x} = -\frac{1}{3}\mathbf{p}$

d $\mathbf{q} + 2\mathbf{x} = \mathbf{r}$

$\therefore\ 2\mathbf{x} = \mathbf{r} - \mathbf{q}$

$\therefore\ \mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q})$

e $4\mathbf{s} - 5\mathbf{x} = \mathbf{t}$

$\therefore\ -5\mathbf{x} = \mathbf{t} - 4\mathbf{s}$

$\therefore\ 5\mathbf{x} = 4\mathbf{s} - \mathbf{t}$

$\therefore\ \mathbf{x} = \frac{4}{5}\mathbf{s} - \frac{1}{5}\mathbf{t}$

f $4\mathbf{m} - \frac{1}{3}\mathbf{x} = \mathbf{n}$

$\therefore\ 4\mathbf{m} - \mathbf{n} = \frac{1}{3}\mathbf{x}$

$\therefore\ \mathbf{x} = 12\mathbf{m} - 3\mathbf{n}$

2 **a** $2\mathbf{y} = \mathbf{r}$

$\therefore\ \mathbf{y} = \frac{1}{2}\mathbf{r}$

$= \frac{1}{2}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix}$

b $\frac{1}{2}\mathbf{y} = \mathbf{s}$

$\therefore\ \mathbf{y} = 2\mathbf{s}$

$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

c $\mathbf{r} + 2\mathbf{y} = \mathbf{s}$

$\therefore\ 2\mathbf{y} = \mathbf{s} - \mathbf{r}$

$\therefore\ \mathbf{y} = \frac{1}{2}\mathbf{s} - \frac{1}{2}\mathbf{r}$

$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$

d $3\mathbf{s} - 4\mathbf{y} = \mathbf{r}$

$\therefore\ 3\mathbf{s} - \mathbf{r} = 4\mathbf{y}$

$\therefore\ \mathbf{y} = \frac{3}{4}\mathbf{s} - \frac{1}{4}\mathbf{r}$

$= \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -\frac{2}{4} \\ \frac{3}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \end{bmatrix}$

3 $k\mathbf{x} = \mathbf{a}$ $\quad \therefore\ k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\therefore\ kx_1 = a_1$ and $kx_2 = a_2$

$\therefore\ x_1 = \frac{1}{k}a_1$ and $x_2 = \frac{1}{k}a_2$ $\quad \therefore\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{bmatrix} = \frac{1}{k}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

4 **a** $2\mathbf{a} + \mathbf{x} = \mathbf{b}$

$$\therefore\ \mathbf{x} = \mathbf{b} - 2\mathbf{a} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix}$$

b $3\mathbf{x} - \mathbf{a} = 2\mathbf{b}$

$\therefore\ 3\mathbf{x} = \mathbf{a} + 2\mathbf{b}$

$$\therefore\ \mathbf{x} = \tfrac{1}{3}(\mathbf{a} + 2\mathbf{b}) = \tfrac{1}{3}\left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right) = \tfrac{1}{3}\left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}\right) = \tfrac{1}{3}\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

c $$2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$$
$$\therefore \quad \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$$
$$\therefore \quad \mathbf{x} = \tfrac{1}{2}(\mathbf{a} + 2\mathbf{b}) = \tfrac{1}{2}\begin{bmatrix}3\\-2\\5\end{bmatrix} \quad \{\text{using } \mathbf{b}\} \quad = \begin{bmatrix}\frac{3}{2}\\-1\\\frac{5}{2}\end{bmatrix}$$

5 $$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\begin{bmatrix}-2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}2\\1\\-1\end{bmatrix} + \begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}3\\4\\-2\end{bmatrix}$$

$$\therefore \quad |\,\overrightarrow{AB}\,| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9+16+4} = \sqrt{29} \text{ units}$$

6 $$\overrightarrow{OA} = \begin{bmatrix}2\\1\\-2\end{bmatrix}, \quad \overrightarrow{OB} = \begin{bmatrix}0\\3\\-4\end{bmatrix}, \quad \overrightarrow{OC} = \begin{bmatrix}1\\-2\\1\end{bmatrix}, \quad \overrightarrow{OD} = \begin{bmatrix}-2\\-3\\2\end{bmatrix}$$

$$\therefore \quad \overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD} = -\overrightarrow{OB} + \overrightarrow{OD} = -\begin{bmatrix}0\\3\\-4\end{bmatrix} + \begin{bmatrix}-2\\-3\\2\end{bmatrix} = \begin{bmatrix}-2\\-6\\6\end{bmatrix}$$

and $$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\begin{bmatrix}2\\1\\-2\end{bmatrix} + \begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}-1\\-3\\3\end{bmatrix} \qquad \therefore \quad \overrightarrow{BD} = \begin{bmatrix}-2\\-6\\6\end{bmatrix} = 2\begin{bmatrix}-1\\-3\\3\end{bmatrix} = 2\overrightarrow{AC}$$

7 **a** $$\overrightarrow{BD} = \tfrac{1}{2}\overrightarrow{OA}$$
$$= \tfrac{1}{2}\mathbf{a}$$

b $$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\mathbf{a} + \mathbf{b}$$
$$= \mathbf{b} - \mathbf{a}$$

c $$\overrightarrow{BA} = -\overrightarrow{AB}$$
$$= -(\mathbf{b} - \mathbf{a})$$
$$= -\mathbf{b} + \mathbf{a} \quad \text{or} \quad \mathbf{a} - \mathbf{b}$$

d $$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$$
$$= \mathbf{b} + \tfrac{1}{2}\mathbf{a}$$

e $$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD}$$
$$= -\mathbf{a} + \mathbf{b} + \tfrac{1}{2}\mathbf{a}$$
$$= -\tfrac{1}{2}\mathbf{a} + \mathbf{b}$$

f $$\overrightarrow{DA} = -\overrightarrow{AD}$$
$$= \tfrac{1}{2}\mathbf{a} - \mathbf{b}$$

8 **a** $$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \begin{bmatrix}-1\\3\\2\end{bmatrix} + \begin{bmatrix}0\\2\\-3\end{bmatrix} = \begin{bmatrix}-1\\5\\-1\end{bmatrix}$$

b $$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = -\begin{bmatrix}2\\-1\\4\end{bmatrix} + \begin{bmatrix}-1\\3\\2\end{bmatrix} = \begin{bmatrix}-2\\1\\-4\end{bmatrix} + \begin{bmatrix}-1\\3\\2\end{bmatrix} = \begin{bmatrix}-3\\4\\-2\end{bmatrix}$$

c $$\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD} = \begin{bmatrix}-3\\4\\-2\end{bmatrix} + \begin{bmatrix}0\\2\\-3\end{bmatrix} \quad (\text{using } \mathbf{b}) \quad = \begin{bmatrix}-3\\6\\-5\end{bmatrix}$$

9 **a** $$\mathbf{a} + \mathbf{b} = \begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\2\\-3\end{bmatrix} = \begin{bmatrix}3\\1\\-2\end{bmatrix}$$

b $$\mathbf{a} - \mathbf{b} = \begin{bmatrix}2\\-1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\-3\end{bmatrix} = \begin{bmatrix}1\\-3\\4\end{bmatrix}$$

c $$\mathbf{b} + 2\mathbf{c} = \begin{bmatrix}1\\2\\-3\end{bmatrix} + 2\begin{bmatrix}0\\1\\-3\end{bmatrix} = \begin{bmatrix}1\\2\\-3\end{bmatrix} + \begin{bmatrix}0\\2\\-6\end{bmatrix} = \begin{bmatrix}1\\4\\-9\end{bmatrix}$$

d $\mathbf{a}-3\mathbf{c}=\begin{bmatrix}2\\-1\\1\end{bmatrix}-3\begin{bmatrix}0\\1\\-3\end{bmatrix}=\begin{bmatrix}2\\-1\\1\end{bmatrix}-\begin{bmatrix}0\\3\\-9\end{bmatrix}=\begin{bmatrix}2\\-4\\10\end{bmatrix}$

e $\mathbf{a}+\mathbf{b}+\mathbf{c}=\begin{bmatrix}2\\-1\\1\end{bmatrix}+\begin{bmatrix}1\\2\\-3\end{bmatrix}+\begin{bmatrix}0\\1\\-3\end{bmatrix}=\begin{bmatrix}3\\2\\-5\end{bmatrix}$

f $\mathbf{c}-\frac{1}{2}\mathbf{a}=\begin{bmatrix}0\\1\\-3\end{bmatrix}-\frac{1}{2}\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\-3\end{bmatrix}-\begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}=\begin{bmatrix}-1\\\frac{3}{2}\\-\frac{7}{2}\end{bmatrix}$

g $\mathbf{a}-\mathbf{b}-\mathbf{c}=\begin{bmatrix}2\\-1\\1\end{bmatrix}-\begin{bmatrix}1\\2\\-3\end{bmatrix}-\begin{bmatrix}0\\1\\-3\end{bmatrix}=\begin{bmatrix}1\\-4\\7\end{bmatrix}$

h $2\mathbf{b}-\mathbf{c}+\mathbf{a}=2\begin{bmatrix}1\\2\\-3\end{bmatrix}-\begin{bmatrix}0\\1\\-3\end{bmatrix}+\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}2\\4\\-6\end{bmatrix}-\begin{bmatrix}0\\1\\-3\end{bmatrix}+\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}4\\2\\-2\end{bmatrix}$

10 **a** $|\mathbf{a}| = \sqrt{(-1)^2+1^2+3^2}$
$= \sqrt{11}$ units

b $|\mathbf{b}| = \sqrt{1^2+(-3)^2+2^2}$
$= \sqrt{14}$ units

c $|\mathbf{b}+\mathbf{c}| = \left|\begin{bmatrix}1\\-3\\2\end{bmatrix}+\begin{bmatrix}-2\\2\\4\end{bmatrix}\right| = \left|\begin{bmatrix}-1\\-1\\6\end{bmatrix}\right|$
$= \sqrt{(-1)^2+(-1)^2+6^2}$
$= \sqrt{1+1+36}$
$= \sqrt{38}$ units

d $|\mathbf{a}-\mathbf{c}| = \left|\begin{bmatrix}-1\\1\\3\end{bmatrix}-\begin{bmatrix}-2\\2\\4\end{bmatrix}\right| = \left|\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right|$
$= \sqrt{1^2+(-1)^2+(-1)^2}$
$= \sqrt{3}$ units

e $|\mathbf{a}|\mathbf{b} = \sqrt{11}\begin{bmatrix}1\\-3\\2\end{bmatrix} = \begin{bmatrix}\sqrt{11}\\-3\sqrt{11}\\2\sqrt{11}\end{bmatrix}$

f $\dfrac{1}{|\mathbf{a}|}\times\mathbf{a} = \frac{1}{\sqrt{11}}\begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{3}{\sqrt{11}}\end{bmatrix}$

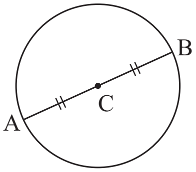
EXERCISE 15G

1 **a** M is $\left(\frac{3+-1}{2}, \frac{6+2}{2}\right)$
i.e., M is (1, 4)

b $\overrightarrow{CA} = \begin{bmatrix}3--4\\6-1\end{bmatrix} = \begin{bmatrix}7\\5\end{bmatrix}$
$\overrightarrow{CM} = \begin{bmatrix}1--4\\4-1\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$
$\overrightarrow{CB} = \begin{bmatrix}-1--4\\2-1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$

c $\frac{1}{2}\overrightarrow{CA}+\frac{1}{2}\overrightarrow{CB}$
$= \frac{1}{2}\begin{bmatrix}7\\5\end{bmatrix}+\frac{1}{2}\begin{bmatrix}3\\1\end{bmatrix}$
$= \begin{bmatrix}5\\3\end{bmatrix}$ which is $\overrightarrow{CM}$

2



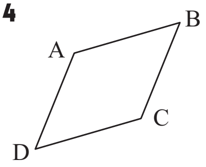
a $\overrightarrow{AC} = \begin{bmatrix}1-3\\4--2\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$ $\therefore$ B is $(1-2, 4+6)$, i.e., $(-1, 10)$

b $\overrightarrow{AC} = \begin{bmatrix}-1-0\\-2-5\end{bmatrix} = \begin{bmatrix}-1\\-7\end{bmatrix}$ $\therefore$ B is $(-1-1, -2-7)$, i.e., $(-2, -9)$

c $\overrightarrow{AC} = \begin{bmatrix}3--1\\0--4\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}$ $\therefore$ B is $(3+4, 0+4)$, i.e., $(7, 4)$

3 $\overrightarrow{AB} = \begin{bmatrix}2--1\\3-5\\-3-2\end{bmatrix} = \begin{bmatrix}3\\-2\\-5\end{bmatrix}$ $\therefore$ C is $(2+3, 3-2, -3-5)$ i.e., $(5, 1, -8)$
D is $(5+3, 1-2, -8-5)$ i.e., $(8, -1, -13)$
E is $(8+3, -1-2, -13-5)$ i.e., $(11, -3, -18)$

4



a $\overrightarrow{AB} = \begin{bmatrix} 4-3 \\ 2--1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} -1--2 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Now $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore$ sides AB and DC are equal in length and parallel.
This is sufficient to deduce that ABCD is a parallelogram.

b $\overrightarrow{AB} = \begin{bmatrix} -1-5 \\ 2-0 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} 4-10 \\ -3--5 \\ 6-5 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$

So $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore$ sides AB and DC are equal in length and parallel.
This is sufficient to deduce that ABCD is a parallelogram.

c $\overrightarrow{AB} = \begin{bmatrix} 1-2 \\ 4--3 \\ -1-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} -2--1 \\ 6--1 \\ -2-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix}$

So, $\overrightarrow{AB} \neq \overrightarrow{DC}$

$\therefore$ ABCD cannot be a parallelogram.

5 **a** Let D be (a, b)

Now $\overrightarrow{CD} = \overrightarrow{BA}$

$\therefore \begin{bmatrix} a-8 \\ b--2 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0--1 \end{bmatrix}$

$\therefore \begin{bmatrix} a-8 \\ b+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \ a = 9, \quad b = -1$

So, D is (9, −1)

b Let R be (a, b, c)

Now $\overrightarrow{SR} = \overrightarrow{PQ}$

$\therefore \begin{bmatrix} a-4 \\ b-0 \\ c-7 \end{bmatrix} = \begin{bmatrix} -2--1 \\ 5-4 \\ 2-3 \end{bmatrix}$

$\therefore \begin{bmatrix} a-4 \\ b \\ c-7 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$\therefore \ a = 3, \quad b = 1, \quad c = 6$

So, R is (3, 1, 6)

c Let X be (a, b, c)

Now $\overrightarrow{WX} = \overrightarrow{ZY}$

$\therefore \begin{bmatrix} a--1 \\ b-5 \\ c-8 \end{bmatrix} = \begin{bmatrix} 3-0 \\ -2-4 \\ -2-6 \end{bmatrix}$

$\therefore \begin{bmatrix} a+1 \\ b-5 \\ c-8 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -8 \end{bmatrix}$

$\therefore \ a = 2, \quad b = -1, \quad c = 0$

So, X is (2, −1, 0)

6 **a**

$$\begin{aligned} r\begin{bmatrix} 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 5 \end{bmatrix} &= \begin{bmatrix} -8 \\ -27 \end{bmatrix} \\ \therefore \begin{bmatrix} r+2s \\ -r+5s \end{bmatrix} &= \begin{bmatrix} -8 \\ -27 \end{bmatrix} \\ \therefore \ r + 2s &= -8 \quad \text{......} \ (1) \\ -r + 5s &= -27 \\ \hline \text{adding} \quad 7s &= -35 \\ \therefore \ s &= -5 \\ \text{and in (1)} \quad r + 2(-5) &= -8 \\ \therefore \ r - 10 &= -8 \\ \therefore \ r &= 2 \end{aligned}$$

So, $r = 2, \quad s = -5$

b

$$r\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \therefore \ 2r + s &= 7 \quad \text{......} \ (1) \\ -3r + 7s &= -19 \quad \text{......} \ (2) \\ r + 2s &= 2 \quad \text{......} \ (3) \end{aligned}$$

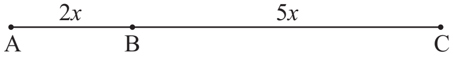
$$\begin{aligned} \text{From (1) and (2)} \quad -4r - 2s &= -14 \\ r + 2s &= 2 \\ \hline \therefore -3r \qquad &= -12 \\ \therefore \ r &= 4 \\ \text{In} \quad (1) \quad 2(4) + s &= 7 \\ \therefore \ s &= -1 \\ \text{Checking in (3)} \quad r + 2s &= 4 + 2(-1) = 2 \end{aligned}$$

$\therefore \ r = 4, \quad s = -1$ satisfies all equations

7 **a** $\overrightarrow{AB} = \begin{bmatrix} 4-(-2) \\ 3-1 \\ 0-4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

$\overrightarrow{BC} = \begin{bmatrix} 19-4 \\ 8-3 \\ -10-0 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ -10 \end{bmatrix} = 5\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, and since B is a common point, A, B and C are collinear.

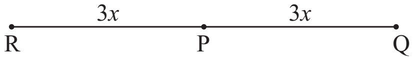


A divides CB in the ratio 7 : 2 externally.

b $\overrightarrow{RP} = \begin{bmatrix}2-(-1)\\1-7\\1-4\end{bmatrix} = \begin{bmatrix}3\\-6\\-3\end{bmatrix} = 3\begin{bmatrix}1\\-2\\-1\end{bmatrix}$ and

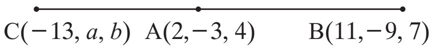
$\overrightarrow{PQ} = \begin{bmatrix}5-2\\-5-1\\-2-1\end{bmatrix} = \begin{bmatrix}3\\-6\\-3\end{bmatrix} = 3\begin{bmatrix}1\\-2\\-1\end{bmatrix}$

$\therefore$ $\overrightarrow{RP}$ is parallel to $\overrightarrow{PQ}$, and since P is a common point, P, Q and R are collinear.



$\therefore$ Q divides PR in the ratio $1:2$ externally.

8 **a**

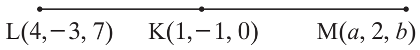


Since A, B and C are collinear, $\overrightarrow{CA}$ is parallel to $\overrightarrow{AB}$.

$$\therefore \quad \begin{bmatrix}15\\-3-a\\4-b\end{bmatrix} = k\begin{bmatrix}9\\-6\\3\end{bmatrix}$$

$$\begin{aligned} \therefore \quad 9k &= 15,\\ -3-a &= -6k\\ \text{and} \quad 4-b &= 3k\\ \therefore \quad k &= \tfrac{5}{3},\\ \text{and so} \quad a &= -3+6k\\ &= -3+10\\ &= 7\\ \text{and} \quad b &= 4-3k = 4-5 = -1 \end{aligned}$$

b



Since K, L and M are collinear, $\overrightarrow{LK}$ is parallel to $\overrightarrow{KM}$.

$$\therefore \quad \begin{bmatrix}-3\\2\\-7\end{bmatrix} = k\begin{bmatrix}a-1\\3\\b\end{bmatrix}$$

$$\begin{aligned} \therefore \quad 3k &= 2,\\ k(a-1) &= -3\\ \text{and} \quad kb &= -7\\ \therefore \quad k &= \tfrac{2}{3},\\ a-1 &= -\tfrac{3}{k} = -\tfrac{9}{2}\\ \text{i.e.,} \quad a &= -\tfrac{7}{2}\\ \text{and} \quad b &= -\tfrac{7}{k} = -\tfrac{21}{2} \end{aligned}$$

EXERCISE 15H

1 Since **a** and **b** are parallel, then $\mathbf{b} = k\mathbf{a}$ $\quad \therefore \begin{bmatrix}-6\\r\\s\end{bmatrix} = k\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}2k\\-k\\3k\end{bmatrix}$

$\therefore \quad 2k = -6, \quad r = -k, \quad s = 3k \qquad \therefore \quad k = -3, \quad r = 3, \quad s = -9$

2 If $\begin{bmatrix}a\\2\\b\end{bmatrix}$ and $\begin{bmatrix}3\\-1\\2\end{bmatrix}$ are parallel, then $\begin{bmatrix}a\\2\\b\end{bmatrix} = k\begin{bmatrix}3\\-1\\2\end{bmatrix}$

i.e., $2 = -k, \quad a = 3k, \quad b = 2k \qquad \therefore\ k = -2, \quad a = -6$ and $b = -4$

3 **a** Let the vector parallel to **a** be $k\mathbf{a}$

i.e., $k\mathbf{a} = k\begin{bmatrix}2\\-1\\-2\end{bmatrix} = \begin{bmatrix}2k\\-k\\-2k\end{bmatrix}$

$k\mathbf{a}$ has length $= 1$,

$$\begin{aligned} \text{so} \quad \sqrt{(2k)^2+(-k)^2+(-2k)^2} &= 1\\ \therefore \quad 4k^2+k^2+4k^2 &= 1\\ \therefore \quad 9k^2 &= 1\\ \therefore \quad k &= \pm\tfrac{1}{3} \end{aligned}$$

Choosing $k = \frac{1}{3}$,

the vector is $\begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}$

b Let the vector parallel to **b** be $k\mathbf{b}$

i.e., $k\mathbf{b} = k\begin{bmatrix}-2\\-1\\2\end{bmatrix} = \begin{bmatrix}-2k\\-k\\2k\end{bmatrix}$

$k\mathbf{b}$ has length $= 2$,

$$\begin{aligned} \text{so} \quad \sqrt{(-2k)^2+(-k)^2+(2k)^2} &= 2\\ \therefore \quad 4k^2+k^2+4k^2 &= 4\\ \therefore \quad 9k^2 &= 4\\ \therefore \quad k &= \pm\tfrac{2}{3} \end{aligned}$$

Choosing $k = \frac{2}{3}$,

the vector is $\begin{bmatrix}-\frac{4}{3}\\-\frac{2}{3}\\\frac{4}{3}\end{bmatrix}$

4 **a** $\overrightarrow{AB} = 3\overrightarrow{CD}$ means that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and 3 times its length.

b $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ means that $\overrightarrow{RS}$ is parallel to $\overrightarrow{KL}$, half its length and in the opposite direction.

c

$\overrightarrow{AB} = 2\overrightarrow{BC}$ means that A, B and C are collinear and the length of $\overrightarrow{AB}$ is twice the length of $\overrightarrow{BC}$ (i.e., B cuts $\overrightarrow{AC}$ internally in the ratio 2 : 1).

d

$\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AC}$ means that A, B and C are collinear and the length of $\overrightarrow{BC}$ is one third the length of $\overrightarrow{AC}$ (i.e., B cuts $\overrightarrow{AC}$ internally in the ratio 2 : 1).

5 $\overrightarrow{OP} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\overrightarrow{OQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\overrightarrow{OR} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\overrightarrow{OS} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$

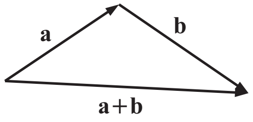
a $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = -\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ and

$\overrightarrow{QS} = \overrightarrow{QO} + \overrightarrow{OS} = -\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix} = 2\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = 2\overrightarrow{PR}$ and so QS $\parallel$ PR

$\therefore$

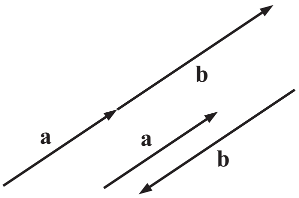
b As $\overrightarrow{QS} = 2\overrightarrow{PR}$ then $|\,\overrightarrow{QS}\,| = 2\,|\,\overrightarrow{PR}\,|$, i.e., QS is twice as long as PR.

6 **a** Consider **a** not parallel to **b**:



Clearly, $|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

b Consider **a** parallel to **b**:



$|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

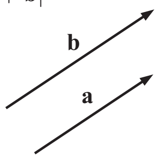
or

$|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

c If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{b}$ $\therefore$ $|\mathbf{a}| + |\mathbf{b}| = \mathbf{0} + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

Similarly if $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{a}$

$\therefore$ $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}| + \mathbf{0} = |\mathbf{a}| = |\mathbf{a} + \mathbf{b}|$



If $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{0}$

$\therefore$ $|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

Combining **all** possibilities, $|\mathbf{a}| + |\mathbf{b}| \geqslant |\mathbf{a} + \mathbf{b}|$ i.e., $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$

EXERCISE 15I

1 **a** $\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $|\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{3}$ units

b $3\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $|3\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{9+1+1} = \sqrt{11}$ units

c $\mathbf{i} - 5\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$

$\therefore$ $|\mathbf{i} - 5\mathbf{k}| = \sqrt{1+25}$
$= \sqrt{26}$ units

d $\frac{1}{2}(\mathbf{j} + \mathbf{k}) = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ $\therefore$ $\left|\frac{1}{2}(\mathbf{j} + \mathbf{k})\right|$
$= \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$
$= \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ units

2 **a**
$$\begin{aligned} \text{length} &= 1\\ \therefore\ \sqrt{0^2+k^2} &= 1\\ \therefore\ k^2 &= 1\\ \therefore\ k &= \pm 1 \end{aligned}$$

b
$$\begin{aligned} \text{length} &= 1\\ \therefore\ \sqrt{k^2+0} &= 1\\ \therefore\ k^2 &= 1\\ \therefore\ k &= \pm 1 \end{aligned}$$

c
$$\begin{aligned} \text{length} &= 1\\ \therefore\ \sqrt{k^2+1} &= 1\\ \therefore\ k^2+1 &= 1\\ \therefore\ k^2 &= 0\\ \therefore\ k &= 0 \end{aligned}$$

d
$$\begin{aligned} \text{length} &= 1\\ \therefore\ \sqrt{\left(\tfrac{1}{4}\right)+k^2+\tfrac{1}{16}} &= 1\\ \therefore\ \sqrt{k^2+\tfrac{5}{16}} &= 1\\ \therefore\ k^2 &= \tfrac{11}{16}\\ \therefore\ k &= \pm\tfrac{\sqrt{11}}{4} \end{aligned}$$

e
$$\begin{aligned} \text{length} &= 1\\ \therefore\ \sqrt{k^2+\tfrac{4}{9}+\tfrac{1}{9}} &= 1\\ \therefore\ \sqrt{k^2+\tfrac{5}{9}} &= 1\\ \therefore\ k^2 &= \tfrac{4}{9}\\ \therefore\ k &= \pm\tfrac{2}{3} \end{aligned}$$

3 **a**
$$\begin{aligned} &\text{length}\\ &= \sqrt{3^2+4^2}\\ &= 5 \text{ units} \end{aligned}$$

b
$$\begin{aligned} &\text{length}\\ &= \sqrt{2^2+(-1)^2+1^2}\\ &= \sqrt{4+1+1}\\ &= \sqrt{6} \text{ units} \end{aligned}$$

c
$$\begin{aligned} &\text{length}\\ &= \sqrt{1^2+2^2+(-2)^2}\\ &= \sqrt{1+4+4}\\ &= \sqrt{9}\\ &= 3 \text{ units} \end{aligned}$$

d length $= \sqrt{(-2.36)^2+(5.65)^2} \doteqdot 6.12$ units

4 **a** $\mathbf{i}+2\mathbf{j}$ has length $\sqrt{1^2+2^2}=\sqrt{5}$ units $\therefore$ unit vector $= \frac{1}{\sqrt{5}}(\mathbf{i}+2\mathbf{j})$

b $2\mathbf{i}-3\mathbf{k}$ has length $\sqrt{2^2+0^2+(-3)^2}=\sqrt{4+9}=\sqrt{13}$ units

$\therefore$ unit vector is $\frac{1}{\sqrt{13}}(2\mathbf{i}-3\mathbf{k})$

c $-2\mathbf{i}-5\mathbf{j}-2\mathbf{k}$ has length $\sqrt{(-2)^2+(-5)^2+(-2)^2}=\sqrt{4+25+4}=\sqrt{33}$ units

$\therefore$ unit vector is $\frac{1}{\sqrt{33}}(-2\mathbf{i}-5\mathbf{j}-2\mathbf{k})$

5 **a** $\begin{bmatrix}2\\-1\end{bmatrix}$ has length $\sqrt{2^2+(-1)^2}=\sqrt{5}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$

$\therefore$ the vector of length 3 units in the same direction is $\frac{3}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}\frac{6}{\sqrt{5}}\\-\frac{3}{\sqrt{5}}\end{bmatrix}$

b $\begin{bmatrix}-1\\-4\end{bmatrix}$ has length $\sqrt{(-1)^2+(-4)^2}=\sqrt{17}$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{\sqrt{17}}\begin{bmatrix}-1\\-4\end{bmatrix}=\frac{1}{\sqrt{17}}\begin{bmatrix}1\\4\end{bmatrix}$

$\therefore$ the vector of length 2 units in the opposite direction is $\frac{2}{\sqrt{17}}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt{17}}\\\frac{8}{\sqrt{17}}\end{bmatrix}$

c $\begin{bmatrix}-1\\4\\1\end{bmatrix}$ has length $\sqrt{(-1)^2+4^2+1^2}=\sqrt{18}=3\sqrt{2}$ units

$\therefore$ the unit vector in the same direction is $\frac{1}{3\sqrt{2}}\begin{bmatrix}-1\\4\\1\end{bmatrix}$

$\therefore$ the vector of length 6 units in the same direction is $\frac{6}{3\sqrt{2}}\begin{bmatrix}-1\\4\\1\end{bmatrix}=\begin{bmatrix}-\sqrt{2}\\4\sqrt{2}\\\sqrt{2}\end{bmatrix}$

d $\begin{bmatrix}-1\\-2\\-2\end{bmatrix}$ has length $\sqrt{(-1)^2+(-2)^2+(-2)^2}=\sqrt{9}=3$ units

$\therefore$ the unit vector in the opposite direction is $-\frac{1}{3}\begin{bmatrix}-1\\-2\\-2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$

$\therefore$ the vector of length 5 units in the opposite direction is $\frac{5}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}\frac{5}{3}\\\frac{10}{3}\\\frac{10}{3}\end{bmatrix}$

EXERCISE 15J.1

1 **a**
$$\begin{aligned} &\mathbf{q} \bullet \mathbf{p} \\ &= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= -3 + 10 \\ &= 7 \end{aligned}$$

b
$$\begin{aligned} &\mathbf{q} \bullet \mathbf{r} \\ &= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ &= 2 + 20 \\ &= 22 \end{aligned}$$

c
$$\begin{aligned} &\mathbf{q} \bullet (\mathbf{p} + \mathbf{r}) \\ &= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ &= -1 + 30 \\ &= 29 \end{aligned}$$

d
$$\begin{aligned} &3\mathbf{r} \bullet \mathbf{q} \\ &= \begin{bmatrix} -6 \\ 12 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 5 \end{bmatrix} \\ &= 6 + 60 \\ &= 66 \end{aligned}$$

e
$$\begin{aligned} &2\mathbf{p} \bullet 2\mathbf{p} \\ &= \begin{bmatrix} 6 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ 4 \end{bmatrix} \\ &= 36 + 16 \\ &= 52 \end{aligned}$$

f
$$\begin{aligned} &\mathbf{i} \bullet \mathbf{p} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &= 3 + 0 \\ &= 3 \end{aligned}$$

g
$$\begin{aligned} &\mathbf{q} \bullet \mathbf{j} \\ &= \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 0 + 5 \\ &= 5 \end{aligned}$$

h
$$\begin{aligned} &\mathbf{i} \bullet \mathbf{i} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

2 **a**
$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\ &= 2(-1) + 1(1) + 3(1) \\ &= -2 + 1 + 3 \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} \mathbf{b} \bullet \mathbf{a} &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= (-1)(2) + 1(1) + 1(3) \\ &= -2 + 1 + 3 \\ &= 2 \end{aligned}$$

c
$$\begin{aligned} |\mathbf{a}|^2 &= \left(\sqrt{2^2 + 1^2 + 3^2}\right)^2 \\ &= 14 \end{aligned}$$

d
$$\begin{aligned} \mathbf{a} \bullet \mathbf{a} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\ &= 2(2) + 1(1) + 3(3) \\ &= 14 \end{aligned}$$

e
$$\begin{aligned} &\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) \\ &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\ &= 2(-1) + 1(0) + 3(2) \\ &= 4 \end{aligned}$$

f
$$\begin{aligned} &\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} \\ &= 2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \{\text{using } \mathbf{a}\} \\ &= 2 + 2(0) + 1(-1) + 3(1) \\ &= 4 \end{aligned}$$

3 **a**
$$\begin{aligned} &(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k}) \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\ &= 1(0) + 1(2) - 1(1) \\ &= 1 \end{aligned}$$

b
$$\begin{aligned} \mathbf{i} \bullet \mathbf{i} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= 1 \end{aligned}$$

c
$$\begin{aligned} \mathbf{i} \bullet \mathbf{j} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= 0 \end{aligned}$$

4
$$\begin{aligned} \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix} \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} \end{aligned}$$

$\therefore \ \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d}$ and if we let $\mathbf{p} = \mathbf{a} + \mathbf{b}$,

then
$$\begin{aligned} (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) &= \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) \\ &= \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d} \end{aligned}$$

$$\begin{aligned}&= (\mathbf{a}+\mathbf{b})\bullet\mathbf{c} + (\mathbf{a}+\mathbf{b})\bullet\mathbf{d}\\ &= \mathbf{c}\bullet(\mathbf{a}+\mathbf{b}) + \mathbf{d}\bullet(\mathbf{a}+\mathbf{b})\\ &= \mathbf{c}\bullet\mathbf{a} + \mathbf{c}\bullet\mathbf{b} + \mathbf{d}\bullet\mathbf{a} + \mathbf{d}\bullet\mathbf{b}\\ &= \mathbf{a}\bullet\mathbf{c} + \mathbf{b}\bullet\mathbf{c} + \mathbf{a}\bullet\mathbf{d} + \mathbf{b}\bullet\mathbf{d}\end{aligned}$$

5 **a**
$$\begin{aligned}\begin{bmatrix}3\\t\end{bmatrix}\bullet\begin{bmatrix}-2\\1\end{bmatrix} &= 0\\ \therefore\quad -6+t &= 0\\ \therefore\quad t &= 6\end{aligned}$$

b
$$\begin{aligned}\begin{bmatrix}t\\t+2\end{bmatrix}\bullet\begin{bmatrix}3\\-4\end{bmatrix} &= 0\\ \therefore\quad 3t-4(t+2) &= 0\\ \therefore\quad 3t-4t-8 &= 0\\ \therefore\quad -t &= 8\\ \therefore\quad t &= -8\end{aligned}$$

c
$$\begin{aligned}\begin{bmatrix}t\\t+2\end{bmatrix}\bullet\begin{bmatrix}2-3t\\t\end{bmatrix} &= 0\\ \therefore\quad 2t-3t^2+t^2+2t &= 0\\ \therefore\quad -2t^2+4t &= 0\\ \therefore\quad t^2-2t &= 0\\ \therefore\quad t(t-2) &= 0\\ \therefore\quad t &= 0 \text{ or } 2\end{aligned}$$

d
$$\begin{aligned}\begin{bmatrix}3\\-1\\t\end{bmatrix}\bullet\begin{bmatrix}2t\\-3\\-4\end{bmatrix} = 0 \quad \therefore\quad 3(2t)+(-1)(-3)+t(-4) &= 0\\ \therefore\quad 6t+3-4t &= 0\\ \therefore\quad 2t+3 &= 0\\ \therefore\quad t &= -\tfrac{3}{2}\end{aligned}$$

6 **a** If $\mathbf{p}\parallel\mathbf{q}$ then $\begin{bmatrix}3\\t\end{bmatrix} = k\begin{bmatrix}-2\\1\end{bmatrix}$ where $k\neq 0$ $\quad\therefore\ 3=-2k$ and $t=k$

$\therefore\ k=-\frac{3}{2}$ and $t=-\frac{3}{2}$

b If $\mathbf{r}\parallel\mathbf{s}$ then $\begin{bmatrix}t\\t+2\end{bmatrix} = k\begin{bmatrix}3\\-4\end{bmatrix}$ where $k\neq 0$ $\quad\therefore\ t=3k$ and $t+2=-4k$

$$\begin{aligned}\therefore\quad t+2 &= -4\left(\tfrac{t}{3}\right)\\ \therefore\quad 3t+6 &= -4t\\ \therefore\quad 7t &= -6\\ \therefore\quad t &= -\tfrac{6}{7}\end{aligned}$$

c If $\mathbf{a}\parallel\mathbf{b}$ then $\begin{bmatrix}t\\t+2\end{bmatrix} = k\begin{bmatrix}2-3t\\t\end{bmatrix}$

$\therefore\ t = k(2-3t)$ and $t+2=kt$

$$\begin{aligned}\therefore\quad \frac{t}{2-3t} &= \frac{t+2}{t} \qquad \{\text{equating } k\text{'s}\}\\ \therefore\quad t^2 &= (t+2)(2-3t)\\ \therefore\quad t^2 &= 2t-3t^2+4-6t\\ \therefore\quad 4t^2+4t-4 &= 0\\ \therefore\quad t^2+t-1 &= 0 \quad \text{which has } \Delta = 1^2-4(1)(-1)=5\\ \therefore\quad t &= \tfrac{-1\pm\sqrt{5}}{2}\end{aligned}$$

d If $\mathbf{a}\parallel\mathbf{b}$ then $\begin{bmatrix}3\\-1\\t\end{bmatrix} = k\begin{bmatrix}2t\\-3\\-4\end{bmatrix}$ where $k\neq 0$

$\therefore\ 3=2kt,\ -1=-3k$ and $t=-4k$

$\therefore\ k=\frac{1}{3}$ and $3=\frac{2}{3}t,\ t=-\frac{4}{3}$

i.e., $t=\frac{9}{2}$ and $-\frac{4}{3}$ simultaneously which is impossible.

$\therefore$ the vectors can never be parallel.

7 $\mathbf{a}\bullet\mathbf{b}$
$$\begin{aligned}&= \begin{bmatrix}3\\1\\2\end{bmatrix}\bullet\begin{bmatrix}-1\\1\\1\end{bmatrix}\\ &= 3(-1)+1(1)+2(1)\\ &= 0\end{aligned}$$

$\mathbf{b}\bullet\mathbf{c}$
$$\begin{aligned}&= \begin{bmatrix}-1\\1\\1\end{bmatrix}\bullet\begin{bmatrix}1\\5\\-4\end{bmatrix}\\ &= (-1)(1)+1(5)+1(-4)\\ &= 0\end{aligned}$$

$\mathbf{a}\bullet\mathbf{c}$
$$\begin{aligned}&= \begin{bmatrix}3\\1\\2\end{bmatrix}\bullet\begin{bmatrix}1\\5\\-4\end{bmatrix}\\ &= (3)(1)+1(5)+2(-4)\\ &= 0\end{aligned}$$

$\therefore$ **a**, **b** and **c** are mutually perpendicular.

8 **a**

$$\begin{bmatrix}1\\1\\5\end{bmatrix} \bullet \begin{bmatrix}2\\3\\-1\end{bmatrix}$$

$$= 1(2) + 1(3) + 5(-1)$$

$$= 0$$

$$\therefore \quad \begin{bmatrix}1\\1\\5\end{bmatrix} \text{ and } \begin{bmatrix}2\\3\\-1\end{bmatrix} \text{ are perpendicular}$$

b

$$\begin{bmatrix}3\\t\\-2\end{bmatrix} \bullet \begin{bmatrix}1-t\\-3\\4\end{bmatrix} = 0$$

$$\therefore \quad 3(1-t) + t(-3) + (-2)4 = 0$$

$$\therefore \quad 3 - 3t - 3t - 8 = 0$$

$$\therefore \quad -6t = 5$$

$$\therefore \quad t = -\tfrac{5}{6}$$

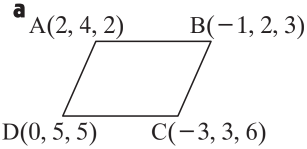
9 We have three points: A(5, 1, 2) B(6, −1, 0) C(3, 2, 0)

Then $\overrightarrow{\text{AB}} = \begin{bmatrix}1\\-2\\-2\end{bmatrix}$, $\overrightarrow{\text{AC}} = \begin{bmatrix}-2\\1\\-2\end{bmatrix}$ and $\overrightarrow{\text{BC}} = \begin{bmatrix}-3\\3\\0\end{bmatrix}$

Now $\overrightarrow{\text{AB}} \bullet \overrightarrow{\text{AC}} = \begin{bmatrix}1\\-2\\-2\end{bmatrix} \bullet \begin{bmatrix}-2\\1\\-2\end{bmatrix} = (-2) + (-2) + 4 = 0$

$\therefore$ $\overrightarrow{\text{AB}}$ is perpendicular to $\overrightarrow{\text{AC}}$ and so ΔABC is right-angled at A.

10 **a**



$\overrightarrow{\text{AB}} = \begin{bmatrix}-3\\-2\\1\end{bmatrix}$ $\quad \overrightarrow{\text{BC}} = \begin{bmatrix}-2\\1\\3\end{bmatrix}$

$\overrightarrow{\text{DC}} = \begin{bmatrix}-3\\-2\\1\end{bmatrix}$ $\quad \overrightarrow{\text{AD}} = \begin{bmatrix}-2\\1\\3\end{bmatrix}$

$\therefore$ $\overrightarrow{\text{AB}}$ is parallel to $\overrightarrow{\text{DC}}$ and $\overrightarrow{\text{BC}}$ is parallel to $\overrightarrow{\text{AD}}$.

$\therefore$ ABCD is a parallelogram.

b $|\,\overrightarrow{\text{AB}}\,| = \sqrt{14}$ units and $|\,\overrightarrow{\text{BC}}\,| = \sqrt{14}$ units $\quad \therefore$ ABCD is a rhombus.

c $\overrightarrow{\text{AC}} \bullet \overrightarrow{\text{BD}} = \begin{bmatrix}-5\\-1\\4\end{bmatrix} \bullet \begin{bmatrix}1\\3\\2\end{bmatrix} = (-5) \times 1 + (-1) \times 3 + 4(2) = 0$

$\therefore$ $\overrightarrow{\text{AC}}$ is perpendicular to $\overrightarrow{\text{BD}}$ which illustrates that the diagonals of a rhombus are perpendicular.

11 **a** $x - y = 3$ has gradient $+\frac{1}{1}$ and so has direction vector $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.

$3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\left[\begin{smallmatrix}2\\-3\end{smallmatrix}\right]$.

$$\therefore \quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \bullet \left[\begin{smallmatrix}2\\-3\end{smallmatrix}\right] = \sqrt{1+1}\sqrt{4+9}\cos\theta$$

$$\therefore \quad 2 - 3 = \sqrt{2}\sqrt{13}\cos\theta$$

$$\therefore \quad \tfrac{-1}{\sqrt{26}} = \cos\theta$$

$$\therefore \quad \theta \doteqdot 101.3 \qquad \therefore \text{ the angle is } 101.3^o \text{ or } 78.7^o$$

b $y = x + 2$ has slope $1 = \frac{1}{1}$ $\therefore$ direction vector is $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.

$y = 1 - 3x$ has slope $-3 = \frac{-3}{1}$ $\therefore$ direction vector is $\left[\begin{smallmatrix}1\\-3\end{smallmatrix}\right]$.

$$\therefore \quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \bullet \left[\begin{smallmatrix}1\\-3\end{smallmatrix}\right] = \sqrt{1+1}\sqrt{1+9}\cos\theta$$

$$\therefore \quad 1 - 3 = \sqrt{2}\sqrt{10}\cos\theta$$

$$\therefore \quad \tfrac{-2}{\sqrt{20}} = \cos\theta$$

$$\therefore \quad \theta \doteqdot 116.6 \qquad \therefore \text{ the angle is } 116.6^o \text{ or } 63.4^o$$

c $y + x = 7$ has slope $-1 = \frac{-1}{1}$ $\therefore$ direction vector is $\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$.

$x - 3y + 2 = 0$ has slope $\frac{1}{3}$ $\therefore$ direction vector is $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$.

$$\begin{aligned}
\therefore\quad \begin{bmatrix}1\\-1\end{bmatrix}\bullet\begin{bmatrix}3\\1\end{bmatrix} &= \sqrt{1+1}\sqrt{9+1}\cos\theta\\
\therefore\quad 3-1 &= \sqrt{2}\sqrt{10}\cos\theta\\
\therefore\quad \tfrac{2}{\sqrt{20}} &= \cos\theta\\
\therefore\quad \theta &\doteqdot 63.4 \qquad \therefore\ \text{the angle is } 63.4^o \text{ or } 116.6^o
\end{aligned}$$

d $y = 2 - x$ has slope $-1 = \frac{-1}{1}$ $\therefore$ has direction vector $\begin{bmatrix}1\\-1\end{bmatrix}$.

$x - 2y = 7$ has slope $\frac{1}{2}$ $\therefore$ has direction vector $\begin{bmatrix}2\\1\end{bmatrix}$

$$\begin{aligned}
\therefore\quad \begin{bmatrix}1\\-1\end{bmatrix}\bullet\begin{bmatrix}2\\1\end{bmatrix} &= \sqrt{1+1}\sqrt{4+1}\cos\theta\\
\therefore\quad 2-1 &= \sqrt{2}\sqrt{5}\cos\theta\\
\therefore\quad \cos\theta &= \tfrac{1}{\sqrt{10}}\\
\therefore\quad \theta &= \cos^{-1}\left(\tfrac{1}{\sqrt{10}}\right) \doteqdot 71.6 \qquad \therefore\ \text{the angle is } 71.6^o \text{ or } 108.4^o
\end{aligned}$$

12 **a**
$$\begin{aligned}
\mathbf{p}\bullet\mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}|\cos\theta\\
&= 2\times 5\times\cos 60^o\\
&= 5
\end{aligned}$$

b
$$\begin{aligned}
\mathbf{p}\bullet\mathbf{q} &= |\mathbf{p}|\,|\mathbf{q}|\cos\theta\\
&= 6\times 3\times\cos 120^o\\
&= -9
\end{aligned}$$

13 **a** $\begin{bmatrix}5\\2\end{bmatrix}\bullet\begin{bmatrix}-2\\5\end{bmatrix} = -10+10 = 0,$ so $\begin{bmatrix}-2\\5\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}-2\\5\end{bmatrix}$ where $k \neq 0$.

Note: $k\begin{bmatrix}2\\-5\end{bmatrix}$, $k\neq 0$ is also ok.

b $\begin{bmatrix}-1\\-2\end{bmatrix}\bullet\begin{bmatrix}-2\\1\end{bmatrix} = 2-2 = 0,$ so $\begin{bmatrix}-2\\1\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}-2\\1\end{bmatrix}$, $k\neq 0$.

c $\begin{bmatrix}3\\-1\end{bmatrix}\bullet\begin{bmatrix}1\\3\end{bmatrix} = 3-3 = 0,$ so $\begin{bmatrix}1\\3\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}1\\3\end{bmatrix}$, $k\neq 0$.

d $\begin{bmatrix}-4\\3\end{bmatrix}\bullet\begin{bmatrix}3\\4\end{bmatrix} = -12+12 = 0,$ so $\begin{bmatrix}3\\4\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}3\\4\end{bmatrix}$, $k\neq 0$

e $\begin{bmatrix}2\\0\end{bmatrix}\bullet\begin{bmatrix}0\\1\end{bmatrix} = 0+0 = 0,$ so $\begin{bmatrix}0\\1\end{bmatrix}$ is one such vector.

$\therefore$ required vectors have form $k\begin{bmatrix}0\\1\end{bmatrix}$, $k\neq 0$.

EXERCISE 15J.2

1 **a**
$$\mathbf{a} = -\mathbf{i}-\mathbf{j}+\mathbf{k} = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$$\mathbf{b} = \mathbf{i}+\mathbf{j}+\mathbf{k} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{aligned}
\mathbf{a}\bullet\mathbf{b} &= \begin{bmatrix}-1\\-1\\1\end{bmatrix}\bullet\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= (-1)+(-1)+1\\
&= -1
\end{aligned}$$

b
$$\begin{aligned}
\cos\theta &= \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\\
&= \frac{-1}{\sqrt{3}\times\sqrt{3}}\\
&= -\tfrac{1}{3}\\
\therefore\quad \theta &\doteqdot 109.5^o
\end{aligned}$$

(acute 70.5^o)

c the projection vector of **a** on **b** $= \left(\dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}\right)\dfrac{1}{|\mathbf{b}|}\mathbf{b} = \frac{-1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$

d the projection vector has length $\left|-\frac{1}{3}\right|\sqrt{1^2+1^2+1^2} = \frac{1}{3}\sqrt{3} = \frac{1}{\sqrt{3}}$ units (as expected)

2 Given A(3, 0, 1), B(−3, 1, 2) and C(−2, 1, −1),

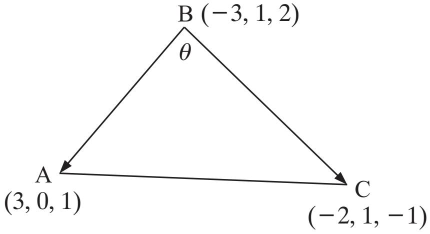
$\overrightarrow{BC} = \begin{bmatrix}1\\0\\-3\end{bmatrix}$ and $\overrightarrow{BA} = \begin{bmatrix}6\\-1\\-1\end{bmatrix}$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{BC} \bullet \overrightarrow{BA}}{|\overrightarrow{BC}|\ |\overrightarrow{BA}|}$$

$$= \frac{\begin{bmatrix}1\\0\\-3\end{bmatrix} \bullet \begin{bmatrix}6\\-1\\-1\end{bmatrix}}{\sqrt{1+9} \times \sqrt{36+1+1}}$$

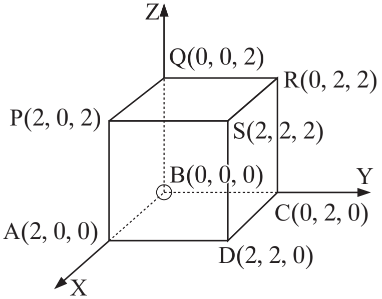
$$= \frac{6+0+3}{\sqrt{10} \times \sqrt{38}}$$

$$= \frac{9}{\sqrt{380}} \quad \text{and so} \quad \theta \doteqdot 62.5^o$$



If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, i.e., 117.5^o.

3



a Suppose the origin is at B.

Now $\overrightarrow{BA} = \begin{bmatrix}2\\0\\0\end{bmatrix}$ and $\overrightarrow{BS} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$$\therefore \quad \overrightarrow{BA} \bullet \overrightarrow{BS} = \begin{bmatrix}2\\0\\0\end{bmatrix} \bullet \begin{bmatrix}2\\2\\2\end{bmatrix} = 4+0+0 = 4$$

$$\therefore \quad \cos\measuredangle ABS = \frac{4}{\sqrt{4+0+0} \times \sqrt{4+4+4}}$$

$$= \frac{4}{2 \times 2\sqrt{3}}$$

$$= \tfrac{1}{\sqrt{3}}$$

$$\therefore \quad \measuredangle ABS \doteqdot 54.7^o$$

b Consider vectors away from B.

$\overrightarrow{BR} = \begin{bmatrix}0\\2\\2\end{bmatrix}$ and $\overrightarrow{BP} = \begin{bmatrix}2\\0\\2\end{bmatrix}$

$$\overrightarrow{BR} \bullet \overrightarrow{BP} = \begin{bmatrix}0\\2\\2\end{bmatrix} \bullet \begin{bmatrix}2\\0\\2\end{bmatrix} = 0+0+4 = 4$$

$$\therefore \quad \cos\measuredangle RBP = \frac{4}{\sqrt{0+4+4}\sqrt{4+0+4}}$$

$$= \frac{4}{\sqrt{8} \times \sqrt{8}}$$

$$= \tfrac{1}{2} \quad \text{and so} \quad \measuredangle RBP = 60^o$$

c $\overrightarrow{BP} = \begin{bmatrix}2\\0\\2\end{bmatrix}$ and $\overrightarrow{BS} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$$\therefore \quad \overrightarrow{BP} \bullet \overrightarrow{BS} = \begin{bmatrix}2\\0\\2\end{bmatrix} \bullet \begin{bmatrix}2\\2\\2\end{bmatrix}$$

$$= 4+0+4$$

$$= 8$$

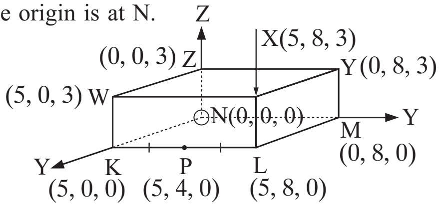
$$\therefore \quad \cos\measuredangle PBS = \frac{8}{\sqrt{4+4}\sqrt{4+4+4}}$$

$$= \frac{8}{\sqrt{96}}$$

$$\therefore \quad \measuredangle PBS \doteqdot 35.3^o$$

4 Suppose the origin is at N.

a



$$\overrightarrow{NY} = \begin{bmatrix}0\\8\\3\end{bmatrix} \quad \text{and} \quad \overrightarrow{NX} = \begin{bmatrix}5\\8\\3\end{bmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NX} = \begin{bmatrix}0\\8\\3\end{bmatrix} \bullet \begin{bmatrix}5\\8\\3\end{bmatrix} = 0 + 64 + 9 = 73$$

$$\therefore \quad \cos\measuredangle YNX = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\tfrac{73}{98}} \qquad \therefore \quad \measuredangle YNX \doteqdot 30.3^o$$

b

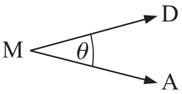
$$\overrightarrow{NY} = \begin{bmatrix}0\\8\\3\end{bmatrix} \quad \text{and} \quad \overrightarrow{NP} = \begin{bmatrix}5\\4\\0\end{bmatrix}$$

$$\overrightarrow{NY} \bullet \overrightarrow{NP} = \begin{bmatrix}0\\8\\3\end{bmatrix} \bullet \begin{bmatrix}5\\4\\0\end{bmatrix} = 0 + 32 + 0 = 32$$

$$\therefore \quad \cos\measuredangle YNP = \frac{32}{\sqrt{64+9}\sqrt{25+16}} = \frac{32}{\sqrt{73}\sqrt{41}} \qquad \therefore \quad \measuredangle YNP \doteqdot 54.2^o$$

5 **a** M is the midpoint of BC $\therefore$ M is at $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$ i.e., $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$

b



Now $\overrightarrow{MD} = \begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\\-\frac{3}{2}\end{bmatrix}$ and $\overrightarrow{MA} = \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{MD} \bullet \overrightarrow{MA}}{|\,\overrightarrow{MD}\,| \;\; |\,\overrightarrow{MA}\,|} = \frac{\begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\\-\frac{3}{2}\end{bmatrix} \bullet \begin{bmatrix}\frac{1}{2}\\-\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}}{\sqrt{\frac{9}{4}+\frac{1}{4}+\frac{9}{4}}\sqrt{\frac{1}{4}+\frac{9}{4}+\frac{1}{4}}}$$

$$\therefore \quad \cos\theta = \frac{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}}{\sqrt{\frac{19}{4}}\sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{201}}{4}} = \frac{9}{\sqrt{201}} \qquad \text{and so} \quad \theta \doteqdot 51.5^o$$

6 **a**

$$\begin{bmatrix}2\\t\\t-2\end{bmatrix} \bullet \begin{bmatrix}t\\3\\t\end{bmatrix} = 0 \quad \therefore \quad 2t + 3t + t(t-2) = 0$$

$$\therefore \quad 5t + t^2 - 2t = 0$$

$$\therefore \quad t^2 + 3t = 0$$

$$\therefore \quad t(t+3) = 0 \quad \text{and so} \quad t = 0 \quad \text{or} \quad t = -3$$

b Given that $\mathbf{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}2\\2\\r\end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix}s\\t\\1\end{bmatrix}$ are mutually perpendicular

$\therefore$ $\mathbf{a} \bullet \mathbf{b} = 0$, $\mathbf{b} \bullet \mathbf{c} = 0$ and $\mathbf{a} \bullet \mathbf{c} = 0$

$$\therefore \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \bullet \begin{bmatrix}2\\2\\r\end{bmatrix} = 0 \qquad \therefore \quad 2 + 4 + 3r = 0$$

$$\therefore \quad 3r = -6$$

$$\therefore \quad r = -2$$

$$\text{and} \quad \begin{bmatrix}2\\2\\-2\end{bmatrix} \bullet \begin{bmatrix}s\\t\\1\end{bmatrix} = 0 \qquad \therefore \quad 2s + 2t - 2 = 0$$

$$\therefore \quad s + t = 1 \quad ......(1)$$

$$\text{and} \quad \begin{bmatrix}1\\2\\3\end{bmatrix} \bullet \begin{bmatrix}s\\t\\1\end{bmatrix} = 0 \qquad \therefore \quad s + 2t + 3 = 0$$

$$\therefore \quad s + 2t = -3 \quad ......(2)$$

(2) − (1) gives $t = -4$ and so $s = 5$ i.e., $r = -2$, $s = 5$ and $t = -4$

7 **a** Choose any vector in the direction of the X-axis, e.g., $\mathbf{i}=\begin{bmatrix}1\\0\\0\end{bmatrix}$.

$$\text{Then}\quad \cos\theta=\frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}1\\2\\3\end{bmatrix}}{\left|\begin{bmatrix}1\\0\\0\end{bmatrix}\right|\left|\begin{bmatrix}1\\2\\3\end{bmatrix}\right|}$$
$$=\frac{1}{\sqrt{1}\sqrt{1+4+9}}$$
$$=\frac{1}{\sqrt{14}}\qquad\text{and so}\quad\theta=74.5^o$$

b A line parallel to the Y-axis is $\mathbf{j}=\begin{bmatrix}0\\1\\0\end{bmatrix}$.

$$\text{Then}\quad \cos\theta=\frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix}-1\\1\\3\end{bmatrix}}{\sqrt{1}\sqrt{1+1+9}}$$
$$=\frac{1}{\sqrt{11}}$$
$$\therefore\quad\theta\doteqdot 72.45^o$$

8 We want vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a}\neq\mathbf{0}$, $\mathbf{a}\bullet\mathbf{b}=\mathbf{a}\bullet\mathbf{c}$ and $\mathbf{b}\neq\mathbf{c}$

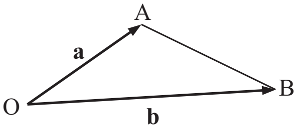
For example, $\mathbf{a}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}0\\0\\1\end{bmatrix}$

In this case, $\mathbf{a}\bullet\mathbf{b}=\mathbf{a}\bullet\mathbf{c}=0$

9 **a**
$$\begin{aligned}|\mathbf{a}+\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2&=(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b})+(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})\\&=\mathbf{a}\bullet\mathbf{a}+\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{a}-\mathbf{a}\bullet\mathbf{b}-\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}\\&=2\mathbf{a}\bullet\mathbf{a}+2\mathbf{b}\bullet\mathbf{b}\\&=2\,|\mathbf{a}|^2+2\,|\mathbf{b}|^2\quad\text{as required}\end{aligned}$$

b
$$\begin{aligned}|\mathbf{a}+\mathbf{b}|^2-|\mathbf{a}-\mathbf{b}|^2&=(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b})-(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})\\&=\mathbf{a}\bullet\mathbf{a}+\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}-\mathbf{a}\bullet\mathbf{a}+\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}-\mathbf{b}\bullet\mathbf{b}\\&=2\mathbf{b}\bullet\mathbf{a}+2\mathbf{a}\bullet\mathbf{b}\quad\text{......(2)}\\&=4\mathbf{a}\bullet\mathbf{b}\quad\text{as required}\quad\{\text{since}\ \mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}\}\end{aligned}$$

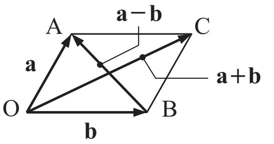
10 We are given that $\mathbf{a}\neq\mathbf{b}$, $\mathbf{a}\neq 0$ and $\mathbf{b}\neq 0$



a
$$\begin{aligned}\text{Now if}\quad|\mathbf{a}+\mathbf{b}|&=|\mathbf{a}-\mathbf{b}|\\\text{then}\quad|\mathbf{a}+\mathbf{b}|^2&=|\mathbf{a}-\mathbf{b}|^2\\\text{i.e.,}\quad(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}+\mathbf{b})&=(\mathbf{a}-\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})\\\therefore\quad\mathbf{a}\bullet\mathbf{a}+\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}&=\mathbf{a}\bullet\mathbf{a}-\mathbf{a}\bullet\mathbf{b}-\mathbf{b}\bullet\mathbf{a}+\mathbf{b}\bullet\mathbf{b}\\\therefore\quad 2\mathbf{a}\bullet\mathbf{b}+2\mathbf{b}\bullet\mathbf{a}&=0\\\therefore\quad 4\mathbf{a}\bullet\mathbf{b}&=0\qquad\{\text{as}\ \mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}\}\\\therefore\quad\mathbf{a}\bullet\mathbf{b}&=0,\quad\text{and since neither } \mathbf{a} \text{ nor } \mathbf{b}=\mathbf{0}\end{aligned}$$

$\mathbf{a}$ is perpendicular to $\mathbf{b}$.

b Consider the following diagram representing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$:



We let $\overrightarrow{AC}=\mathbf{b}$

so $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\mathbf{a}+\mathbf{b}$

and $\overrightarrow{BA}=\overrightarrow{BO}+\overrightarrow{OA}=-\mathbf{b}+\mathbf{a}=\mathbf{a}-\mathbf{b}$

$\therefore$ $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ represent the diagonals of the parallelogram OACB

But if $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, then the diagonals must be equal in length.

This is only possible if OACB is a square, i.e., $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

11 $$\begin{aligned}(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\bullet\mathbf{a}-\mathbf{a}\bullet\mathbf{b}+\mathbf{b}\bullet\mathbf{a}-\mathbf{b}\bullet\mathbf{b}\\ &= \mathbf{a}\bullet\mathbf{a}-\mathbf{b}\bullet\mathbf{b} \qquad \{\text{since}\quad \mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}\}\\ &= |\mathbf{a}|^2-|\mathbf{b}|^2\\ &= 9-16\\ &= -7\end{aligned}$$

12 The scalar product is only defined between two *vectors*.
Hence $\underbrace{(\mathbf{a}\bullet\mathbf{b})}_{\text{a scalar}}\bullet\underbrace{\mathbf{c}}_{\text{a vector}}$ is meaningless.

EXERCISE 15K.1

1 **a** $[2,\,-3,\,1]\times[1,\,4,\,-2]$

$$\begin{aligned}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&1\\1&4&-2\end{vmatrix}=\mathbf{i}\begin{vmatrix}-3&1\\4&-2\end{vmatrix}+\mathbf{j}\begin{vmatrix}1&2\\-2&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}2&-3\\1&4\end{vmatrix}\\&=(6-4)\mathbf{i}+(1-(-4))\mathbf{j}+(8-(-3))\mathbf{k}\\&=2\mathbf{i}+5\mathbf{j}+11\mathbf{k}\end{aligned}$$

b $[-1,\,0,\,2]\times[3,\,-1,\,-2]$

$$\begin{aligned}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&0&2\\3&-1&-2\end{vmatrix}=\mathbf{i}\begin{vmatrix}0&2\\-1&-2\end{vmatrix}+\mathbf{j}\begin{vmatrix}2&-1\\-2&3\end{vmatrix}+\mathbf{k}\begin{vmatrix}-1&0\\3&-1\end{vmatrix}\\&=(0-(-2))\mathbf{i}+(6-2)\mathbf{j}+(1-0)\mathbf{k}\\&=2\mathbf{i}+4\mathbf{j}+\mathbf{k}\end{aligned}$$

c $(\mathbf{i}+\mathbf{j}-2\mathbf{k})\times(\mathbf{i}-\mathbf{k})$

$$\begin{aligned}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&-2\\1&0&-1\end{vmatrix}=\mathbf{i}\begin{vmatrix}1&-2\\0&-1\end{vmatrix}+\mathbf{j}\begin{vmatrix}-2&1\\-1&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&1\\1&0\end{vmatrix}\\&=(-1+0)\mathbf{i}+(-2-(-1))\mathbf{j}+(0-1)\mathbf{k}\\&=-\mathbf{i}-\mathbf{j}-\mathbf{k}\end{aligned}$$

d $(2\mathbf{i}-\mathbf{k})\times(\mathbf{j}+3\mathbf{k})$

$$\begin{aligned}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-1\\0&1&3\end{vmatrix}=\mathbf{i}\begin{vmatrix}0&-1\\1&3\end{vmatrix}+\mathbf{j}\begin{vmatrix}-1&2\\3&0\end{vmatrix}+\mathbf{k}\begin{vmatrix}2&0\\0&1\end{vmatrix}\\&=(0-(-1))\mathbf{i}+(0-6)\mathbf{j}+(2-0)\mathbf{k}\\&=\mathbf{i}-6\mathbf{j}+2\mathbf{k}\end{aligned}$$

2 If $\mathbf{a}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}-1\\3\\-1\end{bmatrix}$,

then $$\begin{aligned}\mathbf{a}\times\mathbf{b}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&3\\-1&3&-1\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&3\\3&-1\end{vmatrix}+\mathbf{j}\begin{vmatrix}3&1\\-1&-1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&2\\-1&3\end{vmatrix}\\&=-11\mathbf{i}-2\mathbf{j}+5\mathbf{k}\end{aligned}$$

$$\therefore\quad \mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=\begin{bmatrix}1\\2\\3\end{bmatrix}\bullet\begin{bmatrix}-11\\-2\\5\end{bmatrix}\begin{aligned}&=-11-4+15,\\&=0\end{aligned}\qquad \mathbf{b}\bullet(\mathbf{a}\times\mathbf{b})=\begin{bmatrix}-1\\3\\-1\end{bmatrix}\begin{bmatrix}-11\\-2\\5\end{bmatrix}\begin{aligned}&=11-6-5\\&=0\end{aligned}$$

$\therefore$ $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

3 **a** $\mathbf{i}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&0&0\end{vmatrix}=\mathbf{0}$ $\quad\mathbf{j}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&1&0\end{vmatrix}=\mathbf{0}$ $\quad\mathbf{k}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&0&1\end{vmatrix}=\mathbf{0}$

b $\mathbf{i}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&1&0\end{vmatrix}=\mathbf{k}$ $\quad\mathbf{j}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\1&0&0\end{vmatrix}=-\mathbf{k}$

$\mathbf{j}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&0&1\end{vmatrix}=\mathbf{i}$ $\quad\mathbf{k}\times\mathbf{j}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&1&0\end{vmatrix}=-\mathbf{i}$

$\mathbf{i}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&0&1\end{vmatrix}=-\mathbf{j}$ $\quad\mathbf{k}\times\mathbf{i}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&0&0\end{vmatrix}=\mathbf{j}$

If **a** is space vector, then $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ and $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

4 **a**
$$\begin{aligned}\mathbf{a}\times\mathbf{a}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\a_1&a_2&a_3\end{vmatrix}=\mathbf{i}\begin{vmatrix}a_2&a_3\\a_2&a_3\end{vmatrix}+\mathbf{j}\begin{vmatrix}a_3&a_1\\a_3&a_1\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1&a_2\\a_1&a_2\end{vmatrix}\\&=\mathbf{i}\times0+\mathbf{j}\times0+\mathbf{k}\times0\\&=\mathbf{0}\end{aligned}$$

b
$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\mathbf{i}\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}+\mathbf{j}\begin{vmatrix}a_3&a_1\\b_3&b_1\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\\&=\mathbf{i}\,(a_2b_3-a_3b_2)+\mathbf{j}\,(a_3b_1-a_1b_3)+\mathbf{k}\,(a_1b_2-a_2b_1)\\&=-[\mathbf{i}\,(a_3b_2-a_2b_3)+\mathbf{j}\,(a_1b_3-a_3b_1)+\mathbf{k}\,(a_2b_1-a_1b_2)]\\&=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\b_1&b_2&b_3\\a_1&a_2&a_3\end{vmatrix}\\&=-\mathbf{b}\times\mathbf{a}\end{aligned}$$

5 **a**
$$\begin{aligned}\mathbf{b}\times\mathbf{c}&=\begin{bmatrix}2\\-1\\1\end{bmatrix}\times\begin{bmatrix}0\\1\\-2\end{bmatrix}\\&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&1\\0&1&-2\end{vmatrix}=\mathbf{i}\begin{vmatrix}-1&1\\1&-2\end{vmatrix}+\mathbf{j}\begin{vmatrix}1&2\\-2&0\end{vmatrix}+\mathbf{k}\begin{vmatrix}2&-1\\0&1\end{vmatrix}\\&=\mathbf{i}+4\mathbf{j}+2\mathbf{k}\\&=\begin{bmatrix}1\\4\\2\end{bmatrix}\end{aligned}$$

b
$$\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})=\begin{bmatrix}1\\3\\2\end{bmatrix}\bullet\begin{bmatrix}1\\4\\2\end{bmatrix}=1+12+4=17$$

c
$$\begin{aligned}\begin{vmatrix}1&3&2\\2&-1&1\\0&1&-2\end{vmatrix}&=1\begin{vmatrix}-1&1\\1&-2\end{vmatrix}+3\begin{vmatrix}1&2\\-2&0\end{vmatrix}+2\begin{vmatrix}2&-1\\0&1\end{vmatrix}\\&=1(1)+3(4)+2(2)\\&=1+12+4\\&=17\end{aligned}$$

7 **a**
$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$
$$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

b
$$\mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$$
$$= 0\mathbf{i} + 5\mathbf{j} + 0\mathbf{k}$$
$$= 5\mathbf{j}$$

c
$$(\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c}) = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + 5\mathbf{j}$$
$$= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} \quad \{\text{using } \textbf{a} \text{ and } \textbf{b}\}$$

d
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times(2\mathbf{i} - \mathbf{j})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$
$$= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

8 We suspect that $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$.

9 $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$
$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$
$$= \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} a_3 & a_1 \\ b_3 + c_3 & b_1 + c_1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix}$$
$$= \mathbf{i}\,(a_2(b_3 + c_3) - a_3(b_2 + c_2)) + \mathbf{j}\,(a_3(b_1 + c_1) - a_1(b_3 + c_3)) + \mathbf{k}\,(a_1(b_2 + c_2) - a_2(b_1 + c_1))$$
$$= \mathbf{i}\,(a_2b_3 - b_2a_3) + \mathbf{j}\,(a_3b_1 - a_1b_3) + \mathbf{k}\,(a_1b_2 - a_2b_1)$$
$$+\, \mathbf{i}\,(a_2c_3 - a_3c_2) + \mathbf{j}\,(a_3c_1 - a_1c_3) + \mathbf{k}\,(a_1c_2 - a_2c_1)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} \quad \text{as required}$$

10 Now $\mathbf{p}\times(\mathbf{c}+\mathbf{d}) = \mathbf{p}\times\mathbf{c} + \mathbf{p}\times\mathbf{d}$

$\therefore$ if we let $\mathbf{p} = (\mathbf{a}+\mathbf{b})$,

then
$$(\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d}) = \mathbf{p}\times(\mathbf{c}+\mathbf{d})$$
$$= \mathbf{p}\times\mathbf{c} + \mathbf{p}\times\mathbf{d}$$
$$= (\mathbf{a}+\mathbf{b})\times\mathbf{c} + (\mathbf{a}+\mathbf{b})\times\mathbf{d}$$
$$= -\mathbf{c}\times(\mathbf{a}+\mathbf{b}) - \mathbf{d}\times(\mathbf{a}+\mathbf{b}) \quad \{\text{since } \mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}\}$$
$$= -\mathbf{c}\times\mathbf{a} - \mathbf{c}\times\mathbf{b} - \mathbf{d}\times\mathbf{a} - \mathbf{d}\times\mathbf{b} \quad \{\text{since } \mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}\}$$
$$= \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c} + \mathbf{a}\times\mathbf{d} + \mathbf{b}\times\mathbf{d} \quad \{\text{since } \mathbf{x}\times\mathbf{y} = -\mathbf{y}\times\mathbf{x}\}$$

11 **a**
$$\begin{aligned}&\mathbf{a}\times(\mathbf{a}+\mathbf{b})\\ &=\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\\ &=\mathbf{0}+\mathbf{a}\times\mathbf{b}\\ &=\mathbf{a}\times\mathbf{b}\end{aligned}$$

b
$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b})\\ &=\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}\\ &=\mathbf{0}+\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}+\mathbf{0}\\ &=\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}\\ &=\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}\\ &=\mathbf{0}\end{aligned}$$

c **a** is perpendicular to $(\mathbf{a}\times\mathbf{b})$,
2a is perpendicular to $(\mathbf{a}\times\mathbf{b})$.
$\therefore\ \ 2\mathbf{a}\bullet(\mathbf{a}\times\mathbf{b})=0$

12 **a**
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&3\\1&1&1\end{vmatrix}=\mathbf{i}\begin{vmatrix}-1&3\\1&1\end{vmatrix}+\mathbf{j}\begin{vmatrix}3&2\\1&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}2&-1\\1&1\end{vmatrix}=-4\mathbf{i}+\mathbf{j}+3\mathbf{k}$$

$\therefore$ the vectors are $k(-4\mathbf{i}+\mathbf{j}+3\mathbf{k}),\ \ k\neq 0,\ \ k\in\mathbf{R}$

b
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&3&4\\5&0&2\end{vmatrix}=\mathbf{i}\begin{vmatrix}3&4\\0&2\end{vmatrix}+\mathbf{j}\begin{vmatrix}4&-1\\2&5\end{vmatrix}+\mathbf{k}\begin{vmatrix}-1&3\\5&0\end{vmatrix}=6\mathbf{i}+22\mathbf{j}-15\mathbf{k}$$

$\therefore$ the vectors are $k(6\mathbf{i}+22\mathbf{j}-15\mathbf{k}],\ \ k\neq 0,\ \ k\in\mathbf{R}$

c
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\1&-1&-1\end{vmatrix}=\mathbf{i}\begin{vmatrix}1&0\\-1&-1\end{vmatrix}+\mathbf{j}\begin{vmatrix}0&1\\-1&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&1\\1&-1\end{vmatrix}=-\mathbf{i}+\mathbf{j}-2\mathbf{k}$$

$\therefore$ the vectors are $n(-\mathbf{i}+\mathbf{j}-2\mathbf{k}),\ \ n\neq 0,\ \ n\in\mathbf{R}$

d
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&-1\\2&2&-3\end{vmatrix}=\mathbf{i}\begin{vmatrix}-1&-1\\2&-3\end{vmatrix}+\mathbf{j}\begin{vmatrix}-1&1\\-3&2\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&-1\\2&2\end{vmatrix}=5\mathbf{i}+\mathbf{j}+4\mathbf{k}$$

$\therefore$ the vectors are $n(5\mathbf{i}+\mathbf{j}+4\mathbf{k}),\ \ n\neq 0,\ \ n\in\mathbf{R}$

13
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&-1\\1&-2&2\end{vmatrix}=\mathbf{i}\begin{vmatrix}3&-1\\-2&2\end{vmatrix}+\mathbf{j}\begin{vmatrix}-1&2\\2&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}2&3\\1&-2\end{vmatrix}=4\mathbf{i}-5\mathbf{j}-7\mathbf{k}$$

$\therefore$ The vectors $k(4\mathbf{i}-5\mathbf{j}-7\mathbf{k}),\ \ k\neq 0$ are all perpendicular to both **a** and **b**.
However, we require the vector to have length 5.

$$\begin{aligned}\therefore\ \ \sqrt{(4k)^2+(-5k)^2+(-7k)^2}&=5\\ \therefore\ \ 16k^2+25k^2+49k^2&=25\\ \therefore\ \ 90k^2&=25\\ \therefore\ \ k^2&=\tfrac{25}{90}=\tfrac{5}{18}\\ \therefore\ \ k&=\pm\tfrac{\sqrt{5}}{3\sqrt{2}}=\pm\tfrac{\sqrt{10}}{6}\end{aligned}$$

$\therefore$ the possible vectors are $\pm\frac{\sqrt{10}}{6}[4,\,-5,\,-7]$

14 **a** Given A(1, 3, 2) B(0, 2, −5) and C(3, 1, −4),

$$\overrightarrow{AB}=\begin{bmatrix}-1\\-1\\-7\end{bmatrix}\quad\text{and}\quad\overrightarrow{AC}=\begin{bmatrix}2\\-2\\-6\end{bmatrix}$$

$$\therefore\ \ \mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&-1&-7\\2&-2&-6\end{vmatrix}=\mathbf{i}\begin{vmatrix}-1&-7\\-2&-6\end{vmatrix}+\mathbf{j}\begin{vmatrix}-7&-1\\-6&2\end{vmatrix}+\mathbf{k}\begin{vmatrix}-1&-1\\2&-2\end{vmatrix}$$

$$\begin{aligned}&=(6-14)\mathbf{i}+(-14-6)\mathbf{j}+(2+2)\mathbf{k}\\ &=-8\mathbf{i}-20\mathbf{j}+4\mathbf{k}]\\ &=-4(2\mathbf{i}+5\mathbf{j}-\mathbf{k})\end{aligned}$$

$\therefore\ \ 2\mathbf{i}+5\mathbf{j}-\mathbf{k}$ will do.

b Given P(2, 0, −1), Q(0, 1, 3) and R(1, −1, 1), $\overrightarrow{PQ} = \begin{bmatrix}-2\\1\\4\end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$

$$\begin{aligned}\therefore \quad \mathbf{n} &= \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&1&4\\-1&-1&2\end{vmatrix} = \mathbf{i}\begin{vmatrix}1&4\\-1&2\end{vmatrix}+\mathbf{j}\begin{vmatrix}4&-2\\2&-1\end{vmatrix}+\mathbf{k}\begin{vmatrix}-2&1\\-1&-1\end{vmatrix}\\ &= 6\mathbf{i}+3\mathbf{k}\\ &= 3(2\mathbf{i}+\mathbf{k}) \quad \text{and so} \quad 2\mathbf{i}+\mathbf{k} \quad \text{will do.}\end{aligned}$$

EXERCISE 15K.2

1 **a**

$$\mathbf{i}\times\mathbf{k} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&0&1\end{vmatrix} = \mathbf{i}\begin{vmatrix}0&0\\0&1\end{vmatrix}+\mathbf{j}\begin{vmatrix}0&1\\1&0\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&0\\0&0\end{vmatrix} \quad \begin{aligned}&= 0\mathbf{i}-\mathbf{j}+0\mathbf{k}\\&= -\mathbf{j}\end{aligned}$$

$$\mathbf{k}\times\mathbf{i} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&0&0\end{vmatrix} = \mathbf{i}\begin{vmatrix}0&1\\0&0\end{vmatrix}+\mathbf{j}\begin{vmatrix}1&0\\0&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}0&0\\1&0\end{vmatrix} \quad \begin{aligned}&= 0\mathbf{i}+\mathbf{j}+0\mathbf{k}\\&= \mathbf{j}\end{aligned}$$

b Using the RH rule these two results check.

c

$$\begin{aligned}\mathbf{i}\times\mathbf{k} &= |\mathbf{i}|\,|\mathbf{k}|\sin 90^o \times (-\mathbf{j}) \quad \{\text{using the RH rule}\}\\ &= 1\times 1\times 1\times(-\mathbf{j})\\ &= -\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{k}\times\mathbf{i} &= |\mathbf{k}|\,|\mathbf{i}|\sin 90^o \times \mathbf{u} \qquad \{\text{using the RH rule}\}\\ &= 1\times 1\times 1\times\mathbf{j}\\ &= \mathbf{j}\end{aligned}$$

2 **a**

$$\begin{aligned}\mathbf{a}\bullet\mathbf{b} &= 2\times 1+(-1)\times 0+3\times(-1)\\ &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&3\\1&0&-1\end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix}-1&3\\0&-1\end{vmatrix}+\mathbf{j}\begin{vmatrix}3&2\\-1&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}2&-1\\1&0\end{vmatrix}\\ &= \mathbf{i}+5\mathbf{j}+\mathbf{k}\end{aligned}$$

b

$$\begin{aligned}&\cos\theta\\ &= \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\\ &= \frac{-1}{\sqrt{4+1+9}\sqrt{1+1}}\\ &= \frac{-1}{\sqrt{14}\sqrt{2}}\\ &= \tfrac{-1}{\sqrt{28}}\end{aligned}$$

c

$$\begin{aligned}\sin^2\theta+\cos^2\theta &= 1\\ \therefore \quad \sin^2\theta+\tfrac{1}{28} &= 1\\ \therefore \quad \sin^2\theta &= \tfrac{27}{28}\\ \therefore \quad \sin\theta &= \pm\sqrt{\tfrac{27}{28}}\end{aligned}$$

But $0 \leqslant \theta \leqslant \pi$

$$\therefore \quad \sin\theta = \sqrt{\tfrac{27}{28}}$$

d

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}| &= |\mathbf{a}|\,|\mathbf{b}|\sin\theta\\ \therefore \quad \sin\theta &= \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}\\ &= \frac{\sqrt{1^2+5^2+1^2}}{\sqrt{14}\sqrt{2}}\\ &= \sqrt{\tfrac{27}{28}}\end{aligned}$$

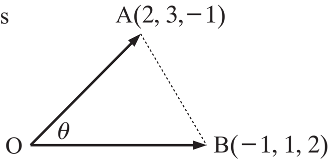
3

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = \mathbf{0} \quad &\Leftrightarrow |\mathbf{a}|\,|\mathbf{b}|\sin\theta\times\mathbf{u} = \mathbf{0}\\ &\Leftrightarrow |\mathbf{a}|\,|\mathbf{b}|\sin\theta = 0 \qquad \{\text{since } |\mathbf{a}| \neq 0 \text{ and } |\mathbf{b}| \neq 0, \ \mathbf{u} \text{ exists and } \neq 0\}\\ &\Leftrightarrow \sin\theta = 0 \qquad \{\text{since } |\mathbf{a}| \neq 0, \text{ and } |\mathbf{b}| \neq 0\}\\ &\Leftrightarrow \theta = 0 \quad \text{or} \quad \pi\\ &\Leftrightarrow \quad \mathbf{a} \text{ is parallel to } \mathbf{b}\end{aligned}$$

4 **a** $\overrightarrow{OA} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

b
$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$$
$$= 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$
$$\therefore \ |\, \overrightarrow{OA} \times \overrightarrow{OB} \,| = \sqrt{7^2 + (-3)^2 + 5^2} = \sqrt{83} \text{ units}$$

c
$$\text{Area } \Delta\text{AOB} = \tfrac{1}{2}\,|\,\overrightarrow{OA}\,|\,|\,\overrightarrow{OB}\,| \sin\theta$$
$$= \tfrac{1}{2}\,|\,\overrightarrow{OA} \times \overrightarrow{OB}\,|$$
$$= \tfrac{1}{2}\sqrt{83} \text{ units}^2$$



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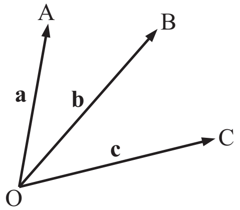


a
$$\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$$
$$\therefore \ \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{c} = \mathbf{0}$$
$$\therefore \ (\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0}$$
$$\overrightarrow{BA} \times \mathbf{c} = \mathbf{0}$$
$$\therefore \ \overrightarrow{OC} \text{ must be parallel to } \overrightarrow{AB}.$$

b
$$\text{Since } \ \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0},$$
$$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{b} \times \mathbf{0}$$
$$\therefore \ \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$
$$\therefore \ \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$
$$\therefore \ \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$$
$$\{\text{since } \ \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}\}$$

c
$$\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \quad \text{where} \quad \mathbf{c} \neq \mathbf{0}$$
$$\therefore \ \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{c}$$
$$\therefore \ \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$
$$\therefore \ (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{0}$$
$$\therefore \ \mathbf{a} + \mathbf{b} \text{ is parallel to } \mathbf{c}$$
$$\text{i.e., } \ \mathbf{a} + \mathbf{b} = k\mathbf{c} \quad \text{for some scalar } k.$$

EXERCISE 15K.3

1 **a** Given A(2, 1, 1) B(4, 3, 0) and C(1, 3, −2), $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$

$$\therefore \ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 2 \\ -3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}$$
$$= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$
$$\therefore \ \text{Area} = \tfrac{1}{2}\,|[-4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}]| \qquad \{\text{as area} = \tfrac{1}{2}\,|\,\overrightarrow{AB} \times \overrightarrow{AC}\,|\}$$
$$= \tfrac{1}{2}\sqrt{(-4)^2 + 7^2 + 6^2}$$
$$= \tfrac{1}{2}\sqrt{101} \text{ units}^2$$

b Given A(0, 0, 0) B(−1, 2, 3) and C(1, 2, 6), $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

$$\therefore \ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 3 & -1 \\ 6 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}$$
$$= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$$
$$\therefore \ \text{Area} = \tfrac{1}{2}\,|[6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}]| \qquad \{\text{as area} = \tfrac{1}{2}\,|\,\overrightarrow{AB} \times \overrightarrow{AC}\,|\}$$
$$= \tfrac{1}{2}\sqrt{6^2 + 9^2 + (-4)^2}$$
$$= \tfrac{1}{2}\sqrt{133} \text{ units}^2$$

c Given A(1, 3, 2) B(2, −1, 0) and C(1, 10, 6), $\overrightarrow{AB}=\begin{bmatrix}1\\-4\\-2\end{bmatrix}$ and $\overrightarrow{AC}=\begin{bmatrix}0\\7\\4\end{bmatrix}$

$$\therefore\quad \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&-2\\0&7&4\end{vmatrix}=\mathbf{i}\begin{vmatrix}-4&-2\\7&4\end{vmatrix}+\mathbf{j}\begin{vmatrix}-2&1\\4&0\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&-4\\0&7\end{vmatrix}$$

$$=-2\mathbf{i}-4\mathbf{j}+7\mathbf{k}$$

$$\therefore\quad \text{area}=\tfrac{1}{2}\,|[-2\mathbf{i}-4\mathbf{j}+7\mathbf{k}]|=\tfrac{1}{2}\sqrt{(-2)^2+(-4)^2+7^2}=\tfrac{1}{2}\sqrt{69}\text{ units}^2$$

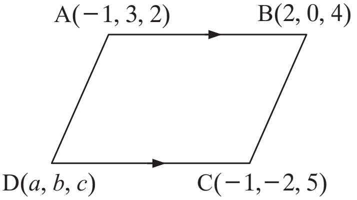
2 Given A(−1, 2, 2) B(2, −1, 4) and C(0, 1, 0), $\overrightarrow{AB}=\begin{bmatrix}3\\-3\\2\end{bmatrix}$ and $\overrightarrow{AC}=\begin{bmatrix}1\\-1\\-2\end{bmatrix}$

$$\therefore\quad \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-3&2\\1&-1&-2\end{vmatrix}=\mathbf{i}\begin{vmatrix}-3&2\\-1&-2\end{vmatrix}+\mathbf{j}\begin{vmatrix}2&3\\-2&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}3&-3\\1&-1\end{vmatrix}$$

$$=8\mathbf{i}+8\mathbf{j}$$

$$\therefore\ \text{area of parallelogram}=|[8\mathbf{i}+8\mathbf{j}]|=\sqrt{8^2+8^2}=8\sqrt{2}\text{ units}^2$$

3 **a**



Suppose D is at (a, b, c)

Since $\overrightarrow{AB}=\overrightarrow{DC}$,

$$\begin{bmatrix}3\\-3\\2\end{bmatrix}=\begin{bmatrix}-1-a\\-2-b\\5-c\end{bmatrix}$$

$\therefore\quad -1-a=3,\quad -2-b=-3$ and $5-c=2$

$\therefore\quad a=-4,\quad b=1$ and $c=3$

i.e., D is at (−4, 1, 3).

b $\overrightarrow{BC}=\begin{bmatrix}-3\\-2\\1\end{bmatrix}$ and $\overrightarrow{BA}=\begin{bmatrix}-3\\3\\-2\end{bmatrix}$

$$\therefore\quad \overrightarrow{BC}\times\overrightarrow{BA}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-3&-2&1\\-3&3&-2\end{vmatrix}=\mathbf{i}\begin{vmatrix}-2&1\\3&-2\end{vmatrix}+\mathbf{j}\begin{vmatrix}1&-3\\-2&-3\end{vmatrix}+\mathbf{k}\begin{vmatrix}-3&-2\\-3&3\end{vmatrix}$$

$$=[\mathbf{i}-9\mathbf{j}-15\mathbf{k}]$$

$$\therefore\quad \text{area}=|[\mathbf{i}-9\mathbf{j}-15\mathbf{k}]|=\sqrt{(1^2+(-9)^2+(-15)^2}=\sqrt{307}\text{ units}^2$$

4 **a** Now $\overrightarrow{AB}=[1, 2, -1]$, $\overrightarrow{AC}=[-1, 2, -3]$ and $\overrightarrow{AD}=[-2, 2, 2]$

$\therefore$ the volume of the tetrahedron

$$=\tfrac{1}{6}\left|\overrightarrow{AB}\bullet\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right|$$

$$=\tfrac{1}{6}\left\|\begin{matrix}1&2&-1\\-1&2&-3\\-2&2&2\end{matrix}\right\|=\tfrac{1}{6}\left|1\begin{vmatrix}2&-3\\2&2\end{vmatrix}+2\begin{vmatrix}-3&-1\\2&-2\end{vmatrix}-1\begin{vmatrix}-1&2\\-2&2\end{vmatrix}\right|$$

$$=\tfrac{1}{6}\,|10+2(8)-1(2)|$$

$$=\tfrac{1}{6}\,|24|$$

$$=4\text{ units}^3$$

b The total surface area of the tetrahedron is the sum of the four triangular faces forming it.

Face 1 $$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= -4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\therefore \quad A_1 = \tfrac{1}{2}\,|\,\overrightarrow{AB}\times\overrightarrow{AC}\,| = \tfrac{1}{2}\sqrt{4^2+4^2+4^2} = \tfrac{1}{2}\left(4\sqrt{3}\right) = 2\sqrt{3} \text{ units}^2$$

Face 2 $$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= 6\mathbf{i} + 6\mathbf{k}$$

$$\therefore \quad A_2 = \tfrac{1}{2}\,|\,\overrightarrow{AB}\times\overrightarrow{AD}\,| = \tfrac{1}{2}\sqrt{6^2+6^2} = \tfrac{1}{2}6\sqrt{2} = 3\sqrt{2} \text{ units}^2$$

Face 3 $$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ -2 & 2 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -3 & -1 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= 10\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}$$

$$\therefore \quad A_3 = \tfrac{1}{2}\,|\,\overrightarrow{AC}\times\overrightarrow{AD}\,| = \tfrac{1}{2}\sqrt{100+64+4} = \tfrac{1}{2}\sqrt{168} = \sqrt{42} \text{ units}^2$$

Face 4 $$\overrightarrow{BC}\times\overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -2 \\ -3 & 0 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -2 & -2 \\ 3 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 0 \\ -3 & 0 \end{vmatrix}$$

$$= 12\mathbf{j}$$

$$\therefore \quad A_4 = \tfrac{1}{2}\,|\,\overrightarrow{BC}\times\overrightarrow{BD}\,| = \tfrac{1}{2}\times 12 = 6 \text{ units}^2$$

$$\therefore \quad \text{total surface area } = (\sqrt{42} + 2\sqrt{3} + 3\sqrt{2} + 6) \text{ units}^2$$

5 **a** Given A(3, 0, 0) B(0, 1, 0) C(1, 2, 3) O(0, 0, 0),

We label the other vertices as shown below.

$$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{OC} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

so X is at (1, 3, 3)

$$\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{OC} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

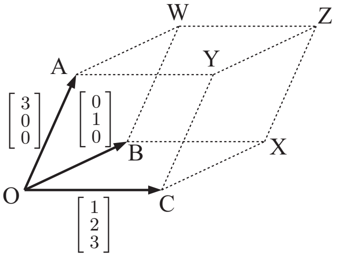
so Y is at (4, 2, 3)

$$\overrightarrow{OW} = \overrightarrow{OA} + \overrightarrow{OB} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

so W is at (3, 1, 0)

$$\overrightarrow{OZ} = \overrightarrow{OW} + \overrightarrow{OC} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$$

so Z is at (4, 3, 3).



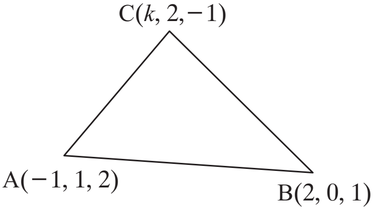
b $$\overrightarrow{BA} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \quad\text{and}\quad \overrightarrow{BC} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\therefore \quad \cos\measuredangle ABC = \frac{\overrightarrow{BA}\bullet\overrightarrow{BC}}{|\,\overrightarrow{BA}\,|\;\;|\,\overrightarrow{BC}\,|} = \frac{2}{\sqrt{110}} \qquad \therefore \quad \theta \doteqdot 79.01^o$$

c Volume

$$= \left\| \begin{matrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{matrix} \right\| = \left| 3 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \right| \begin{aligned} &= 3 \times 3 \\ &= 9 \text{ units}^3 \end{aligned}$$

6 Now $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} k+1 \\ 1 \\ -3 \end{bmatrix}$



$$\text{Area of } \Delta\text{ABC} = \tfrac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$$

$$\therefore \quad \sqrt{88} = \tfrac{1}{2} \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ k+1 & 1 & -3 \\ 3 & -1 & -1 \end{matrix} \right\|$$

$$2\sqrt{88} = \left| \mathbf{i} \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -3 & k+1 \\ -1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} k+1 & 1 \\ 3 & -1 \end{vmatrix} \right|$$

$$= |(-1-3)\mathbf{i} \; + \; (-9+(k+1))\mathbf{j} \; + \; (-(k+1)-3)\mathbf{k}|$$

$$= |-4\mathbf{i} + (k-8)\mathbf{j} + (-k-4)\mathbf{k}|$$

$$\therefore \quad \sqrt{352} = \sqrt{16 + (k-8)^2 + (-k-4)^2}$$

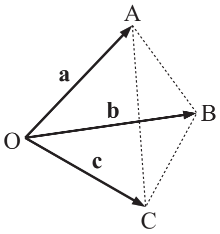
$$\therefore \quad 352 = 16 + k^2 - 16k + 64 + k^2 + 8k + 16$$

$$\therefore \quad 2k^2 - 8k - 256 = 0$$

$$\therefore \quad k^2 - 4k - 128 = 0$$

$$\therefore \quad k = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2} = 2 \pm \sqrt{132} = 2 \pm 2\sqrt{33}$$

7



Total surface area S of the tetrahedron is the sum of the areas of the 4 triangular faces.

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$

and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$

$$\therefore \quad S = \tfrac{1}{2}\,|\mathbf{a} \times \mathbf{b}| + \tfrac{1}{2}\,|\mathbf{a} \times \mathbf{c}| + \tfrac{1}{2}\,|\mathbf{b} \times \mathbf{c}| + \tfrac{1}{2}\,|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$

8

$$\begin{aligned} \overrightarrow{BA} &= \overrightarrow{BO} + \overrightarrow{OA} \\ &= -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b} \end{aligned}$$

$$\text{and} \quad \begin{aligned} \overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} \\ &= -\mathbf{b} + \mathbf{c} = \mathbf{c} - \mathbf{b} \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{Area } \Delta \text{ ABC} &= \tfrac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| \\ &= \tfrac{1}{2}\,|(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b})| \\ &= \tfrac{1}{2}\,|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b})| \end{aligned}$$

Now if A, B and C are collinear, then area ΔABC $= 0$

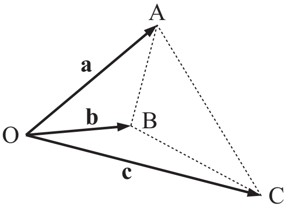
$$\therefore \quad (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$$

or $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \mathbf{c} - \mathbf{b}$

If A, B and C are collinear, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$

$\therefore$ $\overrightarrow{AB} \times \overrightarrow{BC} = \mathbf{0}$, i.e., $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$



9 **a** $\mathbf{b}-\mathbf{a}=\overrightarrow{\mathrm{AB}}=\begin{bmatrix}1\\3\\-2\end{bmatrix}$ $\mathbf{c}-\mathbf{a}=\overrightarrow{\mathrm{AC}}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$ $\mathbf{d}-\mathbf{a}=\overrightarrow{\mathrm{AD}}=\begin{bmatrix}3\\-1\\-1\end{bmatrix}$

$\therefore$ $(\mathbf{b}-\mathbf{a})\bullet(\mathbf{c}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$

$$=\begin{vmatrix}1&3&-2\\2&0&-1\\3&-1&-1\end{vmatrix}=1\begin{vmatrix}0&-1\\-1&-1\end{vmatrix}+3\begin{vmatrix}-1&2\\-1&3\end{vmatrix}-2\begin{vmatrix}2&0\\3&-1\end{vmatrix}$$

$=(-1)+3(-1)-2(-2)$

$=0$ and so A, B, C and D are coplanar

b $\mathbf{q}-\mathbf{p}=\overrightarrow{\mathrm{PQ}}=\begin{bmatrix}-2\\-1\\-1\end{bmatrix}$ $\mathbf{r}-\mathbf{p}=\overrightarrow{\mathrm{PR}}=\begin{bmatrix}0\\1\\-5\end{bmatrix}$ $\mathbf{s}-\mathbf{p}=\overrightarrow{\mathrm{PS}}=\begin{bmatrix}-1\\1\\-4\end{bmatrix}$

$\therefore$ $(\mathbf{q}-\mathbf{p})\bullet(\mathbf{r}-\mathbf{p})\times(\mathbf{s}-\mathbf{p})$

$$=\begin{vmatrix}-2&-1&-1\\0&1&-5\\-1&1&-4\end{vmatrix}=-2\begin{vmatrix}1&-5\\1&-4\end{vmatrix}-1\begin{vmatrix}-5&0\\-4&-1\end{vmatrix}-1\begin{vmatrix}0&1\\-1&1\end{vmatrix}$$

$=-2(1)-1(5)-1(1)$

$=-8$

$\neq 0$ and so P, Q, R and S are not coplanar

10 $\mathbf{b}-\mathbf{a}=\overrightarrow{\mathrm{AB}}=\begin{bmatrix}2\\-1\\-2\end{bmatrix}$ $\mathbf{c}-\mathbf{a}=\overrightarrow{\mathrm{AC}}=\begin{bmatrix}-2\\k-1\\-1\end{bmatrix}$ $\mathbf{d}-\mathbf{a}=\overrightarrow{\mathrm{AD}}=\begin{bmatrix}-1\\1\\-4\end{bmatrix}$

$\therefore$ $(\mathbf{b}-\mathbf{a})\bullet(\mathbf{c}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})$

$$=\begin{vmatrix}2&-1&-2\\-2&k-1&-1\\-1&1&-4\end{vmatrix}=2\begin{vmatrix}k-1&-1\\1&-4\end{vmatrix}-1\begin{vmatrix}-1&-2\\-4&-1\end{vmatrix}-2\begin{vmatrix}-2&k-1\\-1&1\end{vmatrix}$$

$=2(-4k+4+1)-1(1-8)-2(-2+k-1)$

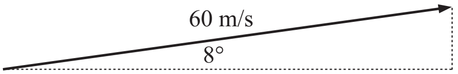
$=-8k+10+7-2k+6$

$=-10k+23$

$\therefore$ A, B, C and D are coplanar when $-10k+23=0$, i.e., when $k=\frac{23}{10}$.

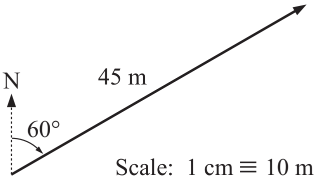
REVIEW SET 15A

1 **a**



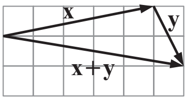
Scale: 1 cm ≡ 10 m/s

b

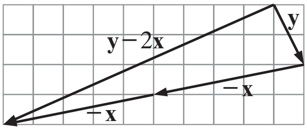


Scale: 1 cm ≡ 10 m

2 **a**



b



3 **a** $\overrightarrow{\mathrm{PR}}+\overrightarrow{\mathrm{RQ}}$
$=\overrightarrow{\mathrm{PQ}}$

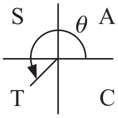
b $\overrightarrow{\mathrm{PS}}+\overrightarrow{\mathrm{SQ}}+\overrightarrow{\mathrm{QR}}$
$=\overrightarrow{\mathrm{PQ}}+\overrightarrow{\mathrm{QR}}$
$=\overrightarrow{\mathrm{PR}}$

4 Dino's first displacement vector is $9\begin{bmatrix}\cos 246^o\\ \sin 246^o\end{bmatrix}$, Dino's second displacement vector is $6\begin{bmatrix}\cos 96^o\\ \sin 96^o\end{bmatrix}$

$\therefore$ Dino's resultant displacement vector is $\begin{bmatrix}9\cos 246^o\\ 9\sin 246^o\end{bmatrix}+\begin{bmatrix}6\cos 96^o\\ 6\sin 96^o\end{bmatrix}\doteqdot\begin{bmatrix}-4.288\\ -2.255\end{bmatrix}$

which has length $\sqrt{(-4.288)^2+(-2.255)^2} \doteqdot 4.845$

$\therefore$ the resultant displacement vector $= 4.845\begin{bmatrix}-0.8850\\-0.4654\end{bmatrix}\begin{array}{l}\leftarrow\cos\theta\\ \leftarrow\sin\theta\end{array}$



If $\cos\theta = -0.8850$ and $\sin\theta = -0.4654$

θ is in Quadrant 3 $\therefore$ $\theta = 180^o + \cos^{-1}(0.8850) \doteqdot 207.7^o$

$\therefore$ Dino is 4.845 km from the start at bearing 208^o.

5 **a** $\overrightarrow{AB} - \overrightarrow{CB}$
$= \overrightarrow{AB} + \overrightarrow{BC}$
$= \overrightarrow{AC}$

b $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$
$= \overrightarrow{AC} + \overrightarrow{CD}$
$= \overrightarrow{AD}$

6 **a** If $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{CD}$ then
AB $\parallel$ CD and AB $= \frac{1}{2}$(CD)

b If $\overrightarrow{AB} = 2\overrightarrow{AC}$ then
AB $\parallel$ AC and AB $=$ 2(AC)
i.e., A, B and C are collinear and
AB $=$ 2(AC).
So, C is the midpoint of AB.

7 **a** $\mathbf{p} + \mathbf{r} - \mathbf{q} = 0$
$\therefore$ $\mathbf{p} + \mathbf{r} = \mathbf{q}$

b $\mathbf{l} + \mathbf{m} - \mathbf{n} + \mathbf{j} - \mathbf{k} = 0$
$\therefore$ $\mathbf{l} + \mathbf{m} + \mathbf{j} = \mathbf{n} + \mathbf{k}$

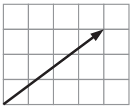
8 **a** $\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q}$

b $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q}$

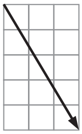
c $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q}$

d $\overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN}$
$= \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR}$
$= \frac{1}{2}(-\mathbf{p} + \mathbf{r} + \mathbf{q}) + \frac{1}{2}(-\mathbf{q})$
$= -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q}$
$= \frac{1}{2}\mathbf{r} - \frac{1}{2}\mathbf{p}$

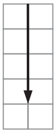
9 **a** $\begin{bmatrix}4\\3\end{bmatrix}$



b $\begin{bmatrix}3\\-5\end{bmatrix}$



c $\begin{bmatrix}0\\-4\end{bmatrix}$



10 **a** $2\mathbf{p} + \mathbf{q}$
$= 2\begin{bmatrix}-3\\1\end{bmatrix} + \begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}-6\\2\end{bmatrix} + \begin{bmatrix}2\\-4\end{bmatrix}$
$= \begin{bmatrix}-4\\-2\end{bmatrix}$

b $\mathbf{q} - 3\mathbf{r}$
$= \begin{bmatrix}2\\-4\end{bmatrix} - 3\begin{bmatrix}1\\3\end{bmatrix}$
$= \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}3\\9\end{bmatrix}$
$= \begin{bmatrix}-1\\-13\end{bmatrix}$

c $\mathbf{p} - \mathbf{q} + \mathbf{r}$
$= \begin{bmatrix}-3\\1\end{bmatrix} - \begin{bmatrix}2\\-4\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix}$
$= \begin{bmatrix}-5\\5\end{bmatrix} + \begin{bmatrix}1\\3\end{bmatrix}$
$= \begin{bmatrix}-4\\8\end{bmatrix}$

11 $\overrightarrow{SP}$
$= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP}$
$= -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ}$
$= -\begin{bmatrix}2\\-3\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}-4\\1\end{bmatrix}$
$= \begin{bmatrix}1\\4\end{bmatrix}$

12 **a** $|\mathbf{r}|$
$= \sqrt{4^2 + 1^2}$
$= \sqrt{17}$ units

b $|\mathbf{s}|$
$= \sqrt{(-3)^2 + 2^2}$
$= \sqrt{13}$ units

c $\mathbf{r} + \mathbf{s}$
$= \begin{bmatrix}4\\1\end{bmatrix} + \begin{bmatrix}-3\\2\end{bmatrix}$
$= \begin{bmatrix}1\\3\end{bmatrix}$
$\therefore$ $|\mathbf{r} + \mathbf{s}|$
$= \sqrt{1^2 + 3^2}$
$= \sqrt{10}$ units

d $2\mathbf{s} - \mathbf{r}$
$= 2\begin{bmatrix}-3\\2\end{bmatrix} - \begin{bmatrix}4\\1\end{bmatrix}$
$= \begin{bmatrix}-10\\3\end{bmatrix}$
$\therefore$ $|2\mathbf{s} - \mathbf{r}|$
$= \sqrt{100 + 9}$
$= \sqrt{109}$ units

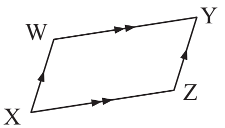
13 **a** $\overrightarrow{BC} = 2\overrightarrow{OA} = 2\mathbf{p}$
Now $\overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BC}$
$= -\mathbf{p} + \mathbf{q} + 2\mathbf{p}$
$= \mathbf{p} + \mathbf{q}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \mathbf{p} + \frac{1}{2}\overrightarrow{AC}$
$= \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q})$
$= \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$

14 **a**
$$\begin{aligned} \mathbf{p} - 3\mathbf{x} &= \mathbf{0} \\ \therefore\ \mathbf{p} &= 3\mathbf{x} \\ \therefore\ \tfrac{1}{3}\mathbf{p} &= \mathbf{x} \\ \therefore\ \mathbf{x} &= \begin{bmatrix} -1 \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

b
$$\begin{aligned} 2\mathbf{q} - \mathbf{x} &= \mathbf{r} \\ \therefore\ 2\mathbf{q} - \mathbf{r} &= \mathbf{x} \\ \therefore\ \mathbf{x} &= 2\begin{bmatrix} 2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ \therefore\ \mathbf{x} &= \begin{bmatrix} 1 \\ -10 \end{bmatrix} \end{aligned}$$

15



$$\overrightarrow{WY} = \begin{bmatrix} 3--3 \\ 4--1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$
$$\overrightarrow{XZ} = \begin{bmatrix} 4--2 \\ 10-5 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

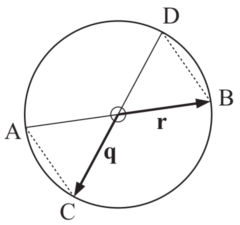
So, $\overrightarrow{WY} = \overrightarrow{XZ}$

∴ WY is parallel to XZ and they are equal in length. This is sufficient to deduce that WYZX is a parallelogram.

16
$$\begin{aligned} r\begin{bmatrix} -2 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -4 \end{bmatrix} &= \begin{bmatrix} 13 \\ -24 \end{bmatrix} \\ \therefore\ \begin{bmatrix} -2r+3s \\ r-4s \end{bmatrix} &= \begin{bmatrix} 13 \\ -24 \end{bmatrix} \\ \therefore\ -2r + 3s &= 13 \\ r - 4s &= -24 \quad \Big|\ \times 2 \ \ldots\ (1) \\ \therefore\ -2r + 3s &= 13 \\ 2r - 8s &= -48 \\ \hline -5s &= -35 \\ \therefore\ s &= 7 \end{aligned}$$

$$\begin{aligned} \text{and in (1)} \quad r - 4(7) &= -24 \\ \therefore\ r &= -24 + 28 \\ \therefore\ r &= 4 \\ \text{i.e.,}\quad r &= 4 \quad \text{and} \quad s = 7 \end{aligned}$$

17



a
$$\begin{aligned} \overrightarrow{DB} &= \overrightarrow{DO} + \overrightarrow{OB} \\ &= \overrightarrow{OC} + \overrightarrow{OB} \\ &= \mathbf{q} + \mathbf{r} \end{aligned}$$

b
$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= \overrightarrow{OB} + \overrightarrow{OC} \\ &= \mathbf{r} + \mathbf{q} \end{aligned}$$

We see that $\overrightarrow{DB} = \overrightarrow{AC}$

∴ DB is parallel to AC and equal in length.

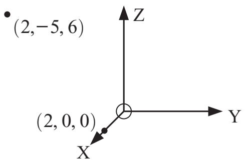
This is sufficient to deduce that ACBD is a parallelogram.

REVIEW SET 15B

1 **a**
$$\overrightarrow{PQ} = \begin{bmatrix} -1-2 \\ 7--5 \\ 9-6 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ 3 \end{bmatrix}$$

b
$$\begin{aligned} PQ &= \sqrt{(-3)^2 + 12^2 + 3^2} \\ &= \sqrt{162} \text{ units} \end{aligned}$$

c



$$\begin{aligned} &\therefore\ \text{distance} \\ &= \sqrt{(2-2)^2 + (0--5)^2 + (0-6)^2} \\ &= \sqrt{0 + 25 + 36} \\ &= \sqrt{61} \text{ units} \end{aligned}$$

2 **a**
$$\mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 11 \end{bmatrix}$$

b
$$2\mathbf{n} - 3\mathbf{p} = 2\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} - \begin{bmatrix} -3 \\ 9 \\ 18 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -26 \end{bmatrix}$$

c
$$\mathbf{m} + \mathbf{p} = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix} \qquad \therefore\ \begin{aligned} |\mathbf{m} + \mathbf{p}| &= \sqrt{25 + 0 + 49} \\ &= \sqrt{74} \text{ units} \end{aligned}$$

3
$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 7 \end{bmatrix}$$

4 The vectors are parallel $\therefore\ \begin{bmatrix} -12 \\ -20 \\ 2 \end{bmatrix} = k\begin{bmatrix} 3 \\ m \\ n \end{bmatrix}$

$\therefore\ 3k = -12, \quad km = -20, \quad kn = 2$

$\therefore\ k = -4, \quad m = 5, \quad n = -\frac{1}{2}$

5 $\overrightarrow{PQ} = \begin{bmatrix} 4 - -6 \\ 6 - 8 \\ 8 - 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$ $\quad\therefore$ Both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel to $\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

$\overrightarrow{QR} = \begin{bmatrix} 19 - 4 \\ 3 - 6 \\ 17 - 8 \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$ $\quad\therefore$ PQ $\parallel$ QR with Q common to both.

$\therefore$ P, Q, R are collinear.

$$\overrightarrow{PQ} : \overrightarrow{QR} = 2\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} : 3\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} = 2 : 3$$

$\therefore$ Q divides PR internally in the ratio 2 : 3.

6 As the vectors are perpendiculars,

$$\begin{aligned} \begin{bmatrix} -4 \\ t+2 \\ t \end{bmatrix} \bullet \begin{bmatrix} t \\ 1+t \\ -3 \end{bmatrix} &= 0 \\ \therefore \quad 4t + (t+2)(1+t) - 3t &= 0 \\ \therefore \quad -4t + t + t^2 + 2 + 2t - 3t &= 0 \\ \therefore \quad t^2 - 4t + 2 &= 0 \end{aligned}$$

$$\therefore \quad t = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2}$$

$$\therefore \quad t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

7 If θ is the angle then $\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \sqrt{4+16+9}\sqrt{1+1+9}\cos\theta$

$$\therefore \quad -2 - 4 + 9 = \sqrt{29}\sqrt{11}\cos\theta$$

$$\therefore \quad \frac{3}{\sqrt{29 \times 11}} = \cos\theta \quad \text{and so} \quad \theta \doteqdot 80.3^o$$

8 If D is the origin, DA the x-axis, DC the y-axis and DE the z-axis, then A is (4, 0, 0), C is (0, 8, 0) and G is (4, 8, 5)

$\overrightarrow{AG} = \begin{bmatrix} 4 - 4 \\ 8 - 0 \\ 5 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix}$ $\quad \overrightarrow{AC} = \begin{bmatrix} 0 - 4 \\ 8 - 0 \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix}$

If the required angle is θ then $\begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix} = \sqrt{0+64+25}\sqrt{16+64+0}\cos\theta$

$$\therefore \quad 0 + 64 + 0 = \sqrt{89}\sqrt{80}\cos\theta$$

$$\therefore \quad \cos\theta = \frac{64}{\sqrt{89 \times 80}} \quad \text{and so} \quad \theta \doteqdot 40.7^o$$

9 **a** $\overrightarrow{PQ} = \begin{bmatrix} -4 - 2 \\ 4 - 3 \\ 2 - -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}$ **b** PQ $= |\,\overrightarrow{PQ}\,| = \sqrt{36+1+9} = \sqrt{46}$ units

c M is $\left(\dfrac{2 + -4}{2}, \dfrac{3+4}{2}, \dfrac{-1+2}{2}\right)$ i.e., $(-1, \frac{7}{2}, \frac{1}{2})$

10 **a** $\mathbf{p} \bullet \mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = -3 - 2 + 4 = -1$

b $\mathbf{p} + 2\mathbf{q} - \mathbf{r} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$

c $\mathbf{p} \bullet \mathbf{r} = |\mathbf{p}|\,|\mathbf{r}|\cos\theta \quad \therefore \quad \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \sqrt{1+4+1}\sqrt{1+1+4}\,\cos\theta$

$$\therefore \quad -1+2+2 = \sqrt{6}\sqrt{6}\cos\theta$$
$$\therefore \quad 3 = 6\cos\theta$$
$$\therefore \quad \cos\theta = \tfrac{1}{2}$$
$$\therefore \quad \theta = 60^o$$

11

$$\overrightarrow{\text{KL}} = \begin{bmatrix} -2-3 \\ 1-1 \\ 3-4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix} \qquad \overrightarrow{\text{LK}} = \begin{bmatrix} 3--2 \\ 1-1 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\overrightarrow{\text{KM}} = \begin{bmatrix} 4-3 \\ 1-1 \\ 3-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \overrightarrow{\text{LM}} = \begin{bmatrix} 4--2 \\ 1-1 \\ 3-3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{\text{KL}} \bullet \overrightarrow{\text{KM}} = |\,\overrightarrow{\text{KL}}\,||\,\overrightarrow{\text{KM}}\,|\cos K$$
$$\therefore \quad -5+0+1 = \sqrt{25+0+1}\sqrt{1+0+1}\cos K$$
$$\therefore \quad -4 = \sqrt{26}\sqrt{2}\cos K$$
$$\therefore \quad \cos K = -\tfrac{4}{\sqrt{52}}$$
$$\therefore \quad K \doteqdot 123.7^o$$
$$\text{and} \quad M \doteqdot 180^o - 123.7^o - 11.3^o \doteqdot 45.0^o$$

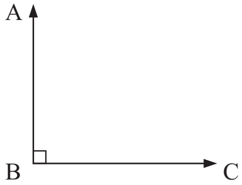
$$\therefore \quad \overrightarrow{\text{LK}} \bullet \overrightarrow{\text{LM}} = |\,\overrightarrow{\text{KL}}\,||\,\overrightarrow{\text{KM}}\,|\cos L$$
$$\therefore \quad 30+0+0 = \sqrt{25+0+1}\sqrt{36+0+0}\cos L$$
$$\therefore \quad 30 = \sqrt{26} \times 6\cos L$$
$$\therefore \quad \tfrac{5}{\sqrt{26}} = \cos L$$
$$\therefore \quad L \doteqdot 11.3^o$$

and $M \doteqdot 180^o - 123.7^o - 11.3^o \doteqdot 45.0^o$

12 If the angle is θ then, $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \sqrt{9+1+4}\ \sqrt{4+25+1}\ \cos\theta$

$$\therefore \quad 6+5-2 = \sqrt{14}\sqrt{30}\cos\theta$$
$$\therefore \quad \frac{9}{\sqrt{14 \times 30}} = \cos\theta$$
$$\therefore \quad \theta \doteqdot 63.95^o$$

13



$$\overrightarrow{\text{BA}} = \begin{bmatrix} 4--1 \\ 2-5 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -3 \end{bmatrix}$$

$$\overrightarrow{\text{BC}} = \begin{bmatrix} 3--1 \\ -3-5 \\ c-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ c-2 \end{bmatrix}$$

$$\text{But} \quad \overrightarrow{\text{BA}} \bullet \overrightarrow{\text{BC}} = 0$$
$$\therefore \quad 20+24-3(c-2) = 0$$
$$\therefore \quad 44 = 3(c-2)$$
$$\therefore \quad 3c-6 = 44$$
$$\therefore \quad 3c = 50$$
$$\therefore \quad c = \tfrac{50}{3}$$

14 **a** $\mathbf{a} \bullet \mathbf{b}$ is a scalar, so in $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ we would be trying to find the scalar product of a scalar and a vector which is impossible.

b $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}$ does not need a bracket about the $\mathbf{b} \times \mathbf{c}$ as this must be performed first before it can be 'dotted' with $\mathbf{a}$.

15 **a** Is a unit vector if

$$\sqrt{\left(\tfrac{4}{7}\right)^2 + \left(\tfrac{1}{k}\right)^2} = 1$$
$$\therefore \quad \tfrac{16}{49} + \frac{1}{k^2} = 1$$
$$\therefore \quad \frac{1}{k^2} = \tfrac{33}{49}$$
$$\therefore \quad k = \pm\tfrac{7}{\sqrt{33}}$$

b Is a unit vector if

$$\sqrt{k^2+k^2} = 1$$
$$\therefore \quad 2k^2 = 1$$
$$\therefore \quad k^2 = \tfrac{1}{2}$$
$$\therefore \quad k = \pm\tfrac{1}{\sqrt{2}}$$

REVIEW SET 15C

1 **a**

$$\begin{aligned}&\mathbf{p}\bullet\mathbf{q}\\&=\begin{bmatrix}3\\-2\end{bmatrix}\bullet\begin{bmatrix}-1\\5\end{bmatrix}\\&=-3+(-10)\\&=-13\end{aligned}$$

b

$$\begin{aligned}&\mathbf{p}-\mathbf{r}\\&=\begin{bmatrix}3\\-2\end{bmatrix}-\begin{bmatrix}-3\\4\end{bmatrix}\\&=\begin{bmatrix}6\\-6\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\therefore\ \ &\mathbf{q}\bullet(\mathbf{p}-\mathbf{r})\\&=\begin{bmatrix}-1\\5\end{bmatrix}\bullet\begin{bmatrix}6\\-6\end{bmatrix}\\&=-6-30\\&=-36\end{aligned}$$

2

$$\begin{aligned}\text{LHS}\ &=\mathbf{p}\bullet(\mathbf{q}-\mathbf{r})\\&=\begin{bmatrix}3\\-2\end{bmatrix}\bullet\left(\begin{bmatrix}-2\\5\end{bmatrix}-\begin{bmatrix}1\\-3\end{bmatrix}\right)\\&=\begin{bmatrix}3\\-2\end{bmatrix}\bullet\begin{bmatrix}-3\\8\end{bmatrix}\\&=-9-16\\&=-25\end{aligned}$$

$$\begin{aligned}\text{RHS}\ &=\mathbf{p}\bullet\mathbf{q}-\mathbf{p}\bullet\mathbf{r}\\&=\begin{bmatrix}3\\-2\end{bmatrix}\bullet\begin{bmatrix}-2\\5\end{bmatrix}-\begin{bmatrix}3\\-2\end{bmatrix}\bullet\begin{bmatrix}1\\-3\end{bmatrix}\\&=(-6-10)-(3+6)\\&=-16-9\\&=-25\qquad\therefore\ \ \text{LHS}=\text{RHS}\ \ \checkmark\end{aligned}$$

3 Since they are perpendicular

$$\begin{aligned}\begin{bmatrix}3\\3-2t\end{bmatrix}\bullet\begin{bmatrix}t^2+t\\-2\end{bmatrix}&=0\\\therefore\ \ 3(t^2+t)-2(3-2t)&=0\\\therefore\ \ 3t^2+3t-6+4t&=0\\\therefore\ \ 3t^2+7t-6&=0\\\therefore\ \ (3t-2)(t+3)&=0\\\therefore\ \ t&=\tfrac{2}{3}\text{ or }-3\end{aligned}$$

4

$$\begin{aligned}\overrightarrow{\text{AB}}&=\begin{bmatrix}-1-2\\4-3\end{bmatrix}=\begin{bmatrix}-3\\1\end{bmatrix}\\\overrightarrow{\text{AC}}&=\begin{bmatrix}3-2\\k-3\end{bmatrix}=\begin{bmatrix}1\\k-3\end{bmatrix}\\\text{Now}\ \ \overrightarrow{\text{AB}}\bullet\overrightarrow{\text{AC}}&=0\qquad\{\text{as }\measuredangle\text{BAC}=90^o\}\\\therefore\ \ \begin{bmatrix}-3\\1\end{bmatrix}\bullet\begin{bmatrix}1\\k-3\end{bmatrix}&=0\\\therefore\ \ -3+k-3&=0\\\therefore\ \ k&=6\end{aligned}$$

5 One vector perpendicular to $\begin{bmatrix}-4\\5\end{bmatrix}$ is $\begin{bmatrix}5\\4\end{bmatrix}$ as dot product $=-20+20=0$

$\therefore$ all vectors have form $k\begin{bmatrix}5\\4\end{bmatrix}$, $k\neq 0$

6

$$\begin{aligned}\overrightarrow{\text{KL}}&=\begin{bmatrix}3--2\\2-1\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}\\\overrightarrow{\text{KM}}&=\begin{bmatrix}1--2\\-3-1\end{bmatrix}=\begin{bmatrix}3\\-4\end{bmatrix}\\\text{Now}\ \ \overrightarrow{\text{KL}}\bullet\overrightarrow{\text{KM}}&=|\,\overrightarrow{\text{KL}}\,||\,\overrightarrow{\text{KM}}\,|\cos K\\\therefore\ \ \begin{bmatrix}5\\1\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}&=\sqrt{25+1}\sqrt{9+16}\cos K\\\therefore\ \ 15-4&=\sqrt{26}\sqrt{25}\cos K\\\therefore\ \ \frac{11}{5\sqrt{26}}&=\cos K\\\therefore\ \ K&=\cos^{-1}\left(\frac{11}{5\sqrt{26}}\right)\\\therefore\ \ K&=64.4^o\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{LK}}&=-\overrightarrow{\text{KL}}=\begin{bmatrix}-5\\-1\end{bmatrix}\\\overrightarrow{\text{LM}}&=\begin{bmatrix}1-3\\-3-2\end{bmatrix}=\begin{bmatrix}-2\\-5\end{bmatrix}\\\text{Now}\ \ \overrightarrow{\text{LK}}\bullet\overrightarrow{\text{LM}}&=|\,\overrightarrow{\text{LK}}\,||\,\overrightarrow{\text{LM}}\,|\cos L\\\therefore\ \ \begin{bmatrix}-5\\-1\end{bmatrix}\begin{bmatrix}-2\\-5\end{bmatrix}&=\sqrt{25+1}\sqrt{4+25}\cos L\\\therefore\ \ 10+5&=\sqrt{26}\sqrt{29}\cos L\\\therefore\ \ \cos L&=\frac{15}{\sqrt{26\times 29}}\\\therefore\ \ L&\doteqdot 56.9^o\\\text{and}\ \ \measuredangle\text{M}&\doteqdot 180^o-56.89^o-64.44^o\doteqdot 58.7^o\end{aligned}$$

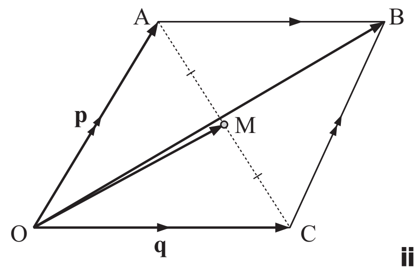
7 $4x-5y=11$ has gradient $\frac{4}{5}$ $\quad\therefore$ direction vector $\begin{bmatrix}5\\4\end{bmatrix}$

$2x+3y=7$ has gradient $-\frac{2}{3}$ $\quad\therefore$ direction vector is $\begin{bmatrix}3\\-2\end{bmatrix}$

If the angle is θ,

$$\begin{aligned}\begin{bmatrix}5\\4\end{bmatrix}\bullet\begin{bmatrix}3\\-2\end{bmatrix}&=\sqrt{5^2+4^2}\sqrt{3^2+(-2)^2}\cos\theta\\\therefore\ \ 15-8&=\sqrt{41}\sqrt{13}\cos\theta\\\therefore\ \ \frac{7}{\sqrt{41\times 13}}&=\cos\theta\\\therefore\ \ \theta&\doteqdot 72.3\qquad\therefore\ \ \text{the angle is }72.3^o\text{ (or }107.7^o)\end{aligned}$$

8 **a**



i (1)
$$\begin{aligned}&\overrightarrow{OB}\\&= \overrightarrow{OA} + \overrightarrow{AB}\\&= \overrightarrow{OA} + \overrightarrow{OC}\\&= \mathbf{p} + \mathbf{q}\end{aligned}$$

(2)
$$\begin{aligned}&\overrightarrow{OM}\\&= \overrightarrow{OA} + \overrightarrow{AM}\\&= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AC}\\&= \mathbf{p} + \tfrac{1}{2}(\overrightarrow{AO} + \overrightarrow{OC})\\&= \mathbf{p} + \tfrac{1}{2}(-\mathbf{p} + \mathbf{q})\\&= \mathbf{p} - \tfrac{1}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{q}\\&= \tfrac{1}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{q}\end{aligned}$$

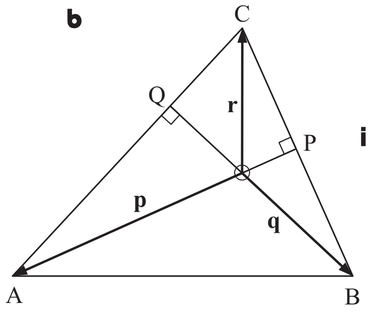
ii We notice that $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$

$\therefore$ $\overline{OM} \parallel \overline{OB}$ and $OM = \frac{1}{2}(OB)$

i.e., O, M and B are collinear (as O is common)

and $\therefore$ M is the midpoint of $\overline{OB}$.

b



i
$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} & \overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\&= -\mathbf{p} + \mathbf{r} & &= -\,\mathbf{q} + \mathbf{r}\\&= \mathbf{r} - \mathbf{p} & &= \mathbf{r} - \mathbf{q}\end{aligned}$$

ii
$$\begin{array}{cc}AP \perp BC & \text{and} \quad BQ \perp AC\\ \therefore\ \mathbf{p} \perp \mathbf{r} - \mathbf{q} & \therefore\ \mathbf{q} \perp (\mathbf{r} - \mathbf{p})\\ \therefore\ \mathbf{p} \bullet (\mathbf{r} - \mathbf{q}) = 0 & \therefore\ \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) = 0\\ \therefore\ \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0 & \therefore\ \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0\\ \therefore\ \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q} & \therefore\ \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}\end{array}$$

iii
$$\begin{aligned}\mathbf{r} \bullet \overrightarrow{AB} &= \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q})\\&= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}\\&= -\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q} \qquad \{\text{from } \textbf{ii}\}\\&= 0 \qquad \text{and so} \quad \mathbf{r} \perp \overrightarrow{AB} \qquad \text{i.e.,} \quad OC \perp AB\end{aligned}$$

9 **a**
$$2\mathbf{a} - 3\mathbf{b} = 2\begin{bmatrix}2\\-3\\1\end{bmatrix} - 3\begin{bmatrix}-1\\2\\3\end{bmatrix} = \begin{bmatrix}4\\-6\\2\end{bmatrix} - \begin{bmatrix}-3\\6\\9\end{bmatrix} = \begin{bmatrix}7\\-12\\-7\end{bmatrix}$$

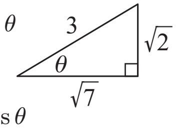
b
$$\mathbf{a} - 3\mathbf{x} = \mathbf{b} \quad \therefore\ \mathbf{a} - \mathbf{b} = 3\mathbf{x} \quad \therefore\ \mathbf{x} = \tfrac{1}{3}(\mathbf{a} - \mathbf{b}) = \tfrac{1}{3}\begin{bmatrix}3\\-5\\-2\end{bmatrix} = \begin{bmatrix}1\\-\frac{5}{3}\\-\frac{2}{3}\end{bmatrix}$$

c The projection of vector **a** on **b**

$$\begin{aligned}&= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}\frac{1}{|\mathbf{b}|}\mathbf{b} \qquad \text{where } \mathbf{a} \bullet \mathbf{b} = \begin{bmatrix}2\\-3\\1\end{bmatrix} \bullet \begin{bmatrix}-1\\2\\3\end{bmatrix} = -2 - 6 + 3 = -5\\&= \frac{-5}{\sqrt{14}}\ \frac{1}{\sqrt{14}}\begin{bmatrix}-1\\2\\3\end{bmatrix}\\&= \tfrac{-5}{14}\begin{bmatrix}-1\\2\\3\end{bmatrix}\end{aligned}$$

10 $|\mathbf{a}| = 3, \quad |\mathbf{b}| = \sqrt{7} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$

a
$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}|\,|\mathbf{b}|\sin\theta \\ \sqrt{1+4+9} &= 3\times\sqrt{7}\sin\theta \\ \sin\theta &= \tfrac{\sqrt{14}}{3\sqrt{7}} = \tfrac{\sqrt{2}}{3} \\ \text{But}\quad \cos^2\theta &= 1-\sin^2\theta \\ \therefore\quad \cos\theta &= \pm\tfrac{\sqrt{7}}{3} \\ \text{Hence}\quad \mathbf{a}\bullet\mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\theta \\ &= 3\times\sqrt{7}\times(\pm\tfrac{\sqrt{7}}{3}) \\ &= \pm 7 \end{aligned}$$



b
$$\begin{aligned} \text{Area}\quad \Delta\text{OAB} &= \tfrac{1}{2}\,|\mathbf{a} \times \mathbf{b}| \\ &= \tfrac{1}{2}\sqrt{14}\ \text{units}^2 \end{aligned}$$

c $V = \frac{1}{6}\,|\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})|$

Now $\mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{a}\times\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$$\begin{aligned} \therefore\quad V &= \tfrac{1}{6}\left|\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right| \\ &= \tfrac{1}{6}\,|1-2-6| \\ &= \tfrac{7}{6}\ \text{units}^3 \end{aligned}$$

REVIEW SET 15D

1 $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ $\quad \mathbf{b} = \mathbf{i} - 2\mathbf{k}$

a
$$\begin{aligned} 3\mathbf{a} - 2\mathbf{b} &= 3[3\mathbf{i} - \mathbf{j} + 2\mathbf{k}] - 2[\mathbf{i} - 2\mathbf{k}] \\ &= 9\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} - 2\mathbf{i} + 4\mathbf{k} \\ &= 7\mathbf{i} - 3\mathbf{j} + 10\mathbf{k} \end{aligned}$$

b
$$\begin{aligned} \mathbf{a} - \mathbf{b} &= 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} - \mathbf{i} + 2\mathbf{k} \\ &= 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} \\ |\mathbf{a} - \mathbf{b}| &= \sqrt{2^2 + (-1)^2 + 4^2} \\ &= \sqrt{21}\ \text{units} \end{aligned}$$

c The vector projection of **b** on **a** has length $= \dfrac{|\mathbf{b} \bullet \mathbf{a}|}{|\mathbf{a}|} = \dfrac{|3+0+(-4)|}{\sqrt{14}} = \frac{1}{\sqrt{14}}$ units

2 Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\quad\therefore\quad k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$

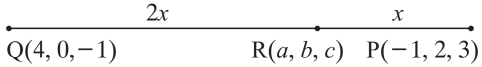
$$\begin{aligned} \therefore\quad |k\mathbf{a}| &= \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2} \\ &= \sqrt{k^2({a_1}^2 + {a_2}^2 + {a_3}^2)} \\ &= \sqrt{k^2}\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} \\ &= |k|\,|\mathbf{a}| \end{aligned}$$

3 Given P(−1, 2, 3) and Q(4, 0, −1)

a $\overrightarrow{\text{PQ}} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix}$

b
$$\begin{aligned} \overrightarrow{\text{PQ}} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= 5 \\ \therefore\quad |\overrightarrow{\text{PQ}}|\sqrt{1^2+0^2+0^2}\cos\alpha &= 5 \\ \therefore\quad \sqrt{25+4+16}\cos\alpha &= 5 \\ \therefore\quad \cos\alpha &= \tfrac{5}{\sqrt{45}} \\ \therefore\quad \cos\alpha &\doteqdot 41.81^o \end{aligned}$$

c

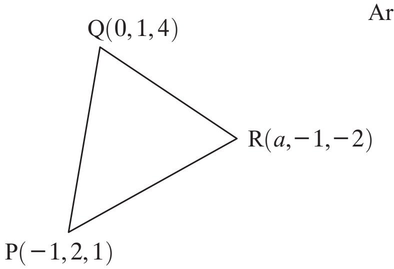


$$\begin{aligned} \overrightarrow{\text{QR}} &= 2\overrightarrow{\text{RP}} \\ [a-4,\ b,\ c+1] &= 2[-1-a,\ 2-b,\ 3-c] \end{aligned}$$

$$\begin{array}{lll} a-4 = -2-2a, & b = 4-2b, & c+1 = 6-2c \\ \therefore\ 3a = 2 & \therefore\ 3b = 4 & \therefore\ 3c = 5 \\ \therefore\ a = \frac{2}{3} & \therefore\ b = \frac{4}{3} & \therefore\ c = \frac{5}{3} \end{array}$$

$\therefore$ R is at $(\frac{2}{3}, \frac{4}{3}, \frac{5}{3})$

4



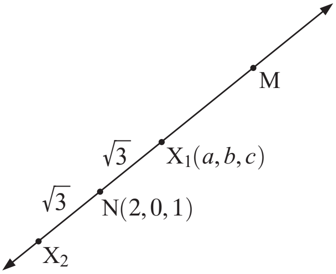
$$\begin{aligned}
\text{Area of } \Delta &= \tfrac{1}{2}\,|\,\overrightarrow{PQ}\times\overrightarrow{PR}\,| \\
&= \tfrac{1}{2}\left|\begin{bmatrix}0--1\\1-2\\4-1\end{bmatrix}\times\begin{bmatrix}a--1\\-1-2\\-2-1\end{bmatrix}\right| \\
&= \tfrac{1}{2}\left|\begin{bmatrix}1\\-1\\3\end{bmatrix}\times\begin{bmatrix}a+1\\-3\\-3\end{bmatrix}\right|
\end{aligned}$$

$$\therefore\ \tfrac{1}{2}\left|\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&3\\a+1&-3&-3\end{bmatrix}\right| = \sqrt{118}$$

$$\therefore\ \left|\begin{vmatrix}-1&3\\-3&-3\end{vmatrix}\mathbf{i}+\begin{vmatrix}3&1\\-3&a+1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&-1\\a+1&-3\end{vmatrix}\mathbf{k}\right| = 2\sqrt{118}$$

$$\begin{aligned}
|12\mathbf{i}\ +\ (3a+3+3)\mathbf{j}\ +\ (-3+a+1)\mathbf{k}| &= 2\sqrt{118} \\
\therefore\ \sqrt{144+(3a+6)^2+(a-2)^2} &= 2\sqrt{118} \\
\therefore\ 144+9a^2+36a+36+a^2-4a+4 &= 472 \\
\therefore\ 10a^2+32a-288 &= 0 \\
\therefore\ 5a^2+16a-144 &= 0 \\
\therefore\ (5a+36)(a-4) &= 0 \\
\therefore\ a &= -\tfrac{36}{5} \quad \text{or} \quad 4
\end{aligned}$$

5 **a**



$$\overrightarrow{MN} = \begin{bmatrix}2--1\\0-3\\1-4\end{bmatrix} = \begin{bmatrix}3\\-3\\-3\end{bmatrix} = 3\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

$$\overrightarrow{NX} = \begin{bmatrix}a-2\\b-0\\c-1\end{bmatrix} \quad \text{and} \quad |\,\overrightarrow{NX}\,| = \sqrt{3}$$

$$\therefore\ \begin{bmatrix}a-2\\b-0\\c-1\end{bmatrix} = \pm\begin{bmatrix}1\\-1\\-1\end{bmatrix} \quad \{\text{as this vector has length } \sqrt{3}\}$$

$\therefore\ a-2=\pm 1,\quad b=\mp 1,\quad c-1=\mp 1$

$\therefore\ a=3$ or 1, $\quad b=-1$ or 1, $\quad c=0$ or 2

$\therefore$ X is at (3, −1, 0) or (1, 1, 2)

b The unit vector in the direction of $\overrightarrow{MN}$ is $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$

$\therefore$ the required vector is $\frac{2}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ i.e., $\begin{bmatrix}\frac{2}{\sqrt{3}}\\-\frac{2}{\sqrt{3}}\\-\frac{2}{\sqrt{3}}\end{bmatrix}$

6 $\overrightarrow{BC} = \begin{bmatrix}-1-2\\-9-0\\4-1\end{bmatrix} = \begin{bmatrix}-3\\-9\\3\end{bmatrix}$ and $\overrightarrow{CA} = \begin{bmatrix}1--1\\-3--9\\2-4\end{bmatrix} = \begin{bmatrix}2\\6\\-2\end{bmatrix}$

$\therefore\ -\frac{1}{3}\overrightarrow{BC} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$ and $\frac{1}{2}\overrightarrow{CA} = \begin{bmatrix}1\\3\\-1\end{bmatrix}$

So, $-\frac{1}{3}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{CA}$

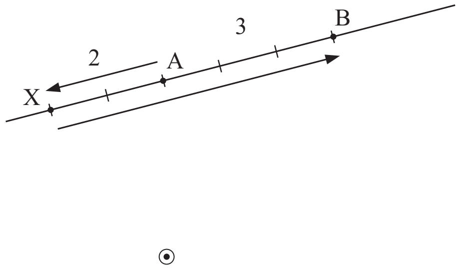
or $\overrightarrow{BC} = -\frac{3}{2}\overrightarrow{CA}$

$\therefore\ \overrightarrow{BC} \parallel \overrightarrow{CA}$ i.e., B, C and A are collinear

For C divides BA we need $\overrightarrow{BC} : \overrightarrow{CA} = -3\begin{bmatrix}1\\3\\-1\end{bmatrix} : 2\begin{bmatrix}1\\3\\-1\end{bmatrix} = -3 : 2$

$\therefore$ C divides BA externally in the ratio $3 : 2$.

7



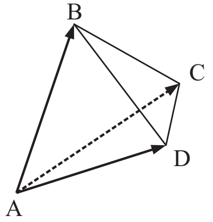
Let X divide AB externally in the ratio $2 : 5$.

$$\therefore\quad \overrightarrow{AX} : \overrightarrow{XB} = -2 : 5$$

$$\begin{aligned}\text{Now}\quad \overrightarrow{OX} &= \overrightarrow{OA} + \overrightarrow{AX}\\ &= \overrightarrow{OA} + \tfrac{2}{3}\overrightarrow{BA}\\ &= \begin{bmatrix}-2\\3\\5\end{bmatrix} + \tfrac{2}{3}\begin{bmatrix}-2-3\\3--1\\5-1\end{bmatrix}\\ &= \begin{bmatrix}-2\\3\\5\end{bmatrix} + \tfrac{2}{3}\begin{bmatrix}-5\\4\\4\end{bmatrix}\\ &= \begin{bmatrix}-2-\frac{10}{3}\\3+\frac{8}{3}\\5+\frac{8}{3}\end{bmatrix} \quad\text{or}\quad \begin{bmatrix}-\frac{16}{3}\\\frac{17}{3}\\\frac{23}{3}\end{bmatrix}\end{aligned}$$

$\therefore$ X is $\left(-\frac{16}{3}, \frac{17}{3}, \frac{23}{3}\right)$

8



$$\overrightarrow{AB} = \begin{bmatrix}-1-3\\2-1\\1-2\end{bmatrix} = \begin{bmatrix}-4\\1\\-1\end{bmatrix}$$

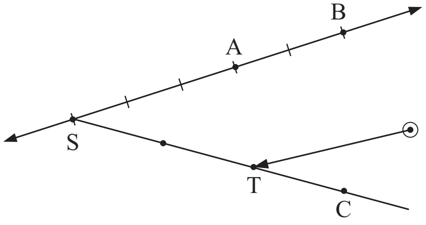
$$\overrightarrow{AC} = \begin{bmatrix}-2-3\\0-1\\3-2\end{bmatrix} = \begin{bmatrix}-5\\-1\\1\end{bmatrix}$$

$$\overrightarrow{AD} = \begin{bmatrix}4-3\\3-1\\-1-2\end{bmatrix} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$$

Volume

$$\begin{aligned}&= \tfrac{1}{6}\ |\ \overrightarrow{AB} \bullet (\overrightarrow{AC} \times \overrightarrow{AD})\ |\\ &= \tfrac{1}{6}\left|\begin{bmatrix}-4 & 1 & -1\\-5 & -1 & 1\\1 & 2 & -3\end{bmatrix}\right| \text{units}^3\\ &= \tfrac{1}{6}\left|-4\begin{vmatrix}-1 & 1\\2 & -3\end{vmatrix} + 1\begin{vmatrix}1 & -5\\-3 & 1\end{vmatrix} - 1\begin{vmatrix}-5 & -1\\1 & 2\end{vmatrix}\right|\\ &= \tfrac{1}{6}\,|-4(1) + 1(-14) - 1(-10+1)|\\ &= \tfrac{1}{6}\,|-9|\\ &= 1\tfrac{1}{2}\ \text{units}^3\end{aligned}$$

9 $\overrightarrow{AS} : \overrightarrow{SB} = -3 : 5$ and $\overrightarrow{CT} : \overrightarrow{TS} = 1 : 2$



$$\begin{aligned}\overrightarrow{OT} &= \overrightarrow{OC} + \overrightarrow{CT}\\ &= \mathbf{c} + \tfrac{1}{3}\overrightarrow{CS}\\ &= \mathbf{c} + \tfrac{1}{3}\left[\overrightarrow{CB} + \overrightarrow{BS}\right]\\ &= \mathbf{c} + \tfrac{1}{3}\left[\overrightarrow{CO} + \overrightarrow{OB}\right] + \tfrac{1}{3} \times \tfrac{5}{2}\overrightarrow{BA}\\ &= \mathbf{c} + \tfrac{1}{3}\left[-\mathbf{c}\ + \mathbf{b}\right] + \tfrac{5}{6}\left[\overrightarrow{BO} + \overrightarrow{OA}\right]\\ &= \mathbf{c} - \tfrac{1}{3}\mathbf{c} + \tfrac{1}{3}\mathbf{b} + \tfrac{5}{6}\left[-\mathbf{b}\ + \mathbf{a}\right]\\ &= \mathbf{c} - \tfrac{1}{3}\mathbf{c} + \tfrac{1}{3}\mathbf{b} - \tfrac{5}{6}\mathbf{b} + \tfrac{5}{6}\mathbf{a}\\ &= \tfrac{5}{6}\mathbf{a} - \tfrac{1}{2}\mathbf{b} + \tfrac{2}{3}\mathbf{c}\end{aligned}$$

10 $$\begin{bmatrix}1\\r\\2\end{bmatrix} \bullet \begin{bmatrix}2\\2\\-1\end{bmatrix} = 0 \qquad \begin{aligned} \therefore \quad 2+2r-2 &= 0\\ \therefore \quad 2r &= 0\\ \therefore \quad r &= 0\end{aligned}$$

So, we want a unit vector parallel to $\begin{bmatrix}1\\0\\2\end{bmatrix}$ (which has length $\sqrt{1+0+4}=\sqrt{5}$)

$\therefore$ vectors are $\pm\frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\2\end{bmatrix}$ i.e., $\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{k}$ or $\frac{-1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{k}$

11 $$\begin{aligned} |\mathbf{u}\ \times \mathbf{v}| &= |\mathbf{u}|\,|\mathbf{v}|\sin\theta\\ \therefore \quad \sqrt{1+9+16} &= 3\times 5\times\sin\theta\\ \therefore \quad \tfrac{\sqrt{26}}{15} &= \sin\theta \end{aligned}$$

$$\begin{aligned} \text{But} \quad \cos^2\theta+\sin^2\theta &= 1\\ \therefore \quad \cos^2\theta+\tfrac{26}{225} &= 1\\ \therefore \quad \cos^2\theta &= \tfrac{199}{225}\\ \therefore \quad \cos\theta &= \pm\tfrac{\sqrt{199}}{15} \end{aligned}$$

So, if θ is acute, $\cos\theta=\frac{\sqrt{149}}{15}$ and $\mathbf{u}\bullet\mathbf{v}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta=3\times5\times\frac{\sqrt{199}}{15}=\sqrt{199}$

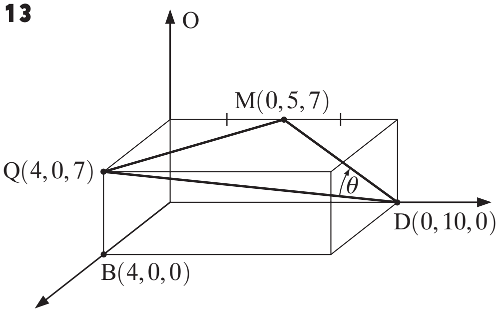
and, if θ is obtuse, $\cos\theta=-\frac{\sqrt{199}}{15}$ and $\mathbf{u}\bullet\mathbf{v}=-\sqrt{199}$

12 3 vectors are coplanar if the volume of the tetrahedron defined by them is 0.

$$\therefore \begin{vmatrix}1&2&-3\\2&2&3\\1&2-t&t+1\end{vmatrix}=0 \quad \text{i.e.,} \quad 1\begin{vmatrix}2&3\\2-t&t+1\end{vmatrix}+2\begin{vmatrix}3&2\\t+1&1\end{vmatrix}-3\begin{vmatrix}2&2\\1&2-t\end{vmatrix}=0$$

$$\begin{aligned} \therefore \quad 1(2t+2-6+3t)+2(3-2t-2)-3(4-2t-2) &= 0\\ \therefore \quad 5t-4+2-4t-6+6t &= 0\\ \therefore \quad 7t &= 8\\ \therefore \quad t &= \tfrac{8}{7} \end{aligned}$$

13



Placing a set of axes with origin at A, as showing gives Q(4, 0, 7), M(0, 5, 7), D(0, 10, 0).

$$\overrightarrow{DQ}=\begin{bmatrix}4-0\\0-10\\7-0\end{bmatrix}=\begin{bmatrix}4\\-10\\7\end{bmatrix}$$

$$\overrightarrow{DM}=\begin{bmatrix}0-0\\5-10\\7-0\end{bmatrix}=\begin{bmatrix}0\\-5\\7\end{bmatrix}$$

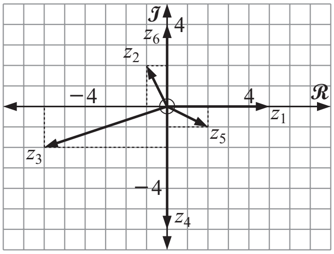
$$\begin{aligned} \overrightarrow{DQ}\bullet\overrightarrow{DM} &= |\,\overrightarrow{DQ}\,||\,\overrightarrow{DM}\,|\cos\theta\\ \therefore \quad 0+50+49 &= \sqrt{16+100+49}\sqrt{0+25+49}\cos\theta\\ \therefore \quad 99 &= \sqrt{165}\sqrt{74}\cos\theta\\ \therefore \quad \cos\theta &= \frac{99}{\sqrt{165\times74}}\\ \therefore \quad \theta &\doteqdot 26.4^o \end{aligned}$$

Chapter 16

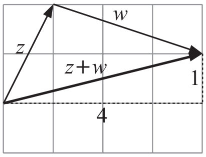
COMPLEX NUMBERS

EXERCISE 16A.1

1

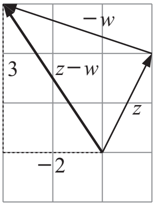


2 **a**



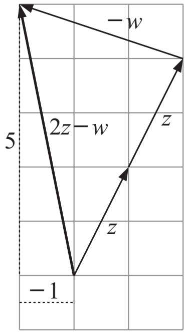
$$\begin{aligned} & z+w \\ &= (1+2i)+(3-i) \\ &= 4+i \end{aligned}$$

b



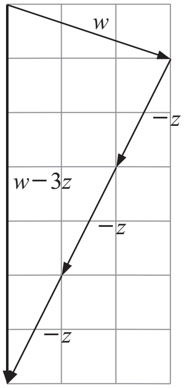
$$\begin{aligned} & z-w \\ &= (1+2i)-(3-i) \\ &= -2+3i \end{aligned}$$

c



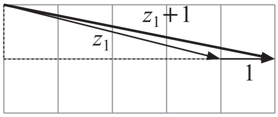
$$\begin{aligned} & 2z-w \\ &= 2(1+2i)-(3-i) \\ &= 2+4i-3+i \\ &= -1+5i \end{aligned}$$

d



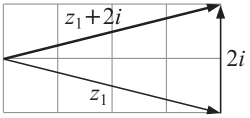
$$\begin{aligned} & w-3z \\ &= (3-i)-3(1+2i) \\ &= 3-i-3-6i \\ &= -7i \end{aligned}$$

3 **a**



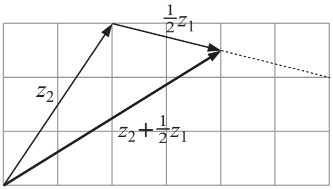
$$\begin{aligned} & z_1+1 \\ &= 4-i+1 \\ &= 5-i \end{aligned}$$

b



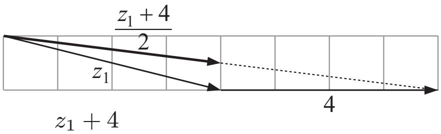
$$\begin{aligned} & z_1+2i \\ &= 4-i+2i \\ &= 4+i \end{aligned}$$

c



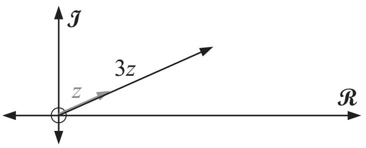
$$\begin{aligned} & z_2+\tfrac{1}{2}z_1 \\ &= (2+3i)+\tfrac{1}{2}(4-i) \\ &= 2+3i+2-\tfrac{1}{2}i \\ &= 4+\tfrac{5}{2}i \end{aligned}$$

d



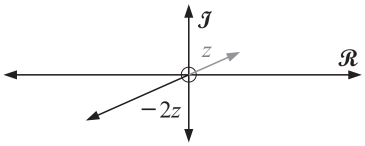
$$\begin{aligned} & \frac{z_1+4}{2} \\ &= \frac{4-i+4}{2} \\ &= 4-\tfrac{1}{2}i \end{aligned}$$

4 **a**



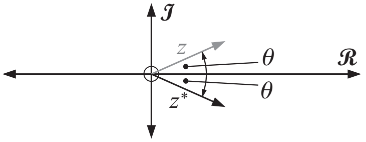
$3z$ is parallel to z and 3 times its length

b



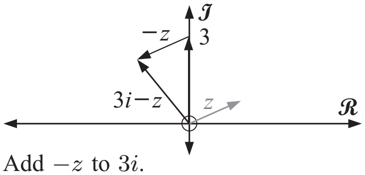
$-2z$ is parallel to z, in the opposite direction and twice its length.

c



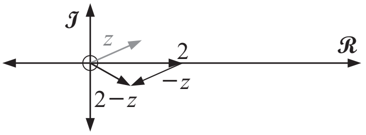
Reflect z in a horizontal line through the start of z.

d



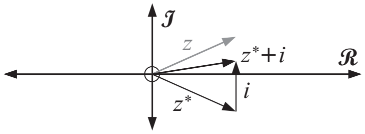
Add $-z$ to $3i$.

e



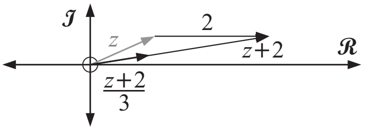
Add $-z$ to 2.

f



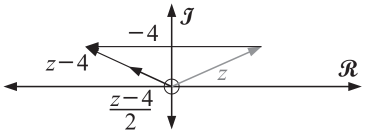
Reflect z in a horizontal line through the start of z and then add i.

g



Add z and 2 and find the vector $\frac{1}{3}$ the length of the result.

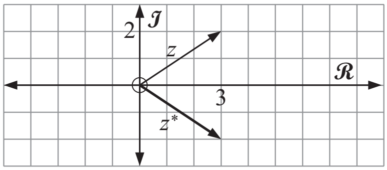
h



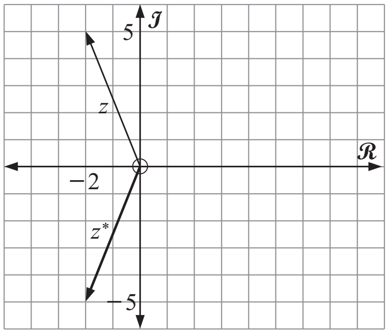
Add z and -4 and find the vector $\frac{1}{2}$ of the length of the result.

EXERCISE 16A.2

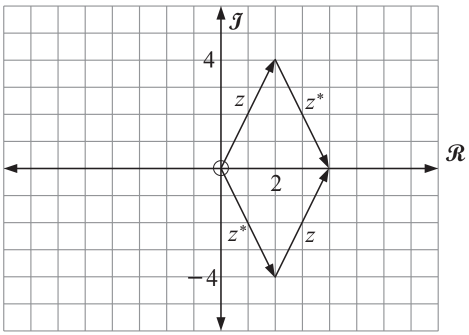
1 **a**



b



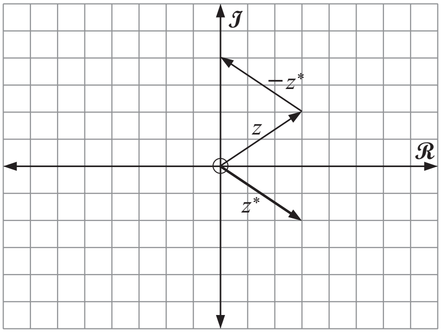
2



The figure formed by $z + z^*$ and $z^* + z$ is a rhombus.

No matter where z is placed, $z + z^*$ lies on the Real axis. i.e., is always real.

3



The resulting vector will always lie along the Imaginary axis.
i.e., will be purely imaginary *or* zero
i.e., when $z = 0$, $z - z^* = 0$.

4 If z is real then z^* will also be real. In fact if $z = a$, $z^* = a$ i.e., $z^* = z$ if z is real.

EXERCISE 16B.1

1 **a**
$$\begin{aligned} &|3-4i| \\ &= \sqrt{3^2+(-4)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

b
$$\begin{aligned} &|5+12i| \\ &= \sqrt{5^2+12^2} \\ &= \sqrt{25+144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

c
$$\begin{aligned} &|-8+2i| \\ &= \sqrt{(-8)^2+2^2} \\ &= \sqrt{64+4} \\ &= \sqrt{68} \\ &= 2\sqrt{17} \end{aligned}$$

d
$$\begin{aligned} &|3i| \\ &= \sqrt{0^2+3^2} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

e
$$\begin{aligned} &|-4| \\ &= \sqrt{(-4)^2+0^2} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

2 **a**
$$\begin{aligned} &|z| \\ &= |2+i| \\ &= \sqrt{2^2+1^2} \\ &= \sqrt{5} \end{aligned}$$

b
$$\begin{aligned} &|z^*| \\ &= |2-i| \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{5} \end{aligned}$$

c
$$\begin{aligned} &|z^*|^2 \\ &= (\sqrt{5})^2 \\ &= 5 \end{aligned}$$

d
$$\begin{aligned} &z \times z^* \\ &= (2+i)(2-i) \\ &= 4-i^2 \\ &= 5 \end{aligned}$$

e
$$\begin{aligned} &|zw| \\ &= |(2+i)(-1+3i)| \\ &= \left|-2+6i-i+3i^2\right| \\ &= |-5+5i| \\ &= \sqrt{(-5)^2+5^2} \\ &= \sqrt{50} \ \text{ or } \ 5\sqrt{2} \end{aligned}$$

f
$$\begin{aligned} &|z|\,|w| \\ &= \sqrt{2^2+1^2}\sqrt{(-1)^2+3^2} \\ &= \sqrt{5} \times \sqrt{10} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

g
$$\begin{aligned} &\left|\frac{z}{w}\right| \\ &= \left|\frac{2+i}{-1+3i}\right| \\ &= \left|\frac{(2+i)}{(-1+3i)} \times \frac{(-1-3i)}{(-1-3i)}\right| \\ &= \left|\frac{-2-6i-i-3i^2}{(-1)^2-(3i)^2}\right| \\ &= \left|\frac{-2+3-7i}{10}\right| \\ &= \left|\tfrac{1}{10}-\tfrac{7}{10}i\right| \\ &= \sqrt{\left(\tfrac{1}{10}\right)^2+\left(\tfrac{-7}{10}\right)^2} \\ &= \sqrt{\tfrac{1+49}{100}} \\ &= \sqrt{\tfrac{50}{100}} \\ &= \tfrac{1}{\sqrt{2}} \end{aligned}$$

h
$$\begin{aligned} &\frac{|z|}{|w|} \\ &= \tfrac{\sqrt{5}}{\sqrt{10}} \\ &= \sqrt{\tfrac{5}{10}} \\ &= \tfrac{1}{\sqrt{2}} \end{aligned}$$

i
$$\begin{aligned} z^2 &= (2+i)^2 \\ &= 4+4i+i^2 \\ &= 3+4i \\ \therefore \quad \left|z^2\right| &= \sqrt{3^2+4^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

j $|z|^2$
$= (\sqrt{5})^2$ {from **a**}
$= 5$

k $z^3 = z^2 \times z$
$= (3+4i)(2+i)$
$= 6+3i+8i+4i^2$
$= 2+11i$
$\therefore \quad \left|z^3\right| = \sqrt{2^2+11^2}$
$= \sqrt{4+121}$
$= \sqrt{125}$
$= 5\sqrt{5}$

l $|z|^3$
$= (\sqrt{5})^3$ {from **a**}
$= \sqrt{125}$
$= 5\sqrt{5}$

3 **Rules for Modulus**

(1) $z \times z^* = |z|^2$ **(2)** $|z^*| = |z|$ **(3)** $|zw| = |z|\,|w|$

(4) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ **(5)** $|z^n| = |z|^n$

4 **a** Let $z = a+bi$
$\therefore \quad |z^*| = |a-bi|$
$= \sqrt{a^2+(-b)^2}$
$= \sqrt{a^2+b^2}$
$= |a+bi|$
$= |z|$ as required

Let $z = a+bi$
$\therefore \quad z \times z^* = (a+bi)(a-bi)$
$= a^2 - b^2 i^2$
$= a^2 + b^2$
$= (\sqrt{a^2+b^2})^2$
$= |z|^2$ as required

5 **a** $|z_1 z_2 z_3| = |(z_1 z_2) z_3|$
$= |z_1 z_2|\,|z_3|$ {as $|zw| = |z|\,|w|$}
$= |z_1|\,|z_2|\,|z_3|$ {$|zw| = |z|\,|w|$ again}

Letting $z_1 = z_2 = z_3 = z$ we get the result that $\left|z^3\right| = |z \times z \times z|$
$= |z| \times |z| \times |z|$
$= |z|^3$

b Now extending this result by the same argument

$|z_1 z_2 z_3 z_4| = |(z_1 z_2 z_3) z_4|$
$= |z_1 z_2 z_3|\,|z_4|$
$= |z_1|\,|z_2|\,|z_3|\,|z_4|$

and putting $z_1 = z_2 = z_3 = z_4 = z$ we get
$\left|z^4\right| = |z \times z \times z \times z| = |z| \times |z| \times |z| \times |z| = |z|^4$

6 The generalisation of **5** is: $|z_1 z_2 z_3 ......z_n| = |z_1|\,|z_2|\,|z_3|\,.....\,|z_n|$ and that $|z^n| = |z|^n$

7 $\left|\dfrac{z}{w}\right| \times |w| = \left|\dfrac{z}{w} \times w\right|$ {using $|z_1|\,|z_2| = |z_1 z_2|$}
$= |z|$

i.e., $\left|\dfrac{z}{w}\right| \times |w| = |z|$

$\therefore \quad \left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ provided $w \neq 0$ {dividing both sides by $|w|$}

8 **a** $|2z|$
$= |2|\,|z|$
$= 2 \times 3$
$= 6$

b $|-3z|$
$= |-3|\,|z|$
$= 3 \times 3$
$= 9$

c $|(1+2i)z|$
$= |1+2i| \times |z|$
$= \sqrt{1+4} \times 3$
$= 3\sqrt{5}$

d $$\begin{aligned}&|iz|\\&=|i|\,|z|\\&=1\times 3\\&=3\end{aligned}$$

e $$\begin{aligned}&\left|\frac{1}{z}\right|\\&=\frac{|1|}{|z|}\\&=\tfrac{1}{3}\end{aligned}$$

f $$\begin{aligned}&\left|\frac{2i}{z^2}\right|\\&=\frac{|2i|}{|z|^2}\\&=\frac{2}{3^2}\\&=\tfrac{2}{9}\end{aligned}$$

9 $z=\cos\theta+i\sin\theta$, $\therefore$ $$\begin{aligned}|z|&=\sqrt{\cos^2\theta+\sin^2\theta}\\&=\sqrt{1}\\&=1\end{aligned}$$

10 If $z=1-i\sqrt{3}$ then $$\begin{aligned}|z|&=\sqrt{1^2+(-\sqrt{3})^2}\\&=\sqrt{4}\\&=2\end{aligned}$$ $\therefore$ $$\begin{aligned}\left|z^{20}\right|&=|z|^{20}\\&=2^{20}\\&=1\,048\,576\end{aligned}$$

11 **a** $w=\dfrac{z+1}{z-1}$. Let $z=a+ib$, $\therefore$ $$\begin{aligned}w&=\frac{a+bi+1}{a+bi-1}\\&=\frac{(a+1)+bi}{(a-1)+bi}\\&=\frac{(a+1)+bi}{(a-1)+bi}\times\frac{(a-1)-bi}{(a-1)-bi}\\&=\frac{(a+1)(a-1)-b(a+1)i+b(a-1)i-b^2i^2}{(a-1)^2-(bi)^2}\\&=\frac{a^2-1-abi-bi+abi-bi+b^2}{(a-1)^2+b^2}\\&=\left(\frac{a^2+b^2-1}{(a-1)^2+b^2}\right)+\left(\frac{-2b}{(a-1)^2+b^2}\right)i\end{aligned}$$

b $\therefore$ $\text{Re}\,(w)=\dfrac{a^2+b^2-1}{(a-1)^2+b^2}=\dfrac{a^2+b^2-1}{a^2-2a+1+b^2}=\dfrac{a^2+b^2-1}{a^2+b^2-2a+1}$

Since $|z|=1$ $\sqrt{a^2+b^2}=1$ $\therefore$ $a^2+b^2=1$

$\therefore$ $\text{Re}(w)=\dfrac{1-1}{1-2a+1}=0$ provided $a\neq 1$

if $a=1$, then $\text{Re}(w)$ is undefined

EXERCISE 16B.2

1 **a** A(3, 6) B(−1, 2) $z=3+6i$ $w=-1+2i$

i $$\begin{aligned}z-w&=(3+6i)-(-1+2i)\\&=4+4i\\|z-w|&=\sqrt{4^2+4^2}\\&=\sqrt{32}\\&=4\sqrt{2}\end{aligned}$$ $\therefore$ AB $=4\sqrt{2}$ units

ii $$\begin{aligned}\frac{z+w}{2}&=\frac{(3+6i)+(-1+2i)}{2}\\&=\frac{2+8i}{2}\\&=1+4i,\end{aligned}$$ and so M is at (1, 4)

b A(−4, 7) B(1, −3) $z = -4 + 7i \quad w = 1 - 3i$

i
$$\begin{aligned} z - w &= (-4+7i) - (1-3i) \\ &= -5 + 10i \\ |z - w| &= \sqrt{5^2 + 10^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \\ \therefore \quad \text{AB} &= 5\sqrt{5} \text{ units} \end{aligned}$$

ii
$$\begin{aligned} \frac{z+w}{2} &= \frac{(-4+7i)+(1-3i)}{2} \\ &= \frac{-3+4i}{2} \\ &= -\tfrac{3}{2} + 2i, \quad \text{and so M is at } \left(-\tfrac{3}{2}, 2\right) \end{aligned}$$

2 **a** **i** $\overrightarrow{\text{OQ}} = z + w$ **ii** $\overrightarrow{\text{PR}} = w - z$

b In ΔOPQ $|z+w|$ represents the length of OQ, $|z| =$ length of OP and $|w|$ the length of PQ. Now if w, z are not parallel we will form the ΔOPQ and this means OQ < OP + PQ

i.e., $|z+w| < |z| + |w|$

if w and z are parallel then we form a straight line and OQ = OP + PQ

i.e., $|z+w| = |z| + |w|$

Consequently $|z+w| \leqslant |z| + |w|$

c In ΔOPR, the length of RP is represented by $|z-w|$ provided w and z not parallel we form a triangle and OR + OP > OR

$|z-w| + |z| > |w|$

$\therefore \quad |z-w| > |w| - |z|$

Equality will occur when w is parallel to z, in which case a straight line OPR is formed and

$|z-w| = |w| - |z|$

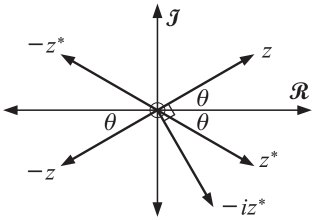
Consequently $|z-w| \geqslant |w| - |z|$

3 **a** $z \to z^*$. Reflection in the Real Axis.

b $z \to -z^*$. Reflection in the Imaginary Axis.

c $z \to -z$. Rotation about O, anticlockwise π.

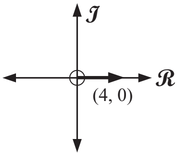
d $z \to -iz$. Rotation about O, clockwise $\frac{\pi}{2}$.



EXERCISE 16B.3

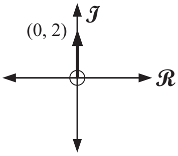
1 **a** $z = 4$

$\arg z = 0$
$|z| = 4$
$\therefore \quad z = 4 \text{ cis } 0$



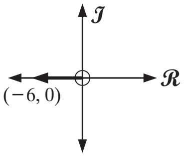
b $z = 2i$

$\arg z = \frac{\pi}{2}$
$|z| = 2$
$\therefore \quad z = 2 \text{ cis } \left(\frac{\pi}{2}\right)$



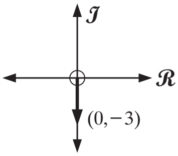
c $z = -6$

$\arg z = \pi$
$|z| = 6$
$\therefore \quad z = 6 \text{ cis } \pi$



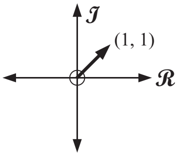
d $z = -3i$

$\arg z = -\frac{\pi}{2}$
$|z| = 3$
$\therefore \quad z = 3 \text{ cis } \left(-\frac{\pi}{2}\right)$



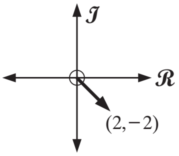
e $z = 1 + i$

$$\begin{aligned} \arg z &= \tfrac{\pi}{4} \\ |z| &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \\ \therefore \quad z &= \sqrt{2}\ \text{cis}\left(\tfrac{\pi}{4}\right) \end{aligned}$$



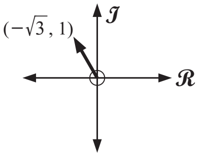
f $z = 2 - 2i$

$$\begin{aligned} \arg z &= -\tfrac{\pi}{4} \\ |z| &= \sqrt{2^2 + 2^2} \\ &= 2\sqrt{2} \\ \therefore \quad z &= 2\sqrt{2}\ \text{cis}\left(-\tfrac{\pi}{4}\right) \end{aligned}$$



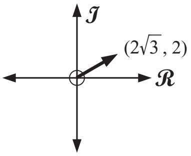
g $z = -\sqrt{3} + i,$

$$\begin{aligned} \arg z &= \tfrac{5\pi}{6} \\ |z| &= \sqrt{(-\sqrt{3})^2 + 1^2} \\ &= \sqrt{4} \\ &= 2 \\ \therefore \quad z &= 2\ \text{cis}\left(\tfrac{5\pi}{6}\right) \end{aligned}$$



h $z = 2\sqrt{3} + 2i$

$$\begin{aligned} \arg z &= \tfrac{\pi}{6} \\ |z| &= \sqrt{12 + 4} \\ &= \sqrt{16} \\ &= 4 \\ \therefore \quad z &= 4\ \text{cis}\left(\tfrac{\pi}{6}\right) \end{aligned}$$

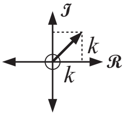


2 $z = 0$, i.e., $0 + 0i$ cannot be written in polar form. The vector representing $\overrightarrow{\text{OP}}$ has length zero, i.e., is the origin and an argument is not defined (i.e., no angle can be formed with the positive x-axis).

3 If $k = 0$ it is not possible

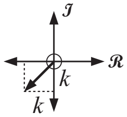
If $k > 0$

$$\begin{aligned} |z| &= \sqrt{k^2 + k^2} \\ &= k\sqrt{2} \\ \arg z &= \tfrac{\pi}{4} \\ \therefore \quad z &= k\sqrt{2}\ \text{cis}\left(\tfrac{\pi}{4}\right) \end{aligned}$$



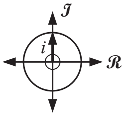
If $k < 0$

$$\begin{aligned} |z| &= \sqrt{k^2 + k^2} \\ &= |k|\sqrt{2} \\ \text{Since } k &< 0 \\ |z| &= -k\sqrt{2} \\ \arg z &= -\tfrac{3\pi}{4} \\ \therefore \quad z &= -k\sqrt{2}\ \text{cis}\left(-\tfrac{3\pi}{4}\right) \end{aligned}$$



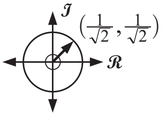
4 **a**

$$\begin{aligned} &2\ \text{cis}\left(\tfrac{\pi}{2}\right) \\ &= 2(\cos\left(\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\pi}{2}\right)) \\ &= 2i \end{aligned}$$



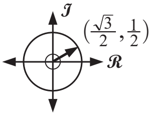
b

$$\begin{aligned} &8\ \text{cis}\left(\tfrac{\pi}{4}\right) \\ &= 8\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right) \\ &= 8\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) \\ &= 4\sqrt{2} + 4\sqrt{2}i \end{aligned}$$



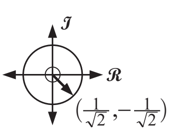
c

$$\begin{aligned} &4\ \text{cis}\left(\tfrac{\pi}{6}\right) \\ &= 4\left(\cos\left(\tfrac{\pi}{6}\right) + i\sin\left(\tfrac{\pi}{6}\right)\right) \\ &= 4\left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right) \\ &= 2\sqrt{3} + 2i \end{aligned}$$



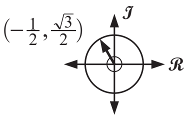
d

$$\begin{aligned} &\sqrt{2}\ \text{cis}\left(-\tfrac{\pi}{4}\right) \\ &= \sqrt{2}(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)) \\ &= \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right) \\ &= 1 - i \end{aligned}$$



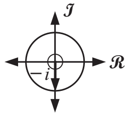
e

$$\begin{aligned} &\sqrt{3}\ \text{cis}\left(\tfrac{2\pi}{3}\right) \\ &= \sqrt{3}\left(\cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right)\right) \\ &= \sqrt{3}\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) \\ &= -\tfrac{\sqrt{3}}{2} + \tfrac{3}{2}i \end{aligned}$$



f

$$\begin{aligned} &5\ \text{cis}\,\pi \\ &= 5(\cos\pi + i\sin\pi) \\ &= 5(-1) \\ &= -5 \end{aligned}$$



5 **a**
$$\begin{aligned}&\text{cis } 0\\ &= \cos 0 + i\sin 0\\ &= 1\end{aligned}$$

b
$$\begin{aligned}&|\text{cis } \theta|\\ &= |\cos\theta + i\sin\theta|\\ &= \sqrt{\cos^2\theta + \sin^2\theta}\\ &= \sqrt{1}\\ &= 1\end{aligned}$$

c
$$\begin{aligned}\text{cis } \alpha \times \text{cis } \beta &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\\ &= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta\\ &= [\cos\alpha\cos\beta - \sin\alpha\sin\beta] + i\,[\sin\alpha\cos\beta + \sin\beta\cos\alpha]\\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta)\\ &= \text{cis } (\alpha+\beta)\end{aligned}$$

EXERCISE 16B.4

1 **a**
$$\begin{aligned}&\text{cis } \theta \text{ cis } 2\theta\\ &= \text{cis } (\theta + 2\theta)\\ &= \text{cis } 3\theta\end{aligned}$$

b
$$\begin{aligned}&\frac{\text{cis } 3\theta}{\text{cis } \theta}\\ &= \text{cis } (3\theta - \theta)\\ &= \text{cis } 2\theta\end{aligned}$$

c
$$\begin{aligned}&[\text{cis } \theta]^3\\ &= (\text{cis } \theta)(\text{cis } \theta)(\text{cis } \theta)\\ &= (\text{cis } 2\theta)(\text{cis } \theta)\\ &= \text{cis } 3\theta\end{aligned}$$

d
$$\begin{aligned}&\text{cis } \left(\tfrac{\pi}{18}\right) \times \text{cis } \left(\tfrac{\pi}{9}\right)\\ &= \text{cis } \left(\tfrac{\pi}{18} + \tfrac{\pi}{9}\right)\\ &= \text{cis } \left(\tfrac{3\pi}{18}\right)\\ &= \text{cis } \left(\tfrac{\pi}{6}\right)\\ &= \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\end{aligned}$$

e
$$\begin{aligned}&2 \text{ cis } \left(\tfrac{\pi}{12}\right) \text{ cis } \left(\tfrac{\pi}{6}\right)\\ &= 2 \text{ cis } \left(\tfrac{\pi}{12} + \tfrac{\pi}{6}\right)\\ &= 2 \text{ cis } \left(\tfrac{3\pi}{12}\right)\\ &= 2 \text{ cis } \left(\tfrac{\pi}{4}\right)\\ &= 2\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\\ &= \tfrac{2}{\sqrt{2}} + \tfrac{2}{\sqrt{2}}i\\ &= \sqrt{2} + i\sqrt{2}\end{aligned}$$

f
$$\begin{aligned}&2 \text{ cis } \left(\tfrac{2\pi}{5}\right) \times 4 \text{ cis } \left(\tfrac{8\pi}{5}\right)\\ &= 8 \text{ cis } \left(\tfrac{2\pi}{5} + \tfrac{8\pi}{5}\right)\\ &= 8 \text{ cis } 2\pi\\ &= 8\end{aligned}$$

g
$$\begin{aligned}&\frac{4 \text{ cis } \left(\frac{\pi}{12}\right)}{2 \text{ cis } \left(\frac{7\pi}{12}\right)}\\ &= 2 \text{ cis } \left(\tfrac{\pi}{12} - \tfrac{7\pi}{12}\right)\\ &= 2 \text{ cis } \left(-\tfrac{6\pi}{12}\right)\\ &= 2 \text{ cis } \left(-\tfrac{\pi}{2}\right)\\ &= -2i\end{aligned}$$

h
$$\begin{aligned}&\frac{\sqrt{32} \text{ cis } \left(\frac{\pi}{8}\right)}{\sqrt{2} \text{ cis } \left(\frac{-7\pi}{8}\right)}\\ &= \tfrac{\sqrt{32}}{\sqrt{2}} \text{ cis } \left(\tfrac{\pi}{8} - \tfrac{-7\pi}{8}\right)\\ &= \sqrt{16} \text{ cis } \left(\tfrac{8\pi}{8}\right)\\ &= 4 \text{ cis } (\pi)\\ &= -4\end{aligned}$$

i
$$\begin{aligned}&\left[\sqrt{2} \text{ cis } \left(\tfrac{\pi}{8}\right)\right]^4\\ &= \sqrt{2} \text{ cis } \left(\tfrac{\pi}{8}\right) \times \sqrt{2} \text{ cis } \left(\tfrac{\pi}{8}\right) \times \sqrt{2} \text{ cis } \left(\tfrac{\pi}{8}\right) \times \sqrt{2} \text{ cis } \left(\tfrac{\pi}{8}\right)\\ &= (\sqrt{2})^4 \text{ cis } \left(\tfrac{\pi}{8} + \tfrac{\pi}{8} + \tfrac{\pi}{8} + \tfrac{\pi}{8}\right)\\ &= 4 \text{ cis } \left(\tfrac{\pi}{2}\right)\\ &= 4(0 + 1i) \quad \text{i.e.,} \quad 4i\end{aligned}$$

2 **a**
$$\begin{aligned}&\text{cis } 17\pi\\ &= \text{cis } (\pi + 8(2\pi))\\ &= \text{cis } \pi\\ &= -1\end{aligned}$$

b
$$\begin{aligned}&\text{cis } -37\pi\\ &= \text{cis } (\pi - 19(2\pi))\\ &= \text{cis } (\pi)\\ &= -1\end{aligned}$$

c
$$\begin{aligned}&\text{cis } \left(\tfrac{91\pi}{3}\right)\\ &= \text{cis } \left(\tfrac{\pi}{3} + 15(2\pi)\right)\\ &= \text{cis } \left(\tfrac{\pi}{3}\right)\\ &= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\end{aligned}$$

3 **a**
$$\begin{aligned}z &= 2 \text{ cis } \theta\\ |z| &= 2\\ \arg z &= \theta\end{aligned}$$

b $z^* = 2 \text{ cis } (-\theta)$ **c** $-z = 2 \text{ cis } (\theta + \pi)$ **d** $-z^* = 2 \text{ cis } (\pi - \theta)$

4 **a** $i = 1 \text{ cis } \left(\frac{\pi}{2}\right) = \text{cis } \left(\frac{\pi}{2}\right)$

b $\therefore \quad iz = \text{cis } \left(\frac{\pi}{2}\right) \times r \text{ cis } \theta$

$= r \text{ cis } \left(\theta + \frac{\pi}{2}\right)$

c iz has the same length as z and its argument is $\frac{\pi}{2}$ more than the argument of z, i.e., an anticlockwise rotation about O of $\frac{\pi}{2}$.

d $z \to -iz \qquad -iz = \text{cis } \left(-\frac{\pi}{2}\right) \times r \text{ cis } \theta$

$= r \text{ cis } \left(\theta - \frac{\pi}{2}\right)$ i.e., a clockwise rotation of $\frac{\pi}{2}$ about O.

5 **a**
$$\begin{aligned} &\cos\theta - i\sin\theta \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \text{cis } (-\theta) \end{aligned}$$

b
$$\begin{aligned} &\sin\theta - i\cos\theta \\ &= -i\cos\theta - i^2\sin\theta \\ &= -i(\cos\theta + i\sin\theta) \\ &= \text{cis } \left(-\tfrac{\pi}{2}\right) \text{ cis } \theta \\ &= \text{cis } \left(\theta - \tfrac{\pi}{2}\right) \end{aligned}$$

If $z = r \text{ cis } \theta$
then $z^* = r \text{ cis } (-\theta)$

6 **a** Consider
$$\begin{aligned} \cos\left(\tfrac{\pi}{12}\right) + i\sin\left(\tfrac{\pi}{12}\right) &= \text{cis } \left(\tfrac{\pi}{12}\right) \\ &= \text{cis } \left(\tfrac{4\pi}{12} - \tfrac{3\pi}{12}\right) \\ &= \text{cis } \left(\tfrac{\pi}{3} - \tfrac{\pi}{4}\right) \\ &= \frac{\text{cis } \left(\frac{\pi}{3}\right)}{\text{cis } \left(\frac{\pi}{4}\right)} \\ &= \left(\frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}\right) \times \frac{2}{2} \\ &= \left(\frac{1 + i\sqrt{3}}{\sqrt{2} + i\sqrt{2}}\right)\left(\frac{\sqrt{2} - i\sqrt{2}}{\sqrt{2} - i\sqrt{2}}\right) \\ &= \frac{\sqrt{2} - i\sqrt{2} + i\sqrt{6} - i^2\sqrt{6}}{2 + 2} \\ &= \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) + \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)i \end{aligned}$$

$\therefore \quad \cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4} \qquad \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$ {equating real and imaginary parts}

b Consider
$$\begin{aligned} \cos\left(\tfrac{11\pi}{12}\right) + i\sin\left(\tfrac{11\pi}{12}\right) &= \text{cis } \left(\tfrac{11\pi}{12}\right) \\ &= \text{cis } \left(\tfrac{3\pi}{12} + \tfrac{8\pi}{12}\right) \\ &= \text{cis } \left(\tfrac{\pi}{4} + \tfrac{2\pi}{3}\right) \\ &= \text{cis } \left(\tfrac{\pi}{4}\right) \times \text{cis } \left(\tfrac{2\pi}{3}\right) \\ &= \left(\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2}\right)\left(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right) \\ &= -\tfrac{\sqrt{2}}{4} + i\tfrac{\sqrt{6}}{4} - i\tfrac{\sqrt{2}}{4} + i^2\tfrac{\sqrt{6}}{4} \\ &= \left(\frac{-\sqrt{2} - \sqrt{6}}{4}\right) + \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)i \end{aligned}$$

Note: $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\therefore \quad \cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{2} - \sqrt{6}}{4} \qquad \sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$ {equating real and imaginary parts}

EXERCISE 16B.5

1 Let $z = R \text{ cis } \theta$ and $w = r \text{ cis } \phi$ $\quad w \neq 0$

$$\frac{z}{w} = \frac{R \text{ cis } \theta}{r \text{ cis } \phi} = \frac{R}{r} \text{ cis } (\theta - \phi) \quad \text{...... } (*)$$

$$\therefore \quad \left|\frac{z}{w}\right| = \frac{R}{r} = \frac{|z|}{|w|}$$

also from $*$ $\quad \arg\left(\frac{z}{w}\right) = \theta - \phi = \arg z - \arg w \quad$ if $\quad w \neq 0$

$$\therefore \quad \arg\left(\frac{z}{w}\right) = \arg z - \arg w$$

2 **a**

$$\begin{aligned} z &= 3 \text{ cis } \theta \\ -z &= -1 \times 3 \text{ cis } \theta \\ &= \text{cis } (\pi) \times 3 \text{ cis } \theta \\ &= 3 \text{ cis } (\theta + \pi) \qquad \therefore \quad |-z| = 3 \quad \text{and} \quad \arg(-z) = \theta + \pi \qquad -\pi \leqslant \arg z \leqslant \pi \end{aligned}$$

b $z^* = 3 \text{ cis } (-\theta) \qquad \therefore \quad |z^*| = 3 \quad$ and $\quad \arg z^* = -\theta$

c

$$\begin{aligned} iz &= \text{cis } \left(\tfrac{\pi}{2}\right) \times 3 \text{ cis } \theta \\ &= 3 \text{ cis } \left(\tfrac{\pi}{2} + \theta\right) \qquad \therefore \quad |iz| = 3 \quad \text{and} \quad \arg iz = \theta + \tfrac{\pi}{2} \end{aligned}$$

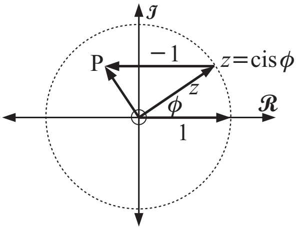
d

$$\begin{aligned} (1+i)z &= \sqrt{2} \text{ cis } \left(\tfrac{\pi}{4}\right) \times 3 \text{ cis } \theta \\ &= 3\sqrt{2} \text{ cis } \left(\theta + \tfrac{\pi}{4}\right) \end{aligned}$$

$$\therefore \quad |(1+i)\,z| = 3\sqrt{2} \quad \text{and} \quad \arg\left[(1+i)\,z\right] = \theta + \tfrac{\pi}{4}$$

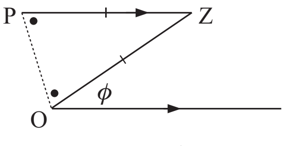
3 **a**

$$\begin{aligned} & z - 1 \\ &= z + (-1) \end{aligned}$$



$\therefore \quad z - 1$ is represented by the vector $\overrightarrow{\text{OP}}$

Considering the ΔOZP



$\measuredangle$PZO $= \phi$ {alternate $\measuredangle$'s}

OZ $=$ ZP $= 1$

$\therefore \quad \Delta$OPZ is isosceles

$\therefore \quad \measuredangle$POZ $= \measuredangle$OPZ $= \alpha$

$$\therefore \quad 2\alpha + \phi = \pi \quad \text{and so} \quad \alpha = \frac{\pi - \phi}{2}$$

$$\begin{aligned} \therefore \quad \arg(z-1) &= \frac{\pi - \phi}{2} + \phi \\ &= \tfrac{\pi}{2} - \tfrac{\phi}{2} + \phi \\ \arg(z-1) &= \tfrac{\pi}{2} + \tfrac{\phi}{2} \text{ ...... } (*) \end{aligned}$$

Using the Cosine Rule in ΔOZP

$$\begin{aligned} \text{OP}^2 &= 1^2 + 1^2 = 2(1)(1)\cos\phi \\ \therefore \quad \text{OP}^2 &= 2 - 2\cos\phi \\ \therefore \quad \text{OP}^2 &= 2 - 2(1 - 2\sin^2\left(\tfrac{\phi}{2}\right)) \\ \therefore \quad \text{OP}^2 &= 2 - 2 + 4\sin^2\left(\tfrac{\phi}{2}\right) \\ \therefore \quad \text{OP}^2 &= 4\sin^2\left(\tfrac{\phi}{2}\right) \end{aligned}$$

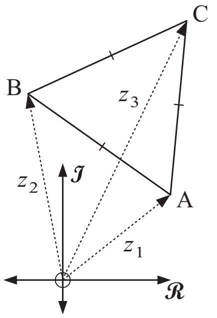
$$\therefore \quad |z - 1| = 2\sin\left(\tfrac{\phi}{2}\right) \quad \text{...... } (**)$$

b $\therefore \quad z - 1 = 2\sin\left(\frac{\phi}{2}\right)\text{cis}\left(\frac{\pi}{2} + \frac{\phi}{2}\right) \qquad$ {using $*$ and $**$}

c $\therefore \quad (z-1)^* = 2\sin\left(\frac{\phi}{2}\right)\text{cis}\left(-\frac{\pi}{2} - \frac{\phi}{2}\right)$

4 **a** Now $z_2 - z_1 = \overrightarrow{AB}$

$z_3 - z_2 = \overrightarrow{BC}$



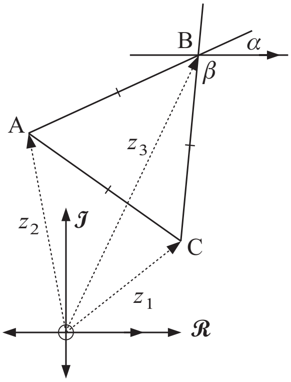
b $$\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = \frac{|z_2 - z_1|}{|z_3 - z_2|} = \frac{|\,\overrightarrow{AB}\,|}{|\,\overrightarrow{BC}\,|}$$

But ΔABC is equilateral

$\therefore \quad |\,\overrightarrow{AB}\,| = |\,\overrightarrow{BC}\,|$

$\therefore \quad \left|\dfrac{z_2 - z_1}{z_3 - z_2}\right| = 1$

c



Now $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$

Since $z_2 - z_1 = \overrightarrow{AB}$ and $z_3 - z_2 = \overrightarrow{BC}$,

$\arg(z_2 - z_1) = \alpha$

and $\arg(z_3 - z_2) = -\beta$

$\therefore \quad \arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right) = \alpha - (-\beta) = \alpha + \beta$

But $\measuredangle ABC = \frac{\pi}{3}$ since the triangle is equilateral

$\therefore \quad \alpha + \beta = \frac{2\pi}{3}$ {angles in a line}

$\therefore \quad \arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right) = \frac{2\pi}{3}$

d $$\left(\frac{z_2 - z_1}{z_3 - z_2}\right)^3 = \left(1 \text{ cis } \tfrac{2\pi}{3}\right)^3 = 1(\text{cis } 2\pi) = 1$$

EXERCISE 16B.6

1 Using technology

a $\sqrt{3} \text{ cis } (2.5187) = -1.4068 + 1.0105i$

b $\sqrt{11} \text{ cis } (-\frac{3\pi}{8}) = 1.2692 - 3.0642i$

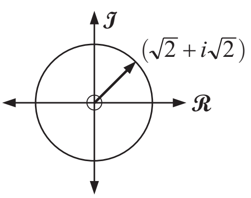
c $2.836\,49 \text{ cis } (-2.684\,32) = -2.5451 - 1.2523i$

2 **a** $3 - 4i = 5 \text{ cis } (-0.9273)$

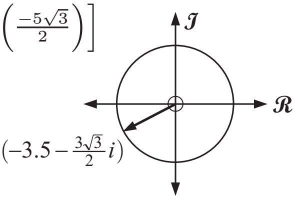
b $-5 - 12i = 13 \text{ cis } (-1.9656)$

c $-11.6814 + 13.2697i = 17.6788 \text{ cis } (2.2926)$

3 **a**
$$\begin{aligned} & 3 \text{ cis } \left(\tfrac{\pi}{4}\right) + \text{cis } \left(\tfrac{-3\pi}{4}\right) \\ &= 3\cos\left(\tfrac{\pi}{4}\right) + 3i\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{-3\pi}{4}\right) + i\sin\left(\tfrac{-3\pi}{4}\right) \\ &= 3\cos\left(\tfrac{\pi}{4}\right) + \cos\left(-\tfrac{3\pi}{4}\right) + i\left[3\sin\left(\tfrac{\pi}{4}\right) + \sin\left(-\tfrac{3\pi}{4}\right)\right] \\ &= \left[\tfrac{3}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right] + i\left[\tfrac{3}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right] \\ &= \sqrt{2} + i\sqrt{2} \\ &= 2\left(\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}\right) \\ &= 2 \text{ cis } \tfrac{\pi}{4} \end{aligned}$$



b
$$\begin{aligned} & 2 \text{ cis } \left(\tfrac{2\pi}{3}\right) + 5 \text{ cis } \left(\tfrac{-2\pi}{3}\right) \\ &= 2\cos\left(\tfrac{2\pi}{3}\right) + 5\cos\left(\tfrac{-2\pi}{3}\right) + i\left[2\sin\left(\tfrac{2\pi}{3}\right) + 5\sin\left(\tfrac{-2\pi}{3}\right)\right] \\ &= \left[-1 + (-\tfrac{5}{2})\right] + i\left[\sqrt{3} + \left(\tfrac{-5\sqrt{3}}{2}\right)\right] \\ &= -\tfrac{7}{2} - \tfrac{3\sqrt{3}}{2}i \\ &= \sqrt{19} \text{ cis } (-2.5030) \end{aligned}$$



4 **a** Sum of roots

$$\begin{aligned}&= 2 \text{ cis } \left(\tfrac{2\pi}{3}\right) + 2 \text{ cis } \left(\tfrac{4\pi}{3}\right)\\ &= 2\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) + 2\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\\ &= -1 + \sqrt{3}i - 1 - \sqrt{3}i\\ &= -2\end{aligned}$$

Product of roots

$$\begin{aligned}&= 2 \text{ cis } \left(\tfrac{2\pi}{3}\right) \times 2 \text{ cis } \left(\tfrac{4\pi}{3}\right)\\ &= 4 \text{ cis } \left(\tfrac{6\pi}{3}\right)\\ &= 4 \text{ cis } 0\\ &= 4\end{aligned}$$

$\therefore$ equations are $k\left(x^2 - (-2)x + 4\right) = 0$

i.e., $k\left(x^2 + 2x + 4\right) = 0, \quad k \neq 0$

b Sum of roots

$$\begin{aligned}&= \sqrt{2} \text{ cis } \left(\tfrac{\pi}{4}\right) + \sqrt{2} \text{ cis } \left(-\tfrac{\pi}{4}\right)\\ &= \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) + \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\\ &= (1+i) + (1-i)\\ &= 2\end{aligned}$$

Product of roots

$$\begin{aligned}&= \sqrt{2} \text{ cis } \left(\tfrac{\pi}{4}\right) \times \sqrt{2} \text{ cis } \left(-\tfrac{\pi}{4}\right)\\ &= 2 \text{ cis } 0\\ &= 2\end{aligned}$$

$\therefore$ equations are $k(x^2 - 2x + 2) = 0, \quad k \neq 0$

EXERCISE 16C

1 **a**

$$\begin{aligned}&(\sqrt{2} \text{ cis } \left(\tfrac{\pi}{5}\right))^{10}\\ &= (\sqrt{2})^{10} \text{ cis } \left(\tfrac{10\pi}{5}\right)\\ &= 2^5 \text{ cis } 2\pi\\ &= 2^5\\ &= 32\end{aligned}$$

b

$$\begin{aligned}&\left(\text{cis } \left(\tfrac{\pi}{12}\right)\right)^{36}\\ &= \text{cis } \left(\tfrac{36\pi}{12}\right)\\ &= \text{cis } 3\pi\\ &= \text{cis } \pi\\ &= -1\end{aligned}$$

c

$$\begin{aligned}&(\sqrt{2} \text{ cis } \left(\tfrac{\pi}{8}\right))^{12}\\ &= 2^6 \text{ cis } \left(\tfrac{12\pi}{8}\right)\\ &= 64 \text{ cis } \left(\tfrac{3\pi}{2}\right)\\ &= -64i\end{aligned}$$

d

$$\begin{aligned}&\sqrt{5 \text{ cis } \left(\tfrac{\pi}{7}\right)}\\ &= \left(5 \text{ cis } \left(\tfrac{\pi}{7}\right)\right)^{\frac{1}{2}}\\ &= \sqrt{5} \text{ cis } \left(\tfrac{1}{2} \times \tfrac{\pi}{7}\right)\\ &= \sqrt{5} \text{ cis } \left(\tfrac{\pi}{14}\right)\end{aligned}$$

e

$$\begin{aligned}&\sqrt[3]{8 \text{ cis } \left(\tfrac{\pi}{2}\right)}\\ &= \left(8 \text{ cis } \left(\tfrac{\pi}{2}\right)\right)^{\frac{1}{3}}\\ &= 2 \text{ cis } \left(\tfrac{\pi}{6}\right)\\ &= 2\left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right)\\ &= \sqrt{3} + i\end{aligned}$$

f

$$\begin{aligned}&\left(8 \text{ cis } \left(\tfrac{\pi}{5}\right)\right)^{\frac{5}{3}}\\ &= 8^{\frac{5}{3}} \text{ cis } \left(\tfrac{5}{3} \times \tfrac{\pi}{5}\right)\\ &= 2^5 \text{ cis } \left(\tfrac{\pi}{3}\right)\\ &= 32\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)\\ &= 16 + 16\sqrt{3}i\end{aligned}$$

2 **a**

$$\begin{aligned}&(1+i)^{15}\\ &= \left(\sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\right)^{15}\\ &= \left(\sqrt{2} \text{ cis } \left(\tfrac{\pi}{4}\right)\right)^{15}\\ &= \sqrt{2}^{14}\sqrt{2} \text{ cis } \left(\tfrac{15\pi}{4}\right)\\ &= 2^7\sqrt{2} \text{ cis } \left(\tfrac{7\pi}{4}\right)\\ &= 128\sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\\ &= 128 - 128i\end{aligned}$$

b

$$\begin{aligned}&(1 - i\sqrt{3})^{11}\\ &= \left(2\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)\right)^{11}\\ &= \left(2 \text{ cis } \left(-\tfrac{\pi}{3}\right)\right)^{11}\\ &= 2^{11} \text{ cis } \left(\tfrac{-11\pi}{3}\right)\\ &= 2048 \text{ cis } \left(\tfrac{\pi}{3}\right)\\ &= 2048\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)\\ &= 1024 + 1024\sqrt{3}i\end{aligned}$$

c

$$\begin{aligned}&(\sqrt{2} - i\sqrt{2})^{-19}\\ &= \left(2\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\right)^{-19}\\ &= \left(2 \text{ cis } \left(-\tfrac{\pi}{4}\right)\right)^{-19}\\ &= 2^{-19} \text{ cis } \left(\tfrac{19\pi}{4}\right)\\ &= 2^{-19} \text{ cis } \left(\tfrac{3\pi}{4}\right)\\ &= 2^{-19}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\\ &= 2^{-19}2^{-\frac{1}{2}}(-1+i)\\ &= 2^{-\frac{39}{2}}(-1+i)\end{aligned}$$

d

$$\begin{aligned}&(-1+i)^{-11}\\ &=\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)^{-11}\\ &=\left(\sqrt{2}\text{ cis }\left(\tfrac{3\pi}{4}\right)\right)^{-11}\\ &=\left(\sqrt{2}\right)^{-11}\text{ cis }\left(-\tfrac{33\pi}{4}\right)\\ &=(\sqrt{2})^{-11}\text{ cis }\left(-\tfrac{\pi}{4}\right)\\ &=(\sqrt{2})^{-11}\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)\\ &=(\sqrt{2})^{-12}(1-i)\\ &=\tfrac{1}{64}(1-i)\end{aligned}$$

e

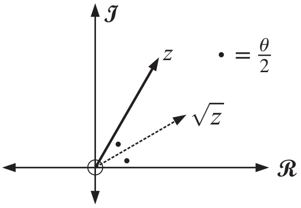
$$\begin{aligned}&(\sqrt{3}-i)^{\frac{1}{2}}\\ &=2\left(\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right)^{\frac{1}{2}}\\ &=\left(2\text{ cis }\left(-\tfrac{\pi}{6}\right)\right)^{\frac{1}{2}}\\ &=2^{\frac{1}{2}}\text{ cis }\left(-\tfrac{\pi}{12}\right)\\ &=\sqrt{2}\text{ cis }\left(-\tfrac{\pi}{12}\right)\end{aligned}$$

f

$$\begin{aligned}&(2+2i\sqrt{3})^{-\frac{5}{2}}\\ &=4\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)^{-\frac{5}{2}}\\ &=\left(4\text{ cis }\left(\tfrac{\pi}{3}\right)\right)^{-\frac{5}{2}}\\ &=2^{-\frac{5}{2}\times 2}\text{ cis }\left(-\tfrac{5}{2}\times\tfrac{\pi}{3}\right)\\ &=2^{-5}\text{ cis }\left(-\tfrac{5\pi}{6}\right)\\ &=\tfrac{1}{32}\left(-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right)\\ &=-\tfrac{1}{64}\left(\sqrt{3}+i\right)\end{aligned}$$

4 **a**

$$\begin{aligned}z&=|z|\text{ cis }\theta\\ \sqrt{z}&=(|z|\text{ cis }\theta)^{\frac{1}{2}}\\ \sqrt{z}&=|z|^{\frac{1}{2}}\text{ cis }\left(\tfrac{\theta}{2}\right)\quad\{\text{De Moivre}\}\end{aligned}$$



b $-\frac{\pi}{2}<\theta\leqslant\frac{\pi}{2}$

c Yes: This can be seen in the given illustration.

5 If $z=r\text{ cis }\theta$, then

De Moivre's Theorem states that $z^n=r^n\text{ cis }n\theta$ for all rational values of n.

and $\therefore\ |z^n|=r^n=|z|^n$,

$\arg(z^n)=n\theta=n\arg z$

6

$$\begin{aligned}\text{cis}(-\theta)&=\cos(-\theta)+i\sin(-\theta)\\ &=\cos\theta-i\sin\theta\\ \therefore\quad(\cos\theta-i\sin\theta)^{-3}&=[\text{cis }(-\theta)]^{-3}\\ &=\text{cis }3\theta\quad\{\text{De Moivre}\}\end{aligned}$$

7

$$\begin{aligned}z&=1+i=\sqrt{2}\text{ cis }\left(\tfrac{\pi}{4}\right)\\ \therefore\quad z^n&=(\sqrt{2})^n\text{ cis }\left(\tfrac{n\pi}{4}\right)\quad\{\text{De Moivre}\}\\ &=\left(\sqrt{2}\right)^n\left[\cos\left(\tfrac{n\pi}{4}\right)+i\sin\left(\tfrac{n\pi}{4}\right)\right]\end{aligned}$$

a If z^n is real then $\sin\left(\frac{n\pi}{4}\right)=0\quad\therefore\ \frac{n\pi}{4}=0+k\pi$

$\therefore\ n=4k$ where k is any integer

b If z^n is purely imaginary then $\cos\left(\frac{n\pi}{4}\right)=0$

i.e., $\frac{n\pi}{4}=\frac{\pi}{2}+k\pi$

$n=2+4k$ where k is any integer

8 **a**

$$\begin{aligned}z&=2\text{ cis }\theta\\ \therefore\quad z^3&=2^3\text{ cis }3\theta\\ \therefore\quad\left|z^3\right|&=8\\ \text{and}\quad\arg z^3&=3\theta\end{aligned}$$

b

$$\begin{aligned}z&=2\text{ cis }\theta\\ \therefore\quad z^2&=4\text{ cis }2\theta\\ \therefore\quad iz^2&=i(4\text{ cis }2\theta)\\ \therefore\quad iz^2&=\text{cis }\left(\tfrac{\pi}{2}\right)(4\text{ cis }2\theta)\\ \therefore\quad iz^2&=4\text{ cis }\left(\tfrac{\pi}{2}+2\theta\right)\\ \therefore\quad\left|iz^2\right|&=4\\ \text{and}\quad\arg(iz^2)&=\tfrac{\pi}{2}+2\theta\end{aligned}$$

c

$$\begin{aligned} z &= 2 \text{ cis } \theta \\ \text{Now } \frac{1}{z} &= z^{-1}, \\ \text{so } \frac{1}{z} &= (2 \text{ cis } \theta)^{-1} \\ &= \tfrac{1}{2} \text{ cis } (-\theta) \\ \therefore \left|\frac{1}{z}\right| &= \tfrac{1}{2} \\ \text{and } \arg\left(\frac{1}{z}\right) &= -\theta \end{aligned}$$

d

$$\begin{aligned} z &= 2 \text{ cis } \theta \\ \text{Now } -\frac{i}{z^2} &= -i \times z^{-2}, \\ \text{so } -\frac{i}{z^2} &= \text{cis } \left(-\tfrac{\pi}{2}\right) \times (2 \text{ cis } \theta)^{-2} \\ &= 2^{-2} \text{ cis } \left(-\tfrac{\pi}{2}\right) \text{ cis } (-2\theta) \\ &= \tfrac{1}{4} \text{ cis } \left(-\tfrac{\pi}{2} - 2\theta\right) \end{aligned}$$

$$\therefore \left|\frac{-i}{z^2}\right| = \tfrac{1}{4} \text{ and } \arg\left(\frac{-i}{z^2}\right) = -\tfrac{\pi}{2} - 2\theta$$

9 If $z = \text{cis } \theta$, then

$$\begin{aligned} \frac{z^2-1}{z^2+1} &= \frac{(\text{cis } \theta)^2 - 1}{(\text{cis } \theta)^2 + 1} \\ &= \frac{\text{cis } 2\theta - 1}{\text{cis } 2\theta + 1} \quad \{\text{De Moivre}\} \\ &= \frac{\cos 2\theta + i \sin 2\theta - 1}{\cos 2\theta + i \sin 2\theta + 1} \\ &= \frac{-2\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) + i \sin 2\theta}{2\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) + i \sin 2\theta} \\ &= \frac{-2\sin^2\theta + 2i\cos\theta\sin\theta}{2\cos^2\theta + 2i\cos\theta\sin\theta} \\ &= \frac{2\sin\theta(i\cos\theta + i^2\sin\theta)}{2\cos\theta(\cos\theta + i\sin\theta)} \\ &= \frac{i\sin\theta \text{ cis } \theta}{\cos\theta \text{ cis } \theta} \\ &= i\tan\theta \end{aligned}$$

10

$$\begin{aligned} \text{cis } 3\theta &= \cos 3\theta + i \sin 3\theta \\ \text{But } \text{cis } 3\theta &= (\text{cis } \theta)^3 \quad \{\text{De Moivre's Theorem}\} \\ &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \left[\cos^3\theta - 3\cos\theta\sin^2\theta\right] + i\left[3\cos^2\theta\sin\theta - \sin^3\theta\right] \end{aligned}$$

a

$$\begin{aligned} \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \quad \{\text{equating imaginary parts}\} \\ &= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta \end{aligned}$$

b

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \quad \{\text{equating real parts}\} \\ &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

11 **a** If $z = \text{cis } \theta$, then

$$\begin{aligned} z^n + \frac{1}{z^n} &= z^n + z^{-n} \\ &= (\text{cis } \theta)^n + (\text{cis } \theta)^{-n} \\ &= \text{cis } (n\theta) + \text{cis } (-n\theta) \\ &= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta)) \\ &= (\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta) \quad \{\cos(-\alpha) = \cos\alpha, \quad \sin(-\alpha) = -\sin\alpha\} \\ &= 2\cos n\theta \end{aligned}$$

b In **a** if we let $n = 1$ we get $z + \dfrac{1}{z} = 2\cos\theta$

c $$\left(z+\frac{1}{z}\right)^3 = z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3$$
$$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$$

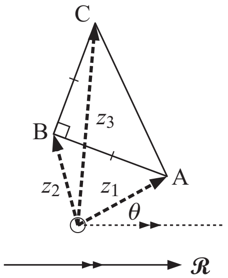
d From **c** $\left(z+\dfrac{1}{z}\right)^3 = \left(z^3+\dfrac{1}{z^3}\right) + 3\left(z+\dfrac{1}{z}\right)$

Using **a** and **b** $$(2\cos\theta)^3 = 2\cos 3\theta + 3(2\cos\theta)$$
$$\therefore \quad 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$$
$$\therefore \quad \cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta$$

e If we let $\theta = \frac{13\pi}{12}$ in **d** we get

$$\begin{aligned}
\cos^3\left(\tfrac{13\pi}{12}\right) &= \tfrac{1}{4}\cos\left(\tfrac{39\pi}{12}\right) + \tfrac{3}{4}\cos\left(\tfrac{13\pi}{12}\right)\\
&= \tfrac{1}{4}\cos\left(\tfrac{13\pi}{4}\right) + \tfrac{3}{4}\cos\left(\tfrac{3\pi}{4}+\tfrac{\pi}{3}\right)\\
&= \tfrac{1}{4}\cos\left(\tfrac{5\pi}{4}\right) + \tfrac{3}{4}\left[\cos\left(\tfrac{3\pi}{4}\right)\cos\left(\tfrac{\pi}{3}\right) - \sin\left(\tfrac{3\pi}{4}\right)\sin\left(\tfrac{\pi}{3}\right)\right]\\
&= \tfrac{1}{4}\left(-\tfrac{1}{\sqrt{2}}\right) + \tfrac{3}{4}\left[\left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right) - \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right)\right]\\
&= -\tfrac{1}{4\sqrt{2}} - \tfrac{3}{8\sqrt{2}} - \tfrac{3\sqrt{3}}{8\sqrt{2}}\\
&= -\tfrac{1}{4\sqrt{2}}\left(\tfrac{2\sqrt{2}}{2\sqrt{2}}\right) - \tfrac{3}{8\sqrt{2}}\left(\tfrac{\sqrt{2}}{\sqrt{2}}\right) - \tfrac{3\sqrt{3}}{8\sqrt{2}}\left(\tfrac{\sqrt{2}}{\sqrt{2}}\right)\\
&= \tfrac{-2\sqrt{2}-3\sqrt{2}-3\sqrt{6}}{16}\\
&= \tfrac{-5\sqrt{2}-3\sqrt{6}}{16}
\end{aligned}$$

12 **a**



$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -z_2 + z_3 = z_3 - z_2$$
$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -z_2 + z_1 = z_1 - z_2$$

Now $\arg\left(\overrightarrow{BC}\right) - \arg\left(\overrightarrow{BA}\right) = \frac{\pi}{2}$

$$\therefore \quad \arg(z_3 - z_2) - \arg(z_1 - z_2) = \tfrac{\pi}{2}$$
$$\therefore \quad \arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \tfrac{\pi}{2}$$

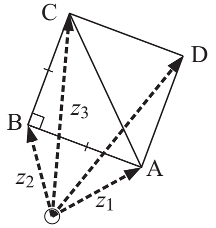
Also $\left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| = \dfrac{|z_3 - z_2|}{|z_1 - z_2|} = \dfrac{\text{BC}}{\text{AB}} = 1$

So, $\dfrac{z_3 - z_2}{z_1 - z_2}$ has modulus 1 and argument $\frac{\pi}{2}$

$$\therefore \quad \frac{z_3 - z_2}{z_1 - z_2} = 1 \text{ cis } \tfrac{\pi}{2} = i$$

$$\therefore \quad \left(\frac{z_3 - z_2}{z_1 - z_2}\right)^2 = i^2 = -1 \quad \text{and so} \quad \therefore \quad (z_1 - z_2)^2 = -(z_3 - z_2)^2$$

b



$\overrightarrow{BA} = \overrightarrow{CD}$ and $\overrightarrow{BA} = z_1 - z_2$

$\therefore \quad \overrightarrow{CD} = z_1 - z_2$

Now $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$\therefore \quad \overrightarrow{OD} = z_3 + z_1 - z_2$

$\therefore \quad z_3 + z_1 - z_2$ represents D.

13 **a**
$$\begin{aligned}&\cos 4\theta + i\sin 4\theta\\ &= \text{cis } 4\theta\\ &= (\text{cis } \theta)^4 \qquad \{\text{De Moivre's theorem in reverse}\}\\ &= (\cos\theta + i\sin\theta)^4\\ &= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4\\ &= \cos^4\theta + [4\cos^3\theta\sin\theta]i - 6\cos^2\theta\sin^2\theta - [4\cos\theta\sin^3\theta]i + \sin^4\theta\\ &= [\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta] + [4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta]i\end{aligned}$$

i Equating real parts gives $\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$\therefore \quad \cos 4\theta = \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

i.e., $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

ii Equating imaginary parts gives $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

b If $z = \text{cis } \theta$,
$$\begin{aligned} z^n - \frac{1}{z^n} &= z^n - z^{-n}\\ &= [\text{cis } \theta]^n - [\text{cis } \theta]^{-n}\\ &= \text{cis } n\theta - \text{cis } (-n\theta) \qquad \{\text{De Moivre}\}\\ &= \cos n\theta + i\sin n\theta - [\cos(-n\theta) + i\sin(-n\theta)]\\ &= \cos n\theta + i\sin n\theta - \cos(-n\theta) - i\sin(-n\theta)\\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\\ &\qquad \{\text{as } \cos(-\phi) = \cos\phi, \ \sin(-\phi) = -\sin\phi\}\\ &= 2i\sin n\theta \quad \text{......} \ (*)\end{aligned}$$

If we let $n = 1$, $z - \frac{1}{z} = 2i\sin\theta$
$$\begin{aligned}\therefore\quad [2i\sin\theta]^3 &= \left(z - \frac{1}{z}\right)^3\\ \therefore\quad 8i^3\sin^3\theta &= z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3\\ &= z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right)\\ &= 2i\sin 3\theta - 3\times 2i\sin\theta \quad \{\text{using } (*)\}\\ \therefore\quad -8i\sin^3\theta &= 2i\sin 3\theta - 6i\sin\theta\\ \therefore\quad \sin^3\theta &= -\tfrac{1}{4}\sin 3\theta + \tfrac{3}{4}\sin\theta\\ \text{i.e.,}\quad \sin^3\theta &= \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta\end{aligned}$$

EXERCISE 16D

1 **a** The cube roots of 1 are solutions to $z^3 = 1$ i.e., $z^3 - 1 = 0$

Now $z = 1$ is a solution, so $z-1$ is a factor.

$\therefore \quad (z-1)(z^2+z+1) = 0$

$\therefore \quad z = 1$ or $\dfrac{-1 \pm \sqrt{1-4}}{2}$

$\therefore \quad z = 1$ or $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

1	1	0	0	−1
	0	1	1	1
	1	1	1	0

b
$$\begin{aligned} z^3 &= 1\\ \therefore\quad z^3 &= 1 \text{ cis } (0 + k2\pi)\\ \therefore\quad z &= (1 \text{ cis } (k2\pi))^{\frac{1}{3}}\\ \therefore\quad z &= 1 \text{ cis } \left(\tfrac{k2\pi}{3}\right) \qquad \{\text{De Moivre}\}\\ \therefore\quad z &= \text{cis } 0, \ \text{cis } \left(\tfrac{2\pi}{3}\right) \text{ or cis } \left(\tfrac{4\pi}{3}\right) \qquad \{\text{when } k = 0, 1, 2\}\end{aligned}$$

i.e., $z = 1, \ -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

2 **a**

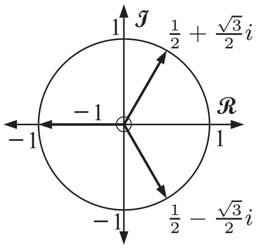
$$\begin{aligned}
z^3 &= -8i\\
z^3 &= 8 \text{ cis } \left(-\tfrac{\pi}{2} + k2\pi\right)\\
\therefore\quad z &= \left(8 \text{ cis } \left(-\tfrac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{3}}\\
\therefore\quad z &= 8^{\frac{1}{3}} \text{ cis } \left(-\tfrac{\pi}{6} + \tfrac{k2\pi}{3}\right) \qquad \{\text{De Moivre}\}\\
\therefore\quad z &= 2 \text{ cis } \left(-\tfrac{\pi}{6}\right),\ 2 \text{ cis } \left(\tfrac{3\pi}{6}\right) \text{ or } 2 \text{ cis } \left(\tfrac{7\pi}{6}\right) \qquad \{\text{when } \ k = 0,\, 1,\, 2\}\\
\therefore\quad z &= 2\left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right),\ \ 2 \text{ cis } \tfrac{\pi}{2} \ \text{ or } \ 2\left(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i\right)\\
\therefore\quad z &= \sqrt{3} - i,\ \ 2i \ \text{ or } \ -\sqrt{3} - i
\end{aligned}$$

b

$$\begin{aligned}
z^3 &= -27i\\
\therefore\quad z^3 &= 27 \text{ cis } \left(-\tfrac{\pi}{2} + k2\pi\right)\\
\therefore\quad z &= 3\left(\text{cis } \left(-\tfrac{\pi}{2} + k2\pi\right)\right)^{\frac{1}{3}}\\
\therefore\quad z &= 3 \text{ cis } \left(-\tfrac{\pi}{6} + \tfrac{k4\pi}{6}\right) \qquad \{\text{De Moivre}\}\\
\therefore\quad z &= 3 \text{ cis } \left(-\tfrac{\pi}{6}\right),\ 3 \text{ cis } \left(\tfrac{3\pi}{6}\right) \text{ or } 3 \text{ cis } \left(\tfrac{7\pi}{6}\right) \qquad \{\text{when } \ k = 0,\, 1,\, 2\}\\
\therefore\quad z &= 3(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i),\ \ 3 \text{ cis } \left(\tfrac{\pi}{2}\right) \ \text{ or } \ 3(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i)\\
\therefore\quad z &= \tfrac{3\sqrt{3}}{2} - \tfrac{3}{2}i,\ \ 3i \ \text{ or } \ \tfrac{-3\sqrt{3}}{2} - \tfrac{3}{2}i
\end{aligned}$$

3

$$\begin{aligned}
z^3 &= -1\\
\therefore\quad z^3 &= 1 \text{ cis } (\pi + k2\pi)\\
\therefore\quad z &= (\text{cis } (\pi + k2\pi))^{\frac{1}{3}}\\
\therefore\quad z &= \text{cis } \left(\tfrac{\pi}{3} + \tfrac{k2\pi}{3}\right) \qquad \{\text{De Moivre}\}\\
\therefore\quad z &= \text{cis } \left(\tfrac{\pi}{3}\right),\ \text{cis } \pi,\ \text{cis } \left(\tfrac{5\pi}{3}\right) \qquad \{\text{when } \ k = 0,\, 1,\, 2\}\\
\therefore\quad z &= \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2},\ -1,\ \tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}
\end{aligned}$$



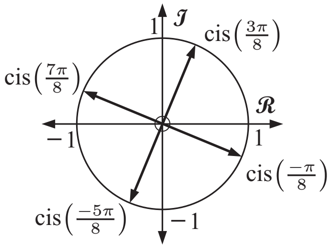
4 **a**

$$\begin{aligned}
z^4 &= 16\\
\therefore\quad z^4 &= 16 \text{ cis } (0 + k2\pi)\\
\therefore\quad z &= (16 \text{ cis } (k2\pi))^{\frac{1}{4}}\\
\therefore\quad z &= 16^{\frac{1}{4}} \text{ cis } \left(\tfrac{k\pi}{2}\right) \qquad \{\text{De Moivre}\}\\
\therefore\quad z &= 2 \text{ cis } \left(\tfrac{k\pi}{2}\right)\\
\therefore\quad z &= 2 \text{ cis } 0,\ 2 \text{ cis } \left(\tfrac{\pi}{2}\right),\ 2 \text{ cis } \pi,\ 2 \text{ cis } \left(\tfrac{3\pi}{2}\right) \qquad \{\text{when } \ k = 0,\, 1,\, 2 \text{ or } 3\}\\
\therefore\quad z &= \pm 2 \ \text{ and } \ \pm 2i
\end{aligned}$$

b

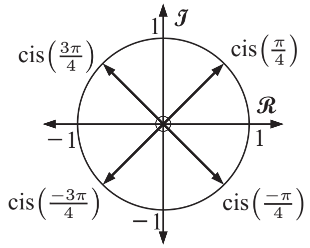
$$\begin{aligned}
z^4 &= 16\\
\therefore\quad z^4 &= 16 \text{ cis } (\pi + k2\pi)\\
\therefore\quad z &= (16 \text{ cis } (\pi + k2\pi))^{\frac{1}{4}}\\
\therefore\quad z &= 16^{\frac{1}{4}} \text{ cis } \left(\tfrac{\pi}{4} + \tfrac{k2\pi}{4}\right) \qquad \{\text{De Moivre}\}\\
\therefore\quad z &= 2 \text{ cis } \left(\tfrac{\pi}{4} + \tfrac{k2\pi}{4}\right)\\
\therefore\quad z &= 2 \text{ cis } \left(\tfrac{\pi}{4}\right),\ 2 \text{ cis } \left(\tfrac{3\pi}{4}\right),\ 2 \text{ cis } \left(\tfrac{5\pi}{4}\right),\ 2 \text{ cis } \left(\tfrac{7\pi}{4}\right) \qquad \{\text{when } \ k = 0,\, 1,\, 2 \text{ or } 3\}\\
\therefore\quad z &= \sqrt{2} \pm i\sqrt{2},\ \ -\sqrt{2} \pm i\sqrt{2}
\end{aligned}$$

5



$$\begin{aligned} z^4 &= -i \\ \therefore\quad z^4 &= \text{cis}\left(-\tfrac{\pi}{2}+k2\pi\right) \\ \therefore\quad z &= \left(\text{cis}\left(-\tfrac{\pi}{2}+k2\pi\right)\right)^{\frac{1}{4}} \\ \therefore\quad z &= \text{cis}\left(-\tfrac{\pi}{8}+\tfrac{k\pi}{2}\right) \qquad \{\text{De Moivre}\} \\ \therefore\quad z &= \text{cis}\left(-\tfrac{\pi}{8}+\tfrac{k4\pi}{8}\right) \\ \therefore\quad z &= \text{cis}\left(-\tfrac{\pi}{8}\right),\ \text{cis}\left(\tfrac{3\pi}{8}\right),\ \text{cis}\left(\tfrac{7\pi}{8}\right),\ \text{cis}\left(\tfrac{11\pi}{8}\right) \qquad \{\text{when}\quad k=0,\,1,\,2 \text{ or } 3\} \end{aligned}$$

6



$$\begin{aligned} z^4+1 &= 0 \\ \therefore\quad z^4 &= -1 \\ \therefore\quad z^4 &= \text{cis}\,(\pi+k2\pi) \\ \therefore\quad z &= [\text{cis}\ (\pi+k2\pi)]^{\frac{1}{4}} \\ \therefore\quad z &= \text{cis}\left(\tfrac{\pi}{4}+\tfrac{k\pi}{2}\right) \qquad \{\text{De Moivre}\} \\ \therefore\quad z &= \text{cis}\left(\tfrac{\pi}{4}+\tfrac{k2\pi}{4}\right) \\ \therefore\quad z &= \text{cis}\left(\tfrac{\pi}{4}\right),\ \text{cis}\left(\tfrac{3\pi}{4}\right),\ \text{cis}\left(\tfrac{5\pi}{4}\right),\ \text{cis}\left(\tfrac{7\pi}{4}\right) \qquad \{\text{when}\quad k=0,\,1,\,2 \text{ or } 3\} \\ \therefore\quad z &= \tfrac{1}{\sqrt{2}} \pm \tfrac{1}{\sqrt{2}}i,\ -\tfrac{1}{\sqrt{2}} \pm \tfrac{1}{\sqrt{2}}i \end{aligned}$$

For the pair of roots $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$$\begin{aligned} \text{sum } &= \tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i+\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i \\ &= \tfrac{2}{\sqrt{2}} \\ &= \sqrt{2} \end{aligned} \qquad\qquad \begin{aligned} \text{product } &= \left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) \\ &= \tfrac{1}{2}+\tfrac{1}{2} \\ &= 1 \end{aligned}$$

and so we have a quadratic factor of $z^2-\sqrt{2}z+1$

For the pair of roots $-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

sum $= -\sqrt{2}$ and product $= \frac{1}{2}+\frac{1}{2} = 1$

and so we have a quadratic factor of $z^2+\sqrt{2}z+1$

EXERCISE 16E

1 **a** **i**

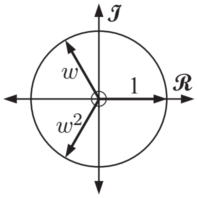
$$\begin{aligned} (z+3)^3 &= 1 \\ \therefore\quad z+3 &= 1,\ w \text{ or } w^2 \quad \text{where}\quad w=\text{cis}\left(\tfrac{2\pi}{3}\right) \\ \therefore\quad z+3 &= w^n \quad \text{where}\quad n=0,\,1,\,2 \\ \therefore\quad z &= w^n-3 \quad \text{where}\quad n=0,\,1,\,2 \quad\text{and}\quad w=\text{cis}\left(\tfrac{2\pi}{3}\right) \end{aligned}$$

ii

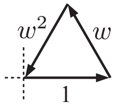
$$\begin{aligned} (z-1)^3 &= 8 \\ \therefore\quad \left[\frac{z-1}{2}\right]^3 &= 1 \\ \therefore\quad \frac{z-1}{2} &= 1,\quad w \quad\text{or}\quad w^2 \quad \text{where}\quad w=\text{cis}\left(\tfrac{2\pi}{3}\right) \\ \therefore\quad \frac{z-1}{2} &= w^n \quad \text{where}\quad n=0,\,1,\,2 \\ \therefore\quad z &= 2w^n+1 \quad \text{where}\quad n=0,\,1,\,2 \quad\text{and}\quad w=\text{cis}\left(\tfrac{2\pi}{3}\right) \end{aligned}$$

iii
$$
\begin{aligned}
(2z-1)^3 &= -1\\
\therefore\ (2z-1)^3 &= (-1)^3\\
\therefore\ \left(\frac{2z-1}{-1}\right)^3 &= 1\\
\therefore\ (1-2z)^3 &= 1\\
\therefore\ 1-2z &= 1,\ w,\ w^2 \quad \text{where} \quad w = \text{cis}\left(\tfrac{2\pi}{3}\right)\\
\therefore\ 1-2z &= w^n \quad \text{where} \quad n = 0,\ 1,\ 2\\
\therefore\ 2z &= 1-w^n\\
\therefore\ z &= \frac{1-w^n}{2} \quad \text{where} \quad n = 0,\ 1,\ 2 \quad \text{and} \quad w = \text{cis}\left(\tfrac{2\pi}{3}\right)
\end{aligned}
$$

2 The following represents the cube roots of unity:



Adding these vectorially



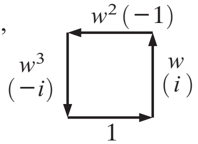
the resultant vector is **0**

$\therefore\quad 1 + w + w^2 = 0$

3 **a**
$$
\begin{aligned}
\text{Yes, as if}\quad z^4 &= 1\\
z^4 - 1 &= 0\\
\therefore\ (z^2+1)(z^2-1) &= 0\\
\therefore\ (z+i)(z-i)(z+1)(z-1) &= 0\\
\therefore\ z &= \pm i,\ \pm 1
\end{aligned}
$$

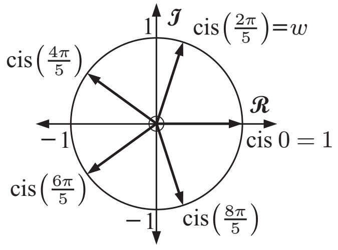
b
$$
\begin{aligned}
1 + w + w^2 + w^3 &= 1 + (i) + (-1) + (-i)\\
&= 1 + i - 1 - i\\
&= 0
\end{aligned}
$$

c Adding the four vectors, vectorially



4 **a** The 5th roots of unity are the solutions to $z^5 = 1$.

$$
\begin{aligned}
\therefore\ z^5 &= \text{cis}\ (0 + k2\pi)\\
\therefore\ z^5 &= \text{cis}\ (k2\pi)\\
\therefore\ z &= [\text{cis}\ \ (k2\pi)]^{\frac{1}{5}}\\
\therefore\ z &= \left[\text{cis}\ \left(\tfrac{k2\pi}{5}\right)\right] \qquad \{\text{De Moivre}\}\\
\therefore\ z &= \text{cis}\ 0 = 1 \quad \text{or}\\
&\quad\ \text{cis}\ \left(\tfrac{2\pi}{5}\right) = w \quad \text{or}\\
&\quad\ \text{cis}\ \left(\tfrac{4\pi}{5}\right) = (\text{cis}\ \tfrac{2\pi}{5})^2 = w^2 \quad \text{or}\\
&\quad\ \text{cis}\ \left(\tfrac{6\pi}{5}\right) = \left(\text{cis}\ \tfrac{2\pi}{5}\right)^2 = w^3 \quad \text{or}\\
&\quad\ \text{cis}\ \left(\tfrac{8\pi}{5}\right) = \text{cis}\ \left(\tfrac{2\pi}{5}\right)^4 = w^4 \qquad \{\text{when} \quad k = 0,\ 1,\ 2,\ 3,\ 4\}
\end{aligned}
$$



b Hence the five roots can be expressed as $1,\ w,\ w^2,\ w^3,\ w^4$ where $w = \text{cis}\ \left(\frac{2\pi}{5}\right)$

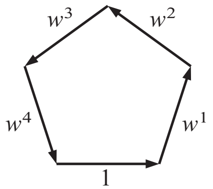
c
$$
\begin{aligned}
&(1 + w + w^2 + w^3 + w^4)(1 - w)\\
&= 1 + w + w^2 + w^3 + w^4 - w - w^2 - w^3 - w^4 - w^5\\
&= 1 - w^5
\end{aligned}
$$

Since w is a solution of $z^5 = 1$, $1 - w^5 = 0$

$\therefore\quad (1 + w + w^2 + w^3 + w^4)(1 - w) = 0$

But $w \neq 1$, so $1 + w + w^2 + w^3 + w^4 = 0$

d



5 **a** The nth roots of unity are the solutions to $z^n = 1$

$$\begin{aligned} z^n &= 1 \\ \therefore\ z^n &= \text{cis}\ (0 + k2\pi) \\ \therefore\ z^n &= \text{cis}\ (k2\pi) \\ \therefore\ z &= [\text{cis}\ (k2\pi)]^{\frac{1}{n}} \\ \therefore\ z &= \text{cis}\ \left(\tfrac{k2\pi}{n}\right) \qquad \{\text{De Moivre}\} \\ \therefore\ z &= \text{cis}\ 0 = 1 \quad \text{or} \end{aligned}$$

$$\text{cis}\ \left(\tfrac{2\pi}{n}\right) = w \quad \text{or} \qquad \{\text{letting}\quad k = 0,\ 1,\ 2,\ 3,\ \ldots\ldots\ 0,\ n-1\}$$

$$\text{cis}\ \left(\tfrac{4\pi}{n}\right) = \left(\text{cis}\ \left(\tfrac{2\pi}{n}\right)\right)^2 = w^2 \quad \text{or}$$

$$\vdots$$

$$\text{cis}\ \left(\tfrac{2\pi}{n}(n-1)\right) = \left[\text{cis}\ \left(\tfrac{2\pi}{n}\right)\right]^{n-1} = w^{n-1}$$

i.e., the n roots of $z^n = 1$ are $1,\ w,\ w^2,\ w^3,\ \ldots\ldots,\ w^{n-1}$ where $w = \text{cis}\ \frac{2\pi}{n}$

b Now $(1 + w + w^2 + \ldots\ldots + w^{n-1})(w-1) = w^n - 1$

But w is a solution to $z^n - 1 = 0$ so $w^n - 1 = 0$

$\therefore\ (1 + w + w^2 + \ldots\ldots + w^{n-1})(w-1) = 0$

$\therefore$ since $w \neq 1$, $1 + w + w^2 + \ldots\ldots + w^{n-1} = 0$

REVIEW SET 16A

1

$$\begin{aligned} (i-\sqrt{3})^2 &= i^2 - 2\sqrt{3}i + 3 \\ &= 2 - 2\sqrt{3}i \\ \therefore\ (i-\sqrt{3})^4 &= (2-2\sqrt{3}i)^2 \\ &= 4 - 8\sqrt{3}i + 12i^2 \\ &= -8 - 8\sqrt{3}i \end{aligned}$$

$$\begin{aligned} \therefore\ (i-\sqrt{3})^5 &= (-8-8\sqrt{3}i)(i-\sqrt{3}) \\ &= -8i + 8\sqrt{3} + 8\sqrt{3} + 24i \\ &= 16\sqrt{3} + 16i \end{aligned}$$

$\therefore$ Real part is $16\sqrt{3}$.
Imaginary part is 16.

2 **a** $|z - i| = |z + 1 + i|$ Since $z = x + iy$,

$$\begin{aligned} |x + iy - i| &= |x + iy + 1 + i| \\ \therefore\ |x + i(y-1)| &= |(x+1) + i(y+1)| \\ \therefore\ \sqrt{x^2 + (y-1)^2} &= \sqrt{(x+1)^2 + (y+1)^2} \\ \therefore\ x^2 + (y-1)^2 &= (x+1)^2 + (y+1)^2 \\ \therefore\ x^2 + y^2 - 2y + 1 &= x^2 + 2x + 1 + y^2 + 2y + 1 \\ \therefore\ 1 - 2y &= 2x + 2y + 2 \\ \therefore\ 2x + 4y &= -1 \end{aligned}$$

b $z^* + iz = 0$

Since $z = x + iy$,

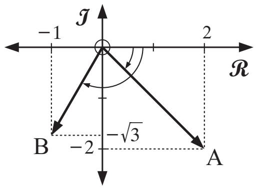
$$\begin{aligned} x - iy + i(x + iy) &= 0 \\ \therefore\ x - iy + ix - y &= 0 \\ \therefore\ (x-y) + i(x-y) &= 0 \\ \therefore\ x - y &= 0 \\ \therefore\ y &= x \end{aligned}$$

3

$$\begin{aligned} |z + 16| &= 4\,|z + 1| \\ \therefore\ |z+16|^2 &= 16\,|z+1|^2 \\ \therefore\ (z+16)(z+16)^* &= 16(z+1)(z+1)^* \\ \therefore\ (z+16)(z^* + 16^*) &= 16(z+1)(z^* + 1^*) \\ \therefore\ zz^* + 16z + 16z^* + 256 &= 16(zz^* + z^* + z + 1) \qquad \{16^* = 16,\ 1^* = 1\} \end{aligned}$$

$$\begin{aligned} \therefore\quad |z|^2+16z+16z^*+256 &= 16\,|z|^2+16z^*+16z+16 \\ 240 &= 15\,|z|^2 \\ \therefore\quad |z|^2 &= 16 \\ \therefore\quad |z| &= 4 \quad \text{as} \quad |z| \geqslant 0 \end{aligned}$$

4



a
$$\begin{aligned} \arg\overrightarrow{\text{OA}} &= \tfrac{-\pi}{4} \\ \arg\overrightarrow{\text{OB}} &= \tfrac{-2\pi}{3} \\ \therefore\quad \measuredangle\text{AOB} &= \tfrac{2\pi}{3}-\tfrac{\pi}{4} \\ &= \tfrac{5\pi}{12} \end{aligned}$$

b
$$\begin{aligned} & zw \\ &= 2\sqrt{2}\ \text{cis}\left(\tfrac{-\pi}{4}\right)\times 2\ \text{cis}\left(\tfrac{-2\pi}{3}\right) \\ &= 4\sqrt{2}\ \text{cis}\left(\tfrac{-\pi}{4}+\tfrac{-2\pi}{3}\right) \\ &= 4\sqrt{2}\ \text{cis}\left(\tfrac{-11\pi}{12}\right) \\ \therefore\quad & \arg(zw) = \tfrac{-11\pi}{12} \end{aligned}$$

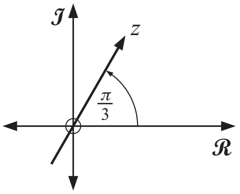
5 **a** $-5i = 5\ \text{cis}\left(-\frac{\pi}{2}\right)$

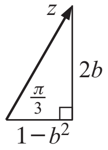
b $$\begin{aligned} 2-2i\sqrt{3} &= 4\left(\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\right) \\ &= 4\ \text{cis}\left(-\tfrac{\pi}{3}\right) \end{aligned}$$

c $$\begin{aligned} k-ki &= -k\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right) \\ &= -k\sqrt{2}\ \text{cis}\left(\tfrac{3\pi}{4}\right) \quad \text{which is in polar form since} \quad k<0 \end{aligned}$$

6 $z=(1+bi)^2 = 1+2bi-b^2$

$$\begin{aligned} \therefore\quad z &= [1-b^2]+2bi \\ \tan\left(\tfrac{\pi}{3}\right) &= \frac{2b}{1-b^2} \\ \therefore\quad \sqrt{3} &= \frac{2b}{1-b^2} \\ \therefore\quad \sqrt{3}-\sqrt{3}b^2 &= 2b \\ \therefore\quad \sqrt{3}b^2+2b-\sqrt{3} &= 0 \\ \therefore\quad b &= \frac{-2\pm\sqrt{4-4(\sqrt{3})(-\sqrt{3})}}{2\sqrt{3}} \\ &= \frac{-2\pm 4}{2\sqrt{3}} \quad \text{but} \quad b>0 \qquad \therefore\ b=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}} \end{aligned}$$





7 **a**
$$\begin{aligned} & \text{cis}\ \theta\times\text{cis}\ \phi \\ &= (\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi) \\ &= (\cos\theta\cos\phi-\sin\theta\sin\phi)+i(\sin\theta\cos\phi+\cos\theta\sin\phi) \\ &= \cos(\theta+\phi)+i\sin(\theta+\phi) \\ &= \text{cis}\ (\theta+\phi) \quad \text{as required} \end{aligned}$$

b
$$\begin{aligned} 1-i &= \sqrt{2}\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = \sqrt{2}\ \text{cis}\left(-\tfrac{\pi}{4}\right) \\ \therefore\quad (1-i)z &= \sqrt{2}\ \text{cis}\left(-\tfrac{\pi}{4}\right)\times 2\sqrt{2}\ \text{cis}\ \alpha \\ &= 4\ \text{cis}\left(\alpha-\tfrac{\pi}{4}\right) \qquad \{\text{using } \mathbf{a}\} \\ \therefore\quad \arg\left[(1-i)z\right] &= \alpha-\tfrac{\pi}{4} \end{aligned}$$

8 **a**

$$\left|\frac{z_1{}^2}{z_2{}^2}\right| = \frac{|z_1|^2}{|z_2|^2} \qquad \text{But} \quad |z_1| = |z_2| \quad \text{since the triangle is isosceles}$$

$$\therefore \quad \left|\frac{z_1{}^2}{z_2{}^2}\right| = 1$$

$$\begin{aligned}\text{Also,} \quad \arg\left(\frac{z_1{}^2}{z_2{}^2}\right) &= \arg(z_1{}^2) - \arg(z_2{}^2)\\ &= 2\arg z_1 - 2\arg z_2\\ &= 2(\arg z_1 - \arg z_2)\\ &= 2 \times \tfrac{\pi}{2} \quad \text{since} \quad z_1 \text{ and } z_2 \quad \text{are perpendicular}\\ &= \pi\end{aligned}$$

b $\dfrac{z_1{}^2}{z_2{}^2} = \text{cis}\ \pi = -1 \qquad \therefore \quad z_1{}^2 = -z_2{}^2 \qquad \therefore \quad z_1{}^2 + z_2{}^2 = 0$

REVIEW SET 16B

1 $z_1 = \text{cis}\left(\frac{\pi}{6}\right)$ and $z_2 = \text{cis}\left(\frac{\pi}{4}\right)$

$$\therefore \quad \left(\frac{z_1}{z_2}\right)^3 = \left[\frac{\text{cis}\left(\frac{\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{4}\right)}\right]^3 = \left[\text{cis}\left(\tfrac{\pi}{6} - \tfrac{\pi}{4}\right)\right]^3$$

$$\therefore \quad \left(\frac{z_1}{z_2}\right)^3 = \left[\text{cis}\left(\tfrac{-\pi}{12}\right)\right]^3 = \text{cis}\left(-\tfrac{3\pi}{12}\right) \qquad \{\text{De Moivre}\}$$

$$\therefore \quad \left(\frac{z_1}{z_2}\right)^3 = \text{cis}\left(-\tfrac{\pi}{4}\right) = \cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)$$

$$\therefore \quad \left(\frac{z_1}{z_2}\right)^3 = \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i$$

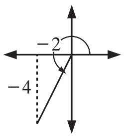
2 $z = 4 + i \qquad w = 2 - 3i$

a
$$\begin{aligned}&2w^* - iz\\ &= 2(2+3i) - i(4+i)\\ &= 4 + 6i - 4i - i^2\\ &= 5 + 2i\end{aligned}$$

b
$$\begin{aligned}|w - z^*| &= |(2-3i) - (4-i)|\\ &= |2 - 3i - 4 + i|\\ &= |-2-2i|\\ &= \sqrt{(-2)^2 + (-2)^2}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}\left|z^{10}\right| &= |z|^{10}\\ &= |4+i|^{10}\\ &= \left(\sqrt{16+1}\right)^{10}\\ &= \sqrt{17}^{10}\\ &= 17^5\end{aligned}$$

d
$$\begin{aligned}\arg(w - z) &= \arg\left[(2-3i) - (4+i)\right]\\ &= \arg\left[-2-4i\right]\\ &\doteqdot -2.034\end{aligned}$$



3 If $\dfrac{2-3i}{2a+bi} = 3 + 2i$, then $\dfrac{2-3i}{3+2i} = 2a + bi$

$$\therefore \quad 2a + bi = \left(\frac{2-3i}{3+2i}\right)\left(\frac{3-2i}{3-2i}\right) = \frac{6 - 4i - 9i - 6}{9+4}$$

$$\therefore \quad 2a + bi = \frac{0 - 13i}{13} = 0 - i$$

$$\therefore \quad 2a = 0 \quad \text{and} \quad b = -1 \qquad \text{i.e.,} \quad a = 0, \quad b = -1$$

4 **a** If $\arg z = \frac{\pi}{2}$, then we have a ray vertically upwards beginning at the origin.

If $\arg(z-i) = \frac{\pi}{2}$, the graph is translated $[0,\ 1]$, and we have a ray vertically upwards beginning at i.

Since $\dfrac{y-1}{x} = \tan\frac{\pi}{2}$, we have $x = 0$,

and geometrically we require $y > 1$.

b

$$\left|\frac{z+2}{z-2}\right| = 2, \qquad \therefore \quad \frac{|\,z+2\,|}{|\,z-2\,|} = 2$$

$$\therefore \quad |z+2| = 2\,|z-2|$$

If $z = x + iy$, then

$$\begin{aligned} \sqrt{(x+2)^2+y^2} &= 2\sqrt{(x-2)^2+y^2} \\ \therefore \quad (x+2)^2+y^2 &= 4(x-2)^2+4y^2 \\ \therefore \quad x^2+4x+4+y^2 &= 4x^2-16x+16+4y^2 \\ \text{i.e.,} \quad 3x^2+3y^2-20x+12 &= 0, \quad \text{which is a circle} \end{aligned}$$

5

$$\begin{aligned} \text{Now} \quad 2-2\sqrt{3}i &= 4\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) \\ &= 4 \text{ cis } \left(-\tfrac{\pi}{3}\right) \\ \therefore \quad (2-2\sqrt{3}i)^n &= 4^n \text{ cis } \left(-\tfrac{n\pi}{3}\right) \qquad \{\text{De Moivre}\} \\ \text{This is real if} \quad \sin\left(-\tfrac{n\pi}{3}\right) &= 0 \\ \text{i.e.,} \quad -\tfrac{n\pi}{3} &= k\pi, \quad k \text{ an integer} \\ \therefore \quad n &= 3k \quad \text{where } k \text{ is an integer} \end{aligned}$$

6 The cube roots of -27 are the solutions to $z^3 = -27$.

$$\begin{aligned} \therefore \quad z^3 &= 27 \text{ cis } (\pi + k2\pi) \\ \therefore \quad z &= [27 \text{ cis } (\pi + k2\pi)]^{\frac{1}{3}} \qquad \text{where } k = 0,\ 1,\ 2 \\ \therefore \quad z &= 27^{\frac{1}{3}} \text{ cis } \left(\tfrac{\pi+k2\pi}{3}\right) \\ \therefore \quad z &= 3 \text{ cis } \left(\tfrac{\pi+k2\pi}{3}\right) \\ \therefore \quad z &= 3 \text{ cis } \left(\tfrac{\pi}{3}\right),\ 3 \text{ cis } \pi,\ 3 \text{ cis } \left(\tfrac{5\pi}{3}\right) \\ \therefore \quad z &= 3\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right),\ 3(-1+0i),\ 3\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) \\ \therefore \quad z &= -3 \ \text{ or } \ \tfrac{3}{2} \pm \tfrac{3\sqrt{3}}{2}i \end{aligned}$$

7 **a**

$$\begin{aligned} z &= 4 \text{ cis } \theta \\ z^3 &= (4 \text{ cis } \theta)^3 \\ &= 4^3 \text{ cis } 3\theta \\ \therefore \quad \left|z^3\right| &= 64 \\ \text{and} \quad \arg\left(z^3\right) &= 3\theta \end{aligned}$$

b

$$\begin{aligned} \frac{1}{z} &= z^{-1} \\ &= (4 \text{ cis } \theta)^{-1} \\ &= 4^{-1} \text{ cis } (-\theta) \end{aligned}$$

$$\therefore \quad \left|\frac{1}{z}\right| = \tfrac{1}{4} \quad \text{and} \quad \arg\left(\frac{1}{z}\right) = -\theta$$

c

$$\begin{aligned} \text{If} \quad z &= 4 \text{ cis } \theta \\ z^* &= 4 \text{ cis } (-\theta) \\ iz^* &= \left(\text{cis } \tfrac{\pi}{2}\right)(4 \text{ cis } (-\theta)) \\ &= 4 \text{ cis } \left(\tfrac{\pi}{2} - \theta\right) \end{aligned}$$

$$\therefore \quad |iz^*| = 4 \quad \text{and} \quad \arg\left(iz^*\right) = \tfrac{\pi}{2} - \theta$$

8 **a**

$$\begin{aligned} \text{If} \quad z &= \text{cis } \phi \\ &= \cos\phi + i\sin\phi \\ \therefore \quad |z| &= \sqrt{\cos^2\phi + \sin^2\phi} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

b

$$\begin{aligned} \text{If} \quad z &= \text{cis } \phi \\ \text{then} \quad z^* &= \text{cis } (-\phi) \\ \therefore \quad z\, z^* &= \text{cis } \phi\, (\text{cis } (-\phi)) \\ &= \text{cis } (\theta + (-\phi)) \\ &= \text{cis } 0 \\ &= 1 \\ \therefore \quad z^* &= \frac{1}{z} \end{aligned}$$

c

$$\begin{aligned} z &= \text{cis } \phi \\ \therefore \quad z^n &= \text{cis } n\phi \\ \therefore \quad z^{-n} &= \text{cis } (-n\phi) \\ \therefore \quad z^n - \frac{1}{z^n} &= \text{cis } n\phi - \text{cis } (-n\phi) \\ &= (\cos n\phi + i\sin n\phi) - (\cos(-n\phi) + i\sin(-n\phi)) \\ &= \cos n\phi + i\sin n\phi - \cos n\phi + i\sin n\phi \\ &= 2i\sin n\phi \end{aligned}$$

9 **a**

$$\begin{aligned} \textit{To prove:} \quad \arg z^n &= n\arg z \\ \text{Let} \quad z &= r \text{ cis } \theta \\ \therefore \quad z^n &= r^n \text{ cis } n\theta \quad \{\text{De Moivre}\} \\ \text{and so} \quad \arg z^n &= n\theta \\ \text{i.e.,} \quad \arg z^n &= n\arg z \quad \text{as required} \end{aligned}$$

b

$$\begin{aligned} \left(\frac{z}{w}\right)^* &= \left(\frac{a+bi}{c+di}\right)^* \\ &= \left(\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right)^* \\ &= \left(\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}\right)^* \\ &= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2} \end{aligned}$$

and also

$$\begin{aligned} \frac{z^*}{w^*} &= \frac{a-bi}{c-di} \\ &= \frac{(a-bi)(c+di)}{(c-di)(c+di)} \\ &= \frac{(ac+bd)-i(bc-ad)}{c^2+d^2} \end{aligned}$$

10 **a**

$$\begin{aligned} &\cos 3\theta + i\sin 3\theta \\ &= \text{cis } 3\theta \\ &= (\text{cis } \theta)^3 \qquad \therefore \quad n = 3 \end{aligned}$$

b

$$\begin{aligned} &\frac{1}{\cos 2\theta + i\sin 2\theta} \times \frac{\cos 2\theta - i\sin 2\theta}{\cos 2\theta - i\sin 2\theta} \\ &= \frac{\cos 2\theta - i\sin 2\theta}{\cos^2 2\theta + \sin^2 2\theta} \\ &= \cos 2\theta - i\sin 2\theta \\ &= \cos(-2\theta) + i\sin(-2\theta) \\ &= \text{cis } (-2\theta) \\ &= (\text{cis } \theta)^{-2} \qquad \therefore \quad n = -2 \end{aligned}$$

or

$$\begin{aligned} \frac{1}{\cos 2\theta + i\sin 2\theta} &= \frac{1}{\text{cis } 2\theta} \\ &= (\text{cis } 2\theta)^{-1} \\ &= [(\text{cis } \theta)^2]^{-1} \\ &= (\text{cis } \theta)^{-2} \\ \therefore \quad n &= -2 \end{aligned}$$

c

$$\begin{aligned} &\cos\theta - i\sin\theta \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \text{cis } (-\theta) \\ &= (\text{cis } \theta)^{-1} \qquad \therefore \quad n = -1 \end{aligned}$$

11 The fifth roots of i are the solutions to $z^5 = i$

$\therefore \quad z^5 = 1 \text{ cis } \left(\frac{\pi}{2} + k2\pi\right)$

$\therefore \quad z = 1^{\frac{1}{5}} \left[\text{cis } \left(\frac{\pi}{2} + k2\pi\right)\right]^{\frac{1}{5}}$

$\therefore \quad z = \text{cis } \left(\frac{\pi}{10} + \frac{k2\pi}{5}\right)$

$\therefore \quad z = \text{cis } \left(\frac{\pi}{10} + \frac{k4\pi}{10}\right)$

$\therefore \quad z_1 = \text{cis } \left(\frac{\pi}{10}\right), \ \text{cis } \left(\frac{5\pi}{10}\right) = i, \ \text{cis } \left(\frac{9\pi}{10}\right), \ \text{cis } \left(\frac{13\pi}{10}\right), \ \text{cis } \left(\frac{17\pi}{10}\right)$ (letting $k = 0, 1, 2, 3, 4$)

12 Let $z = x + iy \quad \therefore \quad z + \frac{1}{z}$

$$\begin{aligned} z + \frac{1}{z} &= (x+iy) + \frac{1}{x+iy} \times \frac{x-iy}{x-iy} \\ &= (x+iy) + \frac{(x-iy)}{x^2+y^2} \\ &= \frac{\left(x^2+y^2\right)(x+iy) + (x-iy)}{x^2+y^2} \\ &= \frac{x^3 + ix^2y + xy^2 + y^3i + x - iy}{x^2+y^2} \\ &= \frac{x(x^2+y^2+1) + i(x^2+y^2-1)y}{x^2+y^2} \end{aligned}$$

which is real if $\frac{(x^2+y^2-1)y}{x^2+y^2} = 0$ i.e., $x^2 + y^2 - 1 = 0$ or $y = 0$

i.e., $x^2 + y^2 = 1$ or $y = 0$

i.e., $|z|^2 = 1$ or $y = 0$

i.e., $|z| = 1$ or z is real

or Let $z = r \text{ cis}\,\theta \quad \therefore \quad z + \frac{1}{z} = r \text{ cis } \theta + \frac{1}{r} \text{ cis } (-\theta)$

$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos(-\theta) + i \times \frac{i}{r}\sin(-\theta)$

This is real if $r\sin\theta + \frac{1}{r}\sin(-\theta) = 0$

$\therefore \quad r\sin\theta - \frac{1}{r}\sin\theta = 0$

$\therefore \quad \sin\theta(r - \frac{1}{r}) = 0$

$\therefore \quad \sin\theta = 0$ or $r - \frac{1}{r} = 0,$

i.e., $\theta = 0$ or $r = 1$

i.e., $z = r$ (which is real) or $|z| = 1$

13 **a** If w is the root of $z^5 = 1$ with smallest positive argument, then $w = \text{cis } \left(\frac{2\pi}{5}\right)$ and $w^4 = \text{cis } \left(\frac{8\pi}{5}\right)$.

These have sum, $\alpha + \beta = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$

$= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)$

$= 2\cos\left(\frac{2\pi}{5}\right)$

and product, $\alpha\beta = \text{cis } \left(\frac{2\pi}{5}\right) \times \text{cis } \left(\frac{8\pi}{5}\right) = \text{cis } \left(\frac{10\pi}{5}\right) = \text{cis } 2\pi = 1$

$\therefore$ a real quadratic with roots w, w^4 is $a\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right) = 0, \quad a \neq 0$

b Let $\alpha = w + w^4$ and $\beta = w^2 + w^3$

$$\begin{aligned} \text{Now we know that}\quad 1 + w + w^2 + w^3 + w^4 &= 0 \quad ......(*) \\ 1 + (w + w^4) + (w^2 + w^3) &= 0 \\ 1 + \alpha + \beta &= 0 \\ \alpha + \beta &= -1 \end{aligned}$$

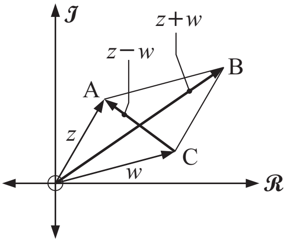
$$\begin{aligned} \text{and}\quad \alpha\beta &= (w + w^4)(w^2 + w^3) \\ &= w^3 + w^4 + w^6 + w^7 \\ &= w^3 + w^4 + w + w^2 \quad \{\text{as } w^5 = 1\} \\ &= w + w^2 + w^3 + w^4 \\ &= -1 \qquad \{\text{from } *\} \end{aligned}$$

$\therefore$ the quadratic factor is $a(z^2 + z - 1) = 0, \quad a \neq 0$

14 Consider the diagram which shows vectors w, z, $z + w$ and $z - w$.

Clearly OABC is a parallelogram with $\overrightarrow{\text{OB}} = z + w$ and $\overrightarrow{\text{CA}} = z - w$

If $|\, z + w \,| = |\, z - w \,|$, the diagonals are equal in length. Hence, OABC is actually a rectangle and so angle COA is a right angle i.e., $\arg z$ and $\arg w$ differ by $\frac{\pi}{2}$.



REVIEW SET 16C

1 **a**
$$\begin{aligned} z &= r \text{ cis } \theta \\ \overline{z} &= r \text{ cis } (-\theta) \end{aligned}$$
$\therefore$ T is a reflection in the R-axis

b
$$\begin{aligned} z &= r \text{ cis } \theta \\ -z &= -r \text{ cis } \theta \\ &= \text{cis } \pi \times r \text{ cis } \theta \\ &= r \text{ cis } (\theta + \pi) \end{aligned}$$
$\therefore$ T is a rotation of π about O.

c
$$\begin{aligned} z &= r \text{ cis } \theta \\ iz &= ir \text{ cis } \theta \\ &= \text{cis } \left(\tfrac{\pi}{2}\right) r \text{ cis } \theta \\ &= r \text{ cis } \left(\theta + \tfrac{\pi}{2}\right) \end{aligned}$$
$\therefore$ T is an anticlockwise rotation of $\frac{\pi}{2}$ about O.

2 Let $z = a + bi$ and $w = c + di$ $\therefore$ $z + w = (a + c) + i(b + d)$, so $b + d = 0$ ...... (1)

$$\begin{aligned} \text{and}\quad zw &= (a + bi)(c + di) \\ &= [ac - bd] + i[ad + bc] \end{aligned}$$

so, $ad + bc = 0$ ...... (2)

$$\begin{aligned} \text{From (1)}\quad d = -b \quad \text{and in (2)}\quad a(-b) + bc &= 0 \\ \therefore\quad b(c - a) &= 0 \\ \therefore\quad a = c \quad \text{as}\quad b &\neq 0 \end{aligned}$$

$$\begin{aligned} \text{So}\quad z &= a + bi \quad \text{and}\quad w = a - bi \\ \therefore\quad z^* &= a - bi = w \end{aligned}$$

3
$$\begin{aligned} \text{If}\quad (x + iy)^n &= X + Yi \\ |(x + yi)^n| &= |X + Yi| \\ |x + iy|^n &= |X + Yi| \\ \left(\sqrt{x^2 + y^2}\right)^n &= \sqrt{X^2 + Y^2} \\ \text{Squaring both sides}\quad X^2 + Y^2 &= (x^2 + y^2)^n \end{aligned}$$

4 $$|z-w|^2+|z+w|^2 = 2\left(|z|^2+|w|^2\right)$$

Let $z=a+bi \qquad w=c+di$

$$\begin{aligned}
\therefore\quad \text{LHS} &= |(a-c)+i(b-d)|^2+|(a+c)+i(b+d)|^2\\
&= \left(\sqrt{(a-c)^2+(b-d)^2}\right)^2+\left(\sqrt{(a+c)^2+(b+d)^2}\right)^2\\
&= (a-c)^2+(b-d)^2+(a+c)^2+(b+d)^2\\
&= a^2-2ac+c^2+b^2-2bd+d^2+a^2+2ac+c^2+b^2+2bd+d^2\\
&= 2(a^2+b^2)+2(c^2+d^2)\\
&= 2\left(\sqrt{a^2+b^2}\right)^2+2\left(\sqrt{c^2+d^2}\right)^2\\
&= 2\,|z|^2+2\,|w|^2\\
&= 2\left[|z|^2+|w|^2\right]
\end{aligned}$$

5 **a** Since 1 is a root of $z^5-1=0$, we find that

$z^5-1=(z-1)(1+z+z^2+z^3+z^4)$

$$\begin{array}{c|cccccc|}
1 & 1 & 0 & 0 & 0 & 0 & -1\\
 & 0 & 1 & 1 & 1 & 1 & 1\\
\hline
 & 1 & 1 & 1 & 1 & 1 & \multicolumn{1}{|c}{0}
\end{array}$$

$\therefore$ since α is a root,

$(\alpha-1)(1+\alpha+\alpha^2+\alpha^3+\alpha^4) = 0$

But $\alpha\neq 1$,

so $1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$

or Note that $1+\alpha+\alpha^2+\alpha^3+\alpha^4$ is the sum of a geometric series.

$$\text{Then}\quad S_n = \frac{a(1-r^n)}{1-r} \qquad a=1,\quad r=\alpha,\quad n=5$$

$$\therefore\quad S_5 = \frac{1(1-\alpha^5)}{1-\alpha} \qquad \text{and}\quad \alpha^5 = \text{cis }(2\pi)=1$$

$$\therefore\quad 1+\alpha+\alpha^2+\alpha^3+\alpha^4 = \frac{1(1-1)}{1-\alpha}=0$$

b Now if $\left(\dfrac{z+2}{z-1}\right)^5=1$ then $\dfrac{z+2}{z-1}=1,\ \alpha,\ \alpha^2,\ \alpha^3$ or α^4 where $\alpha=\text{cis}\left(\frac{2\pi}{5}\right)$

$$\begin{aligned}
\text{i.e.,}\quad \frac{z+2}{z-1} &= \alpha^n \quad \text{where}\quad n=0,1,2,3,4\\
z+2 &= \alpha^n(z-1)\\
z+2 &= \alpha^n z-\alpha^n\\
z(\alpha^n-1) &= \alpha^n+2 \qquad \text{and so}\quad z=\frac{\alpha^n+2}{\alpha^n-1}
\end{aligned}$$

$\therefore$ roots of $\left(\dfrac{z+2}{z-1}\right)^5=1$ are $\dfrac{\alpha+2}{\alpha-1},\ \dfrac{\alpha^2+2}{\alpha^2-1},\ \dfrac{\alpha^3+2}{\alpha^3-1},\ \dfrac{\alpha^4+2}{\alpha^4-1}$, where $\alpha=\text{cis}\left(\frac{2\pi}{5}\right)$

Note: The case where $n=0$ requires $\dfrac{z+2}{z-1}=1$, and so $z+2=z-1$, which has no solution.

6

$$\text{Since}\quad \frac{|\,z+1\,|}{|\,z-1\,|}=1,\quad \text{then}\quad |z+1|=|z-1|$$

$$\begin{aligned}
\text{Letting}\quad z &= x+iy,\\
\therefore\quad |(x+1)+iy| &= |(x-1)+iy|\\
\therefore\quad \sqrt{(x+1)^2+y^2} &= \sqrt{(x-1)^2+y^2}
\end{aligned}$$

Squaring both sides, we get $(x+1)^2+y^2 = (x-1)^2+y^2$

Therefore since $z\neq 0$, z is purely imaginary.

$\therefore\quad x^2+2x+1 = x^2-2x+1 \qquad \therefore\quad 4x=0 \qquad \therefore\quad x=0$

7 $$w = \frac{1+z}{1+z^*} = \frac{1 \;+\; \text{cis }\phi}{1 \;+\; \text{cis }(-\phi)} \times \frac{\text{cis }\phi}{\text{cis }\phi}$$

$$= \frac{(1 \;+\; \text{cis }\phi)\text{ cis }\phi}{\text{cis }\phi \;+\; \text{cis }0}$$

$$= \frac{(1 \;+\; \text{cis}\phi)\text{ cis }\phi}{1 \;+\; \text{cis }\phi}$$

$$= \text{cis }\phi$$

8 The cube roots of $-8i$ are solutions to $z^3 = -8i$

$$z^3 = 8\text{ cis }\left(-\tfrac{\pi}{2} + k2\pi\right) \quad \text{where} \quad k \text{ is an integer}$$

$$\therefore \quad z = 8^{\frac{1}{3}}\text{ cis }\left(\tfrac{-\frac{\pi}{2}+k2\pi}{3}\right)$$

$$\therefore \quad z = 2\text{ cis }\left(-\tfrac{\pi}{6} + \tfrac{k4\pi}{6}\right)$$

$$\therefore \quad z = 2\text{ cis }\left(-\tfrac{\pi}{6}\right),\ 2\text{ cis }\left(\tfrac{\pi}{2}\right),\ 2\text{ cis }\left(\tfrac{7\pi}{6}\right) \quad \{\text{letting} \quad k = 0,\,1,\,2\}$$

$$\therefore \quad z = \sqrt{3} - i,\ 2i,\ -\sqrt{3} - i$$

9 $$-1 + i\sqrt{3} = 2\text{ cis }\left(\tfrac{2\pi}{3}\right)$$

$$\therefore \quad (-1+i\sqrt{3})^m = 2^m\text{ cis }\left(\tfrac{m2\pi}{3}\right) \qquad \{\text{De Moivre}\}$$

$$= 2^m\left[\cos\left(\tfrac{m2\pi}{3}\right) + i\sin\left(\tfrac{m2\pi}{3}\right)\right]$$

This is real provided $\sin\left(\frac{m2\pi}{3}\right) = 0$

$$\therefore \quad \tfrac{m2\pi}{3} = 0 + k\pi$$

$$\therefore \quad m = \tfrac{3k}{2} \quad \text{where } k \text{ is any integer}$$

10 We first note that $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

and $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ ...... (∗)

Now $$\text{cis }\theta + \text{cis }\phi = (\cos\theta + i\sin\theta) + (\cos\phi + i\sin\phi)$$

$$= (\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)$$

$$= \left[\cos\left(\frac{\theta+\phi}{2} + \frac{\theta-\phi}{2}\right) + \cos\left(\frac{\theta+\phi}{2} - \frac{\theta-\phi}{2}\right)\right]$$

$$+i\left[\sin\left(\frac{\theta+\phi}{2} + \frac{\theta-\phi}{2}\right) + \sin\left(\frac{\theta+\phi}{2} - \frac{\theta-\phi}{2}\right)\right]$$

$$= 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + 2i\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \qquad \{\text{using } *\}$$

$$= 2\cos\left(\frac{\theta-\phi}{2}\right)\text{ cis }\left(\frac{\theta+\phi}{2}\right) \qquad \text{as required}$$

Now $Z^5 = 1$ has solutions $Z = 1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4$ where $\alpha = \text{cis }\left(\frac{2\pi}{5}\right)$

i.e., $Z = \text{cis }\left(\frac{2n\pi}{5}\right)$ where $n = 0,\,1,\,2,\,3,\,4$

$$\therefore \quad \text{if} \quad \left(\frac{z+1}{z-1}\right)^5 = 1, \quad \text{then} \quad \frac{z+1}{z-1} = \text{cis }\left(\tfrac{2n\pi}{5}\right)$$

$$\therefore \quad z+1 = \text{cis }\left(\tfrac{2n\pi}{5}\right)(z-1)$$

$$\therefore \quad z\left(\text{cis }\left(\tfrac{2n\pi}{5}\right) - 1\right) = \text{cis }\left(\tfrac{2n\pi}{5}\right) + 1$$

$$\therefore \quad z = \frac{\text{cis }\left(\frac{2n\pi}{5}\right) + 1}{\text{cis }\left(\frac{2n\pi}{5}\right) - 1}$$

$$\therefore \quad z = \frac{\text{cis}\left(\frac{2n\pi}{5}\right) + \text{cis}\ 0}{\text{cis}\left(\frac{2n\pi}{5}\right) + \text{cis}\ \pi}$$

$$\therefore \quad z = \frac{2\cos\left(\dfrac{\frac{2n\pi}{5}-0}{2}\right)\ \text{cis}\left[\dfrac{\frac{2n\pi}{5}+0}{2}\right]}{2\cos\left(\dfrac{\frac{2n\pi}{5}-\pi}{2}\right)\ \text{cis}\left[\dfrac{\frac{2n\pi}{5}+\pi}{2}\right]} \qquad \{\text{using the above identity}\}$$

$$\therefore \quad z = \frac{\cos\frac{n\pi}{5}}{\cos\left(\frac{n\pi}{5}-\frac{\pi}{2}\right)}\ \text{cis}\left[\tfrac{n\pi}{5} - \left(\tfrac{n\pi}{5}+\tfrac{\pi}{2}\right)\right]$$

$$\therefore \quad z = \frac{\cos\frac{n\pi}{5}}{\cos\left[\left(\frac{\pi}{2}-\frac{n\pi}{5}\right)\right]}\ \text{cis}\left(-\tfrac{\pi}{2}\right)$$

$$\therefore \quad z = \frac{\cos\frac{n\pi}{5}}{\sin\frac{n\pi}{5}}(-i)$$

$$\therefore \quad z = -i\cot\tfrac{n\pi}{5}, \quad n = 1, 2, 3, 4 \quad \{\text{neglecting } n=0 \text{ since } \cot 0 \text{ is undefined}\}$$

$$\text{Now} \quad \cot\left(\tfrac{n\pi}{5}\right) = -\cot\left(\pi - \tfrac{n\pi}{5}\right)$$

$$= -\cot\left(\tfrac{(5-n)\pi}{5}\right), \quad n = 1, 2, 3, 4$$

$$= -\cot\left(\tfrac{n\pi}{5}\right), \quad n = 4, 3, 2, 1$$

$$\therefore \quad z = i\cot\left(\tfrac{n\pi}{5}\right), \quad n = 1, 2, 3, 4$$

11 The 3 cube roots of $-64i$ are the solutions to $z^3 = -64i$

$$\therefore \quad z^3 = 64\ \text{cis}\ (-\tfrac{\pi}{2} + k2\pi) \quad \text{for integer } k$$

$$\therefore \quad z = \left[64\ \text{cis}\ (-\tfrac{\pi}{2} + k2\pi)\right]^{\frac{1}{3}}$$

$$\therefore \quad z = 64^{\frac{1}{3}}\ \text{cis}\left(\frac{-\frac{\pi}{2}+k2\pi}{3}\right)$$

$$\therefore \quad z = 4\ \text{cis}\left(-\tfrac{\pi}{6} + \tfrac{k4\pi}{6}\right)$$

$$\therefore \quad z = 4\ \text{cis}\left(-\tfrac{\pi}{6}\right),\ 4\ \text{cis}\left(\tfrac{\pi}{2}\right),\ 4\ \text{cis}\left(\tfrac{7\pi}{6}\right) \quad \{\text{letting } k = 0, 1, 2\}$$

$$\therefore \quad z = 4(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i),\ 4i,\ 4\ \text{cis}\ (-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i)$$

$$\therefore \quad z = 2\sqrt{3} - 2i, \quad 4i, \quad -2\sqrt{3} - 2i$$

12 **a**

$$(2z)^{-1} = (2\ \text{cis}\ \theta)^{-1}$$

$$= 2^{-1}\ \text{cis}\ (-\theta)$$

$$\therefore \quad \left|(2z)^{-1}\right| = \tfrac{1}{2} \quad \text{and}$$

$$\arg\left[(2z)^{-1}\right] = -\theta$$

b

$$1 - z = 1 - \text{cis}\ \theta$$

$$= (1 - \cos\theta) - i\sin\theta$$

$$\therefore \quad |1 - z| = \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

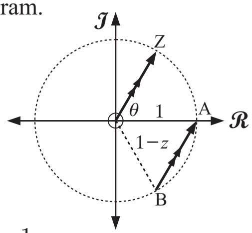
$$= \sqrt{2 - 2\cos\theta}$$

$$= \sqrt{4(\tfrac{1}{2} - \tfrac{1}{2}\cos\theta)}$$

$$= 2\sqrt{\sin^2\left(\tfrac{\theta}{2}\right)}$$

$$= 2\sin\left(\tfrac{\theta}{2}\right)$$

w can be represented on a diagram.



ΔOAB is isosceles since $|z| = 1$,

so we let $\measuredangle\text{AOB} = \measuredangle\text{ABO} = \phi$

Since $\text{OZ} \parallel \text{AB}$, $\measuredangle\text{OAB} = \theta$ {alternate $\measuredangle$'s}

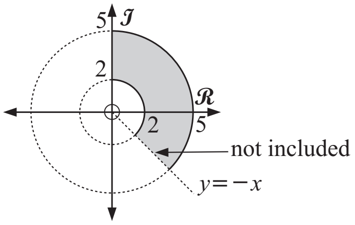
$$\therefore \quad \phi + \phi + \theta = \pi$$
$$\therefore \quad 2\phi = \pi - \theta$$
$$\phi = \tfrac{\pi}{2} - \tfrac{\theta}{2}$$

But $\arg(1 - z) = -\phi$,

so $\arg(1 - z) = -\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$= \frac{\theta}{2} - \frac{\pi}{2}$

13 $\{z\text{: } 2 \leqslant |z| \leqslant 5 \text{ and } -\frac{\pi}{4} < \arg z \leqslant \frac{\pi}{2}\}$



14 If $z = r \text{ cis } \theta$, then $|z| = r$ and $\arg z = \theta$

Now $$\frac{1}{z} = (r \text{ cis } \theta)^{-1} = r^{-1} \text{ cis } (-\theta)$$
$$= \frac{1}{r} \text{ cis } (-\theta)$$
$$= \frac{1}{|z|} \text{ cis } (-\theta)$$

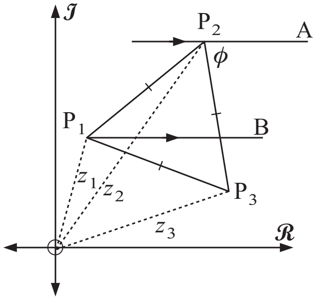
$\left|\frac{1}{z}\right| = \frac{1}{|z|}$ if $z \neq 0$ and $\arg\left(\frac{1}{z}\right) = -\theta = -\arg z$

14

$$z = \text{cis } \alpha$$
$$\therefore \quad 1 + z = 1 + \text{cis } \alpha$$
$$= 1 + \cos\theta + i\sin\theta$$
$$= \left[1 + 2\cos^2\left(\tfrac{\alpha}{2}\right) - 1\right] + i\left[2\sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)\right]$$
$$= 2\cos^2\left(\tfrac{\alpha}{2}\right) + i\left[2\sin\left(\tfrac{\alpha}{2}\right)\cos\left(\tfrac{\alpha}{2}\right)\right]$$
$$= 2\cos\left(\tfrac{\alpha}{2}\right)\left[\cos\left(\tfrac{\alpha}{2}\right) + i\sin\left(\tfrac{\alpha}{2}\right)\right]$$
$$= 2\cos\left(\tfrac{\alpha}{2}\right) \text{ cis } \left(\tfrac{\alpha}{2}\right)$$
$$\therefore \quad |1 + z| = 2\cos\left(\tfrac{\alpha}{2}\right)$$

and $\arg(1 + z) = \frac{\alpha}{2}$

15 **a**



Now $z_2 - z_1 = \overrightarrow{P_1P_2}$

so $\arg(z_2 - z_1) = \alpha$ as shown on the diagram alongside

$z_3 - z_2 = \overrightarrow{P_2P_3}$

so $\arg(z_3 - z_2) = -\phi$

Now $\measuredangle\text{AP}_2\text{P}_1 = \alpha$ since $\text{AP}_2 \parallel \text{P}_1\text{B}$

and $\measuredangle\text{P}_1\text{P}_2\text{P}_3 = \frac{\pi}{3}$ since the Δ is equilateral

Hence $\alpha + \frac{\pi}{3} + \phi = \pi$

$\therefore \quad \phi = -\alpha + \frac{2\pi}{3}$

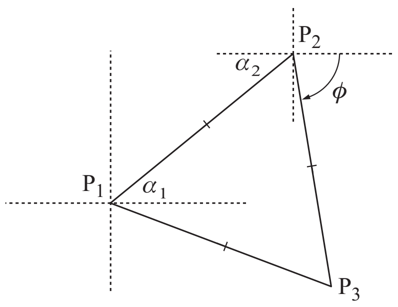
$\therefore \quad \arg(z_3 - z_2) = \alpha - \frac{2\pi}{3}$ as required

b

$$\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = \left|\frac{\overrightarrow{P_1P_2}}{\overrightarrow{P_2P_3}}\right|$$

$= 1$ since the Δ is equilateral

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$$
$$= \alpha - (\alpha - \tfrac{2\pi}{3})$$
$$= \alpha - \alpha + \tfrac{2\pi}{3}$$
$$= \tfrac{2\pi}{3}$$



Chapter 17

LINES AND PLANES IN SPACE

EXERCISE 17A.1

1 **a** **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-4\end{bmatrix}+t\begin{bmatrix}1\\4\end{bmatrix}$ **ii** $x=3+t,\quad y=-4+4t,\quad t\in\mathcal{R}$

b **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}+t\begin{bmatrix}-8\\2\end{bmatrix}$ **ii** $x=5-8t,\quad y=2+2t,\quad t\in\mathcal{R}$

c **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-6\\0\end{bmatrix}+t\begin{bmatrix}3\\7\end{bmatrix}$ **ii** $x=-6+3t,\quad y=7t,\quad t\in\mathcal{R}$

d **i** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\11\end{bmatrix}+t\begin{bmatrix}-2\\1\end{bmatrix}$ **ii** $x=-1-2t,\quad y=11+t,\quad t\in\mathcal{R}$

2 $x=-1+2\lambda,\quad y=4-\lambda,\quad \lambda\in\mathcal{R}$

When $\lambda=0$, $x=-1+2(0)=-1$ and $y=4-0=4$ $\therefore$ point is $(-1,\ 4)$
When $\lambda=1$, $x=-1+2(1)=1$ and $y=4-1=3$ $\therefore$ point is $(1,\ 3)$
When $\lambda=3$, $x=-1+2(3)=5$ and $y=4-3=1$ $\therefore$ point is $(5,\ 1)$
When $\lambda=-1$, $x=-1+2(-1)=-3$ and $y=4--1=5$ $\therefore$ point is $(-3,\ 5)$
When $\lambda=-4$, $x=-1+2(-4)=-9$ and $y=4--4=8$ $\therefore$ point is $(-9,\ 8)$

3 **a** If $t+2=3$ and $1-3t=-2$, we get $t=1$ and $-3t=-3$ i.e., $t=1$
Since $t=1$ in each case, $(3,\ -2)$ lies on the line.

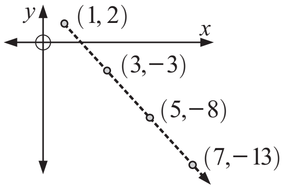
If $t+2=0$ and $1-3t+6$, we get $t=-2$ and $-3t=5$ i.e., $t=-\frac{5}{3}$
$\therefore$ $(0,\ 6)$ does not lie on the line.

b If $(k,\ 4)$ lies on $x=1-2t,\quad y=1+t,$ then

$$\begin{aligned} k &= 1-2t \quad\text{and}\quad 4=1+t\\ \therefore\quad t &= 3 \quad\text{and}\quad k=1-6=-5\\ \text{i.e.,}\quad k &= -5\end{aligned}$$

4 **a** $x(0)=1$ and $y(0)=2$,
$\therefore$ the initial position is $(1,\ 2)$

b



c In 1 second, the
x-step is 2 and y-step is -5, which is
a distance of $\sqrt{2^2+(-5)^2}=\sqrt{29}$
$\therefore$ speed is $\sqrt{29}$ cm/sec.

EXERCISE 17A.2

1 **a** The vector equation is $\begin{bmatrix}x-1\\y-3\\z+7\end{bmatrix}=t\begin{bmatrix}2\\1\\3\end{bmatrix}$ $\therefore$ $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\-7\end{bmatrix}+t\begin{bmatrix}2\\1\\3\end{bmatrix},\quad t\in\mathcal{R}$

b The vector equation is $\begin{bmatrix}x-0\\y-1\\z-2\end{bmatrix}=t\begin{bmatrix}1\\1\\-2\end{bmatrix}$ $\therefore$ $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\1\\2\end{bmatrix}+t\begin{bmatrix}1\\1\\-2\end{bmatrix},\quad t\in\mathcal{R}$

c Since the line is parallel to the X-axis, it has direction vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\therefore$ the vector equation is $\begin{bmatrix}x+2\\y-2\\z-1\end{bmatrix}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\therefore$ $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\2\\1\end{bmatrix}+t\begin{bmatrix}1\\0\\0\end{bmatrix},\quad t\in\mathcal{R}$

2 **a** The parametric equations are:

$x = 5 + (-1)t, \quad y = 2 + 2t, \quad z = -1 + 6t$

i.e., $x = 5 - t, \; y = 2 + 2t, \; z = -1 + 6t, \quad t \in \mathcal{R}$

b The parametric equations are:

$x = 0 + 2t, \quad y = 2 + (-1)t, \quad z = -1 + 3t$

i.e., $x = 2t, \; y = 2 - t, \; z = -1 + 3t, \quad t \in \mathcal{R}$

c Since the line is perpendicular to the XOY plane, it has direction vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore$ the parametric equations are:

$x = 3 + 0t, \quad y = 2 + 0t, \quad z = -1 + 1t$

i.e., $x = 3, \; y = 2, \; z = -1 + t, \quad t \in \mathcal{R}$

3 **a** $\overrightarrow{AB} = \begin{bmatrix} -1-1 \\ 3-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \therefore \; x = 1 - 2t, \quad y = 2 + t, \quad z = 1 + t, \quad t \in \mathcal{R}$

b $\overrightarrow{CD} = \begin{bmatrix} 3-0 \\ 1-1 \\ -1-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \quad \therefore \; x = 3t, \quad y = 1, \quad z = 3 - 4t, \quad t \in \mathcal{R}$

c $\overrightarrow{EF} = \begin{bmatrix} 1-1 \\ -1-2 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \quad \therefore \; x = 1, \quad y = 2 - 3t, \quad z = 5, \quad t \in \mathcal{R}$

d $\overrightarrow{GH} = \begin{bmatrix} 5-0 \\ -1-1 \\ 3--1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \quad \therefore \; x = 5t, \quad y = 1 - 2t, \quad z = -1 + 4t, \quad t \in \mathcal{R}$

4 Given $x = 1 - \lambda, \quad y = 3 + \lambda, \quad z = 3 - 2\lambda$:

a The line meets the XOY plane when $z = 0$ i.e., $3 - 2\lambda = 0 \quad \therefore \; \lambda = \frac{3}{2}$

Then $x = 1 - \frac{3}{2} = -\frac{1}{2}$ and $y = 3 + \frac{3}{2} = \frac{9}{2}$, i.e., the point is $(-\frac{1}{2}, \frac{9}{2}, 0)$

b The line meets the YOZ plane when $x = 0$ i.e., $1 - \lambda = 0$

$\therefore \; \lambda = 1$

Then $y = 3 + 1 = 4$ and $z = 3 - 2 = 1$, i.e., the point is $(0, 4, 1)$

c The line meets the XOZ plane when $y = 0$ i.e., $3 + \lambda = 0 \quad \therefore \; \lambda = -3$

Then $x = 1 - (-3) = 4$ and $z = 3 - 2(-3) = 9$, i.e., the point is $(4, 0, 9)$

5 Given a line with equations $x = 2 - \lambda, \quad y = 3 + 2\lambda$ and $z = 1 + \lambda$

the distance to the point $(1, 0, -2)$ is $\sqrt{(2 - \lambda - 1)^2 + (3 + 2\lambda - 0)^2 + (1 + \lambda + 2)^2}$.

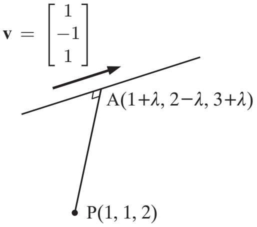
$$\begin{aligned}
\text{But this distance} &= 5\sqrt{3} \text{ units} \\
\therefore \quad \sqrt{(1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2} &= 5\sqrt{3} \\
\therefore \quad (1-\lambda)^2 + (3+2\lambda)^2 + (\lambda+3)^2 &= 75 \\
\therefore \quad 1 - 2\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 + \lambda^2 + 6\lambda + 9 &= 75 \\
\therefore \quad 6\lambda^2 + 16\lambda - 56 &= 0 \\
\therefore \quad 3\lambda^2 + 8\lambda - 28 &= 0 \\
\therefore \quad (3\lambda + 14)(\lambda - 2) &= 0 \\
\therefore \quad \lambda &= -\tfrac{14}{3} \text{ or } \lambda = 2
\end{aligned}$$

When $\lambda = 2$, the point is $(0, 7, 3)$

and when $\lambda = -\frac{14}{3}$, the point is $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$.

6 **a** Let $A(1+\lambda,\ 2-\lambda,\ 3+\lambda)$ be a point on the given line.

Then $\overrightarrow{PA} = \begin{bmatrix}1+\lambda-1\\2-\lambda-1\\3+\lambda-2\end{bmatrix} = \begin{bmatrix}\lambda\\1-\lambda\\1+\lambda\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$ is the direction vector of the line.



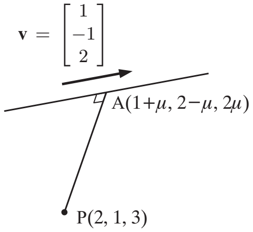
If $\overrightarrow{PA}$ and $\mathbf{v}$ are perpendicular, then $\overrightarrow{PA} \bullet \mathbf{v} = 0$

$$\therefore \begin{bmatrix}\lambda\\1-\lambda\\1+\lambda\end{bmatrix} \bullet \begin{bmatrix}1\\-1\\1\end{bmatrix} = 0 \qquad \begin{aligned}\therefore\quad \lambda-1(1-\lambda)+1(1+\lambda) &= 0\\ \therefore\quad \lambda-1+\lambda+1+\lambda &= 0\\ \therefore\quad 3\lambda &= 0\\ \therefore\quad \lambda &= 0\end{aligned}$$

$\therefore$ A is at (1, 2, 3) i.e., the foot of the perpendicular is (1, 2, 3)

b Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.
Then A is at $(1+\mu,\ 2-\mu,\ 2\mu)$ for some μ.

Now $\overrightarrow{PA} = \begin{bmatrix}1+\mu-2\\2-\mu-1\\2\mu-3\end{bmatrix} = \begin{bmatrix}\mu-1\\1-\mu\\2\mu-3\end{bmatrix}$



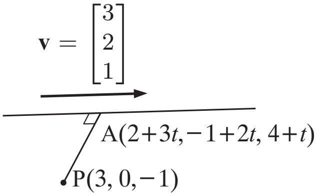
and $\mathbf{v} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$ is the direction vector of the line.

Since $\overrightarrow{PA} \bullet \mathbf{v} = 0$, $\begin{bmatrix}\mu-1\\1-\mu\\2\mu-3\end{bmatrix} \bullet \begin{bmatrix}1\\-1\\2\end{bmatrix} = 0$

$$\begin{aligned}\therefore\quad 1(\mu-1)-1(1-\mu)+2(2\mu-3) &= 0\\ \therefore\quad \mu-1-1+\mu+4\mu-6 &= 0\\ \therefore\quad 6\mu &= 8\\ \therefore\quad \mu &= \tfrac{4}{3}\end{aligned}$$

$\therefore$ A is at $(\frac{7}{3},\ \frac{2}{3},\ \frac{8}{3})$ i.e., the foot of the perpendicular is $(\frac{7}{3},\ \frac{2}{3},\ \frac{8}{3})$

7 **a** Let A $(2+3t,\ -1+2t,\ 4+t)$ be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.



Now $\overrightarrow{PA} = \begin{bmatrix}2+3t-3\\-1+2t-0\\4+t-(-1)\end{bmatrix} = \begin{bmatrix}3t-1\\2t-1\\t+5\end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ is the direction vector of the line.

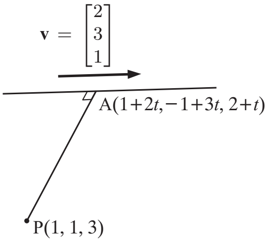
$\therefore$ since $\overrightarrow{PA} \bullet \mathbf{v} = 0$, $\begin{bmatrix}3t-1\\2t-1\\t+5\end{bmatrix} \bullet \begin{bmatrix}3\\2\\1\end{bmatrix} = 0$

$$\begin{aligned}\therefore\quad 3(3t-1)+2(2t-1)+1(t+5) &= 0\\ \therefore\quad 9t-3+4t-2+t+5 &= 0\\ \therefore\quad 14t &= 0\\ \therefore\quad t &= 0\end{aligned}$$

$\therefore$ A is at (2, −1, 4)

$$\therefore \text{ the distance } d = \sqrt{(2-3)^2+(-1-0)^2+(4-(-1))^2} = \sqrt{1+1+25} = \sqrt{27} = 3\sqrt{3} \text{ units}$$

b Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.
Then A is at $(1+2t,\ -1+3t,\ 2+t)$ for some t.



$$\text{Now} \quad \overrightarrow{PA} = \begin{bmatrix} 1+2t-1 \\ -1+3t-1 \\ 2+t-3 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t-2 \\ t-1 \end{bmatrix}$$

$$\text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \text{is the direction vector of the line.}$$

$\therefore$ since $\overrightarrow{PA} \bullet \mathbf{v} = 0$,

$$\begin{bmatrix} 2t \\ 3t-2 \\ t-1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0$$

$$\therefore \quad 4t + 3(3t-2) + 1(t-1) = 0$$
$$\therefore \quad 4t + 9t - 6 + t - 1 = 0$$
$$\therefore \quad 14t = 7$$
$$\therefore \quad t = \tfrac{1}{2}$$
$$\therefore \quad \text{A is at } (2,\ \tfrac{1}{2},\ \tfrac{5}{2})$$

$$\therefore \quad \text{the distance} \quad d = \sqrt{(2-1)^2 + (\tfrac{1}{2}-1)^2 + (\tfrac{5}{2}-3)^2} = \sqrt{1 + \tfrac{1}{4} + \tfrac{1}{4}}$$
$$= \sqrt{\tfrac{3}{2}} \text{ units}$$

EXERCISE 17A.3

1 l_1 has direction vector $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and l_2 has direction vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and if θ is the angle between them,

$$\cos\theta = \frac{\left|\begin{bmatrix} 4 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 4 \end{bmatrix}\right|}{\sqrt{16+9}\sqrt{25+16}} = \frac{|20+(-12)|}{\sqrt{25 \times 41}} = \frac{8}{\sqrt{25 \times 41}}$$

$$\therefore \quad \theta = \arccos\left(\frac{8}{\sqrt{25 \times 41}}\right) \doteqdot 75.5^o$$

$\therefore$ the required angle measures 75.5^o

2 l_1 has direction vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ and l_2 has direction vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and if θ is the angle between them,

$$\cos\theta = \frac{\left|\begin{bmatrix} 12 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -4 \end{bmatrix}\right|}{\sqrt{144+25}\sqrt{9+16}} = \frac{|36+(-20)|}{13 \times 5} = \tfrac{16}{65}$$

$$\therefore \quad \theta = \arccos\left(\tfrac{16}{65}\right) \doteqdot 75.7^o$$

3 Line 1 has direction vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and line 2 has direction vector $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$

and $\begin{bmatrix} 5 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 10 \end{bmatrix} = 20 + (-20) = 0$

$\therefore$ l_1 and l_2 are perpendicular

4 If $\dfrac{x-8}{3} = \dfrac{9-y}{16} = \dfrac{z-10}{7} = \lambda$ say then $x = 8 + 3\lambda$, $y = 9 - 16\lambda$, $z = 10 + 7\lambda$

$\therefore$ line 1 has vector $\begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix}$, line 2 has vector $\begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix}$ and if θ is the angle between them,

$$\cos\theta = \frac{\left|\begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix}\right|}{\sqrt{9+256+49}\sqrt{9+64+25}} = \frac{|9-128-35|}{\sqrt{314}\sqrt{98}} = \frac{154}{\sqrt{314 \times 98}}$$

$$\therefore \quad \theta \doteqdot 28.6^o$$

EXERCISE 17B.1

1 **a** **i** When $t=0$, $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-4\\3\end{bmatrix}$

$\therefore$ the object is at (−4, 3)

ii The velocity vector is $\begin{bmatrix}12\\5\end{bmatrix}$

iii The speed is $\sqrt{12^2+5^2}$
$= 13$ m/s

b **i** When $t=0$, $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\-6\end{bmatrix}$

$\therefore$ the object is at (0, −6)

ii The velocity vector is $\begin{bmatrix}3\\-4\end{bmatrix}$

iii The speed is $\sqrt{3^2+(-4)^2}$
$= 5$ m/s

c **i** When $t=0$, $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\-7\end{bmatrix}$

$\therefore$ the object is at (−2, −7)

ii The velocity vector is $\begin{bmatrix}-6\\-4\end{bmatrix}$

iii The speed is $\sqrt{(-6)^2+(-4)^2}$
$=\sqrt{36+16}$
$=\sqrt{52}$ m/s

2 **a** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\-5\end{bmatrix}+t\begin{bmatrix}8\\4\end{bmatrix}$ **i** velocity vector $=\begin{bmatrix}8\\4\end{bmatrix}$ **ii** speed $=\sqrt{8^2+4^2}=\sqrt{80}$ km/h

b $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}+t\begin{bmatrix}6\\2\end{bmatrix}$ **i** velocity vector $=\begin{bmatrix}6\\2\end{bmatrix}$ **ii** speed $=\sqrt{6^2+2^2}=\sqrt{40}$ km/h

c $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-12\\15\end{bmatrix}+t\begin{bmatrix}7\\24\end{bmatrix}$ **i** velocity vector $=\begin{bmatrix}7\\24\end{bmatrix}$ **ii** speed $=\sqrt{7^2+24^2}=25$ km/h

3 **a** $\begin{bmatrix}4\\-3\end{bmatrix}$ has length $\sqrt{4^2+(-3)^2}=5$

$\therefore$ $30\begin{bmatrix}4\\-3\end{bmatrix}$ has length 150

$\therefore$ velocity vector is $\begin{bmatrix}120\\-90\end{bmatrix}$

b $\begin{bmatrix}24\\7\end{bmatrix}$ has length $\sqrt{24^2+7^2}=25$

$\therefore$ $\frac{1}{2}\begin{bmatrix}24\\7\end{bmatrix}$ has length 12.5

$\therefore$ velocity vector is $\begin{bmatrix}12\\3.5\end{bmatrix}$

c $2\mathbf{i}+\mathbf{j}=\begin{bmatrix}2\\1\end{bmatrix}$ has length

$\sqrt{2^2+1^2}=\sqrt{5}$

$\therefore$ $10\sqrt{5}\begin{bmatrix}2\\1\end{bmatrix}$ has length 50

$\therefore$ velocity vector is $\begin{bmatrix}20\sqrt{5}\\10\sqrt{5}\end{bmatrix}$

d $-3\mathbf{i}+4\mathbf{j}=\begin{bmatrix}-3\\4\end{bmatrix}$ has length

$\sqrt{(-3)^2+4^2}=5$

$\therefore$ $20\begin{bmatrix}-3\\4\end{bmatrix}$ has length 100

$\therefore$ velocity vector is $\begin{bmatrix}-60\\80\end{bmatrix}$.

EXERCISE 17B.2

1 **a** $x=-3+2t,\quad y=-2+4t,\quad t\geqslant 0$

$\therefore$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3+2t\\-2+4t\end{bmatrix}$

b when $t=2.5$,

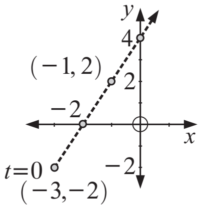
$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3+5\\-2+10\end{bmatrix}=\begin{bmatrix}2\\8\end{bmatrix}$ $\therefore$ position is (2, 8).

c **i** When the car is due north,
$x=0$ and so $-3+2t=0$
$\therefore$ $t=\frac{3}{2}$ sec.

ii When the car is due west,
$y=0$ and so $-2+4t=0$
$\therefore$ $t=\frac{1}{2}$ sec

(O is the observation point)

d



2 **a** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-10\end{bmatrix}+t\begin{bmatrix}4\\-3\end{bmatrix},\quad t\in\mathcal{R}$

b The direction vector is $\begin{bmatrix}18--2\\21-6\end{bmatrix}=\begin{bmatrix}20\\15\end{bmatrix}$ $\therefore$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\6\end{bmatrix}+\frac{t}{10}\begin{bmatrix}20\\15\end{bmatrix},\quad t\in\mathcal{R}$

c Since $2\mathbf{i}+\mathbf{j}=\begin{bmatrix}2\\1\end{bmatrix}$ has length $\sqrt{2^2+1^2}=\sqrt{5}$ then $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-5\\0\end{bmatrix}+t\begin{bmatrix}2\\1\end{bmatrix},\quad t\in\mathcal{R}$

d Since $3\mathbf{i}+4\mathbf{j}=\begin{bmatrix}3\\4\end{bmatrix}$ has length $\sqrt{3^2+4^2}=5$ then $3\begin{bmatrix}3\\4\end{bmatrix}$ has length 15

$\therefore$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}+(t-1)\begin{bmatrix}9\\12\end{bmatrix}$

or $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}-\begin{bmatrix}9\\12\end{bmatrix}+t\begin{bmatrix}9\\12\end{bmatrix}$ i.e., $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-8\\-16\end{bmatrix}+t\begin{bmatrix}9\\12\end{bmatrix},\quad t\in\mathcal{R}$

3 $\mathbf{r} = \begin{bmatrix}-20\\32\end{bmatrix} + t\begin{bmatrix}12\\-5\end{bmatrix}$, where t is the time in hours since 6 am.

a At 6 am, $t = 0$ $\therefore$ $\mathbf{r} = \begin{bmatrix}-20\\32\end{bmatrix}$ and $|\mathbf{r}| = \sqrt{(-20)^2 + (32)^2}$

$\doteq 37.7$ $\therefore$ the ship is 37.7 km away

b Speed $= \left|\begin{bmatrix}12\\-5\end{bmatrix}\right| = \sqrt{12^2 + (-5)^2} = 13$ km/h

c When due north, $x = 0$. $\therefore$ $-20 + 12t = 0$

$$\begin{aligned} \therefore\quad t &= \tfrac{5}{3} \text{ hours} \\ \therefore\quad t &= 1 \text{ hour } 40 \text{ min}, \quad \text{so the time is 7:40 am} \end{aligned}$$

4 Yacht A: $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix}$ Yacht B: $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-8\end{bmatrix} + t\begin{bmatrix}2\\1\end{bmatrix}$, $t \geqslant 0$

a when $t = 0$, $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$ $\therefore$ A is at (4, 5) and $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-8\end{bmatrix}$ and B is at (1, −8)

b For A, the velocity vector is $\begin{bmatrix}1\\-2\end{bmatrix}$, and for B it is $\begin{bmatrix}2\\1\end{bmatrix}$

c Speed of A $= \sqrt{1^2 + (-2)^2} = \sqrt{5}$ km/h Speed of B $= \sqrt{2^2 + 1^2} = \sqrt{5}$ km/h

d The distance between them is
$$\begin{aligned} D &= \sqrt{[(1+2t)-(4+t)]^2 + [(-8+t)-(5-2t)]^2} \\ &= \sqrt{(-3+t)^2 + (-13+3t)^2} \\ &= \sqrt{9 - 6t + t^2 + 169 - 78t + 9t^2} \\ &= \sqrt{10t^2 - 84t + 178} \end{aligned}$$

This is a minimum when $10t^2 - 84t + 178$ is a minimum. This occurs when

$t = \dfrac{-b}{2a} = \dfrac{84}{20} = 4.2$ hours. $\therefore$ the time is 4 h 12 min after 6 am i.e., 10:12 am.

e A has direction vector $\begin{bmatrix}1\\-2\end{bmatrix}$ and B has direction vector $\begin{bmatrix}2\\1\end{bmatrix}$.

Since $\begin{bmatrix}1\\-2\end{bmatrix} \bullet \begin{bmatrix}2\\1\end{bmatrix} = 2 - 2 = 0$, the paths of the yachts are at right angles to each other.

5 **a** P has position $\begin{bmatrix}x_1\\y_1\end{bmatrix} = \begin{bmatrix}-5\\4\end{bmatrix} + t\begin{bmatrix}3\\-1\end{bmatrix}$ and at $t = 0$, the time is 1.34 pm

$\therefore$ $x_1(t) = -5 + 3t$, $y_1(t) = 4 - t$.

b Speed $= \sqrt{3^2 + (-1)^2} = \sqrt{10}$ km/min

c Q fires its torpedo after a minutes.

$\therefore$ at time t, its torpedo has travelled for $(t - a)$ minutes.

$\therefore$ $\begin{bmatrix}x_2\\y_2\end{bmatrix} = \begin{bmatrix}15\\7\end{bmatrix} + (t-a)\begin{bmatrix}-4\\-3\end{bmatrix}$, $t > a$

i.e., $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$$\begin{aligned} \therefore\quad -5 + 3t &= 15 - 4(t-a) \quad\text{and}\quad & 4 - t &= 7 - 3(t-a) \\ \text{i.e.,}\quad 7t - 4a &= 20 \quad\text{and}\quad & 2t - 3a &= 3 \end{aligned}$$

Solving simultaneously,
$$\begin{aligned} 7t - 4a &= 20 \qquad \times 3 \\ 2t - 3a &= 3 \qquad \times(-4) \end{aligned}$$

$$\begin{aligned} \therefore\quad 21t - 12a &= 60 \\ -8t + 12a &= -12 \\ \hline 13t \qquad &= 48 \end{aligned}$$

$$\begin{aligned} \therefore\quad t &= \tfrac{48}{13} \quad\text{and}\quad & 7\left(\tfrac{48}{13}\right) - 4a &= 20 \\ \text{i.e.,}\quad t &\doteq 3.6923 & \therefore\quad 5.8462 &= 4a \\ \text{i.e.,}\quad t &\doteq 3 \text{ min } 41.53 & \therefore\quad a &\doteq 1.4615 \doteq 1 \text{ min } 27.7 \text{ sec} \end{aligned}$$

So, as $a \doteq 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

EXERCISE 17B.3

1 **a** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-3\\-2\end{bmatrix}+t\begin{bmatrix}2\\4\end{bmatrix},\quad t\geqslant 0\quad\therefore$ its position is $(-3+2t,\ -2+4t)$.

b **i** It is due east when $y=0$

$$\therefore\quad -2+4t=0$$
$$\therefore\quad t=\tfrac{1}{2}\text{ sec}$$

ii It is due north when $x=0$

$$\therefore\quad -3+2t=0$$
$$\therefore\quad t=1\tfrac{1}{2}\text{ sec}$$

c When $y=0,\quad t=\frac{1}{2}$ and $x=-3+2\left(\frac{1}{2}\right)=-2$

When $x=0,\quad t=1\frac{1}{2}$ and $y=-2+4\left(1\frac{1}{2}\right)=4$

$\therefore$ axis intercepts are $(-2,\ 0)$ and $(0,\ 4)$

2 **a** $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}+t\begin{bmatrix}-1\\-3\end{bmatrix},\quad t\geqslant 0\quad\therefore$ its position is $(-2-t,\ 1-3t)$

b Crosses the x-axis when $y=0\quad\therefore\quad 1-3t=0\quad\therefore\quad t=\frac{1}{3}$ sec

c When $t=\frac{1}{3}$, the point is at $(-2\frac{1}{3},\ 0)$.

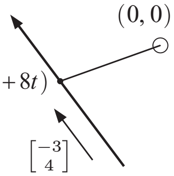
3 **a** $\begin{bmatrix}-3\\4\end{bmatrix}$ has length $\sqrt{(-3)^2+4^2}=5\quad\therefore$ unit direction vector is $\frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix}$

$\therefore$ the velocity vector is $\frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix}\times 10=\begin{bmatrix}-6\\8\end{bmatrix}=-6\mathbf{i}+8\mathbf{j}$

b $\therefore$ the liner's position is given by $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}6\\-6\end{bmatrix}+t\begin{bmatrix}-6\\8\end{bmatrix},\quad t\geqslant 0$

c The liner is due east when $y=0,\quad\therefore\quad 0=-6+8t\quad\therefore\quad t=\frac{3}{4}$ hour

d The liner is nearest to the fishing boat when $\overrightarrow{\text{OP}}\perp\begin{bmatrix}-3\\4\end{bmatrix}$



$$\therefore\quad \begin{bmatrix}6-6t-0\\-6+8t-0\end{bmatrix}\bullet\begin{bmatrix}-3\\4\end{bmatrix}=0$$
$$\therefore\quad -3(6-6t)+4(-6+8t)=0$$
$$\therefore\quad -18+18t-24+32t=0$$
$$\therefore\quad 50t=42$$
$$\therefore\quad t=0.84\text{ hours}$$

i.e., after 50 min 24 sec.

At this time, the liner is at $(6-6(0.84),\ -6+8(0.84))$ i.e., at $(0.96,\ 0.72)$.

4 **a** **i** Initial position vector is $\begin{bmatrix}-8-0\\-5-0\end{bmatrix}$
$=-8\mathbf{i}-5\mathbf{j}$

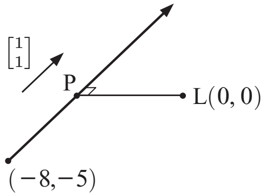
ii $\begin{bmatrix}1\\1\end{bmatrix}$ has length $\sqrt{1^2+1^2}=\sqrt{2}$

$\therefore\quad 3\begin{bmatrix}1\\1\end{bmatrix}$ has length $3\sqrt{2}$

$\therefore$ the direction vector is $\begin{bmatrix}3\\3\end{bmatrix}=3\mathbf{i}+3\mathbf{j}$

iii position vector at time t is $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-8\\-5\end{bmatrix}+t\begin{bmatrix}3\\3\end{bmatrix}$, or $(-8+3t)\mathbf{i}+(-5+3t)\mathbf{j}$

b



P is $(-8+3t,\ -5+3t)$

and $\overrightarrow{\text{LP}}$ is $\begin{bmatrix}-8+3t\\-5+3t\end{bmatrix}$

Now when closest, $\overrightarrow{\text{LP}}\bullet\begin{bmatrix}1\\1\end{bmatrix}=0$

$$\therefore\quad \begin{bmatrix}-8+3t\\-5+3t\end{bmatrix}\bullet\begin{bmatrix}1\\1\end{bmatrix}=0$$
$$\therefore\quad -8+3t-5+3t=0$$
$$\therefore\quad 6t=13$$
$$\therefore\quad t=\tfrac{13}{6}=2\tfrac{1}{6}$$

i.e., at 2 hours 10 min

c When $t = \frac{13}{6}$, P is $\left(-8+3\left(\frac{13}{6}\right), \ -5+3\left(\frac{13}{6}\right)\right)$ i.e., $\left(-\frac{3}{2}, \frac{3}{2}\right)$

$$\begin{aligned} \therefore \quad \text{the closest distance} &= \sqrt{\left(-\tfrac{3}{2}-0\right)^2+\left(\tfrac{3}{2}-0\right)^2} \\ &= \sqrt{\tfrac{9}{4}+\tfrac{9}{4}} \\ &= \sqrt{\tfrac{9}{2}} \\ &\doteqdot 2.12 \text{ km} \qquad \therefore \ \text{the trawler would be breaking the law.} \end{aligned}$$

5 **a** $|\mathbf{b}| = \sqrt{(-3)^2+(-1)^2} = \sqrt{10}$

As the speed is $40\sqrt{10}$ km/h, the velocity vector is $40\begin{bmatrix}-3\\-1\end{bmatrix} = \begin{bmatrix}-120\\-40\end{bmatrix}$.

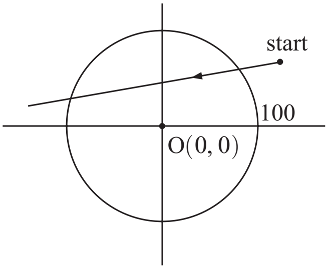
b $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}200\\100\end{bmatrix} + t\begin{bmatrix}-120\\-40\end{bmatrix}, \quad t \geqslant 0 \quad \{t = 0 \text{ at } 12.00 \text{ noon}\}$

c At 1:00 pm, $t = 1$ and $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}200-120\\100-40\end{bmatrix} = \begin{bmatrix}80\\60\end{bmatrix}$

d The distance from O(0, 0) to P_1 (80, 60) is $\left|\begin{bmatrix}80\\60\end{bmatrix}\right| = \sqrt{80^2+60^2} = 100$ km,

which is when it becomes visible to radar. {within 100 km of O(0, 0)}

e



A general point on the path is P(200 – 120t, 100 – 40t).

Now $\overrightarrow{\text{OP}} = \begin{bmatrix}200-120t\\100-40t\end{bmatrix}$,

and for the closest point $\overrightarrow{\text{OP}} \bullet \begin{bmatrix}-3\\-1\end{bmatrix} = 0$

$$\begin{aligned} \therefore \quad -3(200-120t)-1(100-40t) &= 0 \\ \therefore \quad -700+400t &= 0 \\ \therefore \quad t = \tfrac{7}{4} &= 1\tfrac{3}{4} \text{ hours} \end{aligned}$$

The time when the aircraft is closest is 1:45 pm, and

at this time $\overrightarrow{\text{OP}} = \begin{bmatrix}200-120\left(\frac{7}{4}\right)\\100-40\left(\frac{7}{4}\right)\end{bmatrix} = \begin{bmatrix}-10\\30\end{bmatrix}$

$$\therefore \quad d_{\min} = \sqrt{(-10)^2+30^2} \doteqdot 31.6 \text{ km}$$

f It disappears from radar when $|\overrightarrow{\text{OP}}| = 100$ and $t > 1\frac{3}{4}$

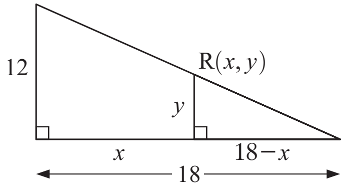
$$\begin{aligned} \therefore \quad \sqrt{(200-120t)^2+(100-40t)^2} &= 100 \\ \therefore \quad 40\,000-48\,000t+14\,400t^2+10\,000-8000t+1400t^2 &= 10\,000 \\ \therefore \quad 16\,000t^2-56\,000t+40\,000 &= 0 \\ \therefore \quad 16t^2-56\,t+40 &= 0 \quad \{\div 1000\} \\ \therefore \quad 2t^2-7t+5 &= 0 \quad \{\div 8\} \\ \therefore \quad (2t-5)(t-1) &= 0 \\ \therefore \quad t &= \tfrac{5}{2} \qquad \{\text{as } t > 1\tfrac{3}{4}\} \end{aligned}$$

i.e., at $2\frac{1}{2}$ hours, i.e., at 2:30 pm

6 **a** When $x = 0$, $3y = 36$, $\therefore$ $y = 12$ $\therefore$ B is (0, 12)

When $y = 0$, $2x = 36$, $\therefore$ $x = 18$ $\therefore$ A is (18, 0)

b



Using the similar triangles,

$$\frac{y}{18-x} = \frac{12}{18} = \frac{2}{3}$$

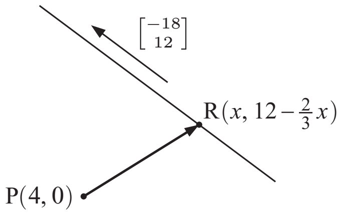
$$\therefore \quad y = \tfrac{2}{3}(18-x)$$

$\therefore$ R is $\left(x, 12-\frac{2}{3}x\right)$

c $\overrightarrow{PR} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x-0 \end{bmatrix} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 0-18 \\ 12-0 \end{bmatrix} = \begin{bmatrix} -18 \\ 12 \end{bmatrix}$

d R is located so the shortest distance PR is when $\overrightarrow{RP} \bullet \begin{bmatrix} -18 \\ 12 \end{bmatrix} = 0$

$$\begin{aligned} \therefore \quad -18(x-4) + 12(12 - \tfrac{2}{3}x) &= 0 \\ \therefore \quad -18x + 72 + 144 - 8x &= 0 \\ \therefore \quad 26x &= 216 \\ \therefore \quad x &= \tfrac{108}{13} \\ \therefore \quad y &= 12 - \tfrac{72}{13} = \tfrac{84}{13} \\ \text{and} \quad y &= \tfrac{2}{3}\left(18 - \tfrac{108}{13}\right) \end{aligned}$$



i.e., R is at $\left(\frac{108}{13}, \frac{84}{13}\right)$ and $\overrightarrow{PR} = \begin{bmatrix} \frac{108}{13}-4 \\ 12-\frac{2}{3}\left(\frac{108}{13}\right) \end{bmatrix} \doteqdot \begin{bmatrix} 4.308 \\ 6.4615 \end{bmatrix}$

$\therefore \quad |\overrightarrow{PR}| \doteqdot \sqrt{(4.308)^2 + (6.4615)^2} \doteqdot 7.77$ km,

so the shortest distance is 7.77 km

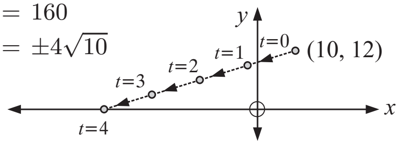
7 **a** $x(0) = 10, \quad y(0) = 12 \quad \therefore$ P is at (10, 12)

b The velocity vector is $\begin{bmatrix} a \\ -3 \end{bmatrix}$ and speed $= \sqrt{a^2+9} = 13$

$$\begin{aligned} \therefore \quad a^2 + 9 &= 169 \\ \therefore \quad a^2 &= 160 \\ \therefore \quad a &= \pm 4\sqrt{10} \end{aligned}$$

c For $a < 0$, i.e., $a = -4\sqrt{10}$,

$x(t) = 10 - 4\sqrt{10}t, \quad y(t) = 12 - 3t.$



8 For A, $x(t) = 3 - t, \quad y(t) = 2t - 4$ For B, $x(t) = 4 - 3t, \quad y(t) = 3 - 2t$

a When $t = 0, \quad x(0) = 3, \quad y(0) = -4$ $\therefore \quad x(0) = 4, \quad y(0) = 3$

$\therefore$ A is at (3, −4) $\therefore$ B is at (4, 3)

b The velocity vector of A is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and the velocity vector of B is $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$.

c If the angle is θ,

$$\begin{aligned} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ -2 \end{bmatrix} &= \sqrt{1+4}\sqrt{9+4}\cos\theta \\ \therefore \quad 3 - 4 &= \sqrt{5}\sqrt{13}\cos\theta \\ \therefore \quad \frac{-1}{\sqrt{65}} &= \cos\theta \quad \text{and so} \quad \theta = 97.1^o \end{aligned}$$

d If D is the distance between them, then

$$\begin{aligned} D &= \sqrt{[(4-3t)-(3-t)]^2 + [(2t-4)-(3-2t)]^2} \\ &= \sqrt{[1-2t]^2 + [-7+4t]^2} \\ &= \sqrt{1 - 4t + 4t^2 + 49 - 56t + 16t^2} \\ &= \sqrt{20t^2 - 60t + 50} \end{aligned}$$

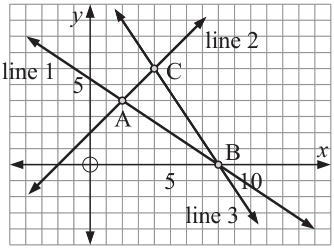
and D is a minimum when

$t = -\dfrac{b}{2a} = \frac{60}{40} = 1\frac{1}{2}$

i.e., $t = 1.5$ hours

EXERCISE 17B.4

1 **a**



b A is (2, 4), B is (8, 0), C is (4, 6)

c $\text{BC} = \sqrt{(8-4)^2 + (0-6)^2} = \sqrt{16+36} = \sqrt{52}$ units

$\text{AC} = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16} = \sqrt{52}$ units

$\therefore$ BC = AC and so ΔABC is isosceles.

d Line 1 and Line 2 meet at A.

$$\therefore \quad \begin{bmatrix}-1\\6\end{bmatrix}+\begin{bmatrix}3r\\-2r\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}+s\begin{bmatrix}1\\1\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}3r-s\\-2r-s\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}$$

$$\begin{aligned}\text{i.e.,}\quad 3r-s&=1\\ \text{and}\quad 2r+s&=4\\ \hline \text{Adding,}\quad 5r\qquad&=5 \quad\therefore\quad r=1\end{aligned}$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\6\end{bmatrix}+\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

Line 2 and Line 3 meet at C.

$$\therefore \quad \begin{bmatrix}0\\2\end{bmatrix}+s\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}10\\-3\end{bmatrix}+t\begin{bmatrix}-2\\3\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}s+2t\\s-3t\end{bmatrix}=\begin{bmatrix}10\\-5\end{bmatrix}$$

$$\begin{aligned}\text{i.e.,}\quad s+2t&=10\\ -s+3t&=5\\ \hline \text{Adding,}\quad 5t&=15 \quad\therefore\quad t=3\end{aligned}$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}10\\-3\end{bmatrix}+3\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$$

Line 1 and Line 3 meet at B.

$$\therefore \quad \begin{bmatrix}-1\\6\end{bmatrix}+r\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}10\\-3\end{bmatrix}+t\begin{bmatrix}-2\\3\end{bmatrix}$$

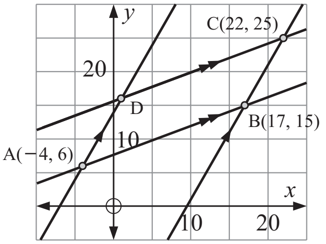
$$\therefore \quad \begin{bmatrix}3r+2t\\-2r-3t\end{bmatrix}=\begin{bmatrix}11\\-9\end{bmatrix}$$

$$\begin{aligned}\therefore\quad 3r+2t&=11 \qquad \times 3\\ -2r-3t&=-9 \qquad \times 2\end{aligned}$$

$$\begin{aligned}\therefore\quad 9r+6t&=33\\ -4r-6t&=-18\\ \hline \text{Adding,}\quad 5r\qquad&=15\\ \therefore\quad r&=3\end{aligned}$$

$$\text{So,}\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\6\end{bmatrix}+3\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}8\\0\end{bmatrix} \quad \checkmark$$

2 **a**



b B(17, 15)
C(22, 25)
D(1, 16)

c Lines 1 and 4 meet at B

$$\therefore \quad \begin{bmatrix}-4\\6\end{bmatrix}+r\begin{bmatrix}7\\3\end{bmatrix}=\begin{bmatrix}22\\25\end{bmatrix}+u\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}7r+u\\3r+2u\end{bmatrix}=\begin{bmatrix}26\\19\end{bmatrix}$$

$$\begin{aligned}\therefore\quad 7r+u&=26\\ 3r+2u&=19\end{aligned}$$

$$\begin{aligned}\therefore\quad -14r-2u&=-52\\ 3r+2u&=19\\ \hline \text{Adding,}\quad -11r\qquad&=-33\\ \therefore\quad r&=3\end{aligned}$$

$$\text{and}\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-4\\6\end{bmatrix}+3\begin{bmatrix}7\\3\end{bmatrix}=\begin{bmatrix}17\\15\end{bmatrix} \quad \checkmark$$

Lines 2 and 3 meet at D

$$\text{and}\quad \begin{bmatrix}-4\\6\end{bmatrix}+s\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}22\\25\end{bmatrix}+t\begin{bmatrix}-7\\-3\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}s+7t\\2s+3t\end{bmatrix}=\begin{bmatrix}26\\19\end{bmatrix}$$

$$\begin{aligned}\therefore\quad -2s-14t&=-52\\ \therefore\quad 2s+3t&=19\\ \hline \text{Adding,}\quad -11t&=-33\\ \therefore\quad t&=3\end{aligned}$$

$$\therefore \quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}22\\25\end{bmatrix}+3\begin{bmatrix}-7\\-3\end{bmatrix}=\begin{bmatrix}1\\16\end{bmatrix} \quad \checkmark$$

Lines 3 and 4 meet at C.

$$\therefore \quad \begin{bmatrix}22\\25\end{bmatrix}+t\begin{bmatrix}-7\\-3\end{bmatrix}=\begin{bmatrix}22\\25\end{bmatrix}+n\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}-7t+u\\-3t+2u\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{aligned}\therefore\quad -7t+u&=0\\ -3t+2u&=0\end{aligned}$$

$$\begin{aligned}\therefore\quad 14t-2u&=0\\ -3t+2u&=0\\ \hline \text{Adding,}\quad 11t\qquad&=0\\ \therefore\quad t&=0\end{aligned}$$

$$\text{and}\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}22\\25\end{bmatrix} \quad \checkmark$$

3 **a** Lines 1 and 3 meet at A.

$$\begin{aligned}\therefore\quad \begin{bmatrix}0\\2\end{bmatrix}+r\begin{bmatrix}2\\1\end{bmatrix} &= \begin{bmatrix}0\\5\end{bmatrix}+t\begin{bmatrix}1\\7\end{bmatrix}\\ \therefore\quad \begin{bmatrix}2r-t\\r+t\end{bmatrix} &= \begin{bmatrix}0\\3\end{bmatrix}\\ \therefore\quad 2r-t &= 0\\ r+t &= 3\\ \text{Adding,}\quad 3r\quad &= 3\\ \therefore\quad r &= 1\\ \therefore\quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}0\\2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}\end{aligned}$$

$\therefore$ A is (2, 3)

Lines 1 and 2 meet at B.

$$\begin{aligned}\therefore\quad \begin{bmatrix}0\\2\end{bmatrix}+r\begin{bmatrix}2\\1\end{bmatrix} &= \begin{bmatrix}8\\6\end{bmatrix}+s\begin{bmatrix}-1\\-2\end{bmatrix}\\ \therefore\quad \begin{bmatrix}2r+s\\r+2s\end{bmatrix} &= \begin{bmatrix}8\\4\end{bmatrix}\\ \therefore\quad -4r-2s &= -16\\ r+2s &= 4\\ \text{Adding,}\quad -3r\quad &= -12\\ \therefore\quad r &= 4\\ \therefore\quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}0\\2\end{bmatrix}+4\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}8\\6\end{bmatrix}\end{aligned}$$

$\therefore$ B is (8, 6)

Lines 2 and 3 meet at C.

$$\begin{aligned}\therefore\quad \begin{bmatrix}8\\6\end{bmatrix}+s\begin{bmatrix}-1\\-2\end{bmatrix} &= \begin{bmatrix}0\\5\end{bmatrix}+t\begin{bmatrix}1\\-1\end{bmatrix}\\ \therefore\quad \begin{bmatrix}-s-t\\-2s+t\end{bmatrix} &= \begin{bmatrix}-8\\-1\end{bmatrix}\\ \therefore\quad -s-t &= -8\\ -2s+t &= -1\\ \text{Adding,}\quad -3s\quad &= -9\\ \therefore\quad s &= 3\\ \therefore\quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}8\\6\end{bmatrix}+3\begin{bmatrix}-1\\-2\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}\end{aligned}$$

$\therefore$ C is (5, 0)

b A (2, 3), B(8, 6), C(5, 0)

$$\begin{aligned}\text{AB} &= \sqrt{(8-2)^2+(6-3)^2}\\ &= \sqrt{36+9}\\ &= \sqrt{45}\end{aligned}$$

$$\begin{aligned}\text{BC} &= \sqrt{(5-8)^2+(0-6)^2}\\ &= \sqrt{9+36}\\ &= \sqrt{45}\end{aligned}$$

The two equal sides are AB and BC and they have length $\sqrt{45}$ units.

4 **a** Lines QP and PR meet at P

$$\begin{aligned}\therefore\quad \begin{bmatrix}3\\-1\end{bmatrix}+r\begin{bmatrix}14\\10\end{bmatrix} &= \begin{bmatrix}0\\18\end{bmatrix}+t\begin{bmatrix}5\\-7\end{bmatrix}\\ \therefore\quad \begin{bmatrix}14r-5t\\10r+7t\end{bmatrix} &= \begin{bmatrix}-3\\19\end{bmatrix}\\ \therefore\quad 14r-5t &= -3 \quad \times 7\\ 10r+7t &= 19 \quad \times 5\\ \therefore\quad 98r-35t &= -21\\ 50r+35t &= 95\\ \text{Adding,}\quad 148r\quad &= 74\\ \therefore\quad r &= \tfrac{1}{2}\end{aligned}$$

$\therefore\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}14\\10\end{bmatrix}=\begin{bmatrix}10\\4\end{bmatrix}$

$\therefore$ P is (10, 4)

Lines QP and PR meet at Q

$$\begin{aligned}\begin{bmatrix}3\\-1\end{bmatrix}+r\begin{bmatrix}14\\10\end{bmatrix} &= \begin{bmatrix}3\\-1\end{bmatrix}+s\begin{bmatrix}17\\-9\end{bmatrix}\\ \therefore\quad r\begin{bmatrix}14\\10\end{bmatrix} &= s\begin{bmatrix}17\\-9\end{bmatrix}\\ \therefore\quad r &= s=0\\ \text{So,}\quad \begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}3\\-1\end{bmatrix}\end{aligned}$$

$\therefore$ Q is (3, -1)

Lines QR and PR meet at R

$$\begin{aligned}\therefore\quad \begin{bmatrix}3\\-1\end{bmatrix}+s\begin{bmatrix}17\\-9\end{bmatrix} &= \begin{bmatrix}0\\18\end{bmatrix}+t\begin{bmatrix}5\\-7\end{bmatrix}\\ \therefore\quad \begin{bmatrix}17s-5t\\-9s+7t\end{bmatrix} &= \begin{bmatrix}-3\\19\end{bmatrix}\\ \therefore\quad 17s-5t &= -3 \quad \times 7\\ -9s+7t &= 19 \quad \times 5\\ \therefore\quad 119s-35t &= -21\\ -45s+35t &= 95\\ \text{Adding,}\quad 74s\quad &= 74\\ \therefore\quad s &= 1\end{aligned}$$

$\therefore\quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}+\begin{bmatrix}17\\-9\end{bmatrix}=\begin{bmatrix}20\\-10\end{bmatrix}$

$\therefore$ R is (20, -10)

b $\overrightarrow{\text{PQ}}=\begin{bmatrix}3-10\\-1-4\end{bmatrix}=\begin{bmatrix}-7\\-5\end{bmatrix}$

$\overrightarrow{\text{PR}}=\begin{bmatrix}20-10\\-10-4\end{bmatrix}=\begin{bmatrix}10\\-14\end{bmatrix}$

and $\overrightarrow{\text{PQ}}\bullet\overrightarrow{\text{PR}}=-70+70=0$

c $\text{PQ}\perp\text{PR}$ i.e., $\measuredangle\text{QPR}=90^o$

d
$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\,|\,\overrightarrow{\text{PQ}}\,||\,\overrightarrow{\text{PR}}\,|\\ &= \tfrac{1}{2}\sqrt{49+25}\sqrt{100+196}\\ &= 74\text{ units}^2\end{aligned}$$

5 **a** Lines 1 and 2 meet at B

$$\therefore\ \begin{bmatrix}2\\5\end{bmatrix}+r\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}18\\9\end{bmatrix}+s\begin{bmatrix}-8\\32\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}4r+8s\\r-32s\end{bmatrix}=\begin{bmatrix}16\\4\end{bmatrix}$$

$$\begin{aligned}\therefore\ 4r+8s&=16\\ r-32s&=4\end{aligned}$$

$$\begin{aligned}\therefore\ r+2s&=4\\ -r+32s&=-4\\ \hline \text{Adding,}\qquad 34s&=0\\ \therefore\ s&=0\end{aligned}$$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}18\\9\end{bmatrix}\qquad\therefore\ \text{B is (18, 9)}$$

Lines 2 and 3 meet at C

$$\therefore\ \begin{bmatrix}18\\9\end{bmatrix}+s\begin{bmatrix}-8\\32\end{bmatrix}=\begin{bmatrix}14\\25\end{bmatrix}+t\begin{bmatrix}-8\\-2\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}-8s+8t\\32s+2t\end{bmatrix}=\begin{bmatrix}-4\\16\end{bmatrix}$$

$$\begin{aligned}2s-2t&=1\\ 32s+2t&=16\\ \hline \text{Adding,}\qquad 34s\qquad&=17\\ \therefore\ s&=\tfrac{1}{2}\end{aligned}$$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}18\\9\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}-8\\32\end{bmatrix}=\begin{bmatrix}14\\25\end{bmatrix}$$

$\therefore$ C is (14, 25)

Lines 3 and 4 meet at D

$$\therefore\ \begin{bmatrix}14\\25\end{bmatrix}+t\begin{bmatrix}-8\\-2\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}+u\begin{bmatrix}-3\\12\end{bmatrix}$$

$$\therefore\ \begin{bmatrix}-8t+3u\\-2t-12u\end{bmatrix}=\begin{bmatrix}-11\\-24\end{bmatrix}$$

$$\begin{aligned}\therefore\ -8t+3u&=-11\\ t+6u&=12\end{aligned}$$

$$\begin{aligned}\therefore\ 16t-6u&=22\\ t+6u&=12\\ \hline \text{Adding,}\qquad 17t\qquad&=34\\ \therefore\ t&=2\end{aligned}$$

$$\therefore\ \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}14\\25\end{bmatrix}+2\begin{bmatrix}-8\\-2\end{bmatrix}=\begin{bmatrix}-2\\21\end{bmatrix}$$

$\therefore$ D is (−2, 21)

b $\overrightarrow{AC}=\begin{bmatrix}14-2\\25-5\end{bmatrix}=\begin{bmatrix}12\\20\end{bmatrix}$

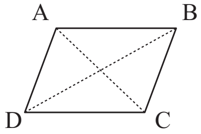
$\overrightarrow{DB}=\begin{bmatrix}18--2\\9-21\end{bmatrix}=\begin{bmatrix}20\\-12\end{bmatrix}$

i $|\overrightarrow{AC}|=\sqrt{12^2+20^2}=\sqrt{544}$ units

ii $|\overrightarrow{DB}|=\sqrt{20^2+(-12)^2}=\sqrt{544}$ units

iii $\overrightarrow{AC}\bullet\overrightarrow{DB}=240-240=0$

c The diagonals are perpendicular and equal in length, so, ABCD is a square.



EXERCISE 17C

1 **a** Line 1 has direction vector $\begin{bmatrix}2\\-1\\1\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\-1\\2\end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $1+2t=-2+3s$ $\qquad$ $2-t=3-s$ $\qquad$ $3+t=1+2s$

$\therefore\ 2t-3s=-3$ ...... (1) $\qquad\therefore\ -t+s=1$ ...... (2) $\qquad\therefore\ t-2s=-2$ ...... (3)

Solving (2) and (3) simultaneously:

$$\begin{aligned}-t+s&=1\\ t-2s&=-2\\ \hline -s&=-1\end{aligned}\qquad\therefore\ s=1\ \text{ and }\ t=0$$

and in (1), LHS $=2t-3s=2(0)-3(1)=-3$ ✓

$\therefore\ s=1,\ \ t=0$ satisfies all three equations

$\therefore$ the two lines meet at (1, 2, 3) $\qquad$ {using $t=0$ or $s=1$}

The acute angle between the lines has $\cos\theta=\dfrac{|\,6+1+2\,|}{\sqrt{4+1+1}\,\sqrt{9+1+4}}=\dfrac{9}{\sqrt{84}}$

and so $\theta \doteqdot 10.9^o$

b Line 1 has direction vector $\begin{bmatrix}2\\-12\\12\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}4\\3\\-1\end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $-1+2t = 4s-3$ $\qquad$ $2-12t = 3s+2$ $\qquad$ $4+12t = -s-1$

$\therefore\ 2t-4s = -2$ $\qquad$ $-12t-3s = 0$ $\qquad$ $12t+s = -5$ ...... (3)

$\therefore\ t-2s = -1$ ...... (1) $\qquad$ $s = -4t$ ...... (2)

Solving (1) and (2) simultaneously: $t-2(-4t) = -1$

$\therefore\ 9t = -1$

$\therefore\ t = -\frac{1}{9}$ and so $s = \frac{4}{9}$

In (3), $12t+s = 12\left(-\frac{1}{9}\right)+\frac{4}{9} = -\frac{12}{9}+\frac{4}{9} = -\frac{8}{9}$, which is not -5.

Since the system is inconsistent, the lines don't intersect. $\therefore$ they are skew.

The acute angle between the lines has $\cos\theta = \dfrac{|8-36-12|}{\sqrt{292}\,\sqrt{26}} = \dfrac{40}{\sqrt{7592}}$ and so $\theta \doteqdot 62.7^o$

c Line 1 has direction vector $\begin{bmatrix}6\\8\\2\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\4\\1\end{bmatrix}$

As $\begin{bmatrix}6\\8\\2\end{bmatrix} = 2\begin{bmatrix}3\\4\\1\end{bmatrix}$ the two lines are parallel. Hence, $\theta = 0^o$.

d Line 1 has direction vector $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\-2\\2\end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $t = 1+3s$ ...... (1) $\qquad$ $2-t = -2-2s$ $\qquad$ $-2+t = 2s+\frac{1}{2}$

$\therefore\ -t+2s = -4$ ...... (2) $\qquad$ $t-2s = 2\frac{1}{2}$ ...... (3)

Solving (1) and (2) simultaneously, $-(1+3s)+2s = -4$

$\therefore\ -1-3s+2s = -4$

$\therefore\ -s = -3$

$\therefore\ s = 3$ and so $t = 1+3(3) = 10$

Substituting in (3), $t-2s = 10-2(3) = 4 \neq 2\frac{1}{2}$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

The acute angle between the lines has $\cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \dfrac{7}{\sqrt{3}\,\sqrt{17}}$

$\therefore\ \theta \doteqdot 11.4^o$

e Line 1 has direction vector $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}3\\-2\\1\end{bmatrix}$

As one vector is not a scalar multiple of the other, the lines are not parallel.

$1+t = 2+3s$ $\qquad$ $2-t = 3-2s$ $\qquad$ $3+2t = s-5$

$t-3s = 1$ ...... (1) $\qquad$ $-t+2s = 1$ ...... (2) $\qquad$ $3t-s = -8$ ...... (3)

Solving (1) and (2) simultaneously, $t-3s = 1$

$-t+2s = 1$

Adding, $-s = 2$

$\therefore\ s = -2$ and $t-3(-2) = 1$ $\therefore\ t = -5$

Checking in (3), $2t-s = 2(-5)-(-2) = -10+2 = -8$ ✓

Since $s=-2$, $t=-5$ satisfies all three equations, the lines meet.

They meet at $x = 1+(-5)$, $y = 2-(-5)$, $z = 3+2(-5)$ i.e., at $(-4, 7, -7)$

The acute angle between the lines has $\cos\theta = \dfrac{|3+2+2|}{\sqrt{1+1+4}\sqrt{9+4+1}}$

$$= \frac{7}{\sqrt{84}} \qquad \text{and so} \quad \theta \doteqdot 40.2^o$$

f Line 1 has direction vector $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix}4\\-2\\0\end{bmatrix}$

Now $\begin{bmatrix}4\\-2\\0\end{bmatrix} = -2\begin{bmatrix}-2\\1\\0\end{bmatrix}$, so the lines are parallel and hence $\theta = 0^o$.

2 $\begin{array}{rcl|l} 3x-y &=& 8 & \times 2 \\ 6x-2y &=& k & - \end{array}$ i.e., $\begin{array}{rcl} 6x-2y &=& 16 \\ 6x-2y &=& k \end{array}$ which are parallel as the slope in each case is 3.

(1) If $k \neq 16$, the lines are parallel, $\therefore$ no solutions exist.

(2) If $k = 16$, the lines are coincident and so have infinitely many solutions of the form $x = t$, $y = 3t-8$, $t \in \mathcal{R}$.

3 $\begin{array}{rcl|l} 4x+8y &=& 1 & - \\ 2x-ay &=& 11 & \times 2 \end{array}$ i.e., $\begin{array}{rcl} 4x+8y &=& 1 \\ 4x-2ay &=& 22 \end{array}$

(1) If $-2a = 8$, i.e., $a = -4$, the equations are $\begin{cases} 4x+8y = 1 \\ 4x+8y = 22 \end{cases}$

The lines are parallel, $\therefore$ no solutions exist.

(2) If $a \neq -4$, the lines are not parallel or coincident.
So, a unique solution exists.

$$\left[\begin{array}{cc|c} 4 & 8 & 1 \\ 2 & -a & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 4 & \frac{1}{2} \\ -2 & a & -11 \end{array}\right] \qquad \begin{array}{l} R_1 \to \frac{1}{2}R_1 \\ R_2 \to -R_1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & 4 & \frac{1}{2} \\ 0 & a+4 & -10\frac{1}{2} \end{array}\right]$$

The second equation is $(a+4)y = -\frac{21}{2}$ $\quad\therefore\quad y = \dfrac{-21}{2(a+4)}$

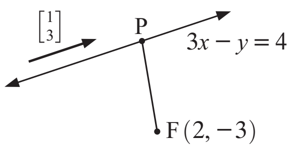
$$\text{and} \quad 4x + 8\left(\frac{-21}{2(a+4)}\right) = 1 \quad \therefore\ 4x - \frac{84}{a+4} = 1$$

$$\therefore\ 4x = 1 + \frac{84}{a+4} = \frac{a+88}{a+4}$$

$$\therefore\ x = \frac{a+88}{4(a+4)}$$

$\therefore$ the unique solution is $x = \dfrac{a+88}{4(a+4)}$, $y = \dfrac{-21}{2(a+4)}$, $a \neq -4$

4



$3x - y = 4$ i.e., $y = 3x - 4$ has slope $3 = \frac{3}{1}$

$\therefore$ direction vector is $\begin{bmatrix}1\\3\end{bmatrix}$

If $x = t$, $y = 3t - 4$

So $P(t, 3t-4)$ is a general point on the line.

Now $\overrightarrow{FP} = \begin{bmatrix} t-2 \\ 3t-4--3 \end{bmatrix} = \begin{bmatrix} t-2 \\ 3t-1 \end{bmatrix}$

But for shortest distance $\overrightarrow{FP} \perp \begin{bmatrix}1\\3\end{bmatrix}$

$$\therefore \quad \begin{bmatrix}t-2\\3t-1\end{bmatrix} \bullet \begin{bmatrix}1\\3\end{bmatrix} = 0$$

$$\therefore \quad t-2+9t-3 = 0$$

$$\therefore \quad 10t = 5$$

$$\therefore \quad t = \tfrac{1}{2}$$

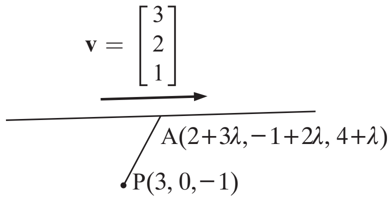
$$\therefore \quad \overrightarrow{FP} = \begin{bmatrix}-\frac{3}{2}\\ \frac{1}{2}\end{bmatrix} \quad \text{and} \quad |\,\overrightarrow{FP}\,| = \sqrt{\tfrac{9}{4}+\tfrac{1}{4}}$$

$$= \sqrt{\tfrac{10}{4}}$$

$$= \tfrac{1}{2}\sqrt{10} \text{ units}$$

Thus, the shortest distance is $\frac{1}{2}\sqrt{10}$ units.

5 Let A $(2+3\lambda, -1+2\lambda, 4+\lambda)$ be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.



$$\text{Now} \quad \overrightarrow{PA} = \begin{bmatrix}2+3\lambda-3\\-1+2\lambda-0\\4+\lambda-(-1)\end{bmatrix} = \begin{bmatrix}3\lambda-1\\2\lambda-1\\\lambda+5\end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ is the direction vector of the line.

$$\therefore \quad \text{since} \quad \overrightarrow{PA} \bullet \mathbf{v} = 0,$$

$$\begin{bmatrix}3\lambda-1\\2\lambda-1\\\lambda+5\end{bmatrix} \bullet \begin{bmatrix}3\\2\\1\end{bmatrix} = 0$$

$$\therefore \quad 3(3\lambda-1)+2(2\lambda-1)+1(\lambda+5) = 0$$

$$\therefore \quad 9\lambda-3+4\lambda-2+\lambda+5 = 0$$

$$\therefore \quad 14\lambda = 0$$

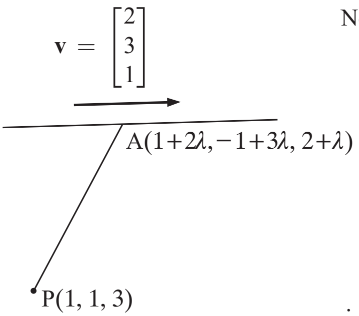
$$\therefore \quad \lambda = 0$$

$\therefore$ A is at (2, −1, 4) and the distance $d = \sqrt{(2-3)^2+(-1-0)^2+(4-(-1))^2}$

$$= \sqrt{1+1+25}$$

$$= \sqrt{27} \quad \text{i.e., } 3\sqrt{3} \text{ units}$$

6 Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.
Then A is at $(1+2\lambda, -1+3\lambda, 2+\lambda)$ for some λ.



$$\text{Now} \quad \overrightarrow{PA} = \begin{bmatrix}1+2\lambda-1\\-1+3\lambda-1\\2+\lambda-3\end{bmatrix} = \begin{bmatrix}2\lambda\\3\lambda-2\\\lambda-1\end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix}2\\3\\1\end{bmatrix}$ is the direction vector of the line.

$$\therefore \quad \text{since} \quad \overrightarrow{PA} \bullet \mathbf{v} = 0,$$

$$\begin{bmatrix}2\lambda\\3\lambda-2\\\lambda-1\end{bmatrix} \bullet \begin{bmatrix}2\\3\\1\end{bmatrix} = 0$$

$$\therefore \quad 4\lambda+3(3\lambda-2)+1(\lambda-1) = 0$$

$$\therefore \quad 4\lambda+9\lambda-6+\lambda-1 = 0$$

$$\therefore \quad 14\lambda = 7$$

$$\therefore \quad \lambda = \tfrac{1}{2}$$

$\therefore$ A is at $(2, \frac{1}{2}, \frac{5}{2})$

$\therefore$ the distance $d = \sqrt{(2-1)^2+(\frac{1}{2}-1)^2+(\frac{5}{2}-3)^2}$

$$= \sqrt{1+\tfrac{1}{4}+\tfrac{1}{4}}$$

$$= \sqrt{\tfrac{3}{2}} \text{ units}$$

7 **a** Line 1 has direction vector $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and letting $t = 0$, it contains the point A(1, 0, 2).

Line 2 has direction vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and contains the point B(0, 0, 0), $\overrightarrow{\mathrm{AB}} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$

$$\text{and} \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

$$\therefore \quad \text{the distance} \quad d = \frac{|\,\overrightarrow{\mathrm{AB}} \bullet (\mathbf{u} \times \mathbf{v})\,|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|4 + 0 - 6|}{\sqrt{(-4)^2 + 1^2 + 3^2}} = \tfrac{2}{\sqrt{26}} \text{ units}$$

b Line 1 contains A(1, 1, 3) and has direction vector $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

Line 2 contains B(2, 1, 0) and has direction vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\overrightarrow{\mathrm{AB}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

$$\text{and} \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= -\mathbf{i} + \mathbf{k}$$

$$\therefore \quad d = \frac{|\,\overrightarrow{\mathrm{AB}} \bullet (\mathbf{u} \times \mathbf{v})\,|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|1 \times -1 \ + \ 0 \times 0 \ + \ -3 \times 1|}{\sqrt{(-1)^2 + 0^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \text{ units}$$

8 **a** The lines intersect $\therefore$ shortest distance = 0 units.

b Line 1 contains A(−1, 2, 4) and has direction vector $\mathbf{u} = \begin{bmatrix} 2 \\ -12 \\ 12 \end{bmatrix}$.

Line 2 contains B(−3, 2, −1) and has $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$, $\overrightarrow{\mathrm{AB}} = \begin{bmatrix} -3 - -1 \\ 2 - 2 \\ -1 - 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -5 \end{bmatrix}$

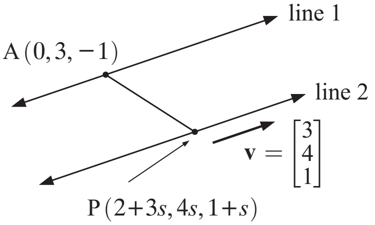
$$\text{and} \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -12 & 12 \\ 4 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -12 & 12 \\ 3 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 12 & 2 \\ -1 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -12 \\ 4 & 3 \end{vmatrix}$$

$$= -24\mathbf{i} + 50\mathbf{j} + 54\mathbf{k}$$

$$\therefore \ \text{shortest distance is} \ \frac{|\,\overrightarrow{\mathrm{AB}} \bullet (\mathbf{u} \times \mathbf{v})\,|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|(-2)(-24) + (0)(50) + (-5)(54)|}{\sqrt{(-24)^2 + 50^2 + 54^2}} = \frac{222}{\sqrt{5992}}$$

$$\doteqdot 2.87 \text{ units}$$

c



$$\overrightarrow{\mathrm{AP}} = \begin{bmatrix} 2 + 3s - 0 \\ 4s - 3 \\ 1 + s - -1 \end{bmatrix} = \begin{bmatrix} 2 + 3s \\ 4s - 3 \\ s + 2 \end{bmatrix}$$

and for the shortest distance $\overrightarrow{\mathrm{AP}} \bullet \mathbf{v} = 0$

$$\begin{aligned} \therefore \quad 3(2 + 3s) + 4(4s - 3) + 1(s + 2) &= 0 \\ \therefore \quad 6 + 9s + 16s - 12 + s + 2 &= 0 \\ \therefore \quad 26s &= 4 \\ \therefore \quad s &= \tfrac{2}{13} \end{aligned}$$

$$\therefore \quad \overrightarrow{\mathrm{AP}} = \begin{bmatrix} \frac{32}{13} \\ -\frac{31}{13} \\ \frac{28}{13} \end{bmatrix} \quad \text{and} \quad |\,\overrightarrow{\mathrm{AP}}\,| = \sqrt{\left(\tfrac{32}{13}\right)^2 + \left(-\tfrac{31}{13}\right)^2 + \left(\tfrac{28}{13}\right)^2} \doteqdot 4.05 \text{ units}$$

d Line 1 contains A(0, 2, -2) and has direction vector $\mathbf{u}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

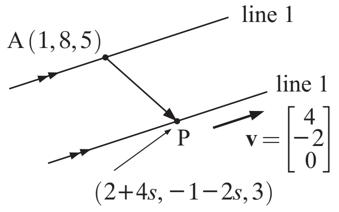
Line 2 contains B(1, -2, $\frac{1}{2}$) and has direction vector $\mathbf{v}=\begin{bmatrix}3\\-2\\2\end{bmatrix}$, $\overrightarrow{AB}=\begin{bmatrix}1-0\\-2-2\\\frac{1}{2}--2\end{bmatrix}=\begin{bmatrix}1\\-4\\\frac{5}{2}\end{bmatrix}$

and
$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\3&-2&2\end{vmatrix}=\mathbf{i}\begin{vmatrix}-1&1\\-2&2\end{vmatrix}+\mathbf{j}\begin{vmatrix}1&1\\2&3\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&-1\\3&-2\end{vmatrix}$$
$$=0\mathbf{i}+\mathbf{j}+\mathbf{k}$$

$\therefore$ shortest distance is
$$\frac{|\,\overrightarrow{AB}\bullet(\mathbf{u}\times\mathbf{v})\,|}{|\mathbf{u}\times\mathbf{v}|}=\frac{\left|(1)(0)+(-4)(1)+(\frac{5}{2})(1)\right|}{\sqrt{0^2+1^2+1^2}}=\frac{1.5}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\tfrac{3\sqrt{2}}{4}\text{ units}$$

e The lines intersect $\therefore$ shortest distance $=0$ units.

f The lines are parallel.



$$\overrightarrow{AP}=\begin{bmatrix}2+4s-1\\-1-2s-8\\3-5\end{bmatrix}=\begin{bmatrix}1+4s\\-2s-9\\-2\end{bmatrix}$$

and for the shortest distance

$\overrightarrow{AP}\bullet\mathbf{v}=0$

$\therefore\ (1+4s)(4)+(-2s-9)(-2)+0=0$

$\therefore\ 4+16s+4s+18=0$

$\therefore\ 20s=-22$ and so $s=-1.1$

Thus $\overrightarrow{AP}=\begin{bmatrix}-3.4\\-6.8\\2\end{bmatrix}$ and $|\,\overrightarrow{AP}\,|=\sqrt{(-3.4)^2+(-6.8)^2+2^2}\doteqdot 7.86$ units

EXERCISE 17D

1 **a** Since $\mathbf{n}=\begin{bmatrix}2\\-1\\3\end{bmatrix}$ and $(-1, 2, 4)$ lies on the plane, the equation is

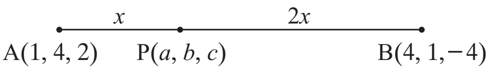
$2x-y+3z=2(-1)-2+3(4)$ i.e., $2x-y+3z=8$

b A vector normal to the plane is $\overrightarrow{AB}=\begin{bmatrix}3\\4\\1\end{bmatrix}$, and $(2, 3, 1)$ lies on the plane.

$\therefore\ 3x+4y+z=3(2)+4(3)+1$

i.e., $3x+4y+z=19$

c



Now $2\overrightarrow{AP}=\overrightarrow{PB}$

$\therefore$ if P has coordinates (a, b, c), then $2\begin{bmatrix}a-1\\b-4\\c-2\end{bmatrix}=\begin{bmatrix}4-a\\1-b\\-4-c\end{bmatrix}$

$\therefore\ 2a-2=4-a$ and $2b-8=1-b$ and $2c-4=-4-c$

$\therefore\ 3a=6$ $\quad 3b=9$ $\quad 3c=0$

$\therefore\ a=2$ $\quad b=3$ $\quad c=0$

$\therefore$ P is at (2, 3, 0)

$\overrightarrow{AB}=\begin{bmatrix}3\\-3\\-6\end{bmatrix}=3\begin{bmatrix}1\\-1\\-2\end{bmatrix}$ so $\begin{bmatrix}1\\-1\\-2\end{bmatrix}$ is a normal vector to the plane.

$\therefore$ the plane is $x-y-2z=2-3-2(0)$ i.e., $x-y-2z=-1$

d The line $x = 1 + t, \quad y = 2 - t, \quad z = 3 + 2t$ has direction vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Also, letting $t = 0$, the point (1, 2, 3) lies on the plane and we call this point B.

$\therefore \quad \overrightarrow{AB} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ and so a vector normal to the plane is $\overrightarrow{AB} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\text{i.e.,} \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} \quad \text{or} \quad 2(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$\therefore$ since A(3, 2, 1) lies on the plane, it has equation $x + 3y + z = 3 + 3(2) + 1$

i.e., $x + 3y + z = 10$

2 **a** $2x + 3y - z = 8$ has $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ **b** $3x - y + 0z = 11$ has $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

c $0x + 0y + z = 2$ has $\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **d** $1x + 0y + 0z = 0$ has $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3 **a** The y-axis is perpendicular to the XOZ plane $\therefore$ a normal vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ since the origin lies on the plane, it has equation $y = 0$.

b Since the plane is perpendicular to the Z-axis, it has normal vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ since (2, −1, 4) lies on the plane, it has equation $z = 4$.

4 **a** If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ -1 & 0 & -2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 1 & -4 \\ 0 & -2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$$

$\therefore$ since A(0, 2, 6) lies on the plane, it has equation

$$-2x + 6y + z = -2(0) + 6(2) + 6$$

$$\text{i.e.,} \quad -2x + 6y + z = 18$$

b If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \times \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & -2 \\ -3 & -1 & -1 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -2 & -3 \\ -1 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -3 & 3 \\ -3 & -1 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$$

$\therefore$ since C(0, 0, 1) lies on the plane, it has equation $-5x + 3y + 12z = 12$.

c If $\mathbf{n}$ is the normal vector, then

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 2 & -3 & -3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & -2 \\ -3 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} = -8\mathbf{j} + 8\mathbf{k} \quad \text{or} \quad -8(\mathbf{j} - \mathbf{k})$$

$\therefore$ since A(2, 0, 3) lies on the plane, it has equation $y - z = -3$

5 **a** The normal to $x - 3y + 4z = 8$ is $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, and this is the direction vector of the line.

$\therefore$ since the line passes through (1, -2, 0), it has equation
$x = 1 + t, \quad y = -2 - 3t, \quad z = 4t, \quad t$ in $\mathcal{R}$.

b The normal to $x - y - 2z = 11$ is $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

$\therefore$ since the line passes through (3, 4, -1), it has equation
$x = 3 + t, \quad y = 4 - t, \quad z = -1 - 2t, \quad t$ in $\mathcal{R}$.

6 The line has direction vector $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$

$\therefore$ since the line passes through A, it has parametric equations
$x = 2 - t, \quad y = -1 + 3t, \quad z = 3 - 3t, \quad t$ in $\mathcal{R}$.

This line meets $x + 2y - z = 5$ when $(2 - t) + 2(-1 + 3t) - (3 - 3t) = 5$

$$\therefore \quad 2 - t - 2 + 6t - 3 + 3t = 5$$

$\therefore \quad 8t = 8 \quad \therefore \quad t = 1,$ and so they meet at (1, 2, 0)

7 The direction vector of the line is $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

$\therefore$ since it passes through P(1, -2, 4), it has parametric equations
$x = 1 + t, \quad y = -2 + 2t, \quad z = 4 - 5t, \quad t$ in $\mathcal{R}$.

a The line meets the YOZ plane when $x = 0$ i.e., when $t = -1$.
This corresponds to the point (0, -4, 9)

b The line meets $y + z = 2$ when $-2 + 2t + 4 - 5t = 2 \quad \therefore \quad -3t = 0 \quad \therefore \quad t = 0$
This corresponds to the point (1, -2, 4)

c The line meets $\dfrac{x-3}{2} = \dfrac{y+2}{3} = \dfrac{z-30}{-1}$ when $\dfrac{1+t-3}{2} = \dfrac{-2+2t+2}{3} = \dfrac{4-5t-30}{-1}$

$$\therefore \quad \frac{t-2}{2} = \frac{2t}{3} = 5t + 26$$

$$\therefore \quad 3t - 6 = 4t = 30t + 156$$

$$\therefore \quad 3t - 6 = 4t \quad \text{and} \quad 4t = 30t + 156$$

$$\therefore \quad t = -6 \quad \text{and} \quad -26t = 156$$

$$\therefore \quad t = -6 \quad \text{is a common solution.}$$

$\therefore$ the lines meet at the point corresponding to $t = -6$, which is (-5, -14, 34).

8 **a** The plane $2x + y - 2z = -11$ has $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ the parametric equations of AN are $x = 1 + 2t, \ y = 0 + t, \ z = 2 - 2t, \ t$ in $\mathcal{R}$.

This line meets the plane when $2(1 + 2t) + t - 2(2 - 2t) = -11$

$$\therefore \quad 2 + 4t + t - 4 + 4t = -11$$

$$\therefore \quad 9t = -9$$

$$\therefore \quad t = -1$$

Thus N is $(-1, -1, 4)$ and $\therefore\ \overrightarrow{AN} = \begin{bmatrix}-2\\-1\\2\end{bmatrix}$ and $\text{AN} = \sqrt{(-2)^2+(-1)^2+2^2}$

$$= \sqrt{9}$$
$$= 3 \text{ units}$$

b The plane $x - y + 3z = -10$ has $\mathbf{n} = \begin{bmatrix}1\\-1\\3\end{bmatrix}$

$\therefore$ the parametric equations of AN are
$x = 2 + t,\quad y = -1 - t,\quad z = 3 + 3t,\quad t \text{ in } \mathcal{R}$

This line meets the plane when
$$\begin{aligned}(2+t) - (-1-t) + 3(3+3t) &= -10\\ \therefore\ 2+t+1+t+9+9t &= -10\\ \therefore\ 11t &= -22\\ \therefore\ t &= -2\end{aligned}$$

$\therefore$ N is $(0, 1, -3)$

$\therefore\ \overrightarrow{AN} = \begin{bmatrix}-2\\2\\-6\end{bmatrix}$ and $\text{AN} = \sqrt{(-2)^2+2^2+(-6)^2} = \sqrt{44} = 2\sqrt{11}$ units

c The plane $4x - y - 2z = 8$ has $\mathbf{n} = \begin{bmatrix}4\\-1\\-2\end{bmatrix}$

$\therefore$ the parametric equations of AN are
$x = 1 + 4t,\quad y = -4 - t,\quad z = -3 - 2t,\quad t \text{ in } \mathcal{R}$

This line meets the plane when
$$\begin{aligned}4(1+4t) - (-4-t) - 2(-3-2t) &= 8\\ \therefore\ 4+16t+4+t+6+4t &= 8\\ \therefore\ 21t &= -6\\ \therefore\ t &= -\tfrac{2}{7}\end{aligned}$$

$\therefore$ N is $(-\frac{1}{7}, -\frac{26}{7}, -\frac{17}{7})$, $\therefore\ \overrightarrow{AN} = \begin{bmatrix}-\frac{8}{7}\\ \frac{2}{7}\\ \frac{4}{7}\end{bmatrix}$ and so $\text{AN} = \sqrt{(-\frac{8}{7})^2+(\frac{2}{7})^2+(\frac{4}{7})^2}$

$$= \sqrt{\tfrac{84}{49}}$$
$$= 2\sqrt{\tfrac{3}{7}} \text{ units}$$

9 The mirror image lies on the normal line to the plane through the object point.

Now $x + 2y + z = 1$ has $\mathbf{n} = \begin{bmatrix}1\\2\\1\end{bmatrix}$

$\therefore$ the normal at A has parametric equations $x = 3 + t,\quad y = 1 + 2t,\quad z = 2 + t$

The line meets the plane when
$$\begin{aligned}(3+t) + 2(1+2t) + 2 + t &= 1\\ \therefore\ 3+t+2+4t+2+t &= 1\\ \therefore\ 6t &= -6\\ \therefore\ t &= -1\end{aligned}$$

$\therefore$ N is $(2, -1, 1)$

If A$'$ is the mirror image of A, then $\overrightarrow{AN} = \overrightarrow{NA'}$

$\therefore$ letting A$'$ have coordinates (a, b, c), $\begin{bmatrix}-1\\-2\\-1\end{bmatrix} = \begin{bmatrix}a-2\\b+1\\c-1\end{bmatrix}$

$\therefore\ a - 2 = -1,\quad b + 1 = -2,\quad c - 1 = -1$ and so A$'$ is at $(1, -3, 0)$.

10 The plane $x+4y-z=-2$ has normal $\mathbf{n}=\begin{bmatrix}1\\4\\-1\end{bmatrix}$ which passes through $(3, 4, -1)$.

$\therefore$ the normal has parametric equations: $x=3+t$, $y=4+4t$, $z=-1-t$
and will meet any of the coordinate axes if any two of the values of x, y, and z are zero at the same time

$\therefore$ since $x=0$ when $t=-3$ and $y=z=0$ when $t=-1$, the normal meets the X-axis when $t=-1$, i.e., at the point (2, 0, 0).

11 **a** Suppose the plane has normal $\mathbf{n}=\begin{bmatrix}a\\b\\c\end{bmatrix}$

Since the plane is parallel to the X-axis, $\mathbf{n}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix}=0 \quad \therefore \quad \begin{bmatrix}a\\b\\c\end{bmatrix}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix}=0$

$\therefore$ $a=0$ and the plane has normal $\mathbf{n}=\begin{bmatrix}0\\b\\c\end{bmatrix}$

$\therefore$ the plane has equation $by+cz=d$

But A and B lie on the plane $\therefore$ $2b+3c=d$ and $-b+2c=d$

$\therefore$ $2b+3c=-b+2c$

$\therefore$ $c=-3b$ and so $\mathbf{n}=\begin{bmatrix}0\\b\\-3b\end{bmatrix}=b\begin{bmatrix}0\\1\\-3\end{bmatrix}$

$\therefore$ a normal is $\begin{bmatrix}0\\1\\-3\end{bmatrix}$, and the plane has equation $y-3z=d$.

Using A(1, 2, 3), $d=-7$ and the equation is $y-3z=-7$

b Suppose $\mathbf{n}=\begin{bmatrix}a\\b\\c\end{bmatrix}$. Since the plane is parallel to the Y-axis, $\mathbf{n}\bullet\begin{bmatrix}0\\1\\0\end{bmatrix}=0$

$\therefore$ $b=0$ and so $\mathbf{n}=\begin{bmatrix}a\\0\\c\end{bmatrix}$ and the plane has equation $ax+cz=d$

But A and B lie on the plane. $\therefore$ $a+3c=d$ and $2c=d$

$\therefore$ $a+3c=2c$

$\therefore$ $c=-a$ and $\mathbf{n}=\begin{bmatrix}a\\0\\-a\end{bmatrix}$

$\therefore$ a normal is $\begin{bmatrix}1\\0\\-1\end{bmatrix}$, so the plane has equation $x-z=d$

Using A(1, 2, 3), $d=-2$ and so the equation is $x-z=-2$

c Suppose $\mathbf{n}=\begin{bmatrix}a\\b\\c\end{bmatrix}$. Since the plane is parallel to the Z-axis, $\mathbf{n}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix}=0$

$\therefore$ $c=0$ and so $\mathbf{n}=\begin{bmatrix}a\\b\\c\end{bmatrix}$ and the plane has equation $ax+by=d$.

But A and B lie on the plane. $\therefore$ $a+2b=d$ and $-b=d$

$\therefore$ $a+2b=-b$

$\therefore$ $a=-3b$ and $\mathbf{n}=\begin{bmatrix}-3b\\b\\0\end{bmatrix}$

$\therefore$ a normal is $\begin{bmatrix}-3\\1\\0\end{bmatrix}$, so the plane has equation $-3x+y=d$.

Using A(1, 2, 3), $d=-1$, $\therefore$ the equation is $-3x+y=-1$
i.e., $3x-y=1$

12 Now $x-1=\dfrac{y-2}{2}=z+3$ has direction vector $\begin{bmatrix}1\\2\\1\end{bmatrix}$

and $x+1=y-3=2z+5$ has direction vector $\begin{bmatrix}1\\1\\\frac{1}{2}\end{bmatrix}$ $\left\{\text{since } 2z+5=\dfrac{z+\frac{5}{2}}{\frac{1}{2}}\right\}$

$\therefore$ a vector perpendicular to both lines is:

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\times\begin{bmatrix}1\\1\\\frac{1}{2}\end{bmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\1&1&\frac{1}{2}\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&1\\1&\frac{1}{2}\end{vmatrix}+\mathbf{j}\begin{vmatrix}1&1\\\frac{1}{2}&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&2\\1&1\end{vmatrix}$$
$$=0\mathbf{i}+\tfrac{1}{2}\mathbf{j}-\mathbf{k}$$

$\therefore$ $\begin{bmatrix}0\\1\\-2\end{bmatrix}$ is perpendicular to both lines.

A plane with normal $\begin{bmatrix}0\\1\\-2\end{bmatrix}$ has equation $y-2z=c$ for some c.

Now for line 1, $\dfrac{y-2}{2}=z+3$
$\therefore$ $y-2=2z+6$
$\therefore$ $y-2z=8$

And for line 2, $y-3=2z+5$
$\therefore$ $y-2z=8$ also.

$\therefore$ $y-2z=8$ is a plane containing both lines, i.e., the lines are coplanar.

13 **a** Since A(1, 2, k) lies on $x+2y-2z=8$, $1+2(2)-2k=8$
$\therefore$ $1+4-2k=8$
$\therefore$ $-2k=3$
$\therefore$ $k=-\frac{3}{2}$

b Since $x+2y-2z=8$, the plane has normal vector $\mathbf{n}=\begin{bmatrix}1\\2\\-2\end{bmatrix}$.

$\therefore$ the normal from A has parametric equations
$x=1+t$, $y=2+2t$, $z=-\frac{3}{2}-2t$, t in $\mathcal{R}$.

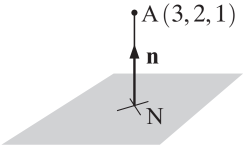
$\therefore$ points of the normal that are 6 units from A have

$$\sqrt{(1+t-1)^2+(2+2t-2)^2+(-\tfrac{3}{2}-2t+\tfrac{3}{2})^2}=6$$
$$\therefore\ \sqrt{t^2+4t^2+4t^2}=6$$
$$\therefore\ 9t^2=36$$
$$\therefore\ t^2=4$$
$$\therefore\ t=\pm 2$$

$\therefore$ B is $(3, 6, -\frac{11}{2})$ or $(-1, -2, \frac{5}{2})$

14 **a** The normal from A(3, 2, 1) to the plane has direction vector

$$\mathbf{n} = (2\mathbf{i}+\mathbf{j}+\mathbf{k}) \times (4\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & 2 & -2 \end{vmatrix}$$



$$= \mathbf{i}\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= -4\mathbf{i} + 8\mathbf{j}$$

$\therefore$ AN has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix}$

So, N has coordinates of the form $(3-4t,\ 2+8t,\ 1)$

But N lies on the plane $\therefore$ $\begin{bmatrix} 3-4t \\ 2+8t \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \mu\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

$$\therefore \begin{aligned} 3-4t &= 3+2\lambda+4\mu \\ 2+8t &= 1+\lambda+2\mu \\ 1 &= 2+\lambda-2\mu \end{aligned} \qquad \therefore \begin{aligned} 2\lambda+4\mu+4t &= 0 \\ \lambda+2\mu-8t &= 1 \\ \lambda-2\mu &= -1 \end{aligned}$$

Solving simultaneously using technology gives $\lambda = -0.4$, $\mu = 0.3$, $t = -0.1$

$\therefore$ N is $(3-4(-0.1),\ 2+8(-0.1),\ 1)$ i.e., $(3.4,\ 1.2,\ 1)$

$\therefore$ $\overrightarrow{\text{AN}} = \begin{bmatrix} 3.4-3 \\ 1.2-2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.8 \\ 0 \end{bmatrix}$ and $|\,\overrightarrow{\text{AN}}\,| = \sqrt{(0.4)^2+(0.8)^2+0^2} \doteq \frac{2}{\sqrt{5}}$ units

b The normal from A(1, 0, −2) to the plane has direction vector

$$\mathbf{n} = (3\mathbf{i}-\mathbf{j}+2\mathbf{k}) \times (-\mathbf{i}+\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$\therefore$ AN has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

So, N has coordinates of the form $(1-t,\ t,\ -2+2t)$

But N lies on the plane $\therefore$ $\begin{bmatrix} 1-t \\ t \\ -2+2t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \lambda\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \mu\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$$\therefore \begin{aligned} 1+3\lambda-\mu &= 1-t \\ -1-\lambda+\mu &= t \\ 1+2\lambda-\mu &= -2+2t \end{aligned} \qquad \text{and so} \quad \begin{aligned} 3\lambda-\mu+t &= 0 \\ -\lambda+\mu-t &= 1 \\ 2\lambda-\mu-2t &= -3 \end{aligned}$$

Solving simultaneously using technology gives $\lambda = \frac{1}{2}$, $\mu = 2\frac{1}{3}$, $t = \frac{5}{6}$

$\therefore$ N is $\left(1-\frac{5}{6},\ \frac{5}{6},\ -2+\frac{5}{3}\right)$ i.e., $\left(\frac{1}{6},\ \frac{5}{6},\ -\frac{1}{3}\right)$

$\therefore$ $\overrightarrow{\text{AN}} = \begin{bmatrix} \frac{1}{6}-1 \\ \frac{5}{6}-0 \\ -\frac{1}{3}--2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$ and $|\,\overrightarrow{\text{AN}}\,| = \sqrt{\frac{25}{36}+\frac{25}{36}+\frac{25}{9}} = \frac{5\sqrt{6}}{6}$ units

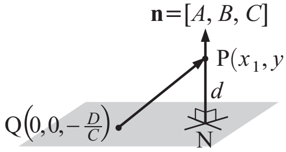
$\doteq 2.04$ units

15 **a** If N is the point on the plane such that NP is a normal to it, then ΔNPQ is right-angled at N.

Since $\overrightarrow{\text{NP}}$ is in the same direction as $\mathbf{n}$, $\overrightarrow{\text{PN}}$ is the projection vector of $\overrightarrow{\text{PQ}}$ on $-\mathbf{n}$.

$$\therefore \quad d = \text{PN} = \frac{|\,\overrightarrow{\text{PQ}} \bullet (-\mathbf{n})\,|}{|-\mathbf{n}|} = \frac{|\,\overrightarrow{\text{QP}} \bullet \mathbf{n}\,|}{|\mathbf{n}|}$$

{property of projection vectors}



b Since Q is any point on the plane, it has coordinates (x, y, z) such that $Ax+By+Cz+D=0$.

The normal vector to the plane is $\mathbf{n} = [A, B, C]$.

$$\therefore \text{ using } \mathbf{a}, \quad d = \frac{|\,\overrightarrow{\text{QP}} \bullet \mathbf{n}\,|}{|\mathbf{n}|} = \frac{|[x_1 - x, y_1 - y, z_1 - z] \bullet [A, B, C]|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax_1 - Ax + By_1 - By + Cz_1 - Cz|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2+B^2+C^2}}$$

c **8 a** *check:* Given A(1, 0, 2) and the plane $2x + y - 2z + 11 = 0$,

$$d = \frac{|2x_1 + y_1 - 2z_1 + 11|}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{|2(1) + 0(1) - 2(2) + 11|}{\sqrt{9}} = \tfrac{9}{3} = 3 \text{ units}$$

8 b *check:* Given A(2, −1, 3) and the plane $x - y + 3z = -10$,

$$d = \frac{|x_1 - y_1 + 3z_1 + 10|}{\sqrt{1^2 + (-1)^2 + 3^2}} = \frac{|2 - (-1) + 3(3) + 10|}{\sqrt{11}} = \frac{22}{\sqrt{11}} = 2\sqrt{11} \text{ units}$$

8 c *check:* Given A(1, −4, −3) and the plane $4x - y - 2z = 8$,

$$d = \frac{|4x_1 - y_1 - 2z_1 - 8|}{\sqrt{4^2 + (-1)^2 + (-2)^2}} = \frac{|4 - (-4) - 2(-3) - 8|}{\sqrt{21}} = \frac{6}{\sqrt{21}} \text{ units}$$

16 Using the formula derived in **15 b**,

a $$d = \frac{|x_1 + 2y_1 - z_1 - 10|}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|0 + 2(0) - 0 - 10|}{\sqrt{6}} = \frac{10}{\sqrt{6}} \text{ units}$$

b $$d = \frac{|x_1 + y_1 - z_1 - 2|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{|1 + (-3) - 2 - 2|}{\sqrt{3}} = \frac{|-6|}{\sqrt{3}} = \frac{6}{\sqrt{3}} \text{ units} \quad (\text{or } 2\sqrt{3} \text{ units})$$

17 **a** First choose a point on the first plane, $x + y + 2z = 4$, for example, (0, 0, 2).

Then find the distance from this point to the other plane, using the formula obtained in **15 b**,

$$\therefore \quad d = \frac{|2x_1 + 2y_1 + 4z_1 + 11|}{\sqrt{2^2 + 2^2 + 4^2}} = \frac{|2(0) + 2(0) + 4(2) + 11|}{\sqrt{24}} = \frac{19}{\sqrt{24}} \text{ units}$$

b Choose a point on the plane $ax + by + cz + d_1 = 0$, for example, $(0, 0, -\frac{d_1}{c})$.

Using the formula obtained in **15 b** to calculate the distance from this pont to the second plane,

$$\therefore \quad d = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{\left|a(0) + b(0) + c(-\frac{d_1}{c}) + d_2\right|}{\sqrt{a^2+b^2+c^2}} = \frac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}} \text{ units}$$

18 The line $x = 2 + t, \quad y = -1 + 2t, \quad z = -3t$ has direction vector $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$,

and $\begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix}$ is a vector normal to the plane $11x - 4y + z = 0$.

But $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix} = 11 - 8 - 3 = 0$

$\therefore$ these vectors are perpendicular and so the line is parallel to the plane.

Choose any point on the line, say $t = 0$, which corresponds to the point $(2, -1, 0)$.

Then the distance $d = \dfrac{|11x_1 - 4y_1 + z_1|}{\sqrt{11^2 + (-4)^2 + 1^2}} = \dfrac{|11(2) - 4(-1) + 0|}{\sqrt{138}} = \dfrac{26}{\sqrt{138}}$ units.

19 Since the planes are parallel to $2x - y + 2z = 5$, they have equation $2x - y + 2z = a$ for some a.

Choose any point on $2x - y + 2z = 5$, for example, $(0, -5, 0)$

Then the distance from this point to the plane $2x - y + 2z = a$ is

$$d = \frac{|2x_1 - y_1 + 2z_1 - a|}{\sqrt{2^2 + (-1)^2 + 2^2}} \qquad \therefore \quad 2 = \frac{|2(0) - (-5) + 2(0) - a|}{3}$$

$\therefore \quad 6 = |5 - a|$ and so $5 - a = \pm 6$

$\therefore \quad a = 5 \pm 6$

$\therefore \quad a = -1$ or $a = 11$

$\therefore$ the planes are $2x - y + 2z = -1$ and $2x - y + 2z = 11$

EXERCISE 17E

1 **a** $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

$$\therefore \quad \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} = \frac{|4 - 3 + 1|}{\sqrt{3}\sqrt{26}} = \frac{2}{\sqrt{78}}$$ and so $\phi \doteqdot 13.09^o$

b $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$\therefore \quad \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} = \frac{|2 - 3 + 1|}{\sqrt{6}\sqrt{11}} = 0$$ and so $\phi = 0^o$,

i.e., the line and plane are parallel

c $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ and if $x - 4 = 3 - y = 2(z + 1) = t$

i.e., $x = 4 + t, \quad y = 3 - t, \quad z = -1 + \frac{1}{2}t$ then $\mathbf{l} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$

$$\therefore \quad \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} = \frac{\left|3 + (-4) + (-\frac{1}{2})\right|}{\sqrt{26}\sqrt{\frac{9}{4}}} = \frac{\left|-\frac{3}{2}\right|}{\frac{3}{2}\sqrt{26}} = \frac{1}{\sqrt{26}}$$ and so $\phi \doteqdot 11.31^o$

d The plane has normal vector

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -1 \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -4 & -1 \\ 1 & -2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and the line has direction vector $\mathbf{l} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\therefore \quad \sin\phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} = \frac{|9 - 5 + 7|}{\sqrt{81 + 25 + 49}\sqrt{1 + 1 + 1}} = \frac{11}{\sqrt{155}\sqrt{3}}$$

$\therefore \quad \phi \doteqdot 30.67^o$

2 **a** $\mathbf{n}_1 = \begin{bmatrix}2\\-1\\1\end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix}1\\3\\2\end{bmatrix}$

$$\therefore \quad \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{|2-3+2|}{\sqrt{6}\sqrt{14}} = \tfrac{1}{\sqrt{84}}$$

$\therefore \quad \theta \doteqdot 83.7^o$

b $\mathbf{n}_1 = \begin{bmatrix}1\\-1\\3\end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix}3\\1\\-1\end{bmatrix}$

$$\therefore \quad \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{|3-1-3|}{\sqrt{11}\sqrt{11}} = \tfrac{1}{11}$$

$\therefore \quad \theta \doteqdot 84.8^o$

c $\mathbf{n}_1 = \begin{bmatrix}3\\-1\\1\end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix}2\\4\\-1\end{bmatrix}$ $\quad\therefore \quad \cos\theta = \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{|6-4-1|}{\sqrt{11}\sqrt{21}} = \dfrac{1}{\sqrt{231}}$

$\therefore \quad \theta \doteqdot 86.2^o$

d
$$\mathbf{n}_1 = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & -1\\ 2 & -4 & 3\end{vmatrix} = \mathbf{i}\begin{vmatrix}1 & -1\\ -4 & 3\end{vmatrix} + \mathbf{j}\begin{vmatrix}-1 & -1\\ 3 & 2\end{vmatrix} + \mathbf{k}\begin{vmatrix}-1 & 1\\ 2 & -4\end{vmatrix}$$
$$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\mathbf{n}_2 = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & -1 & -1\\ 1 & 1 & 1\end{vmatrix} = \mathbf{i}\begin{vmatrix}-1 & -1\\ 1 & 1\end{vmatrix} + \mathbf{j}\begin{vmatrix}-1 & -2\\ 1 & 1\end{vmatrix} + \mathbf{k}\begin{vmatrix}-2 & -1\\ 1 & 1\end{vmatrix}$$
$$= 0\mathbf{i} + 1\mathbf{j} - \mathbf{k}$$

$$\therefore \quad \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{|0+1-2|}{\sqrt{1+1+4}\sqrt{0+1+1}} = \frac{1}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{12}}$$

$\therefore \quad \theta \doteqdot 73.2^o$

e
$$\mathbf{n}_1 = \begin{bmatrix}3\\-4\\1\end{bmatrix} \text{ and } \mathbf{n}_2 = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & 0\\ 2 & 1 & 1\end{vmatrix} = \mathbf{i}\begin{vmatrix}-1 & 0\\ 1 & 1\end{vmatrix} + \mathbf{j}\begin{vmatrix}0 & 3\\ 1 & 2\end{vmatrix} + \mathbf{k}\begin{vmatrix}3 & -1\\ 2 & 1\end{vmatrix}$$
$$= -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$\therefore \quad \cos\theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{|(3)(-1)+(-4)(-3)+(1)(5)|}{\sqrt{9+16+1}\sqrt{1+9+25}} = \frac{14}{\sqrt{26 \times 35}} \quad \therefore \quad \theta \doteqdot 62.3^o$$

EXERCISE 17F

1 **a** Either (1) no solutions or (2) an infinite number of solutions.

b **i** They are parallel if

$a_1 = ka_2$
$b_1 = kb_2$
and $c_1 = kc_2$ for some k.

ii They are coincident if

$a_1 = ka_2$
$b_1 = kb_2$
$c_1 = kc_2$
and $d_1 = kd_2$ for some k.

c **i**
$$\left[\begin{array}{ccc|c}1 & -3 & 2 & 8\\ 3 & -9 & 2 & 4\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & -3 & 2 & 8\\ 0 & 0 & -4 & -20\end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$\therefore \quad -4z = -20$ and $x - 3y + 2z = 8$ i.e., $x = 3y - 2z + 8$

$\therefore \quad z = 5$ and if we let $y = t$, then $x = 3t - 2(5) + 8 = -2 + 3t$

$\therefore$ the planes meet in the line $x = -2 + 3t,\ y = t,\ z = 5,\ t$ in $\mathcal{R}$

ii
$$\left[\begin{array}{ccc|c}2 & 1 & 1 & 5\\ 1 & -1 & 1 & 3\end{array}\right] \sim \left[\begin{array}{ccc|c}2 & 1 & 1 & 5\\ 0 & -3 & 1 & 1\end{array}\right] \quad R_2 \to 2R_2 - R_1$$

$\therefore \quad -3y + z = 1$ and $2x + y + z = 5$

$\therefore$ if we let $y = t$, then $z = 1 + 3y = 1 + 3t$ and
$$\begin{aligned} 2x &= 5 - y - z\\ &= 5 - t - (1 + 3t)\\ &= 4 - 4t \end{aligned}$$

i.e., $x = 2 - 2t$

$\therefore$ the planes meet in the line $x = 2 - 2t,\ \ y = t,\ \ z = 1 + 3t,\ \ t$ in $\mathcal{R}$

iii $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 3 & 6 & -9 & 18 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to R_2 - 3R_1$

$\therefore$ there are infinitely many solutions, as the planes are coincident.

Let $y = s$ and $z = t$ in $x + 2y - 3z = 6$, $s, t \in \mathcal{R}$

$\therefore \quad x = 3t - 2s + 6$

$\therefore \quad x = 3t - 2s + 6, \quad y = s, \quad z = t$ is the general solution of the plane.

2 **a** $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 4 & k & 12 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 0 & k+2 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1$

If $k = -2$, the two planes are coincident.
$\therefore$ infinitely many solutions.

If $k \neq -2$, the two planes meet in a line.
$\therefore$ infinitely many solutions.

b $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 2 & -2 & 6 & k \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 0 & 0 & k-16 \end{array}\right] \quad R_2 \to R_2 - 2R_1$

If $k = 16$, the planes are coincident.
$\therefore$ infinitely many solutions.

If $k \neq 16$, the planes are parallel but not coincident.
$\therefore$ no solutions exist.

3 (1) $P_1 = P_2 = P_3$: infinitely many solutions where x, y and z are in terms of two parameters, s and t say, i.e., solution is a plane.

(2) $P_1 = P_2$ are coincident and cut by P_3: infinitely many solutions where x, y and z are in terms of one parameter, t say, i.e., solution is a line.

(3) $P_1 = P_2$ with P_3 parallel but not coincident: no solutions exist.

(4) P_1 and P_2 are parallel but not coincident, and P_3 cuts both planes: no solutions exist.

(5) P_1, P_2 and P_3 are all parallel but not coincident: no solutions exist.

(6) P_1, P_2 and P_3 meet in a unique point (a, b, c), so that $x = a$, $y = b$, $z = c$.

(7) P_1, P_2 and P_3 meet in a common line: infinitely many solutions where x, y and z are in terms of one parameter, t, say.

(8) P_1, P_2 and P_3 are such that the line of intersection between any two is parallel to the third plane: no solutions exist.

4 **a** The system has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 1 & -1 & 2 & 11 \\ 4 & 1 & -5 & -18 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & -3 & -1 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & 0 & -11 & -44 \end{array}\right] \quad R_3 \to 2R_3 - 3R_2$$

$\therefore$ the planes meet at the unique point $(1, -2, 4)$

Now $-11z = -4 \quad \therefore \quad z = 4$

and $-2y + 3z = 16$

$\therefore \quad -2y + 12 = 16$

$\therefore \quad -2y = 4$

$\therefore \quad y = -2$

and $x + y - z = -5$

$\therefore \quad x = -5 - (-2) + 4$

$\therefore \quad x = 2 + 4 - 5$

$\therefore \quad x = 1$

b The system has augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 8 \\ 5 & -2 & 5 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 3 & -5 & 6 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \end{array}$$

$\therefore$ the three planes meet in a common line

$x = \dfrac{9-t}{3}$, $y = \dfrac{5t+6}{3}$, $z = t$, t in $\mathcal{R}$

Let $z = t$

$$\begin{aligned} \text{As}\quad 3y - 5z &= 6 \\ 3y &= 5t + 6 \\ \therefore\quad y &= \frac{5t+6}{3} \end{aligned}$$

$$\begin{aligned} \text{But}\quad x - y + 2z &= 1 \\ \therefore\quad x &= 1 - \frac{5t+6}{3} - 2t \\ \therefore\quad x &= \frac{3 + 5t + 6 - 6t}{3} \\ \therefore\quad x &= \frac{9-t}{3} \end{aligned}$$

c The system has augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 8 \\ 2 & -1 & -1 & 5 \\ 3 & -4 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & -10 & 2 & -22 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

$\therefore$ the three planes meet in a common line

$\therefore$ $x = 3t - 3$, $y = t$, $z = 5t - 11$, t in $\mathcal{R}$

Let $y = t$

$$\begin{aligned} \text{Now}\quad -5y + z &= -11 \\ \therefore\quad z &= -11 + 5t \\ \text{Also}\quad x + 2y - z &= 8 \\ \therefore\quad x &= 8 - 2y + z \\ \therefore\quad x &= 8 - 2t - 11 + 5t \\ \therefore\quad x &= -3 + 3t \end{aligned}$$

d The system has augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & -2 & 2 & 11 \\ 1 & 3 & -1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 0 & 0 & -5 \\ 0 & 4 & -2 & -10 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

The first two planes are parallel and are cut by the third plane.

$\therefore$ the equations are inconsistent and there are no solutions.

e The system has augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 1 & -1 & 1 & 4 \\ 3 & 3 & -6 & 3 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -2 & 1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

There are two coincident planes cut by a third plane.

$\therefore$ infinitely many solutions in a line:

$x = \dfrac{t+5}{2}$, $y = \dfrac{3t-3}{2}$, $z = t$, t in $\mathcal{R}$

Let $z = t$

$$\begin{aligned} \text{Now}\quad -2y + 3z &= 3 \\ \therefore\quad 2y &= 3z - 3 \\ \therefore\quad y &= \frac{3t-3}{2} \end{aligned}$$

$$\begin{aligned} \text{and as}\quad x + y - 2z &= 1 \\ \therefore\quad x &= 1 - y + 2z \\ \therefore\quad x &= 1 - \frac{3t-3}{2} + 2t \\ \therefore\quad x &= \frac{2 - 3t + 3 + 4t}{2} \\ \therefore\quad x &= \frac{t+5}{2} \end{aligned}$$

f The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 5\\ 1 & 1 & 1 & 1\\ 5 & -1 & 2 & 17\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5\\ 0 & 2 & 2 & -4\\ 0 & 4 & 7 & -8\end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1\\ R_3 \to R_3 - 5R_1\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5\\ 0 & 2 & 2 & -4\\ 0 & 0 & 3 & 0\end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2\end{array}$$

$$\begin{aligned} \text{Now}\quad 3z &= 0\\ \therefore\quad z &= 0\\ \text{As}\quad 2y+2z &= -4\\ \therefore\quad 2y &= -4\\ \therefore\quad y &= -2\\ \text{and as}\quad x-y-z &= 5\\ \therefore\quad x &= 5+(-2)+0\\ \therefore\quad x &= 3 \end{aligned}$$

$\therefore$ the planes meet at the unique point (3, −2, 0).

5 The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 1\\ 2 & -3 & -1 & 3\\ 3 & -5 & -5 & k\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1\\ 0 & -1 & -7 & 1\\ 0 & -2 & -14 & k-3\end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - 3R_1\end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1\\ 0 & -1 & -7 & 1\\ 0 & 0 & 0 & k-5\end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2\end{array}$$

(1) If $k=5$, the planes meet in a line
{as we have a row of zeros}

$$\begin{aligned} \text{Let}\quad z &= t\\ \text{Now}\quad -y-7z &= 1\\ \therefore\quad y &= -1-7t\\ \text{and}\quad x-y+3z &= 1\\ \therefore\quad x &= 1+y-3z\\ \therefore\quad x &= 1-1-7t-3t\\ \therefore\quad x &= -10t \end{aligned}$$

$\therefore$ $x=-10t$, $y=-1-7t$, $z=t$, t in $\mathcal{R}$

(2) If $k \neq 5$ there are no solutions.

Since the first two planes are not parallel, the line of intersection of any two planes is parallel to the third plane.

6 The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & m & -1\\ 2 & 1 & -1 & 3\\ m & -2 & 1 & 1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1\\ 0 & -3 & -1-2m & 5\\ 0 & -2-2m & 1-m^2 & 1+m\end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - mR_1\end{array}$$

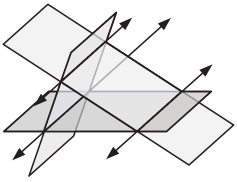
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1\\ 0 & -3 & -1-2m & 5\\ 0 & 0 & -m^2-6m-5 & 7m+7\end{array}\right] \begin{array}{l} \\ \\ R_3 \to -3R_3 - (-2-2m)R_2\end{array}$$

$$\begin{aligned} \therefore\quad -(m^2+6m+5)z &= 7(m+1)\\ \text{i.e.,}\quad -(m+5)(m+1)z &= 7(m+1) \end{aligned}$$

(1) When $m=-5$ we have $0z=-28$.

$\therefore$ the system is inconsistent, and there are no solutions.

The augmented matrix is: $\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1\\ 0 & -3 & 9 & 5\\ 0 & 0 & 0 & -28\end{array}\right]$

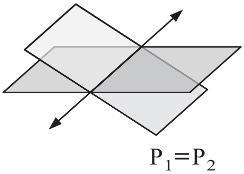


The line of intersection of any two planes is parallel to the third plane.

(2) When $m = -1$, we have $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

i.e., three planes meet in a common line {two coincident planes cut by the third}

Let $z = t$. Then $-3y + t = 5$, $\therefore$ $y = \dfrac{t-5}{3}$



$$\begin{aligned} \text{Also,}\quad x &= -1 - 2y + z \\ &= -1 - 2\left(\frac{t-5}{3}\right) + t \\ &= -1 + \tfrac{1}{3}t + \tfrac{10}{3} \\ &= \frac{t+7}{3} \end{aligned}$$

i.e., $x = \dfrac{t+7}{3}$, $y = \dfrac{t-5}{3}$, $z = t$, t in $\mathcal{R}$

(3) When $m \neq -1$ or -5, the system has a unique solution,
i.e., the three planes meet in a common point.

7 P_1 meets P_2 where

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + r \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \left\{ \begin{aligned} 2 + 3\lambda + \mu &= 3 + 2r + s \\ -1 \quad + \mu &= -1 \quad + s \\ \lambda - \mu &= 3 - r \end{aligned} \right. \qquad \text{i.e.,} \quad \left\{ \begin{aligned} 3\lambda + \mu &= 2r + s + 1 \\ \mu &= s \\ \lambda - \mu &= 3 - r \end{aligned} \right.$$

If $\mu = a$, say then $s = a$, $\left\{ \begin{aligned} 3\lambda + a &= 2r + a + 1 \\ \lambda - a &= 3 - r \quad \text{i.e.,} \quad r = 3 - \lambda + a \end{aligned} \right.$

$\therefore$ $3\lambda + a = 6 - 2\lambda + 2a + a + 1$

i.e., $5\lambda = 2a + 7$

$\lambda = \dfrac{2a+7}{5}$ and $r = 3 + a - \dfrac{2a+7}{5}$

i.e., $r = \dfrac{3a+8}{5}$

i.e., if $\mu = a$, $\lambda = \dfrac{2a+7}{5}$, $r = \dfrac{3a+8}{5}$, $s = a$ ...... (1)

P_2 meets P_3 where

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + r \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - u \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore \left\{ \begin{aligned} 3 + 2r + s &= 2 + t \\ -1 \quad + s &= -1 - t + u \\ 3 - r \quad &= 2 - 2u \end{aligned} \right. \qquad \text{i.e.,} \quad \left\{ \begin{aligned} 2r + s + 1 &= t \\ s &= u - t \\ 2u - r &= -1 \end{aligned} \right.$$

So, if $u = b$, say then $r = 2b + 1$ and $4b + 2 + b - t + 1 = t$

$\therefore$ $5b + 3 = 2t$ i.e., $t = \dfrac{5b+3}{2}$

and $s = u - t = b - \dfrac{5b+3}{2} = \dfrac{-3b-3}{2}$

i.e., if $u = b$, $r = 2b + 1$, $t = \dfrac{5b+3}{2}$, $s = \dfrac{-3b-3}{2}$ ...... (2)

From (1) and (2) $\dfrac{3a+8}{5} = 2b + 1$ and $a = \dfrac{-3b-3}{2}$

$\therefore$ $3a + 8 = 10b + 5$ and $2a = -3b - 3$

i.e., $\left\{\begin{aligned} 3a - 10b &= -3 \\ 2a + 3b &= -3 \end{aligned}\right.$ which has solutions $a = -\frac{39}{29}$, $b = -\frac{3}{29}$

In (2), $u = -\frac{3}{29}$, $t = \dfrac{5\left(-\frac{3}{29}\right) + 3}{2} = 1\frac{7}{29}$ or $\frac{36}{29}$

$$\therefore \quad \mathbf{r}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \tfrac{36}{29}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \tfrac{3}{29}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\frac{7}{29} \\ -2\frac{10}{29} \\ 2\frac{6}{29} \end{bmatrix} = \begin{bmatrix} \frac{94}{29} \\ -\frac{68}{29} \\ \frac{64}{29} \end{bmatrix}$$

$\therefore$ all 3 planes meet at $\left(\frac{94}{29}, -\frac{68}{29}, \frac{64}{29}\right)$.

REVIEW SET 17A

1 **a** The vector equation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

b The parametric equations are $x = -6 + 4t$, $y = 3 - 3t$, $t \in \mathcal{R}$

2 The vector equation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} + t\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $t \in \mathcal{R}$

3 $(-3, m)$ lies on the line, so $\begin{bmatrix} -3 \\ m \end{bmatrix} = \begin{bmatrix} 18 \\ -2 \end{bmatrix} + \begin{bmatrix} -7t \\ 4t \end{bmatrix}$

$$\begin{aligned} \therefore \quad -3 &= 18 - 7t \quad \text{and} \quad m = -2 + 4t \\ \therefore \quad 7t &= 21 \\ \text{i.e.,} \quad t &= 3 \quad \text{and so} \quad m = -2 + 4(3) = 10 \\ \text{i.e.,} \quad m &= 10 \end{aligned}$$

4 **a** $x(0) = -4$ and $y(0) = 3$, so the initial position is $(-4, 3)$.

b $x(4) = -4 + 32 = 28$, $y(4) = 3 + 6(4) = 27$, so at $t = 4$, the position is $(28, 27)$.

c The velocity vector is $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$, so the speed is $\sqrt{8^2 + 6^2} = 10$ m/s **d** $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$

5 The direction vector is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$\therefore$ $2\sqrt{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ has length 20. So, the velocity vector is $\begin{bmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{bmatrix}$ or $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$

6 **a** **i** The yacht is initially at $(-6, 10)$, so its initial position vector is $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$ i.e., $-6\mathbf{i} + 10\mathbf{j}$

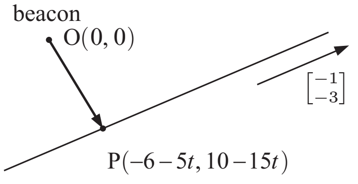
ii $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\therefore$ $5(-\mathbf{i} - 3\mathbf{j})$ has length $5\sqrt{10}$

$\therefore$ the direction vector is $-5\mathbf{i} - 15\mathbf{j}$

iii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix} + t\begin{bmatrix} -5 \\ -15 \end{bmatrix}$ i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = -6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j}) = (-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j}$

b



$\overrightarrow{OP} = \begin{bmatrix} -6-5t \\ 10-15t \end{bmatrix}$ and $\overrightarrow{OP} \bullet \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0$

$$\begin{aligned} \therefore \quad -1(-6 - 5t) - 3(10 - 15t) &= 0 \\ \therefore \quad 6 + 5t - 30 + 45t &= 0 \\ \therefore \quad 50t &= 24 \\ \therefore \quad t &= 0.48 \text{ h} \end{aligned}$$

(i.e., 28.8 min)

c When $t = 0.48$, $\overrightarrow{OP} = \begin{bmatrix} -6-5(0.48) \\ 10-15(0.48) \end{bmatrix} = \begin{bmatrix} -8.4 \\ 2.8 \end{bmatrix}$

and $\text{OP} = \sqrt{(-8.4)^2 + (2.8)^2} \doteqdot 8.85$ km

As the closest distance is 8.85 km and the radius is 8 km, the yacht will miss the reef.

7 **a** $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ where $t \geqslant 0$. When $t = 0$, the time is 2:17 pm

i.e., $x_1(t) = 2 + t, \quad y_1(t) = 4 - 3t$

b Likewise, $x_2(t) = 11 - (t + 2), \quad y_2(t) = 3 + a(t + 2)$

i.e., $x_2(t) = 9 - t \quad y_2(t) = [3 + 2a] + at$

c They meet where $2 + t = 9 - t$ and $4 - 3t = [3 + 2a] + at$

$\therefore \; 2t = 7$

$\therefore \; t = \frac{7}{2}$ $\therefore$ the time would be 2:17 pm plus $3\frac{1}{2}$ min i.e., 2:20:30 pm

d When $t = \frac{7}{2}$,

$$4 - 3\left(\tfrac{7}{2}\right) = \left[3 + 2\left(\tfrac{7}{2}\right)\right] + a\left(\tfrac{7}{2}\right)$$

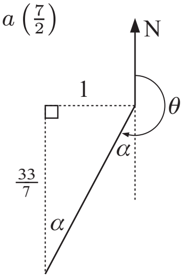
$$\therefore \; -\tfrac{13}{2} = 10 + \frac{7a}{2}$$

$$\therefore \; -13 = 20 + 7a$$

$$\therefore \; 7a = -33$$

$$\therefore \; a = -\tfrac{33}{7}$$

Y18 has velocity vector $\begin{bmatrix} -1 \\ -\frac{33}{7} \end{bmatrix}$



with speed $= \sqrt{(-1)^2 + \left(-\frac{33}{7}\right)^2}$

$\doteqdot 4.82$ km/min

$$\tan\alpha = \frac{1}{\frac{33}{7}} = \tfrac{7}{33}$$

$$\therefore \; \alpha = \tan^{-1}\left(\tfrac{7}{33}\right) \doteqdot 12.0^o$$

$\therefore$ the direction is $180^o + \alpha^o = 192^o$

i.e., the torpedo has speed 4.82 km/min and direction 192^o.

8 **a** Line 1 has direction vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and Line 4 has direction vector $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

Now $\begin{bmatrix} 5 \\ -2 \end{bmatrix} = -\begin{bmatrix} -5 \\ 2 \end{bmatrix}$, so Lines 1 and 4 are parallel, i.e., KL $\parallel$ MN.

b $\overrightarrow{\text{KL}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \quad \overrightarrow{\text{NK}} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}, \quad \overrightarrow{\text{MN}} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \quad \therefore \; \overrightarrow{\text{KL}} \bullet \overrightarrow{\text{NK}} = 20 - 20 = 0,$

and $\overrightarrow{\text{NK}} \bullet \overrightarrow{\text{MN}} = -20 + 20 = 0$

$\therefore$ NK is perpendicular to both KL and MN.

c Lines 1 and 3 meet at K

$$\therefore \; \begin{bmatrix} 2 \\ 19 \end{bmatrix} + p\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r\begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$\therefore \; \begin{bmatrix} 5p - 4r \\ -2p - 10r \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \end{bmatrix}$$

$$\begin{aligned} \therefore \; 5p - 4r &= 1 && \times 5 \\ 2p + 10r &= 12 && \times 2 \\ \therefore \; 25p - 20r &= 5 \\ 4p + 20r &= 24 \\ \hline \text{Adding,} \quad 29p \qquad &= 29 \\ \therefore \; p &= 1 \end{aligned}$$

$$\text{So,} \; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 19 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

$\therefore$ K is (7, 17)

Lines 2 and 4 meet at M.

$$\therefore \; \begin{bmatrix} 33 \\ -5 \end{bmatrix} + r\begin{bmatrix} -11 \\ 16 \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s\begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\therefore \; \begin{bmatrix} -11r + 5s \\ 16r - 2s \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$$

$$\begin{aligned} \therefore \; -11r + 5s &= 10 && \times 2 \\ 16r - 2s &= -4 && \times 5 \\ \therefore \; -22r + 10s &= 20 \\ 80r - 10s &= -20 \\ \hline \therefore \; 58r \qquad &= 0 \\ \therefore \; r &= 0 \end{aligned}$$

$$\text{So,} \; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ -5 \end{bmatrix}$$

$\therefore$ M is (33, −5)

Lines 3 and 4 meet at N

$$\therefore \; \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r\begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s\begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\therefore \; \begin{bmatrix} 4r + 5s \\ 10r - 2s \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix}$$

$$\begin{aligned} \therefore \; 4r + 5s &= 40 && \times 2 \\ 10r - 2s &= -16 && \times 5 \\ \therefore \; 8r + 10s &= 80 \\ 50r - 10s &= -80 \\ \hline \end{aligned}$$

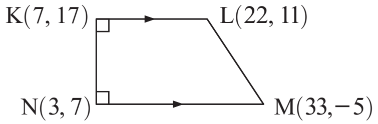
$$\therefore \; 58r \qquad = 0$$

$$\therefore \; r = 0$$

$$\therefore \; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

i.e., N is (3, 7)

d



$$\begin{aligned} \text{KL} &= \sqrt{(22-7)^2+(11-17)^2} \\ &= \sqrt{225+36} \\ &= \sqrt{261} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{NK} &= \sqrt{(33-3)^2+(-5-7)^2} \\ &= \sqrt{900+144} \\ &= \sqrt{1044} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{KN} &= \sqrt{(7-3)^2+(17-7)^2} \\ &= \sqrt{16+100} \\ &= \sqrt{116} \end{aligned}$$

$$\therefore \quad \text{area} = \left(\frac{\sqrt{261}+\sqrt{1044}}{2}\right) \times \sqrt{116} = 261 \text{ units}^2$$

REVIEW SET 17B

1 **a** We first find the plane containing A, B and C.

The plane has normal

$$\mathbf{n} = \overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 8 \\ -2 & 2 & -2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 8 \\ 2 & -2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 8 & 2 \\ -2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}$$

$$= -18\mathbf{i} - 12\mathbf{j} + 6\mathbf{k} \quad \text{or} \quad -6[3\mathbf{i}+2\mathbf{j}-\mathbf{k}]$$

$\therefore$ since A lies on the plane, it has equation $3x+2y-z = 3(1)+2(0)+(-1)(4)$

i.e., $3x+2y-z = -1$

b Since the coordinates of D satisfy this equation, all 4 points are coplanar.

The closest point on the plane to E(3, 3, 2) is the foot of the normal from E.

The equation of the normal through E is $x = 3+3t$, $y = 3+2t$, $z = 2-t$, and this intersects the plane when

$$\begin{aligned} 3(3+3t)+2(3+2t)-(2-t) &= -1 \\ \therefore \quad 9+9t+6+4t-2+t &= -1 \\ \therefore \quad 14t &= -14 \\ \therefore \quad t &= -1 \end{aligned}$$

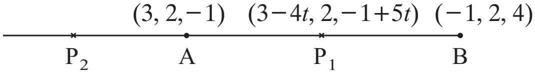
i.e., the nearest point is (0, 1, 3)

2 **a** $\overrightarrow{\text{AB}} = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$ $\quad\therefore$ the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$, t in $\mathcal{R}$

b The equation of the plane is

$-4x+0y+5z = -4(-1)+5(4)$

i.e., $-4x+5z = 24$



c The distance from a point on the line to A is $d = \sqrt{(-4t)^2+0^2+(5t)^2} = \sqrt{41t^2}$

$$\begin{aligned} \therefore \quad \text{since } d = 2\sqrt{41} \text{ units, } \sqrt{41t^2} &= 2\sqrt{41} \\ \therefore \quad t^2 &= 4 \\ \therefore \quad t &= \pm 2 \end{aligned}$$

$\therefore$ the points are (−5, 2, 9) and (11, 2, −11).

3 **a** $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\begin{aligned} \therefore \quad \sin\phi &= \frac{|2+-2+2|}{\sqrt{9}\sqrt{6}} \\ &= \frac{2}{3\sqrt{6}} \\ \therefore \quad \phi &\doteqdot 15.79^o \end{aligned}$$

b The planes have normals

$\mathbf{n}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{aligned} \therefore \quad \cos\theta &= \frac{|2+(-1)+(-4)|}{\sqrt{9}\sqrt{6}} \\ &= \frac{3}{3\sqrt{6}} \quad \text{i.e., } \tfrac{1}{\sqrt{6}} \\ \therefore \quad \theta &\doteqdot 65.91^o \end{aligned}$$

4 **a** $\overrightarrow{AB} = \begin{bmatrix}0-3\\2--1\\-1-1\end{bmatrix} = \begin{bmatrix}-3\\3\\-2\end{bmatrix} \quad \therefore \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\-1\\1\end{bmatrix} + t\begin{bmatrix}-3\\3\\-2\end{bmatrix}, \quad t \in \mathcal{R}$

b If P divides BA in the ratio 2 : 5, then $\overrightarrow{BP} : \overrightarrow{PA} = 2 : 5$

If P is (a, b, c) then $\begin{bmatrix}a-0\\b-2\\c--1\end{bmatrix} : \begin{bmatrix}3-a\\-1-b\\1-c\end{bmatrix} = 2:5$

$$\therefore \quad 5\begin{bmatrix}a\\b-2\\c+1\end{bmatrix} = 2\begin{bmatrix}3-a\\-1-b\\1-c\end{bmatrix}$$

$$\begin{aligned}
\therefore \quad 5a &= 6-2a, & 5b-10 &= -2-2b, & 5c+5 &= 2-2c\\
\therefore \quad 7a &= 6, & 7b &= 8, & 7c &= -3\\
\therefore \quad a &= \tfrac{6}{7}, & b &= \tfrac{8}{7}, & c &= -\tfrac{3}{7}
\end{aligned}$$

So, P is $\left(\frac{6}{7}, \frac{8}{7}, \frac{-3}{7}\right)$

5 Given C$(-3, 2, -1)$ and D$(0, 1, -4)$, $\overrightarrow{CD} = \begin{bmatrix}3\\-1\\-3\end{bmatrix}$

∴ the line passing through C and D has parametric equations

$x = -3+3t, \quad y = 2-t, \quad z = -1-3t$

The line meets $2x - y + z = 3$ when

$$\begin{aligned}
2(-3+3t)-(2-t)+(-1-3t) &= 3\\
\therefore \quad -6+6t-2+t-1-3t &= 3\\
\therefore \quad 4t &= 12\\
\therefore \quad t &= 3
\end{aligned}$$

∴ they meet at $(6, -1, -10)$

6 **a** $\dfrac{x-8}{3} = \dfrac{y+9}{-16} = \dfrac{z-10}{7}$ has direction vector $\begin{bmatrix}3\\-16\\7\end{bmatrix}$

$x = 15+3t, \quad y = 29+8t, \quad z = 5-5t$ has direction vector $\begin{bmatrix}3\\8\\-5\end{bmatrix}$

∴ the lines are not parallel.

If they intersect then $\dfrac{15+3t-8}{3} = \dfrac{29+8t+9}{-16} = \dfrac{5-5t-10}{7}$

$\therefore \quad t + \frac{7}{3} = -\frac{1}{2}t - \frac{38}{16} = -\frac{5}{7}t - \frac{5}{7}$

Now $t + \frac{7}{3} = -\frac{1}{2}t - \frac{38}{16}$ requires $\frac{3}{2}t = -\frac{19}{8} - \frac{7}{3} = -\frac{113}{24}$, i.e., $t = -\frac{113}{36}$

and $t + \frac{7}{3} = -\frac{5}{7}t - \frac{5}{7}$ requires $\frac{12}{7}t = -\frac{5}{7} - \frac{7}{3} = -\frac{64}{21}$, i.e., $t = -\frac{16}{9}$

Hence the lines do not intersect, and since they are not parallel they are skew.

b If θ is the acute angle between the two lines, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are their direction vectors,

then
$$\cos\theta = \frac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|} = \frac{\left|\begin{bmatrix}3\\-16\\7\end{bmatrix} \bullet \begin{bmatrix}3\\8\\-5\end{bmatrix}\right|}{\sqrt{3^2+(-16)^2+7^2}\sqrt{3^2+8^2+(-5)^2}}$$

$$\therefore \quad \cos\theta = \frac{|9-128-35|}{\sqrt{314}\sqrt{98}} = \frac{154}{\sqrt{30\,772}} \qquad \text{and so} \quad \theta \doteqdot 28.6^o$$

c Translate the first line so that it intersects with the second, and find the equation of the plane containing them.

The plane has normal $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$

$$= \mathbf{i}\begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 7 & 3 \\ -5 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix}$$

$$= 24\mathbf{i} + 36\mathbf{j} + 72\mathbf{k} \quad \text{or} \quad 12(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$$

Choose any point on the second line, for example (15, 29, 5).

$\therefore$ the plane has equation $2x + 3y + 6z = 2(15) + 3(29) + 6(5)$

i.e., $2x + 3y + 6z = 147$

Next choose any point on the first line, for example (8, −9, 10).

Then the shortest distance between the two lines is the distance from this point to the plane.

$$\therefore \quad d = \frac{|2x_1 + 3y_1 + 6z_1 - 147|}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{|2(8) + 3(-9) + 6(10) - 147|}{7} = 14 \text{ units}$$

7 **a** The distance of X(−1, 1, 3) from $x - 2y - 2z = 8$

is $d = \dfrac{|x_1 - 2y_1 - 2z_1 - 8|}{\sqrt{1^2 + (-2)^2 + (-2)^2}} = \dfrac{|-1-2-6-8|}{3} = \dfrac{|-17|}{3} = \frac{17}{3}$ units

b Since $2 - x = y - 3 = -\frac{1}{2}z$, $\dfrac{x-2}{-1} = \dfrac{y-3}{1} = \dfrac{z}{-2}$

$\therefore$ the line has direction vector $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, and passes through (2, 3, 0)

$\therefore$ if P is a point on the line with coordinates $(2 - t, 3 + t, -2t)$,

then $\overrightarrow{QP} = \begin{bmatrix} 3-t \\ 1+t \\ -2t-3 \end{bmatrix}$

If P is chosen such that $\overrightarrow{QP}$ is perpendicular to the line,

then $\mathbf{u} \bullet \overrightarrow{QP} = 0$

$$\therefore \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 3-t \\ 1+t \\ -2t-3 \end{bmatrix} = 0$$

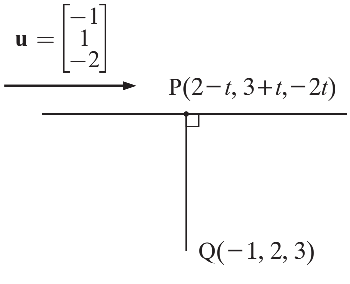
$$\therefore \quad -(3-t) + 1(1+t) - 2(-2t-3) = 0$$

$$\therefore \quad -3 + t + 1 + t + 4t + 6 = 0$$

$$\therefore \quad 6t = -4$$

$$\therefore \quad t = -\tfrac{2}{3}$$

$\therefore$ meet at P$(\frac{8}{3}, \frac{7}{3}, \frac{4}{3})$.



8 P(2, 0, 1) Q(3, 4, −2) R(−1, 3, 2)

a $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

$|\overrightarrow{PQ}| = \sqrt{1 + 16 + 9}$

$= \sqrt{26}$ units

and $\overrightarrow{QR} = \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}$

b Since $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$ and P is at (2, 0, 1),

the line has equation

$x = 2 + t \quad y = 0 + 4t \quad z = 1 - 3t$

i.e., $x = 2 + t, \quad y = 4t, \quad z = 1 - 3t, \quad t \text{ in } \mathcal{R}$

c $\overrightarrow{PQ} = \begin{bmatrix}1\\4\\-3\end{bmatrix} \qquad \overrightarrow{PR} = \begin{bmatrix}-3\\3\\1\end{bmatrix}$

$$\therefore \quad \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 4 & -3\\ -3 & 3 & 1\end{vmatrix} = \mathbf{i}\begin{vmatrix}4 & -3\\ 3 & 1\end{vmatrix} + \mathbf{j}\begin{vmatrix}-3 & 1\\ 1 & -3\end{vmatrix} + \mathbf{k}\begin{vmatrix}1 & 4\\ -3 & 3\end{vmatrix}$$

$$= 13\mathbf{i} + 8\mathbf{j} + 15\mathbf{k}$$

$\therefore$ since P lies on the plane, it has equation

$$13x + 8y + 15z = 13(2) + 8(0) + 15(1)$$

i.e., $13x + 8y + 15z = 41$

$$\therefore \quad \text{vector equation of plane is} \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} + \lambda\begin{bmatrix}1\\4\\-3\end{bmatrix} + \mu\begin{bmatrix}-4\\-1\\4\end{bmatrix}, \quad \lambda,\ \mu \in \mathcal{R}$$

9 **a** Given A(−1, 3, 2) and the plane $2x - y + 2z = 8$,

the distance from A to the plane is
$$d = \frac{|2x_1 - y_1 + 2z_1 - 8|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|2(-1) - 3 + 2(2) - 8|}{3} = \frac{|-9|}{3} = 3 \text{ units}$$

b The point on the plane nearest A is the foot of the normal to the plane that passes through A.

Since the normal has direction vector $\begin{bmatrix}2\\-1\\2\end{bmatrix}$ and passes through (−1, 3, 2),

it has equation $x = -1 + 2t, \quad y = 3 - t, \quad z = 2 + 2t, \quad t$ in $\mathcal{R}$

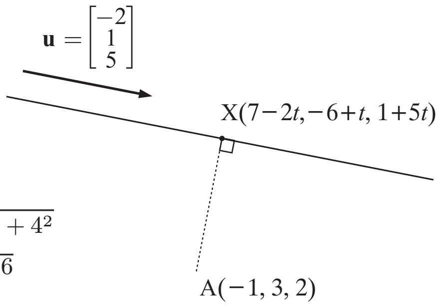
and meets the plane when
$$2(-1 + 2t) - (3 - t) + 2(2 + 2t) = 8$$
$$\therefore \quad -2 + 4t - 3 + t + 4 + 4t = 8$$
$$\therefore \quad 9t = 9$$
$$\therefore \quad t = 1$$

$\therefore$ the point is (1, 2, 4).

c Call the foot of the perpendicular from A to the line X, so X has coordinates $(7 - 2t, -6 + t, 1 + 5t)$ for some t in $\mathcal{R}$. Then the shortest distance from A to the line is AX.

Now $\overrightarrow{AX} = \begin{bmatrix}8-2t\\t-9\\-1+5t\end{bmatrix}$ and since the line has direction vector $\mathbf{u} = \begin{bmatrix}-2\\1\\5\end{bmatrix}$,

$$\mathbf{u} \bullet \overrightarrow{AX} = 0$$
$$\begin{bmatrix}-2\\1\\5\end{bmatrix} \bullet \begin{bmatrix}8-2t\\t-9\\-1+5t\end{bmatrix} = 0$$
$$-16 + 4t + t - 9 - 5 + 25t = 0$$
$$\therefore \quad 30t = 30$$
$$\therefore \quad t = 1$$
$$\therefore \quad |AX| = \sqrt{6^2 + (-8)^2 + 4^2} = \sqrt{36 + 64 + 16} = \sqrt{116} \text{ units}$$



10 Given A(−1, 0, 2), B(0, −1, 1) and C(1, 2, −1)

a $\overrightarrow{\text{AB}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $\overrightarrow{\text{AC}} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$

$$\therefore \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$\therefore$ since A lies on the plane it has equation $5x + y + 4z = 5(-1) + 0 + 4(2)$

i.e., $5x + y + 4z = 3$

b Since the normal has direction $\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$ and passes through (0, 0, 0), it has equation

$x = 0 + 5t, \quad y = 0 + t, \quad z = 0 + 4t$

i.e., $x = 5t, \quad y = t, \quad z = 4t, \quad t$ in $\mathcal{R}$

c The line meets the plane when $5(5t) + t + 4(4t) = 3$

$\therefore \quad 25t + t + 16t = 3$

$\therefore \quad 42t = 3$

$\therefore \quad t = \frac{1}{14}$ i.e., at $(\frac{5}{14}, \frac{1}{14}, \frac{2}{7})$

11 The system has augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 7 & 2 & k & -k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 9 & k-7 & -k-35 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 0 & k+2 & -k-2 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

Thus $(k+2)z = -(k+2)$

If $k \neq -2$ then $z = -1$, and as $3y - 3z = -11$,

then $3y = -14$

$\therefore \quad y = -\frac{14}{3}$

and $x - y + z = 5$,

so $x = 5 - \frac{14}{3} + 1 = \frac{4}{3}$

$\therefore$ we have three planes that meet at the unique point $(\frac{4}{3}, -\frac{14}{3}, -1)$

If $k = -2$, then the 3 planes meet in a common line and hence there are an infinite number of solutions.

In this case, let $z = t, \quad t$ in $\mathcal{R}$ Now $3y - 3z = -11$,

$3y = -11 + 3t$

$\therefore \quad y = -\frac{11}{3} + t$

and as $x - y + z = 5$

$\therefore \quad x = 5 + y - z$

$\therefore \quad x = 5 - \frac{11}{3} + t - t$

$\therefore \quad x = \frac{4}{3}$

$\therefore \quad x = \frac{4}{3} \quad y = -\frac{11}{3} + t \quad z = t, \quad t$ in $\mathcal{R}$

REVIEW SET 17C

1 Given: A(−1, 2, 3), B(2, 0, −1) and C(−3, 2, −4)

a $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix} \quad \overrightarrow{AC} = \begin{bmatrix} -2 \\ 0 \\ -7 \end{bmatrix}$ $\quad\therefore$ a normal vector to the plane is

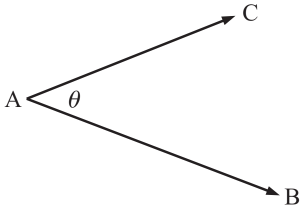
$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -7 \\ 3 & -2 & -4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & -7 \\ -2 & -4 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -7 & -2 \\ -4 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix}$$

$$= -14\mathbf{i} - 29\mathbf{j} + 4\mathbf{k}$$

$\therefore$ since B lies on the plane, it has equation $14x + 29y - 4z = 14(2) + 29(0) - 4(-1)$

i.e., $14x + 29y - 4z = 32$

b

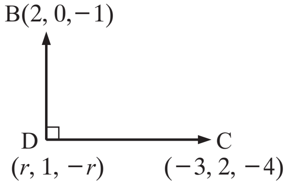


$$\cos\theta = \frac{|\overrightarrow{AB} \bullet \overrightarrow{AC}|}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

$$= \frac{|3 \times -2 + -2 \times 0 + -4 \times 7|}{\sqrt{9+4+16}\sqrt{4+0+49}}$$

$$= \frac{22}{\sqrt{29 \times 53}} \quad \text{and so} \quad \theta \doteqdot 55.9^o$$

c



If D is at $[r, 1, -r]$ then $\overrightarrow{DB} = \begin{bmatrix} 2-r \\ -1 \\ -1+r \end{bmatrix}$

and $\overrightarrow{DC} = \begin{bmatrix} -3-r \\ 1 \\ -4+r \end{bmatrix}$

Now $\measuredangle$BDC is a right-angle, so $\overrightarrow{DB} \bullet \overrightarrow{DC} = 0$

$$\therefore \quad (2-r)(-3-r) + (-1) + (-4+r)(-1+r) = 0$$

$$\therefore \quad -6 - 2r + 3r + r^2 - 1 + 4 - 4r - r + r^2 = 0$$

$$\therefore \quad 2r^2 - 4r - 3 = 0$$

$$\therefore \quad r = \frac{4 \pm \sqrt{16+24}}{4}$$

$$\therefore \quad r = \frac{2 \pm \sqrt{10}}{2}$$

2 **a** $\overrightarrow{LM} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ since L lies on the line, it has parametric equations
$x = 1 + t, \quad y = 0 - t, \quad z = 1 + t, \quad t$ in $\mathcal{R}$

The line meets $x - 2y - 3z = 14$ if

$$(1+t) - 2(-t) - 3(1+t) = 14$$

$$\therefore \quad 1 + t + 2t - 3 - 3t = 14$$

$$\therefore \quad -2 = 14, \quad \text{which is absurd.}$$

$\therefore$ the line and plane do not meet (they are parallel).

b Since the line and the plane are parallel we may use any point on the line to determine the shortest distance to the plane. Hence, using L, the distance

$$d = \frac{|x_1 - 2y_1 - 3z_1 - 14|}{\sqrt{1+4+9}} = \frac{|1 - 2(0) - 3(1) - 14|}{\sqrt{14}} = \tfrac{16}{\sqrt{14}} \text{ units}$$

3 **a** Given A(−1, 2, 3), B(1, 0, −1) and C(1, 3, 0),

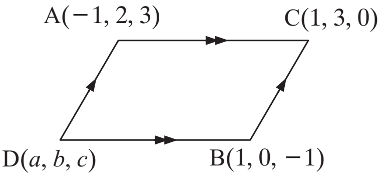
$$\overrightarrow{AB} = \begin{bmatrix}2\\-2\\-4\end{bmatrix} = 2\begin{bmatrix}1\\-1\\-2\end{bmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{bmatrix}2\\1\\-3\end{bmatrix}$$

$\therefore$ a normal to the plane containing A, B and C is

$$\therefore \quad \mathbf{n} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & -2\\ 2 & 1 & -3\end{vmatrix} = \mathbf{i}\begin{vmatrix}-1 & -2\\ 1 & -3\end{vmatrix} + \mathbf{j}\begin{vmatrix}-2 & 1\\ -3 & 2\end{vmatrix} + \mathbf{k}\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}$$

$$\therefore \quad \mathbf{n} = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

b



Suppose D has coordinates (a, b, c)

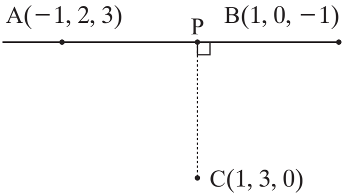
$\therefore$ since $\overrightarrow{AD} = \overrightarrow{CB}$,

$$\begin{bmatrix}a+1\\b-2\\c-3\end{bmatrix} = \begin{bmatrix}0\\-3\\-1\end{bmatrix}$$

$\therefore \quad a = -1, \quad b = -1 \quad$ and $\quad c = 2$

$\therefore$ D is at (−1, −1, 2)

c



From **a**, $\overrightarrow{AB}$ has direction vector $\begin{bmatrix}1\\-1\\-2\end{bmatrix}$

$\therefore$ the line through A and B has parametric equations
$x = 1 + t, \quad y = 0 - t, \quad z = -1 - 2t, \quad t$ in $\mathcal{R}$.

$\therefore$ if $P(1+t, -t, -1-2t)$ is the foot of the perpendicular,

$$\text{then} \quad \overrightarrow{CP} = \begin{bmatrix}t\\-t-3\\-1-2t\end{bmatrix}$$

$$\text{and} \quad \overrightarrow{CP} \bullet \overrightarrow{AB} = 0 \quad \therefore \quad \begin{bmatrix}t\\-t-3\\-1-2t\end{bmatrix} \bullet \begin{bmatrix}1\\-1\\-2\end{bmatrix} = 0$$

$$\therefore \quad t + t + 3 + 2 + 4t = 0$$

$$\therefore \quad 6t = -5$$

$$\therefore \quad t = -\tfrac{5}{6}$$

$\therefore$ P is $(1 - \frac{5}{6}, \frac{5}{6}, -1 + \frac{10}{6})$ i.e., $(\frac{1}{6}, \frac{5}{6}, \frac{2}{3})$

4 $x - 1 = \dfrac{y+2}{2} = \dfrac{z-3}{4}$ has direction vector $\begin{bmatrix}1\\2\\4\end{bmatrix}$ and $6x + 7y - 5z = 8$ has $\mathbf{n} = \begin{bmatrix}6\\7\\-5\end{bmatrix}$.

$$\text{Now} \quad \begin{bmatrix}1\\2\\4\end{bmatrix} \bullet \begin{bmatrix}6\\7\\-5\end{bmatrix} = 6 + 14 - 20 = 0$$

$\therefore$ since these two vectors are perpendicular, the line is parallel to the plane.

Choose any point on the line, for example, (1, −2, 3).

Then the distance from the line to the plane is

$$d = \frac{|6x_1 + 7y_1 - 5z_1 - 8|}{\sqrt{6^2 + 7^2 + (-5)^2}}$$

$$= \frac{|6(1) + 7(-2) - 5(3) - 8|}{\sqrt{110}}$$

$$= \tfrac{31}{\sqrt{110}} \text{ units}$$

5 **a** $\dfrac{x-3}{2} = \dfrac{y-4}{1} = \dfrac{z+1}{-2}$ has direction vector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

while $x = -1 + 3t$, $y = 2 + 2t$, $z = 3 - t$ has direction vector $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

$\therefore$ the lines are not parallel.

If the lines intersect, then $\dfrac{-1+3t-3}{2} = \dfrac{2+2t-4}{1} = \dfrac{3-t+1}{-2}$

i.e., $\frac{3}{2}t - 2 = 2t - 2 = \frac{t}{2} - 2$

Now $t = 0$ satisfies this relation, so the lines intersect at $(-1, 2, 3)$.

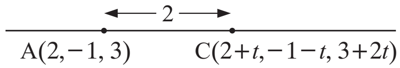
b If θ is the acute angle between the lines, then

$$\cos\theta = \frac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1|\,|\mathbf{v}_2|} = \frac{|2 \times 3 + 1 \times 2 + -2 \times -1|}{\sqrt{9}\sqrt{14}} = \frac{|6+2+2|}{3\sqrt{14}} = \frac{10}{3\sqrt{14}}$$

6 **a** $\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\therefore$ since A lies on the line, it has equations $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

b If C lies on $\overrightarrow{AB}$ and is 2 units from A, then C corresponds to t such that



$$\begin{aligned} \sqrt{(t)^2 + (-t)^2 + (2t)^2} &= 2 \\ \therefore \quad \sqrt{6t^2} &= 2 \\ \therefore \quad 6t^2 &= 4 \\ \therefore \quad t^2 &= \tfrac{4}{6} \\ \therefore \quad t &= \pm\tfrac{2}{\sqrt{6}} \end{aligned}$$

$\therefore$ C is $\left(2 + \frac{2}{\sqrt{6}}, -1 - \frac{2}{\sqrt{6}}, 3 + \frac{4}{\sqrt{6}}\right)$ or $\left(2 - \frac{2}{\sqrt{6}}, -1 + \frac{2}{\sqrt{6}}, 3 - \frac{4}{\sqrt{6}}\right)$

7 Given A(−1, 2, 3) B(1, 0, −1) and C(0, −1, 5)

$\overrightarrow{AB} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ $\therefore$ a normal to the plane is

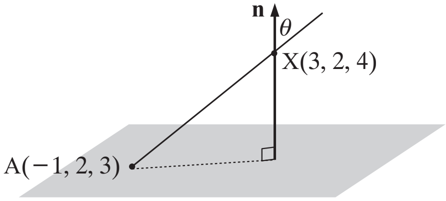
$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -4 \\ 1 & -3 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & -4 \\ -3 & 2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -2 \\ 1 & -3 \end{vmatrix}$$

$= -16\mathbf{i} - 8\mathbf{j} - 4\mathbf{k}$ or $-4(4\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$\therefore$ the plane has equation $4x + 2y + z = 4(1) + 2(0) + 1(-1)$

i.e., $4x + 2y + z = 3$

Given the plane has normal $\mathbf{n} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{AX} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$,



$$\begin{aligned} \sin\phi &= \frac{|\,\mathbf{n} \bullet \overrightarrow{AX}\,|}{|\mathbf{n}|\,|\,\overrightarrow{AX}\,|} \\ &= \frac{|4 \times 4 + 2 \times 0 + 1 \times 1|}{\sqrt{21}\sqrt{17}} \\ &= \frac{17}{\sqrt{21}\sqrt{17}} \quad \text{and so} \quad \phi \doteqdot 64.1^o \end{aligned}$$

8 **a** All vectors normal to $x-y+z=6$ have the form $t\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}t\\-t\\t\end{bmatrix}$, t in $\mathcal{R}$

$\therefore$ if the vector has length 3 units,
$$\begin{aligned}\sqrt{t^2+t^2+t^2}&=3\\ \therefore\ 3t^2&=9\\ \therefore\ t^2&=3\\ \therefore\ t&=\pm\sqrt{3}\end{aligned}$$

$\therefore$ the vectors are $\begin{bmatrix}\sqrt{3}\\-\sqrt{3}\\\sqrt{3}\end{bmatrix}$ and $\begin{bmatrix}-\sqrt{3}\\\sqrt{3}\\-\sqrt{3}\end{bmatrix}$

b Any vector parallel to $\mathbf{i}+r\mathbf{j}+3\mathbf{k}$ has the form $t\begin{bmatrix}1\\r\\3\end{bmatrix}=\begin{bmatrix}t\\rt\\3t\end{bmatrix}$, t in $\mathcal{R}$.

This is perpendicular to $\begin{bmatrix}2\\-1\\2\end{bmatrix}$ if $\begin{bmatrix}2\\-1\\2\end{bmatrix}\bullet\begin{bmatrix}t\\rt\\3t\end{bmatrix}=0$
$$\begin{aligned}\therefore\ 2t-rt+6t&=0\\ \therefore\ 8t-rt&=0\\ \therefore\ t(8-r)&=0\\ \therefore\ t&=0 \text{ or } 8\end{aligned}$$

But if $t=0$, the vector has zero length $\therefore$ $r=8$. $\therefore$ a vector is $\begin{bmatrix}1\\8\\3\end{bmatrix}$,

$\therefore$ unit vectors are $\mathbf{u}=\frac{1}{\sqrt{74}}\mathbf{i}+\frac{8}{\sqrt{74}}\mathbf{j}+\frac{3}{\sqrt{74}}\mathbf{k}$ or $-\frac{1}{\sqrt{74}}\mathbf{i}-\frac{8}{\sqrt{74}}\mathbf{j}-\frac{3}{\sqrt{74}}\mathbf{k}$

c The distance from the plane to A is $d=\dfrac{|2x_1-y_1+2z_1-k|}{\sqrt{9}}$

$$\therefore\ \frac{|2(-1)-(2)+2(3)-k|}{3}=3 \quad \therefore\ |2-k|=9$$
$$\therefore\ 2-k=9 \text{ or } k-2=9$$
$$\therefore\ k=-7 \text{ or } 11$$

9 If A is the origin, AB the x-axis, AD the y-axis, and AP the z-axis, then Q is (4, 0, 7), D is (0, 10, 0) and M is (0, 5, 7).

Now $\overrightarrow{DQ}=\begin{bmatrix}4-0\\0-10\\7-0\end{bmatrix}=\begin{bmatrix}4\\-10\\7\end{bmatrix}$ and $\overrightarrow{DM}=\begin{bmatrix}0-0\\5-10\\7-0\end{bmatrix}=\begin{bmatrix}0\\-5\\7\end{bmatrix}$

So, if the required angle is θ, $\begin{bmatrix}4\\-10\\7\end{bmatrix}\bullet\begin{bmatrix}0\\-5\\7\end{bmatrix}=\sqrt{16+100+49}\sqrt{0+25+49}\cos\theta$
$$\therefore\ 0+50+49=\sqrt{165}\sqrt{74}\cos\theta$$
$$\therefore\ \frac{99}{\sqrt{165\times 74}}=\cos\theta \quad\text{and so}\quad \theta \doteqdot 26.4^o$$

10 **a** $\overrightarrow{PQ}=\begin{bmatrix}4--1\\0-2\\-1-3\end{bmatrix}$
$$=\begin{bmatrix}5\\-2\\-4\end{bmatrix}$$

b For the x-axis, $\mathbf{v}=\begin{bmatrix}1\\0\\0\end{bmatrix}$

Now $\begin{bmatrix}5\\-2\\-4\end{bmatrix}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix}=\sqrt{25+4+16}\sqrt{1+0+0}\cos\theta$
$$\therefore\ 5+0+0=\sqrt{45}\cos\theta$$
$$\therefore\ \cos\theta=\tfrac{5}{\sqrt{45}} \quad\text{and so}\quad \theta\doteqdot 41.8^o$$

REVIEW SET 17D

1 **a** $\overrightarrow{AB} = \begin{bmatrix} 2-4 \\ 1-2 \\ 5--1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 9-4 \\ 4-2 \\ 1--1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$

$$\therefore \quad \overrightarrow{AB} \bullet \overrightarrow{AC} = (-2)(5) + (-1)(2) + (6)(2) = -10 - 2 + 12 = 0$$

$$\therefore \quad \overrightarrow{AB} \perp \overrightarrow{AC}$$

b $$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 6 \\ 5 & 2 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 6 \\ 2 & 2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 6 & -2 \\ 2 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & -1 \\ 5 & 2 \end{vmatrix}$$

$$= -14\mathbf{i} + 34\mathbf{j} + \mathbf{k}$$

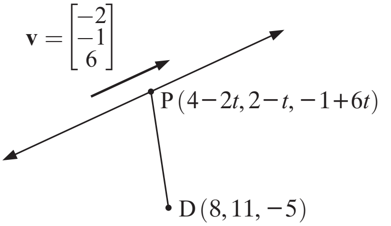
$\therefore$ the equation is $-14x + 34y + z = -14(4) + 34(2) + (-1) = 11$

i.e., $14x - 34y - z = -11$

and distance $= \dfrac{|14(8) - 34(1) - (0)|}{\sqrt{14^2 + (-34)^2 + (-1)^2}} = \dfrac{78}{\sqrt{1353}}$ units ($\doteqdot 2.12$ units)

c The equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}, \quad t \in \mathcal{R}$

d $\mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$



$$\overrightarrow{DP} = \begin{bmatrix} 4-2t-8 \\ 2-t-11 \\ -1+6t--5 \end{bmatrix} = \begin{bmatrix} -4-2t \\ -9-t \\ 4+6t \end{bmatrix}$$

and $\overrightarrow{DP} \bullet \mathbf{v} = 0$

$$\therefore \quad -2(-4-2t) - 1(-9-t) + 6(4+6t) = 0$$

$$\therefore \quad 8 + 4t + 9 + t + 24 + 36t = 0$$

$$\therefore \quad 41t = -41$$

$$\therefore \quad t = -1$$

$\therefore \quad \overrightarrow{DP} = \begin{bmatrix} -2 \\ -8 \\ -2 \end{bmatrix}$ and $|\overrightarrow{DP}| = \sqrt{4 + 64 + 4}$

$= \sqrt{72}$

$= 6\sqrt{2}$ units

2 **a** l_1 meets $2x + y - z = 2$ where $2(3t-4) + (t+2) - (2t-1) = 2$

$$\therefore \quad 6t - 8 + t + 2 - 2t + 1 = 2$$

$$\therefore \quad 5t = 7$$

$$\therefore \quad t = \tfrac{7}{5}$$

i.e., at $(3(\frac{7}{5}) - 4, \frac{7}{5} + 2, 2(\frac{7}{5}) - 1)$

i.e., $(\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$

b l_1 meets l_2 where $3t - 4 = \dfrac{t+2-5}{2} = \dfrac{-(2t-1)-1}{2}$

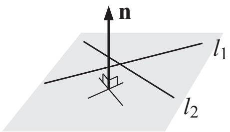
i.e., $6t - 8 = t - 3 = -2t$

$\therefore \quad 5t = 5$ and $3t = 3$

i.e., $t = 1$

$\therefore$ they meet at $(-1, 3, 1)$

c



$$\mathbf{n} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} + \mathbf{j}\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$$

$\therefore$ equation is $-6x + 8y + 5z = -6(-1) + 8(3) + 5(1)$

i.e., $-6x + 8y + 5z = 35$

i.e., $6x - 8y - 5z = -35$

3 **a** We substitute l_1 into the LHS of the plane's equation

$$2(-2t+2) + (t) + (3t+1) = -4t + 4 + t + 3t + 1 = 5 \quad \checkmark$$

$\therefore$ the plane contains the line.

b If $x + ky + z = 3$ contains l_1 then $(-2t+2) + k(t) + 3t + 1 = 3$

$$\therefore \quad t[-2+k+3] + 2 + 1 = 3$$

$$\therefore \quad t[k+1] = 0$$

$$\therefore \quad k = -1 \quad \text{as } t \in \mathcal{R}$$

c From **a** and **b**, both $2x + y + z = 5$ and $x - y + z = 3$ contain l_1.

So, substituting l_1 into plane 3 gives

$$-2(-2t+2) + pt + 2(3t+1) = q \quad \text{for all } t \text{ in } \mathcal{R}$$

$$\therefore \quad 4t - 4 + pt + 6t + 2 = q \quad \text{for all } t \text{ in } \mathcal{R}$$

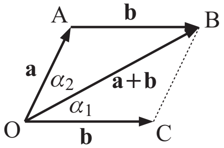
$$\therefore \quad [10+p]t - 2 = q \quad \text{for all } t$$

This equation has infinitely many solutions for b

when $10 + p = 0$ and $-2 = q$ {equating coefficients}

$\therefore$ $p = -10$ and $q = -2$

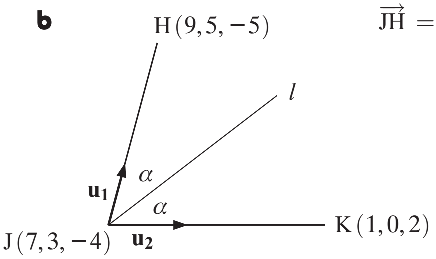
4 **a**



As **a** and **b** are unit vectors, OABC is a rhombus. But the angles of a rhombus are bisected by its diagonals

$\therefore$ $\alpha_1 = \alpha_2$

b



$$\overrightarrow{JH} = \begin{bmatrix} 9-7 \\ 5-3 \\ -5--4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \qquad \overrightarrow{JK} = \begin{bmatrix} 1-7 \\ 0-3 \\ 2--4 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$$

$$\therefore \quad \mathbf{u}_1 = \frac{1}{\sqrt{4+4+1}}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$\text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{36+9+36}}\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\text{and} \quad \mathbf{u}_1 + \mathbf{u}_2 = \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\therefore \text{ the line's equation is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \quad t \in \mathcal{R}$$

c $\overrightarrow{\text{HK}} = \begin{bmatrix} 1-9 \\ 0-5 \\ 2--5 \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix}$ $\therefore$ has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix}$

and this line meets the first line where:

$$7 = 1-8s, \quad 3+\frac{t}{3} = -5s \quad \text{and} \quad -4+\frac{t}{3} = 2+7s \quad \text{......} \ (*)$$

$$\therefore \ 8s = -6 \quad \text{i.e.,} \quad s = -\tfrac{3}{4} \quad \text{and so} \quad 3+\frac{t}{3} = \tfrac{15}{4} \quad \therefore \ \frac{t}{3} = \tfrac{3}{4} \quad \therefore \ t = \tfrac{9}{4}$$

In $(*)$

$$\begin{aligned} \text{LHS} &= -4+\frac{t}{3} \\ &= -4+\tfrac{3}{4} \\ &= -\tfrac{13}{4} \end{aligned} \qquad \begin{aligned} \text{RHS} &= 2+7s \\ &= 2+7(-\tfrac{3}{4}) \\ &= \tfrac{8}{4}-\tfrac{21}{4} \\ &= -\tfrac{13}{4} \ \checkmark \end{aligned}$$

$\therefore$ $s = -\frac{3}{4}$, $t = \frac{9}{4}$ satisfy all 3 equations

So $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 0+\frac{15}{4} \\ 2-\frac{21}{4} \end{bmatrix} = \begin{bmatrix} 7 \\ 3\frac{3}{4} \\ -3\frac{1}{4} \end{bmatrix}$ $\therefore$ M is $(7, 3\frac{3}{4}, -3\frac{1}{4})$

5 If A is $(3, -1, -2)$ and B$(5, 3, -4)$ then $\overrightarrow{\text{AB}} = \begin{bmatrix} 5-3 \\ 3--1 \\ -4--2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

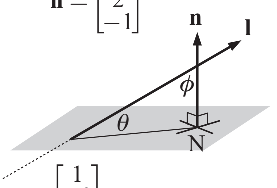
$\therefore$ the line has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $t \in \mathcal{R}$

$$\begin{aligned} \text{and it meets} \quad x^2+y^2+z^2 &= 26 \quad \text{where} \\ (3+t)^2+(-1+2t)^2+(-2-t)^2 &= 26 \\ \therefore \ 9+6t+t^2+1-4t+4t^2+4+4t+t^2-26 &= 0 \\ \therefore \ 6t^2+6t-12 &= 0 \\ \therefore \ t^2+t-2 &= 0 \\ \therefore \ (t+2)(t-1) &= 0 \\ \therefore \ t &= -2 \ \text{or} \ 1 \end{aligned}$$

$$\therefore \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore$ meet at $(1, -5, 0)$ and $(4, 1, -3)$

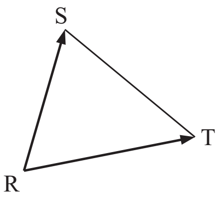
6 $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$



$\mathbf{l} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

$$\begin{aligned} \mathbf{n} \bullet \mathbf{l} &= |\mathbf{n}|\,|\mathbf{l}|\cos\phi \\ \therefore \ \cos\phi &= \frac{\mathbf{n} \bullet \mathbf{l}}{|\mathbf{n}|\,|\mathbf{l}|} \\ \therefore \ \sin\theta &= \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}|\,|\mathbf{l}|} \quad \text{as} \quad \cos\phi = \cos\left(\tfrac{\pi}{2}-\theta\right) \\ & \qquad\qquad\qquad\qquad\quad = \sin\theta \\ &= \frac{|2-4+1|}{\sqrt{4+4+1}\sqrt{1+4+1}} \\ &= \tfrac{1}{\sqrt{54}} \\ \therefore \ \theta &\doteqdot 7.82^o \end{aligned}$$

7



$$\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$\therefore \quad \overrightarrow{RS} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$= 3\mathbf{j} + 3\mathbf{k}$$

Likewise $\overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$= -\mathbf{i} + 4\mathbf{j}$$

Now area $= \frac{1}{2} \,|\, \overrightarrow{RS} \times \overrightarrow{RT} \,|$

$$= \frac{1}{2} \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 3 \\ -1 & 4 & 0 \end{matrix} \right\|$$

$$= \frac{1}{2} \left| \mathbf{i} \begin{vmatrix} 3 & 3 \\ 4 & 0 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & 3 \\ -1 & 4 \end{vmatrix} \right\|$$

$$= \frac{1}{2} \,|-12\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}|$$

$$= \frac{1}{2}\sqrt{144 + 9 + 9}$$

$$= \frac{1}{2}\sqrt{162}$$

$$= \frac{1}{2}\, 9\sqrt{2}$$

$$= \frac{9\sqrt{2}}{2} \text{ units}^2$$

8 **a** X is $\left(\dfrac{4+10}{2}, \dfrac{4+2}{2}, \dfrac{-2+0}{2}\right)$ i.e., (7, 3, −1)

If D is (a, b, c) $\overrightarrow{AD} = \begin{bmatrix} a-1 \\ b-3 \\ c--4 \end{bmatrix} = \begin{bmatrix} a-1 \\ b-3 \\ c+4 \end{bmatrix}$ and $\overrightarrow{BC} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$

$\therefore$ $a - 1 = 6$, $b - 3 = -2$, $c + 4 = 2$
i.e., $a = 7$, $b = 1$ $c = -2$ $\therefore$ D is (7, 1, −2)

b $\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{AY} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \frac{2}{3}\overrightarrow{AX} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 7-1 \\ 3-3 \\ -1--4 \end{bmatrix}$

$\therefore$ $\overrightarrow{OY} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ and so Y is (5, 3, −2)

c $\overrightarrow{BY} = \begin{bmatrix} 5-4 \\ 3-4 \\ -2--2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\overrightarrow{BD} = \begin{bmatrix} 7-4 \\ 1-4 \\ -2--2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$

$\therefore$ $\overrightarrow{BD} = 3\overrightarrow{BY}$ and so $\overrightarrow{BD} \parallel \overrightarrow{BY}$ $\Rightarrow$ B, D, Y are collinear

9 **a** $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$

$$= -3\mathbf{j} - 3\mathbf{k}$$

b $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 0 & -3 & -3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ -3 & -3 \end{vmatrix} + \mathbf{j}\begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 2 \\ 0 & -3 \end{vmatrix}$

$$= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k} \quad \checkmark$$

and $\mathbf{b}\,(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b})$

$$\begin{aligned}
&= [(3)(2) + (2)(-1) + (-1)(1)]\,\mathbf{b} - [(3)(1) + (2)(1) + (-1)(-1)]\,\mathbf{c} \\
&= 3\mathbf{b} - 6\mathbf{c} \\
&= 3(\mathbf{i} + \mathbf{j} - \mathbf{k}) - 6(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \\
&= 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} - 12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} \\
&= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k} \quad \checkmark
\end{aligned}$$

10 **a**

$$\begin{aligned}
\mathbf{a} &\perp \mathbf{b} \\
\therefore \quad \mathbf{a} \bullet \mathbf{b} &= 0 \\
\therefore \quad (1)(-t) + (2)(1+t) + (-2)(2t) &= 0 \\
\therefore \quad -t + 2 + 2t - 4t &= 0 \\
\therefore \quad -3t &= -2 \\
\therefore \quad t &= \tfrac{2}{3}
\end{aligned}$$

b

$$\begin{aligned}
\mathbf{a} &\parallel \mathbf{b} \\
\Rightarrow \quad \mathbf{b} &= k\mathbf{a} \quad \text{for some scalar } k \\
\therefore \quad \begin{bmatrix} -t \\ 1+t \\ 2t \end{bmatrix} &= k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \\
\therefore \quad -t &= k, \quad 1 + t = 2k, \quad 2t = -2k \\
\therefore \quad 1 + t &= 2(-t) \\
\therefore \quad 1 &= -3t \\
\therefore \quad t &= -\tfrac{1}{3}
\end{aligned}$$

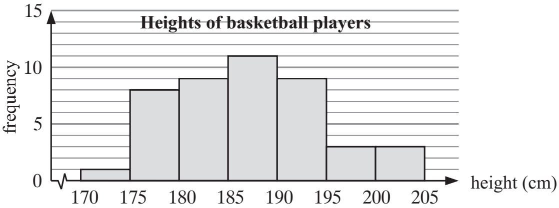
Chapter 18

DESCRIPTIVE STATISTICS

EXERCISE 18A

1 **a** Heights can take any value from 170 cm to 205 cm, including decimal values such as 181.372 cm, i.e., any real number between 170 and 205.

b



c The modal class is the class occurring most often. This is '185 -'.

d The distribution is slightly positively skewed (more of a 'tail' to the right (positive)).

2 **a** The data is continuous numerical. Actual time is continuous and could be measured to the nearest millisecond. After it has been rounded to the nearest minute it becomes discrete numerical data.

b

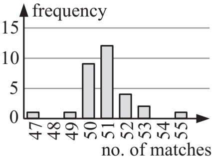
Stem	*Leaf*
0	3 6 8 8 8 8
1	0 0 0 0 0 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6

1 | 2 means 12 minutes

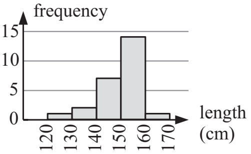
c The distribution is positively skewed (i.e., skewed to the high end).

d "The modal travelling time was between 10 and 20 min" if considering classes. The mode is actually 10.

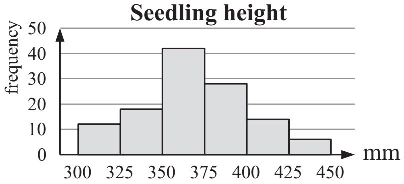
3 **a** The data is discrete numerical, so a column graph should be used.



b The data is continuous, so a histogram should be used.



4 **a**



b Number which are $\geqslant 400$ mm is $14 + 6 = 20$ seedlings.

c $\%$ between 349 and 400 $= \dfrac{42+28}{120} \times 100\%$

$= \dfrac{70}{120} \times 100\%$

$\doteq 58.3\%$

d **i** Number

$= \dfrac{12+18+42+28}{120} \times 1462$

$= \dfrac{100}{120} \times 1462$

$\doteq 1218$ seedlings

ii Number

$= \dfrac{28+14}{120} \times 1462$

$= \dfrac{42}{120} \times 1462$

$\doteq 512$ seedlings

EXERCISE 18B.1

1 **a** **i** mean $= \frac{2+3+3+3+4+\ldots.+9+9}{23}$

$= \frac{129}{23}$

$\doteq 5.61$

ii median = 12th score (when in order)

$= 6$

iii mode $= 6$ (6 occurs most often)

b **i** $\text{mean} = \frac{10+12+12+15+....+20+21}{15} = \frac{245}{15} \doteq 16.3$

i $\text{median} = \text{8th score (when in order)} = 17$

iii $\text{mode} = 18$

c **i** $\text{mean} = \frac{22.4+24.6+21.8+....+23.5}{11} = \frac{273}{11} \doteq 24.8$

ii $\text{median} = \text{6th score (when in order)} = 24.9$

iii $\text{mode} = 23.5$

2 **a** mean of set A $= \dfrac{3+4+4+5+....+10}{13} = 6.46$

mean of set B $= \dfrac{3+4+4+5+....+15}{13} = 6.85$

b median of set A $=$ 7th score $= 7$ median of set B $=$ 7th score $= 7$

c The data sets are the same except for the last value, and the last value of set A is less than that of set B. So, the mean of set A is less than that of set B.

d The middle value of both data sets is the same, so the median is the same.

3 **a** $$\text{mean} = \frac{23\,000 + 46\,000 + 23\,000 + ... + 32\,000}{10} = \$29\,300$$

$$\text{median} = \text{middle score when in order of size} = \frac{\$23\,000 + \$24\,000}{2} = \$23\,500$$

$\text{mode} = \$23\,000$

b The mode is unsatisfactory because it is the lowest salary. It does not take the higher values into account.

c The median is too close to the lower end of the distribution and it does not take the higher values into account. So the median is not a satisfactory measure of the middle.

4 **a** $$\text{mean} = \frac{3+1+0+0+....+1+0+0}{31} = \frac{99}{31} \doteq 3.19$$

$\text{median} = \text{16th score (when in order)} = 0$

$\text{mode} = 0$ (most frequently occurring score)

b The median is not in the centre. It ignores the high upper values of the distribution.

c The mode is the lowest value. It does not take the higher values into account.

d Yes, 42 and 21.

e No, as this would ignore actual valid data.

5 **a** $\text{mean} = \dfrac{43+55+41+37}{4} = \dfrac{176}{4} = 44$

b another 44

c $\text{new mean} = \dfrac{43+55+41+37+25}{5} = 40.2$

d It will increase the new mean to 40.3 as 41 is greater than the old mean of 40.2.

$$\left\{\frac{5 \times 40.2 + 41}{6} \doteq 40.3\right\}$$

6 $\text{mean} = \dfrac{\text{total}}{10}, \quad \therefore \quad 11.6 = \dfrac{\text{total}}{10}, \quad \therefore \quad \text{total} = 11.6 \times 10 = 116$

7 $\text{mean} = \dfrac{\text{total}}{12}, \quad \therefore \quad 262 = \dfrac{\text{total}}{12}, \quad \therefore \quad \text{total} = 262 \times 12 = 3144 \text{ km}$

8 $\text{mean} = \dfrac{\text{total}}{12}, \quad \therefore \quad 15\,467 = \dfrac{\text{total}}{12}, \quad \therefore \quad \text{total} = 15\,467 \times 12 = \$185\,604$

9
$$\begin{aligned} \frac{5+9+11+12+13+14+17+x}{8} &= 12 \\ \therefore \quad \frac{81+x}{8} &= 12 \\ \therefore \quad 81+x &= 96 \\ \therefore \quad x &= 15 \end{aligned}$$

10
$$\begin{aligned} \frac{3+0+a+a+4+a+6+a+3}{9} &= 4 \\ \therefore \quad \frac{4a+16}{9} &= 4 \\ \therefore \quad 4a+16 &= 36 \\ \therefore \quad 4a &= 20 \\ \therefore \quad a &= 5 \end{aligned}$$

11
$$\frac{29+36+32+38+35+34+39+x}{8} = 35$$
$$\begin{aligned} \therefore \quad \frac{243+x}{8} &= 35 \\ \therefore \quad 243+x &= 280 \\ \therefore \quad x &= 37, \quad \text{so her 8th result was 37} \end{aligned}$$

12 Total for first 10 measurements $= 10 \times 15.7 = 157$

Total for next 20 measurements $= 20 \times 14.3 = 286$ $\qquad \therefore \quad \text{mean} = \dfrac{157+286}{30} \doteqdot 14.8$

13 Scores were $\;5 \;\; 7 \;\; 9 \;\; 9 \;\; 10 \;\; a \;\; b\;$ where $a \leqslant b$ say.

$$\text{mean} = \frac{5+7+9+9+10+a+b}{7} = 8 \qquad \therefore \quad \frac{40+a+b}{7} = 8$$
$$\begin{aligned} \therefore \quad 40+a+b &= 56 \\ \therefore \quad a+b &= 16 \quad \{a \leqslant 12, \quad b \leqslant 12\} \end{aligned}$$

Possibilities are:

a	5	6	7	8
b	11	10	9	8
	×	×	✓	×

reject as modes are 8 and 9

reject as modes are 9 and 10

reject as modes are 5 and 9

So, the missing results are 7 and 9.

EXERCISE 18B.2

1 **a** The mode is 1 (occurs most often).

b The median is the average of the 15th and 16th scores $= \dfrac{1+1}{2} = 1$

c

x	f	$f\,x$
0	4	0
1	12	12
2	11	22
3	3	9
$\sum$	30	43

$$\begin{aligned} \text{mean} &= \frac{\sum f\,x}{\sum f} \\ &= \frac{43}{30} \\ &\doteqdot 1.43 \end{aligned}$$

2 **a** **i**

x	f	$f\,x$
0	5	0
1	8	8
2	13	26
3	8	24
4	6	24
5	3	15
6	3	18
7	2	14
8	1	8
9	0	0
10	0	0
11	1	11
$\sum$	50	148

$$\begin{aligned} \text{mean} &= \frac{\sum f\,x}{\sum f} \\ &= \frac{148}{50} \\ &= 2.96 \end{aligned}$$

ii median

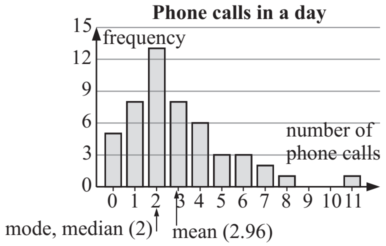
= average of 25th and 26th (when in order)

$$= \frac{2+2}{2} \quad \left\{ \begin{array}{l} \text{13 scores are 1 or 0} \\ \text{26 scores are 2, 1 or 0} \end{array} \right\}$$

$= 2$

iii mode $= 2$ $\quad$ {occurs most often}

b



c The distribution is positively skewed. 11 is an outlier.

d The mean takes into account the larger numbers of phone calls.

e The mean as it best represents all the data.

3 **a** **i** mode $= 49$ {occurs most often}

ii median $=$ average of 15th and 16th values (when in order)

$$= \frac{49+49}{2} = 49 \quad \{9 \text{ are } 47 \text{ or } 48 \text{ and the next } 11 \text{ are } 49\}$$

iii

x	f	fx
47	5	235
48	4	192
49	11	539
50	6	300
51	3	153
52	1	52
$\sum$	30	1471

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{1471}{30} \doteqdot 49.0$$

b No, as they claim the average (mean) is 50 matches/box.

The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

4 **a** **i**

x	f	fx
1	5	5
2	28	56
3	15	45
4	8	32
5	2	10
6	1	6
$\sum$	59	154

$$\text{mean} = \frac{\sum fx}{\sum f} = \frac{154}{59} \doteqdot 2.61$$

ii mode $= 2$ {occurs most often}

iii median $=$ 30th score $= 2$

b This school has more children per family (2.61) than the average Australian family (2.2).

c Positive as the higher values are more spread out.

d The mean is higher than the mode and median.

5 **a** **i** $\text{mean} = \dfrac{53+55+56+60+....+91}{17} = \dfrac{1175}{17} \doteqdot 69.1$

ii median $=$ 9th score (when in order) $= 67$

iii mode $= 73$ (73 is the only one occurring more than once)

b **i** $\text{mean} = \dfrac{37+40+44+48+....+81}{23} = \dfrac{1347}{23} \doteqdot 5.86$

ii median $=$ 12th score (when in order) $= 5.8$ **iii** mode $= 6.7$ {occurs most often}

6 **a** **Without fertiliser**

2	||
3	𝍸 𝍸 |
4	𝍸 𝍸 𝍸 ||||
5	𝍸 𝍸 𝍸 𝍸 𝍸 ||||
6	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |
7	𝍸 𝍸 𝍸 𝍸 𝍸
8	𝍸 𝍸 ||
9	|

x	f	fx	cf
2	2	4	2
3	11	33	13
4	19	76	32
5	29	145	61
6	51	306	112
7	25	175	137
8	12	96	149
9	1	9	150

i $\text{mean} = \dfrac{\sum fx}{\sum f} = \dfrac{845}{150} \doteqdot 5.63$ **ii** mode $= 6$ {occurs most often}

iii median = average of 75th and 76th scores $= \dfrac{6+6}{2} = 6$

b **With fertiliser**

3	||||
4	𝍸 𝍸 |||
5	𝍸 𝍸 |
6	𝍸 𝍸 𝍸 𝍸 𝍸 |||
7	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |||
8	𝍸 𝍸 𝍸 𝍸 𝍸 ||
9	𝍸 𝍸 ||||
10	||||
13	|

x	f	$f\,x$	cf
3	4	12	4
4	13	52	17
5	11	55	28
6	28	168	56
7	48	336	104
8	27	216	131
9	14	126	145
10	4	40	149
13	1	13	150

i mean $= \dfrac{\sum f\,x}{\sum f} = \dfrac{1018}{150} \doteqdot 6.79$ **ii** mode = 7 {occurs most often}

iii median = average of 75th and 76th scores $= \dfrac{7+7}{2} = 7$

c The mean best represents the centre for this data.

d Yes, as 6.79 is significantly greater than 5.63.

Note: The total yield of the crop may not have improved as, for example, the number of pods per plant may have decreased when using the fertiliser.

7 **a** mean selling price $= \dfrac{146\,400 + 127\,600 + 211\,000 + ... + 162\,500}{10} = \$163\,770$

median selling price $= \dfrac{\text{5th} + \text{6th}}{2} = \dfrac{146\,400 + 148\,000}{2} = \$147\,200$

These figures differ by \$16 570. There are more selling prices at the lower end of the market (i.e., smaller prices).

b **i** Use the mean as it tends to inflate the average house value of that district.

ii Use the median as you want to buy at the lowest price possible.

8 **a** mean birth weight $= \dfrac{75 + 70 + 80 + ... + 83}{8} = \dfrac{567}{8} \doteqdot 70.9$ grams

b mean after 2 weeks $= \dfrac{210 + 200 + 200 + ... + 230}{8} = \dfrac{1681}{8} \doteqdot 210$ grams

c mean increase $\doteqdot$ (210.13 – 70.88) grams $\doteqdot$ 139 grams

9 15 of these 10.1, 10.4, 10.7, 10.9, 12 of these

Median = 16th score (when in order) = 10.1 cm

10 **a** **Brand A**

x	f	$f\,x$
46	1	46
47	1	47
48	2	96
49	7	343
50	10	500
51	20	1020
52	15	780
53	3	159
55	1	55
$\sum$	60	3046

$$\text{mean} = \frac{\sum f\,x}{\sum f} = \frac{3046}{60} \doteqdot 50.8$$

Brand B

x	f	$f\,x$
48	3	144
49	17	833
50	30	1500
51	7	357
52	2	104
53	1	53
54	1	54
$\sum$	61	3045

$$\text{mean} = \frac{\sum f\,x}{\sum f} = \frac{3045}{61} \doteqdot 49.9$$

b Based on average contents the CPS should not prosecute either manufacturer. To the nearest match, the average contents of A is 51 and B, 50.

11 Total for first 14 matches $= 14 \times 16.5$ goals $= 231$ goals

$$\therefore \quad \text{new average} = \frac{231+21+24}{16} = \frac{276}{16} = 17.25 \text{ goals/game}$$

12 The measurements are 7, 9, 11, 13, 14, 17, 19, a, b where $a \leqslant b$

$$\text{mean} = \frac{7+9+11+13+ .... +a+b}{9} = \frac{90+a+b}{9} \qquad \therefore \quad \frac{90+a+b}{9} = 12$$

$$\therefore \quad 90+a+b = 108$$

$$\therefore \quad a+b = 18$$

7 9 11 13 14 17 19

If $b \geqslant 13$, then $a \leqslant 5$ and the median $= 13$ $(\times)$

If $b = 12$, then $a = 6$ and the median $= 12$ $(\checkmark)$

The remaining cases are

a	7	8	9
b	11	10	9
median	11	11	11

So the other two data values are 6 and 12.

13 **a** **i** median salary

$$= \frac{10\text{th} + 11\text{th}}{2} \quad \{\text{when in order}\}$$

$$= \frac{35\,000 + 28\,000}{2}$$

$$= \$31\,500$$

ii modal salary

$= \$28\,000$ {occurs most often}

iii

x	f	$f\,x$
50 000	1	50 000
42 000	3	126 000
35 000	6	210 000
28 000	10	280 000
$\sum$	20	666 000

mean

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{666\,000}{20}$$

$$= \$33\,300$$

b The mean as it is the highest value.

EXERCISE 18B.3

1

x (midpoint)	f	$f\,x$
4.5	2	9
14.5	5	72.5
24.5	7	171.5
34.5	27	931.5
44.5	9	400.5
$\sum$	50	1585

$$\therefore \quad \text{mean result} = \frac{1585}{50}$$

$$= 31.7$$

2 **a** 70 **b** $\doteqdot$ 411 000 litres **c**

midpoint (x)	f	$f\,x$
2499.5	4	9998
3499.5	4	13 998
4499.5	9	40 495.5
5499.5	14	76 993
6499.5	23	149 488.5
7499.5	16	119 992
$\sum$	70	410 965

Approximate mean

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{410\,965}{70}$$

$\doteqdot$ 5870 litres

3 **a** $5+10+25+40+10+15+10+10=125$ people

b

midpoint (x)	*frequency* (f)	$f\,x$
85	5	425
95	10	950
105	25	2625
115	40	4600
125	10	1250
135	15	2025
145	10	1450
155	10	1550
$\sum$	125	14 875

Approximate mean

$$= \frac{\sum f\,x}{\sum f}$$

$$= \frac{14\,875}{125}$$

$\doteqdot 119$ people

c $\frac{15}{125}$ scored < 100

i.e., $\frac{3}{25}$ scored < 100

d 20% of 125 people $=$ 25 people and 90 people scored < 130 for the test

$\therefore$ estimate is $130 + \frac{10}{15} \times 10 \doteqdot 137$ marks

EXERCISE 18C

1 **a** Total frequency $= 1+1+0+3+ .... +0+1 = 50$

$\therefore$ the median is the average of the 25th and 26th scores

$$\left.\begin{array}{l}23 \text{ are } 7\frac{1}{2} \text{ or less}\\ 40 \text{ are } 8 \text{ or less}\end{array}\right\} \therefore \text{ median} = \frac{8+8}{2} = 8$$

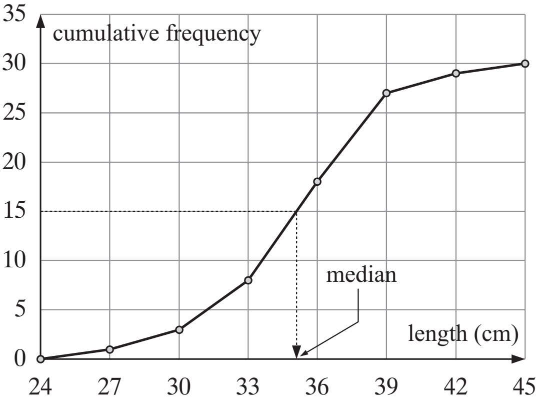
b **i** $13+17+7+2+0+1 = 40$ people

ii $1+1+0+3+5+13+17 = 40$ are 8 or less

2 **a**

Length (x cm)	*Freq.*	*C. freq.*
$24 \leqslant x < 27$	1	1
$27 \leqslant x < 30$	2	3
$30 \leqslant x < 33$	5	8
$33 \leqslant x < 36$	10	18
$36 \leqslant x < 39$	9	27
$39 \leqslant x < 42$	2	29
$42 \leqslant x < 45$	1	30

b



c median $\doteqdot 35$

d There are 30 data values. So, the median is the average of the 15th and 16th scores (when in order).

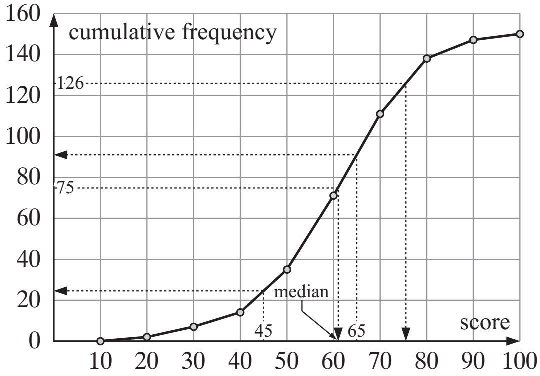
In order they are:

24 27 28 30 31 31 32 32 33 33 33 33 34 34 34 35 35 35 36 .... etc.

$$\text{median} = \frac{34+35}{2} = 34.5$$

So, the median from the graph is a good approximation.

3



a Total frequency $= 150$

$\therefore$ median $\doteqdot 61$ {from the graph}

b When the score $= 65$,

CF $\doteqdot 91$ {from the graph}

i.e., about 91 students scored 65 or less.

c $\doteqdot 36 + 40 = 76$ students scored between 50 and 70

d For a pass mark of 45, CF $\doteqdot 24.5$

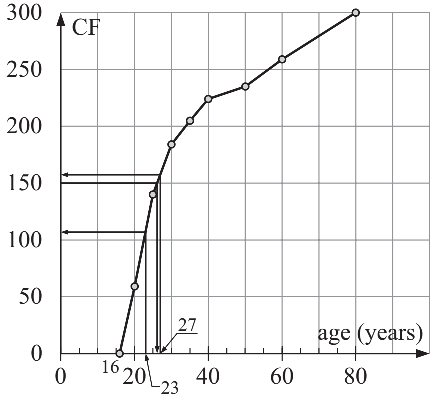
$\therefore$ 24 or 25 students failed the exam.

e 84% of $150 = 126$

So, for a CF of 126, the score value is 76.

So, the minimum credit mark would be 76.

4



Total frequency $= 300$

a median $\doteqdot 26$ years {from the graph}

b when the age $= 23$, CF $\doteqdot 108$

and $\dfrac{108}{300} \times 100\% \doteqdot 36\%$

c **i** when age is 27, CF $\doteqdot 158$

$\therefore$ $\text{P}(\leqslant 27) = \dfrac{158}{300} = 0.527$

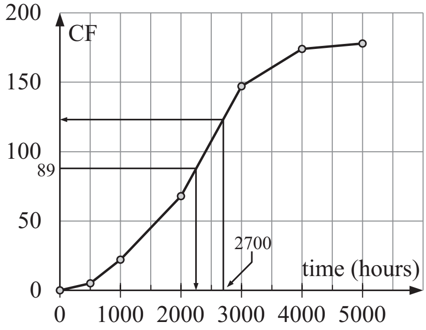
ii when age is 27, CF $\doteqdot 158$

when age is 28, CF $\doteqdot 167$

i.e., 9 died

$\therefore$ $\text{P(aged 27)} \doteqdot \dfrac{9}{300} \doteqdot 0.030$

5



Total frequency $= 178$

a median $\doteqdot 2300$ hours

b For a life of 2700 hours

CF $\doteqdot 123$

and $\dfrac{123}{178} \times 100\% \doteqdot 69\%$

So, about 69% have a life $\leqslant 2700$ h

c For a life of 1500 h, CF $\doteqdot 45$

For a life of 2500 h, CF $\doteqdot 107.5$

i.e., $107.5 - 45 = 62.5$

$\therefore$ 62 or 63 failed

EXERCISE 18D.1

1 **a** $2\ 3\ 3\ 3\ 4\ \underset{\underset{Q_1}{\uparrow}}{4}\ 4\ 5\ 5\ 5\ 5\ \underset{\underset{\text{median}}{\uparrow}}{6}\ 6\ 6\ 6\ 6\ 7\ \underset{\underset{Q_3}{\uparrow}}{7}\ 8\ 8\ 8\ 9\ 9 \quad (n = 23)$

i median $= 6$ **ii** $Q_1 = 4$, $Q_3 = 7$ **iii** range $= 9 - 2 = 7$ **iv** IQR $= Q_3 - Q_1 = 3$

b $10\ 12\ 12\ 14\ \underset{\underset{Q_1}{\uparrow}}{15}\ 15\ 16\ 16\ \underbrace{17\ 18}_{\underset{\text{median}}{\uparrow}}\ 18\ 18\ 18\ \underset{\underset{Q_3}{\uparrow}}{19}\ 20\ 21\ 22\ 24 \quad (n = 18)$

i median $= 17.5$ **ii** $Q_1 = 15$, $Q_3 = 19$ **iii** range $= 24 - 10 = 14$ **iv** IQR $= Q_3 - Q_1 = 4$

c $21.8\ 22.4\ \underset{\underset{Q_1}{\uparrow}}{23.5}\ 23.5\ 24.6\ \underset{\underset{\text{median}}{\uparrow}}{24.9}\ 25.0\ 25.3\ \underset{\underset{Q_3}{\uparrow}}{26.1}\ 26.4\ 29.5 \quad (n = 11)$

i median $= 24.9$ **ii** $Q_1 = 23.5$, $Q_3 = 26.1$ **iii** range $= 29.5 - 21.8 = 7.7$ **iv** IQR $= Q_3 - Q_1 = 2.6$

2 $\underset{\underset{\text{min}}{\uparrow}}{0}\ 0\ 0\ 0.8\ \underbrace{1.4\ 1.5}_{\underset{Q_1}{\uparrow}}\ 1.6\ 1.9\ 2.1\ \underbrace{2.2\ 2.7}_{\underset{\text{median}}{\uparrow}}\ 3.0\ 3.4\ 3.6\ \underbrace{3.8\ 3.8}_{\underset{Q_3}{\uparrow}}\ 4.5\ 4.8\ 5.2\ \underset{\underset{\text{max}}{\uparrow}}{5.2}$

a median $= \dfrac{2.2 + 2.7}{2} = 2.45$ $Q_1 = 1.45$, $Q_3 = 3.8$ **b** range $= 5.2 - 0 = 5.2$ IQR $= Q_3 - Q_1 = 3.8 - 1.45 = 2.35$

c **i** the median, i.e., 2.45 min **ii** Q_3, i.e., 3.8 min

iii The minimum waiting time was 0 minutes and the maximum waiting time was 5.2 minutes. The waiting time was spread over 5.2 minutes.

3 $\underset{\underset{\text{min}}{\uparrow}}{3}\ 4\ 7\ 9\ 10\ \underset{\underset{Q_1}{\uparrow}}{13}\ 14\ 16\ 17\ 18\ 20\ \underset{\underset{\text{median}}{\uparrow}}{20}\ 23\ 25\ 26\ 29\ 29\ \underset{\underset{Q_3}{\uparrow}}{29}\ 31\ 33\ 37\ 38\ \underset{\underset{\text{max}}{\uparrow}}{42} \quad (n = 23)$

a min $= 3$ **b** max $= 42$ **c** median $= 20$ **d** $Q_1 = 13$ **e** $Q_3 = 29$

f range $= 42 - 3 = 39$ **g** IQR $= Q_3 - Q_1 = 29 - 13 = 16$

4 $\overset{\overset{\text{min}}{\downarrow}}{109}\ 111\ 113\ 114\ \underbrace{114\ 118}_{\underset{Q_1}{\uparrow}}\ 119\ 122\ 122\ \underbrace{124\ 124}_{\underset{\text{median}}{\uparrow}}\ 126\ 128\ 129\ \underbrace{129\ 131}_{\underset{Q_3}{\uparrow}}\ 132 \quad (n = 20)$
$135\ 138\ \underset{\underset{\text{max}}{\uparrow}}{138}$

a **i** median $= 124$ cm **ii** $Q_3 = 130$ cm, $Q_1 = 116$ cm

b **i** .... 124 cm tall **ii** .... 130 cm tall

c **i** range $= 138 - 109 = 29$ cm **ii** IQR $= Q_3 - Q_1 = 14$ cm

d the IQR, i.e., over 14 cm

5 **a** **Without fertiliser** - See **Exercise 18B.2** solution to question **6**.

i range $= 9 - 2 = 7$ **ii** median $= 6$

iii lower quartile $=$ 38th score $= 5$ **iv** upper quartile $=$ 113th score $= 7$

v interquartile range $= 7 - 5 = 2$

b **With fertiliser**

i range $= 13 - 3 = 10$ **ii** median $= 7$

iii lower quartile $=$ 38th score $= 6$ **iv** upper quartile $=$ 113th score $= 8$

v interquartile range $= 8 - 6 = 2$

EXERCISE 18D.2

1 **a** **i** median $= 35$ **ii** max. value $= 78$ **iii** min. value $= 13$

iv $\text{Q}_3 = 53$ **v** $\text{Q}_1 = 26$

b **i** range $= 78 - 13 = 65$ **ii** $\text{IQR} = \text{Q}_3 - \text{Q}_1 = 53 - 26 = 27$

2 **a** highest mark was 98, lowest mark was 25 **b** the median which is 70

c Q_3 which is 85 **d** $\text{Q}_1 = 55$ and $\text{Q}_3 = 85$ **e** range $= 98 - 25 = 73$

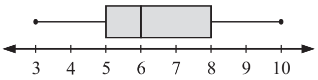
f $\text{IQR} = \text{Q}_3 - \text{Q}_1 = 85 - 55 = 30$

3 **a** **i**

$$\underset{\substack{\uparrow\\ \text{min}}}{3}\quad 4\quad 5\quad \underset{\substack{\uparrow\\ \text{Q}_1}}{5}\quad 5\quad 6\quad \underbrace{6\quad 6}_{\substack{\uparrow\\ \text{median}}}\quad 7\quad 7\quad \underset{\substack{\uparrow\\ \text{Q}_3}}{8}\quad 8\quad 9\quad \underset{\substack{\uparrow\\ \text{max}}}{10}$$

So, min $= 3$, $\text{Q}_1 = 5$, median $= 6$, $\text{Q}_3 = 8$, max $= 10$

ii



iii range $= 10 - 3$
$= 7$

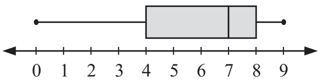
iv $\text{IQR} = \text{Q}_3 - \text{Q}_1$
$= 8 - 5$
$= 3$

b **i**

$$\underset{\substack{\uparrow\\ \text{min}}}{0}\quad 1\quad 2\quad 3\quad \underset{\substack{\uparrow\\ \text{Q}_1}}{4}\quad 5\quad 6\quad 6\quad 7\quad \underset{\substack{\uparrow\\ \text{median}}}{7}\quad 7\quad 8\quad 8\quad 8\quad \underset{\substack{\uparrow\\ \text{Q}_3}}{8}\quad 8\quad 8\quad 9\quad \underset{\substack{\uparrow\\ \text{max}}}{9}$$

So, min $= 0$, $\text{Q}_1 = 4$, median $= 7$, $\text{Q}_3 = 8$, max $= 9$

ii



iii range $= 9 - 0$
$= 9$

iv $\text{IQR} = \text{Q}_3 - \text{Q}_1$
$= 8 - 4$
$= 4$

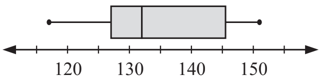
c **i**

$$\overset{\substack{\text{min}\\ \downarrow}}{117}\ \ 120\ \ 123\ \ 126\ \ \underbrace{126\ \ 128}_{\substack{\uparrow\\ \text{Q}_1}}\ \ 130\ \ 131\ \ 131\ \ \underbrace{131\ \ 133}_{\substack{\uparrow\\ \text{median}}}\ \ 135\ \ 135\ \ 137\ \ \underbrace{144\ \ 147}_{\substack{\uparrow\\ \text{Q}_3}}\ \ 147$$

$$149\ \ 149\ \ \underset{\substack{\uparrow\\ \text{max}}}{151}$$

So, min $= 117$, $\text{Q}_1 = 127$, median $= 132$, $\text{Q}_3 = 145.5$, max $= 151$

ii



iii range
$= 151 - 117$
$= 34$

iv $\text{IQR} = \text{Q}_3 - \text{Q}_1$
$= 145.5 - 127$
$= 18.5$

4 **a**

Statistic	*Year 9*	*Year 12*
min value	1	6
Q_1	5	10
median	7.5	14
Q_3	10	16
max value	12	17.5

b For year 9 group

i range $= 12 - 1$
$= 11$

ii IQR $= 10 - 5$
$= 5$

For year 12 group

i range $= 17.5 - 6$
$= 11.5$

ii IQR $= 16 - 10$
$= 6$

c **i** True, indicated by the median.

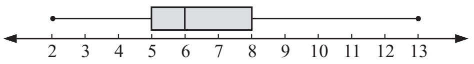
ii True, as Q_1 for year 9 $= 5$ and min for year 12 $= 6$.

5 $\underset{\substack{\uparrow \\ \text{min}}}{2}\ 3\ 3\ 4\ 4\ 4\ 4\ \underset{\substack{\uparrow \\ Q_1}}{\underbrace{5\ 5}}\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ \underset{\substack{\uparrow \\ \text{median}}}{6}\ 6\ 6\ 6\ 7\ 7\ 7\ 7\ \underset{\substack{\uparrow \\ Q_3}}{\underbrace{8\ 8}}\ 8\ 9\ 9\ 9\ 10\ 12\ \underset{\substack{\uparrow \\ \text{max}}}{13}$

a median $= 6$, $Q_1 = 5$, $Q_3 = 8$

b IQR $= 8 - 5 = 3$

c



6 **a**

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

↑ min (33) ↑ median is one of these (36) ↑ max (40)

There are 47 scores

$\therefore$ median $= 24$th $\left(\dfrac{47+1}{2} = 24\right)$

14 scores are 35 or less

27 scores are 36 or less

$\therefore$ median is 36

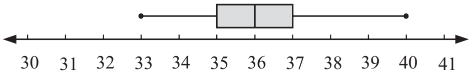
$Q_1 = 12$th $\left(\dfrac{47+1}{4} = 12\right)$
$= 35$

$Q_3 = 36$th
$= 37$

So, min $= 33$, $Q_1 = 35$, median $= 36$, $Q_3 = 37$, max $= 40$

b **i** range $= 40 - 33 = 7$

ii IQR $= 37 - 35 = 2$



7 **a** For $h = 5$, CF $\doteqdot 10$ $\therefore$ 10 seedlings have height 5 cm or less

b For $h = 8$, CF $\doteqdot 43$ $\therefore$ % taller than 8 cm $= \dfrac{60-43}{60} \times 100\% \doteqdot 28.3\%$

c Approx. median occurs at CF $= 30$, i.e., median $\doteqdot 7$ cm.

d IQR $= Q_3 - Q_1 = (h$ when CF $= 45) - (h$ when CF $= 15)$
$\doteqdot 8.4 - 5.8$
$\doteqdot 2.6$ cm

e 90th percentile occurs when CF $= 90\%$ of $60 = 54$

$\therefore$ 90th percentile $= 10$

This means that 90% of the seedlings have a height of 10 cm or less.

8 **a** The lower quartile occurs when CF $= 25\%$ of $80 = 20$

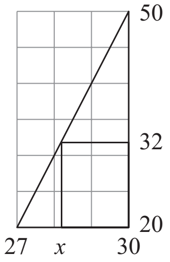
$\therefore$ $Q_1 = 27$ min

b The median occurs when CF $= 50\%$ of $80 = 40$

$\therefore$ median $= 29$ min

c The upper quartile occurs when CF $= 75\%$ of $80 = 60$

$\therefore$ $Q_3 \doteqdot 31\frac{1}{2}$ min



d $\text{IQR} = \text{Q}_3 - \text{Q}_1 = 31\frac{1}{2} - 27 = 4\frac{1}{2}$ min

e For the 40th percentile, $\text{CF} = 40\%$ of $80 = 32$

From the similar Δs, $\dfrac{x-27}{12} = \dfrac{3}{30}$

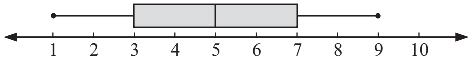
$\therefore \quad x - 27 = 1.2$

$\therefore \quad x = 28.2$ So about 28 min 10 sec.

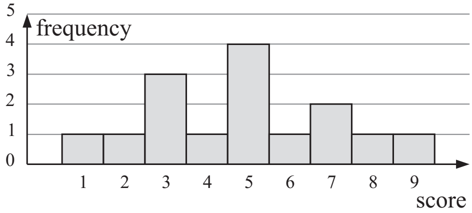
EXERCISE 18E.1

1 **a** $\overline{x} \doteqdot 4.87, \quad \text{Min}_x = 1, \quad \text{Q}_1 = 3, \quad \text{Q}_2 = 5, \quad \text{Q}_3 = 7, \quad \text{Max}_x = 9$

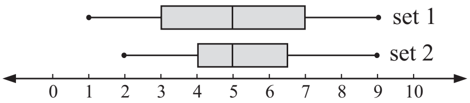
b



c

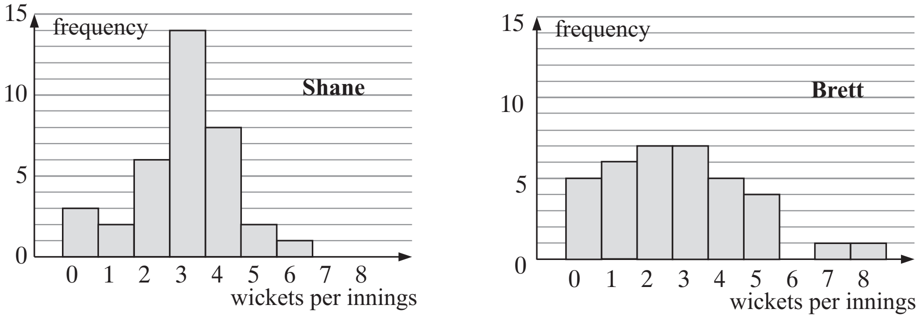


d $\overline{x} \doteqdot 5.24, \quad \text{Min}_x = 2, \quad \text{Q}_1 = 4, \quad \text{Q}_2 = 5,$
$\text{Q}_3 = 6.5, \quad \text{Max}_x = 9$



EXERCISE 18E.2

1 **a** discrete **c**



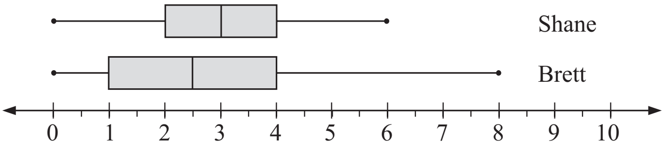
d There are no outliers for Shane. Brett has outliers of 7 and 8 which must not be removed.

e Shane's distribution is reasonably symmetrical. Brett's distribution is positively skewed.

f Shane has a higher mean ($\doteqdot$ 2.89 wickets) compared with Brett ($\doteqdot$ 2.67 wickets). Shane has a higher median (3 wickets) compared with Brett (2.5 wickets). Shane's modal number of wickets is 3 (14 times) compared with Brett, who has a bi-modal distribution of 2 and 3 (7 times each).

g Shane's range is 6 wickets, compared with Brett's range of 8 wickets. Shane's IQR is 2 wickets, compared with Brett's IQR of 3 wickets. Brett's wicket taking shows greater spread or variability.

h



j Generally, Shane takes more wickets than Brett and is a more consistent bowler.

2 **a** continuous

c For the 'New type' globes, 191 hours could be considered an outlier. However, it could be a genuine piece of data, so we will include it in the analysis.

d

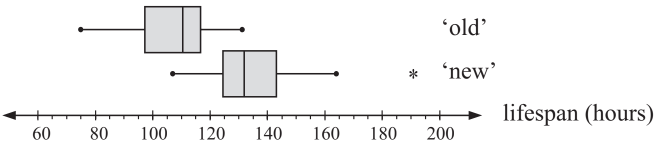
	Old type	*New type*
Mean	107	134
Median	110.5	132
Range	56	84
IQR	19	18.5

The mean and median are $\doteqdot$ 25% and $\doteqdot$ 19% higher for the 'new type' of globe compared with the 'old type'.

The range is higher for the 'new type' of globe (but has been affected by the 191 hours).

The IQR for each type of globe is almost the same.

e



f For the 'old type' of globe, the data is bunched to the right of the median, hence the distribution is negatively skewed. For the 'new type' of globe, the data is bunched to the left of the median, hence the distribution is positively skewed.

g The manufacturer's claim, that the 'new type' of globe has a 20% longer life than the 'old type' seems to be backed up by the 25% higher mean life and 19.5% higher median life.

EXERCISE 18F

1 **a** **Sally**

x	$(x-\bar{x})^2$
23	4
17	64
31	36
25	0
25	0
19	36
28	9
32	49
200	198

$$\bar{x} = \frac{\sum x}{n} = \frac{200}{8} = 25 \qquad s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{198}{8}} \doteqdot 4.97$$

Joanne

x	$(x-\bar{x})^2$
9	$(21.5)^2$
29	$(1.5)^2$
41	$(10.5)^2$
26	$(4.5)^2$
14	$(16.5)^2$
44	$(13.5)^2$
38	$(7.5)^2$
43	$(12.5)^2$
244	1262

$$\bar{x} = \frac{\sum x}{n} = \frac{244}{8} = 30.5 \qquad s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{1262}{8}} \doteqdot 12.56$$

b The standard deviation. The lower the value of s, the more consistent.

2 **a** Glen has mean $= \dfrac{0+10+1+9+11+0+8+5+6+7}{10} = \dfrac{57}{10} = 5.7$

Shane has mean $= \dfrac{4+3+4+1+4+11+7+6+12+5}{10} = \dfrac{57}{10} = 5.7$

Glen's range $= 11 - 0 = 11$ Shane's range $= 12 - 1 = 11$

b We suspect Glen's as he has more high and low values.

c **Glen**

x	$(x-\bar{x})^2$
0	$(5.7)^2$
10	$(4.3)^2$
1	$(4.7)^2$
9	$(3.3)^2$
11	$(5.3)^2$
0	$(5.7)^2$
8	$(2.3)^2$
5	$(0.7)^2$
6	$(0.3)^2$
7	$(1.3)^2$
	154.9

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{154.9}{10}}$$

$$\doteq 3.9$$

↑
greater
variability

Shane

x	$(x-\bar{x})^2$
4	$(1.7)^2$
3	$(2.7)^2$
4	$(1.7)^2$
1	$(4.7)^2$
4	$(1.7)^2$
11	$(5.3)^2$
7	$(1.3)^2$
6	$(0.3)^2$
12	$(6.3)^2$
5	$(0.7)^2$
	108.1

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{108.1}{10}}$$

$$\doteq 3.29$$

d The standard deviation as it takes all values into account, not just the lowest and highest.

3 **a** We suspect variability in standard deviation since the factors may change every day.

b **i** sample mean **ii** sample standard deviation **c** less variability

4 **a**

x	$(x-\bar{x})^2$
79	10^2
64	5^2
59	10^2
71	2^2
68	1^2
68	1^2
74	5^2
483	256

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{483}{7}$$

$$= 69$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{256}{7}}$$

$$\doteq 6.05$$

b

x	$(x-\bar{x})^2$
89	10^2
74	5^2
69	10^2
81	2^2
78	1^2
78	1^2
84	5^2
553	256

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{553}{7}$$

$$= 79$$

$$s \doteq 6.05$$

c The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same.

5 **a**

x	$(x-\bar{x})^2$
0.8	$(0.21)^2$
1.1	$(0.09)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
1.2	$(0.19)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
0.7	$(0.31)^2$
1.0	$(0.01)^2$
1.1	$(0.09)^2$
10.4	0.289

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{10.1}{10}$$

$$= 1.01 \text{ kg}$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{0.289}{10}}$$

$$\doteq 0.17 \text{ kg}$$

b

x	$(x-\bar{x})^2$
1.6	$(0.42)^2$
2.2	$(0.18)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
2.4	$(0.38)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
1.4	$(0.62)^2$
2.0	$(0.02)^2$
2.2	$(0.18)^2$
20.2	1.156

$$\bar{x} = \frac{\sum x}{n}$$

$$= \frac{20.2}{10}$$

$$= 2.02 \text{ kg}$$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{1.156}{10}}$$

$$= 0.34 \text{ kg}$$

c Doubling the values doubles the mean and standard deviation.

6 **a**
$$\begin{aligned} s &= \sqrt{\text{variance}} \\ &= \sqrt{45.9} \text{ kg} \\ &= 6.77 \text{ kg} \end{aligned}$$

b Unbiased estimate of μ is $\overline{x} = 93.8$ kg
Unbiased estimate of σ is

$$\sqrt{\tfrac{87}{86}} \times 6.77 \text{ kg} \doteqdot 6.81 \text{ kg}$$

7 **a** Using technology, $\overline{x} \doteqdot 77.5$ g and $s_n \doteqdot 7.445$ g

b Unbiased estimate of μ is $\overline{x}$ i.e., 77.5 g

Unbiased estimate of σ is $\sqrt{\tfrac{17}{16}} \times 7.445 \doteqdot 7.67$ g

8
$$\begin{aligned} \text{mean} = \frac{1+3+5+7+4+5+p+q}{8} &= 5 \\ 25+p+q &= 40 \\ \therefore \quad p+q &= 15 \\ \therefore \quad q &= 15-p \end{aligned}$$

$$\begin{aligned} \text{and} \quad s = \sqrt{\frac{(-4)^2+(-2)^2+0^2+2^2+(-1)^2+0^2+(p-5)^2+(q-5)^2}{8}} &= \sqrt{5.25} \\ \therefore \quad \frac{16+4+4+1+(p-5)^2+(15-p-5)^2}{8} &= 5.25 \\ \therefore \quad 25+p^2-10p+25+100-20p+p^2 &= 42 \\ \therefore \quad 2p^2-30p+108 &= 0 \\ \therefore \quad p^2-15p+54 &= 0 \\ \therefore \quad (p-6)(p-9) &= 0 \\ \therefore \quad p = 6 \text{ or } 9 \quad \text{and} \quad q &= 9 \text{ or } 6 \\ \text{But} \quad p<q \qquad \therefore \quad p &= 6, \quad q = 9 \end{aligned}$$

9
$$\begin{aligned} \text{mean} = \frac{3+9+5+5+6+4+6+8+a+b}{10} &= 6 \\ \therefore \quad \frac{46+a+b}{10} &= 6 \\ \therefore \quad 46+a+b &= 60 \\ \therefore \quad a+b &= 14 \\ \therefore \quad b &= 14-a \end{aligned}$$

$$\begin{aligned} \text{and} \quad s = \sqrt{\frac{(-3)^2+3^2+(-1)^2+(-1)^2+(-2)^2+(a-6)^2+(b-6)^2+2^2}{10}} &= \sqrt{3.2} \\ \therefore \quad 9+9+1+1+4+4+(a-6)^2+(8-a)^2 &= 32 \\ \therefore \quad 28+a^2-12a+36+64-16a+a^2 &= 32 \\ \therefore \quad 2a^2-28a+96 &= 0 \\ \therefore \quad a^2-14a+48 &= 0 \\ \therefore \quad (a-6)(a-8) &= 0 \\ \therefore \quad a = 6 \text{ or } 8 \quad \text{and} \quad b &= 8 \text{ or } 6 \\ \therefore \quad a &= 8, \quad b = 6 \end{aligned}$$

10 **a**
$$\begin{aligned} \sum_{i=1}^{3} (x_i - \overline{x})^2 &= (x_1-\overline{x})^2 + (x_2-\overline{x})^2 + (x_3-\overline{x})^2 \\ &= x_1^2 - 2\overline{x}x_1 + \overline{x}^2 + x_2^2 - 2\overline{x}x_2 + \overline{x}^2 + x_3^2 - 2\overline{x}x_3 + \overline{x}^2 \\ &= x_1^2 + x_2^2 + x_3^2 - 2\overline{x}(x_1+x_2+x_3) + 3\overline{x}^2 \end{aligned}$$

$$= \sum_{i=1}^{3}(x_i^2) - 2\overline{x}\sum_{i=1}^{3}x_i + 3\overline{x}^2 \quad \text{where} \quad \overline{x} = \frac{\sum_{i=1}^{3}x_i}{3}$$

$$= \sum_{i=1}^{3}(x_i^2) - 2\overline{x}\times 3\overline{x} + 3\overline{x}^2$$

$$= \sum_{i=1}^{3}(x_i^2) - 6\overline{x}^2 + 3\overline{x}^2$$

$$= \sum_{i=1}^{3}(x_i^2) - 3\overline{x}^2 \qquad \text{Generalisation is:} \quad \sum_{i=1}^{n}(x_i-\overline{x})^2 = \sum_{i=1}^{n}(x_i^2) - n\overline{x}^2$$

b Unbiased estimate of μ is $\overline{x} = \dfrac{\sum_{i=1}^{16}x_i}{16} = \frac{519}{16} \doteqdot 32.4$ min

and $$s = \sqrt{\frac{\sum_{i=1}^{16}(x_i-\overline{x})^2}{16}} = \sqrt{\frac{\sum_{i=1}^{16}(x_i)^2 - 16\overline{x}^2}{16}} = \sqrt{\frac{16\,983 - 16\times 32.4375^2}{16}}$$

$$\doteqdot \sqrt{9.246.....}$$

$$\doteqdot 3.041 \text{ min}$$

$\therefore$ unbiased estimate of $\sigma = \sqrt{\frac{16}{15}} \times 3.041 \doteqdot 3.140$ min

$\therefore$ unbiased estimate of σ^2 is $\doteqdot 9.86$ min

11 **a**

x	f	$f\,x$	$f(x-\bar{x})^2$
0	14	0	41.61
1	18	18	9.44
2	13	26	0.99
3	5	15	8.14
4	3	12	15.54
5	2	10	21.46
6	2	12	36.57
7	1	7	27.84
$\sum$	58	100	161.59

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{100}{58} \doteqdot 1.72 \text{ children}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \doteqdot \sqrt{\frac{161.59}{58}} \doteqdot 1.67 \text{ children}$$

b Unbiased estimate of μ is $\overline{x} = 1.72$ children

Unbiased estimate of σ is $\sqrt{\frac{58}{57}} \times s$

$\doteqdot 1.68$ children

12 **a**

x	f	$f\,x$	$f(x-\bar{x})^2$
11	2	22	24.22
12	1	12	6.15
13	4	52	8.76
14	5	70	1.15
15	6	90	1.62
16	4	64	9.24
17	2	34	12.70
18	1	18	12.39
$\sum$	25	362	76.23

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{362}{25} \doteqdot 14.5 \text{ years}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{76.25}{25}} \doteqdot 1.75 \text{ years}$$

b Unbiased estimate of μ is $\overline{x} = 14.5$ years

Unbiased est. of σ is $\sqrt{\frac{25}{24}} \times 1.7464 .... \doteqdot 1.78$ years

13 **a**

x	f	$f\,x$	$f(x-\bar{x})^2$
33	1	33	18.24
35	5	175	25.78
36	7	252	11.31
37	13	481	0.95
38	12	456	6.38
39	8	312	23.92
40	2	80	14.90
$\sum$	48	1789	101.48

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{1789}{48} \doteq 37.3 \text{ toothpicks}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{101.48}{48}} \doteq 1.45 \text{ toothpicks}$$

b Unbiased estimate of μ is $\overline{x} \doteq 37.3$ toothpicks

Unbiased est. of σ is $\sqrt{\frac{48}{47}} \times s \doteq 1.47$ toothpicks

14 **a**

Midpoint (x)	f	$f\,x$	$f(x-\bar{x})^2$
40.5	1	40.5	52.85
42.5	1	42.5	27.77
44.5	3	133.5	32.08
46.5	7	325.5	11.29
48.5	11	533.5	5.86
50.5	5	252.5	37.26
52.5	2	105	44.75
$\sum$	30	1433	211.86

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{1433}{30} = 47.7666.... \doteq 47.8 \text{ cm}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \doteq \sqrt{\frac{211.86}{30}} \doteq 2.657.... \doteq 2.66 \text{ cm}$$

b Unbiased estimate of μ is $\overline{x} \doteq 47.8$ cm

Unbiased estimate of σ is $\sqrt{\frac{30}{29}} \times 2.657..... \doteq 2.70$ cm

15 **a**

Midpoint (x)	f	$f\,x$	$f(x-\bar{x})^2$
364.995	17	6204.9	10 885.83
374.995	38	14 249.8	8901.23
384.995	47	18 094.8	1322.72
394.995	57	22 514.7	1256.45
404.995	18	7289.9	3886.97
414.995	10	4149.9	6098.43
424.995	10	4250.0	12 037.43
434.995	3	1305.0	5992.93
$\sum$	200	78 059	50 381.99

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{78\,059}{200} \doteq \$390.30$$

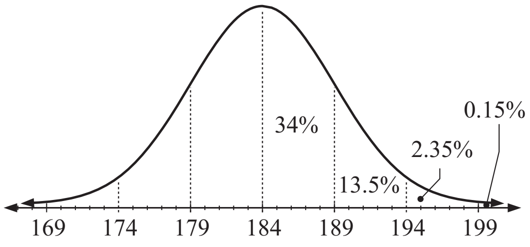
$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\frac{50\,381.99}{200}} \doteq \$15.87$$

b Unbiased estimate of μ is $\overline{x} \doteq \$390.30$

Unbiased estimate of σ is $\sqrt{\frac{200}{199}} \times \$15.87..... \doteq \$15.91$

EXERCISE 18G

1



a $50\% + 34\% = 84\%$

So, 16% are above 189.

b $50\% + 34\% = 84\%$

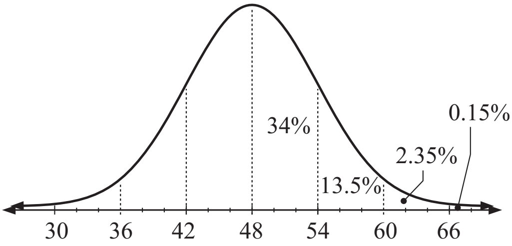
So, 84% are above 179.

c $2 \times 34\% + 2 \times 13.5\% + 2.35\%$

$\doteq 97.4\%$

d 0.15%

2

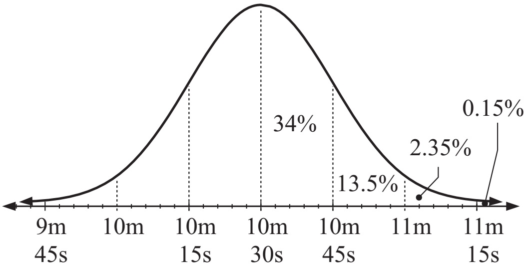


$100\% - 50\% - 34\% = 16\%$

and $20 \times 16\%$
$= 20 \times 0.16$
$= 3.2$

On 3 occasions.

3

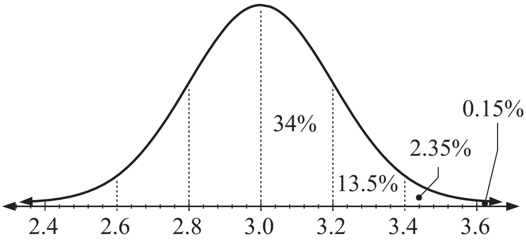


a $200 \times (50 + 34 + 13.5)\%$
$= 200 \times 95.5\%$
$\doteqdot 195$
$\therefore$ $200 - 195 = 5$ took > 11 minutes

b $200 \times (0.15 + 2.35 + 13.5)\%$
$= 200 \times 16\%$
$= 32$ lifesavers

c $200 \times 68\% = 136$ lifesavers

4



a $545 \times (50 + 34)\%$
$= 545 \times 0.84$
$\doteqdot 458$ babies

b $545 \times (68 + 13.5)\%$
$= 545 \times 81.5\%$
$\doteqdot 444$ babies

REVIEW SET 18A

1 **a** Diameter of bacteria colonies

0	4 8 9
1	3 5 5 7
2	1 1 5 6 8 8
3	0 1 2 3 4 5 5 6 6 7 7 9
4	0 1 2 7 9

leaf unit: 0.1 cm

b Using technology

i median = 3.15 cm

ii range = 4.9 – 0.4 = 4.5 cm

c The distribution has a large tail on the left. So, it is negatively skewed.

2 **a** highest = 97.5 m, lowest = 64.6 m

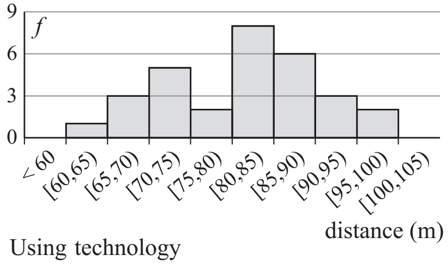
b The range = 97.5 − 64.5 = 33
So, if intervals of length 5 are used we need about 7 of them.
We choose 60 - , 65 - , 70 - , 75 - , 80 - , etc.

c

A frequency distribution table for distances thrown by Thabiso

distance (m)	tally	freq. (f)
60 -	|	1
65 -	|||	3
70 -	~~||||~~	5
75 -	||	2
80 -	~~||||~~ |||	8
85 -	~~||||~~ |	6
90 -	|||	3
95 < 100	||	2
	Total	30

d **i/ii** **Frequency histogram displaying the distance Thabiso throws a baseball**



e Using technology

i $\bar{x} \doteqdot 81.1$ m **ii** median $\doteqdot 83.0$ m

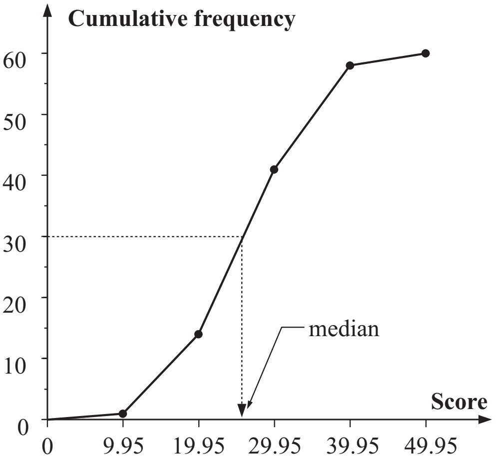
3 Mean $= \dfrac{5+6+8+3+a+b}{6} = 6$

$$\begin{aligned} \therefore\quad 22+a+b &= 36 \\ \therefore\quad a+b &= 14 \\ \therefore\quad b &= 14-a \end{aligned}$$

and $s = \sqrt{\dfrac{(-1)^2+0^2+2^2+(-3)^2+(a-6)^2+(b-6)^2}{6}} = \sqrt{3}$

$$\begin{aligned} \therefore\quad \frac{1+4+9+(a-6)^2+(8-a)^2}{6} &= 3 \\ \therefore\quad 14+a^2-12a+36+64-16a+a^2 &= 18 \\ \therefore\quad 2a^2-28a+96 &= 0 \\ \therefore\quad a^2-14a+48 &= 0 \\ \therefore\quad (a-6)(a-8) &= 0 \\ \therefore\quad a = 6 \text{ or } 8 \quad \text{and} \quad b &= 8 \text{ or } 6 \\ \therefore\quad a = 6,\quad b = 8 \quad \text{or} \quad a &= 8,\quad b = 6 \end{aligned}$$

4 **a**



b median $\doteq$ 25.9 (see graph)

c IQR $=$ Q$_3$ $-$ Q$_1$

$=$ (score for CF of 45)

$-$ (score for CF of 15)

$\doteq 32 - 20 \doteq 12$

d

	f	midpt x	$f\,x$	$(x-\bar{x})^2$
0 – 9.9	1	4.95	4.95	441
10 – 19.9	13	14.95	194.35	121
20 – 29.9	27	24.95	673.65	1
30 – 39.9	17	34.95	594.15	81
40 – 49.9	2	44.95	89.9	361
	60		1557	

$\bar{x} = \dfrac{\sum f\,x}{\sum x}$ $\qquad$ $s = \sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}}$

$= \frac{1557}{60} \doteq 26.0$ $\qquad$ $= \sqrt{\frac{4140}{60}} \doteq 8.31$

5 **a** Using technology

	Girls	*Boys*
shape	pos. skewed	approx. symm.
centre (median)	36.3 sec	34.9 sec
spread (range)	7.7 sec	4.9 sec

b The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median swim times for boys is 1.4 seconds lower than for girls but the range of the girls' swim times is 2.8 seconds higher than for boys. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

6 **a** Reading from the box-plots

	A	B
Min	11	11.2
Q$_1$	11.6	12
Median	12	12.6
Q$_3$	12.6	13.2
Max	13	13.8

b **i** range of A

$= 13 - 11$

$= 2$

range of B

$= 13.8 - 11.2$

$= 2.6$

ii IQR of A

$= 12.6 - 11.6$

$= 1.0$

IQR of B

$= 13.2 - 12$

$= 1.2$

c **i** The members in squad A generally ran faster times.
ii The times in squad B were more varied.

7 Using technology **a** **i** 101.5 **ii** 98 **iii** 105.5 **b** 7.5 **c** $\bar{x} = 100.2$, $s \doteqdot 7.59$

8 **a** This question could be done using technology or

litres (x)	f	$f\,x$	$f(x-\bar{x})^2$
17	5	85	1299.27
22	13	286	1607.50
27	17	459	636.72
32	29	928	36.38
37	27	999	406.47
42	18	756	1419.38
47	7	329	1348.58
$\sum$	116	3842	6754.30

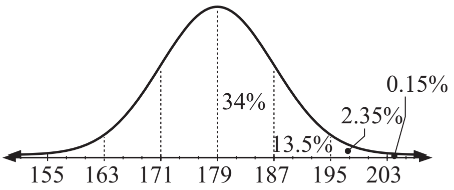
$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{3842}{116} \doteqdot 33.1 \text{ litres}$$

$$s = \sqrt{\frac{\sum f(x-\bar{x})^2}{\sum f}} \doteqdot \sqrt{\frac{6754.30}{116}} \doteqdot 7.63 \text{ litres}$$

b Unbiased estimate of μ is $\overline{x} \doteqdot 33.1$ litres

Unbiased estimate of σ is $\sqrt{\frac{116}{115}} \times 7.63$
$\doteqdot 7.66$ litres.

9



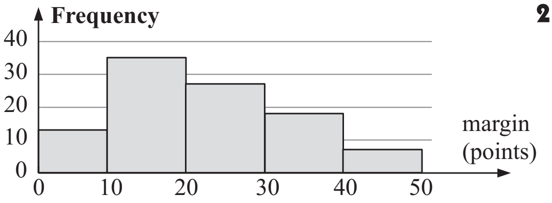
a $\% > 195$ cm
$= (2.35 + 0.15)\%$
$= 2.5\%$

b % between 163 cm and 195 cm
$= 2 \times (34\% + 13.5\%)$
$= 95\%$

c % between 171 cm and 187 cm $= 34\% + 34\% = 68\%$

REVIEW SET 18B

1



2 Using technology,
$\bar{x} \doteqdot 49.6$, $s \doteqdot 1.60$ does not justify claim.
A much greater sample is needed.

3 Use technology or

Midpoint (x)	f	$f\,x$
274.5	14	3843
324.5	34	11 033
374.5	68	25 466
424.5	72	30 564
474.5	54	25 623
524.5	23	12 063.5
574.5	7	4021.5
$\sum$	272	112 613.5

$$\bar{x} = \frac{\sum f\,x}{\sum f} = \frac{112\,613.5}{272} \doteqdot 414 \text{ customers}$$

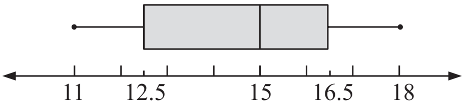
4

$$\underset{\substack{\uparrow\\ \text{min}}}{116}\ \ \underbrace{118\ \ 120}_{\substack{\uparrow\\ \text{Q}_1}}\ \ \underbrace{122\ \ 127}_{\substack{\uparrow\\ \text{median}}}\ \ \underbrace{128\ \ 132}_{\substack{\uparrow\\ \text{Q}_3}}\ \ \underset{\substack{\uparrow\\ \text{max}}}{135}\quad (n = 8)$$

$$\text{range} = 135 - 116 = 19 \qquad \text{Q}_1 = \frac{118+120}{2} = 119 \qquad \text{Q}_3 = \frac{128+132}{2} = 130 \qquad s \doteqdot 6.38 \quad \{\text{technology}\}$$

5

$$\underset{\substack{\uparrow\\ \text{min}}}{11}\ \ 12\ \ \underset{\substack{\uparrow\\ \text{Q}_1}}{\underbrace{12\ \ 13}}\ \ 14\ \ 14\ \ \underset{\substack{\uparrow\\ \text{median}}}{15}\ \ 15\ \ 15\ \ \underset{\substack{\uparrow\\ \text{Q}_3}}{\underbrace{16\ \ 17}}\ \ 17\ \ \underset{\substack{\uparrow\\ \text{max}}}{18}$$



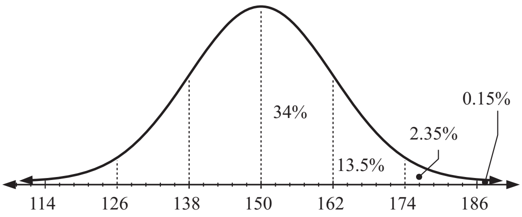
6 **a** Using technology with x values 74.995, 84.995, 94.995, etc.

$\bar{x} = \$103.50$ and $s = \$19.40$

b Unbiased estimate of μ is $\overline{x} = \$103.50$

Unbiased estimate of σ is $\sqrt{\dfrac{n}{n-1}} \times \$19.40 \doteqdot \sqrt{\frac{215}{214}} \times \$19.40 \doteqdot \$19.45$

7



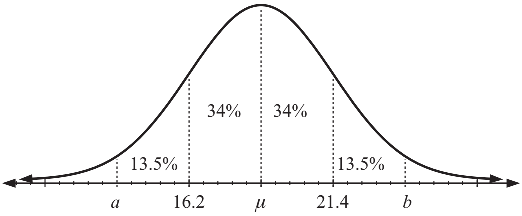
a 68% lie here

b $2 \times (34 + 13.5) = 95\%$ lie here

c $2 \times 34 + 13.5 = 81.5\%$ lie here

d 13.5% lie here

8

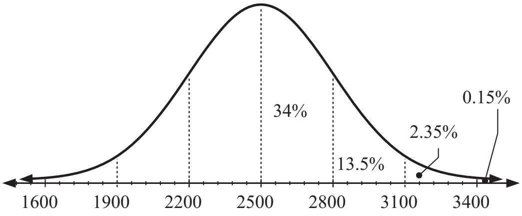


a $\mu = \dfrac{16.2 + 21.4}{2} = \frac{37.6}{2} = 18.8$

$\sigma = 21.4 - 18.8 = 2.6$

b The middle 95% of data lies between a and b
where $a = 16.2 - 2.6 = 13.6$
and $b = 21.4 + 2.6 = 24.0$
i.e., between 13.6 and 24.0

9



a $(50 - 34 - 13.5)\% = 2.5\%$

days are less than 1900

b $(50 + 34)\% = 84\%$

c $(2 \times 34 + 13.5)\% = 81.5\%$

10 **a** **i** mean $= \dfrac{\sum\limits_{i=1}^{30} x_i}{30} = \frac{116.3}{30} \doteqdot 3.877 \text{ cm} \doteqdot 3.88 \text{ cm}$

ii

$$s^2 = \frac{\sum\limits_{i=1}^{30}(x_i - \overline{x})^2}{30} = \frac{\sum\limits_{i=1}^{30}(x_i^2) - 30\overline{x}^2}{30}$$

$$\therefore \quad s^2 = \frac{452.57 - 30 \times 3.877^2}{30} \doteqdot 0.0571 \text{ cm}^2$$

b unbiased estimate of μ is $\overline{x} = 3.88$ cm

unbiased estimate of σ is $\sqrt{\dfrac{n}{n-1}} \times s$

$\doteqdot \sqrt{\frac{30}{29}} \times \sqrt{0.0545}$

$\doteqdot 0.243 \text{ cm}^2$

Chapter 19

PROBABILITY

EXERCISE 19A

1 **a** P (inside a square) $\doteqdot \dfrac{113}{145}$ $\doteqdot 0.780$

b P (on a line) $\doteqdot \dfrac{32}{145}$ $\doteqdot 0.221$

2 Total frequency $= 17 + 38 + 19 + 4 = 78$

a P (20 to 39 seconds) $= \dfrac{38}{78}$ $\doteqdot 0.487$

b P (> 60 seconds) $= \dfrac{4}{78}$ $\doteqdot 0.051$

c P (between 20 and 59 sec (inc)) $= \dfrac{38+19}{78}$ $\doteqdot 0.731$

3

Calls/day	*No. of days*
0	2
1	7
2	11
3	8
4	7
5	4
6	3
7	0
8	1

a Survey lasted $2 + 7 + 11 + 8 + 7 + 4 + 3 + 0 + 1$ $= 43$ days

b **i** P (0 calls) $= \dfrac{2}{43}$ $\doteqdot 0.047$

ii P ($\geqslant$ 5 calls) $= \dfrac{4+3+0+1}{43}$ $\doteqdot 0.186$

iii P (< 3 calls) $= \dfrac{2+7+11}{43}$ $\doteqdot 0.465$

4 Total frequency $= 37 + 81 + 48 + 17 + 6 + 1$ $= 190$

a P (4 days gap) $= \dfrac{17}{190}$ $\doteqdot 0.089$

b P (at least 4 days gap) $= \dfrac{17+6+1}{190}$ $\doteqdot 0.126$

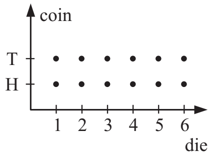
EXERCISE 19B

1 **a** {A, B, C, D} **b** {BB, BG, GB, GG}

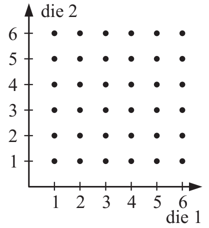
c {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

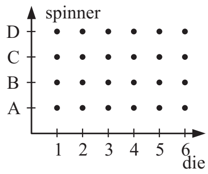
2 **a**



b



c



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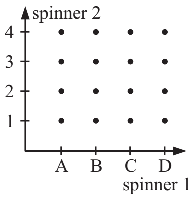
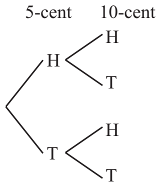


3 **a**



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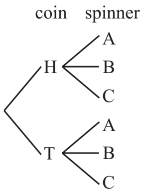
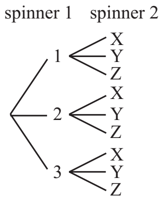
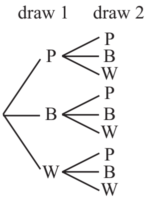


c



d



EXERCISE 19C

1 Total number of marbles $= 5 + 3 + 7 = 15$

a P (red) $= \frac{3}{15}$ $= \frac{1}{5}$

b P (green) $= \frac{5}{15}$ $= \frac{1}{3}$

c P (blue) $= \frac{7}{15}$

d P (not red) $= \dfrac{5+7}{15}$ $= \frac{12}{15}$ or $= \frac{4}{5}$

e P (neither green nor blue) $=$ P (red) $= \frac{1}{5}$

f P (green or red) $= \dfrac{5+3}{15} = \frac{8}{15}$

2 **a** 8 are brown and so 4 are white.

b **i** P (brown) $= \frac{8}{12}$ $= \frac{2}{3}$

ii P (white) $= \frac{4}{12}$ $= \frac{1}{3}$

3 **a** P (multiple of 4) $=$ P (4, 8, 12, 16, 20, 24, 28, 32, 36) $= \frac{9}{36}$ $= \frac{1}{4}$

b P (between 6 and 9 inclusive) $=$ P (6, 7, 8 or 9) $= \frac{4}{36}$ $= \frac{1}{9}$

c P (> 20) $=$ P (21, 22, 23, 24, ..., 35, 36) $= \dfrac{36-20}{36}$ $= \frac{16}{36}$ $= \frac{4}{9}$

d P (9) $= \frac{1}{36}$

e P (multiple of 13) $=$ P (13 or 26) $= \frac{2}{36}$ $= \frac{1}{18}$

f P (odd multiple of 3) $=$ P (3, 9, 15, 21, 27, or 33) $= \frac{6}{36}$ $= \frac{1}{6}$

4 **a** P (on Tuesday) $= \frac{1}{7}$

b P (on a weekend) $= \frac{2}{7}$

c P (in July) $= \dfrac{4 \times 31}{365 \times 3 + 366}$ {over a 4 year period} $= \frac{124}{1461}$

d P (in January or February) $= \dfrac{4 \times 31 + 3 \times 28 + 1 \times 29}{3 \times 365 + 1 \times 366}$ {over a 4 year period} $= \frac{237}{1461}$

5 Let A denote Antti, K denote Kai and N denote Neda.
Possible orders are: {AKN, ANK, KAN, KNA, NAK, NKA}

a P (A in middle) $= \frac{2}{6}$ $= \frac{1}{3}$

b P (A at left end) $= \frac{2}{6}$ $= \frac{1}{3}$

c P (A at right end) $= \frac{2}{6}$ $= \frac{1}{3}$

d P (K and N are together) $= \frac{4}{6} = \frac{2}{3}$

6 Let G denote 'a girl' and B denote 'a boy'.

a Possible orders are: {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

b **i** P (all boys) $= \text{P}(\text{BBB}) = \frac{1}{8}$

ii P (all girls) $= \text{P}(\text{GGG}) = \frac{1}{8}$

iii P (boy, then girl, then girl) $= \text{P}(\text{BGG}) = \frac{1}{8}$

iv P (2 girls and a boy) $= \text{P}(\text{GGB or GBG or BGG}) = \frac{3}{8}$

v P (girl is eldest) $= \text{P}(\text{GGG or GBG or GBB or GGB}) = \frac{4}{8} = \frac{1}{2}$

vi P (at least one boy) $= \frac{7}{8}$ {all except GGG}

7 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

b **i** P (A sits on one end) $= \frac{12}{24} = \frac{1}{2}$

ii P (B sits on one of the two middle seats) $= \frac{12}{24} = \frac{1}{2}$

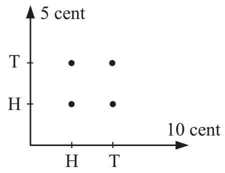
iii P (A and B are together) $= \frac{12}{24} = \frac{1}{2}$

iv P (A, B and C are together) $= \frac{12}{24} = \frac{1}{2}$

8 P (hits the bulls-eye) $= \dfrac{\text{area of bulls-eye}}{\text{area of whole target}} = \dfrac{\pi \times 20^2}{\pi \times 30^2} = \dfrac{400\pi}{900\pi} = \frac{4}{9}$

EXERCISE 19D

1



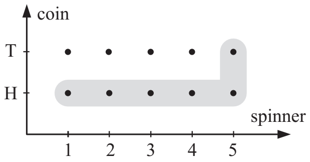
a P (2 heads) $= \frac{1}{4}$

b P (2 tails) $= \frac{1}{4}$

c P (exactly 1 head) $= \text{P}(\text{HT or TH}) = \frac{2}{4}$ or $\frac{1}{2}$

d P (at least one H) $= \text{P}(\text{HT or TH or HH}) = \frac{3}{4}$

2 **a**



b There are $2 \times 5 = 10$ possible outcomes.

c **i** P (T and 3) $= \frac{1}{10}$

ii P (H and even) $= \text{P}(\text{H2 or H4}) = \frac{2}{10}$ i.e., $\frac{1}{5}$

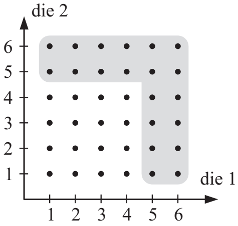
iii P (an odd) $= \text{P}(\text{H1, T1, H3, T3, H5, T5}) = \frac{6}{10} = \frac{3}{5}$

iv P (H or 5) $= \frac{6}{10} = \frac{3}{5}$ {those encircled}

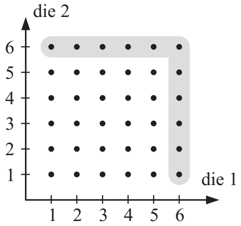
3 **a** P(two 3s)
$= \text{P}((3, 3))$
$= \frac{1}{36}$

b P(5 and a 6)
$= \text{P}((5, 6), (6, 5))$
$= \frac{2}{36}$
$= \frac{1}{18}$

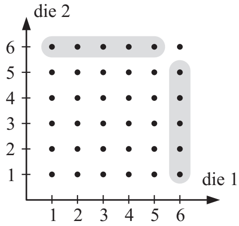
c P(5 or a 6)
$= \frac{20}{36}$
$= \frac{5}{9}$



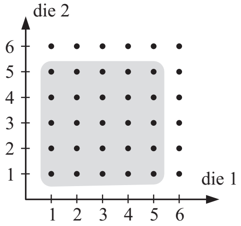
d P(at least one 6)
$= \frac{11}{36}$



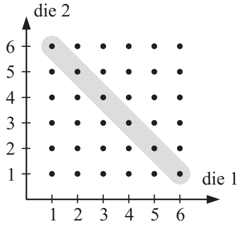
e P(exactly one 6)
$= \frac{10}{36}$
$= \frac{5}{18}$



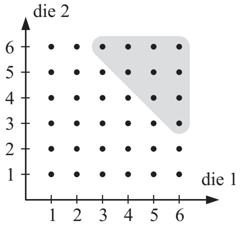
f P(no sixes)
$= \frac{25}{36}$



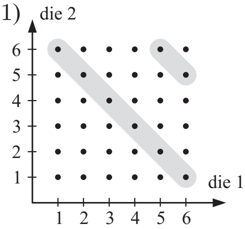
g P(sum of 7)
$= \frac{6}{36}$
$= \frac{1}{6}$



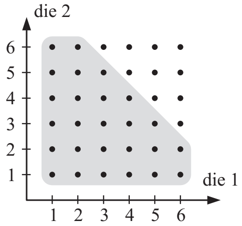
h P(sum > 8)
$= \frac{10}{36}$
$= \frac{5}{18}$



i P(sum of 7 or 11)
$= \dfrac{6+2}{36}$
$= \frac{2}{9}$



j P(sum no more than 8)
$= \text{P}(\text{sum} \leqslant 8)$
$= \frac{26}{36}$
$= \frac{13}{18}$



EXERCISE 19E.1

1 **a** P(rains on any one day)
$= \frac{6}{7}$

b P(rains on 2 successive days)
$= \text{P}(\text{R and R})$
$= \frac{6}{7} \times \frac{6}{7}$
$= \frac{36}{49}$

c P(rains on 3 successive days)
$= \text{P}(\text{R and R and R})$
$= \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7}$ or $\frac{216}{343}$

2 **a** P(H, then H, then H)
$= \text{P}(\text{H and H and H})$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b P(T, then H, then T)
$= \text{P}(\text{T and H and T})$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

3 Let A be the event of photocopier A malfunctioning and
B be the event of photocopier B malfunctioning.

a P (both malfunction)
$= \text{P}(\text{A and B})$
$= 0.08 \times 0.12$
$= 0.0096$

b P (both work)
$= \text{P}(\text{A}' \text{ and B}')$
$= 0.92 \times 0.88$
$= 0.8096$

4 **a** P (they will be happy)
$= \text{P}(\text{B, then G, then B, then G})$
$= \text{P}(\text{B and G and B and G})$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{16}$

b P (they will be unhappy)
$= 1 - \text{P}(\text{they will be happy})$
$= 1 - \frac{1}{16}$
$= \frac{15}{16}$

5 Let J be the event of Jiri hitting the target and
B be the event of Benita hitting the target.

a P (both hit)
$= \text{P}(\text{JB})$
$= 0.7 \times 0.8$
$= 0.56$

b P (both miss)
$= \text{P}(\text{J}'\text{B}')$
$= 0.3 \times 0.2$
$= 0.06$

c P (J hits and B misses)
$= \text{P}(\text{JB}')$
$= 0.7 \times 0.2$
$= 0.14$

d P (B hits and J misses) $= \text{P}(\text{BJ}') = 0.8 \times 0.3 = 0.24$

6 Let H be the event the archer hits the target.
$\therefore \text{P}(\text{H}) = \frac{2}{5}, \quad \text{P}(\text{H}') = \frac{3}{5}$

a P (3 hits)
$= \text{P}(\text{HHH})$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$
$= \frac{8}{125}$

b P (2 hits then a miss)
$= \text{P}(\text{HHH}')$
$= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5}$
$= \frac{12}{125}$

c P (all misses)
$= \text{P}(\text{H}'\text{H}'\text{H}')$
$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$
$= \frac{27}{125}$

EXERCISE 19E.2

1 **a** P (both red)
$= \text{P}(\text{RR})$
$= \frac{7}{10} \times \frac{6}{9}$
$= \frac{7}{15}$

b P (GR)
$= \frac{3}{10} \times \frac{7}{9}$
$= \frac{7}{30}$

c P (a green and a red)
$= \text{P}(\text{GR or RG})$
$= \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{3}{9}$
$= \frac{7}{15}$

2 **a** P (all strawberry creams)
$= \text{P}(\text{SSS})$
$= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$
$= \frac{14}{55}$

b P (none is a strawberry cream)
$= \text{P}(\text{S}'\text{S}'\text{S}')$
$= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$
$= \frac{1}{55}$

3 **a** P (wins first prize)
$= \frac{3}{100}$

b P (wins 1st and 2nd)
$= \text{P}(\text{WW})$
$= \frac{3}{100} \times \frac{2}{99}$
$\doteqdot 0.000\,606$

c P (wins all 3)
$= \text{P}(\text{WWW})$
$= \frac{3}{100} \times \frac{2}{99} \times \frac{1}{98}$
$\doteqdot 0.000\,006\,18$

d P (wins none of them)
$= \text{P}(\text{W}'\text{W}'\text{W}')$
$= \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98}$
$\doteqdot 0.912$

4 **a**

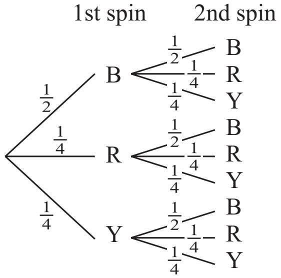
$$\begin{aligned}&\text{P (contains the captain)}\\&= \text{P}(\text{CC}'\text{C}' \text{ or } \text{C}'\text{CC}' \text{ or } \text{C}'\text{C}'\text{C})\\&= \tfrac{1}{7}\times\tfrac{6}{6}\times\tfrac{5}{5}+\tfrac{6}{7}\times\tfrac{1}{6}\times\tfrac{5}{5}+\tfrac{6}{7}\times\tfrac{5}{6}\times\tfrac{1}{5}\\&= 3\left(\tfrac{30}{7\times6\times5}\right)\\&= \tfrac{3}{7}\end{aligned}$$

b

$$\begin{aligned}&\text{P (contains captain and vice captain)}\\&= \text{P}(\text{CVO or COV or VCO or VOC}\\&\qquad \text{or OCV or OVC})\\&= \tfrac{1}{7}\times\tfrac{1}{6}\times\tfrac{5}{5}+\tfrac{1}{7}\times\tfrac{5}{6}\times\tfrac{1}{5}+\tfrac{1}{7}\times\tfrac{1}{6}\times\tfrac{5}{5}\\&\quad+\tfrac{1}{7}\times\tfrac{5}{6}\times\tfrac{1}{5}+\tfrac{5}{7}\times\tfrac{1}{6}\times\tfrac{1}{5}+\tfrac{5}{7}\times\tfrac{1}{6}\times\tfrac{1}{5}\\&= 6\left(\tfrac{5}{7\times6\times5}\right)\\&= \tfrac{1}{7}\end{aligned}$$

EXERCISE 19F

1 **a**



b

$$\begin{aligned}&\text{P (both black)}\\&= \text{P (BB)}\\&= \tfrac{1}{2}\times\tfrac{1}{2}\\&= \tfrac{1}{4}\end{aligned}$$

c

$$\begin{aligned}&\text{P (both yellow)}\\&= \text{P (YY)}\\&= \tfrac{1}{4}\times\tfrac{1}{4}\\&= \tfrac{1}{16}\end{aligned}$$

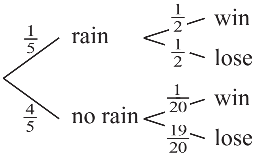
d

$$\begin{aligned}&\text{P (both different)}\\&= \text{P (BR or BY or RB or RY or YB or YR)}\\&= \tfrac{1}{2}\times\tfrac{1}{4}+\tfrac{1}{2}\times\tfrac{1}{4}+\tfrac{1}{4}\times\tfrac{1}{2}+\tfrac{1}{4}\times\tfrac{1}{4}+\tfrac{1}{4}\times\tfrac{1}{2}+\tfrac{1}{4}\times\tfrac{1}{4}\\&= \tfrac{4}{8}+\tfrac{2}{16}\\&= \tfrac{5}{8}\end{aligned}$$

e

$$\begin{aligned}&\text{P (B appears on either spin)}\\&= \text{P (BB or BR or BY or RB or YB)}\\&= \tfrac{1}{2}\times\tfrac{1}{2}+\tfrac{1}{2}\times\tfrac{1}{4}+\tfrac{1}{2}\times\tfrac{1}{4}+\tfrac{1}{4}\times\tfrac{1}{2}+\tfrac{1}{4}\times\tfrac{1}{2}\\&= 4(\tfrac{1}{8})+\tfrac{1}{4}\\&= \tfrac{3}{4}\end{aligned}$$

2

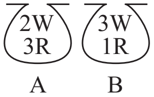


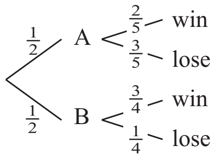
$$\begin{aligned}&\text{P (M wins)}\\&= \text{P (rain and win or no rain and win)}\\&= \tfrac{1}{5}\times\tfrac{1}{2}+\tfrac{4}{5}\times\tfrac{1}{20}\\&= \tfrac{1}{10}\times\tfrac{10}{10}+\tfrac{4}{100}\\&= \tfrac{14}{100}\\&= \tfrac{7}{50}\end{aligned}$$

3 P (next is spoiled)

$$\begin{aligned}&= \text{P (from A and spoiled or from B and spoiled)}\\&= 0.4\times0.05+0.6\times0.02\\&= 0.020+0.012\\&= 0.032 \qquad (3.2\%)\end{aligned}$$

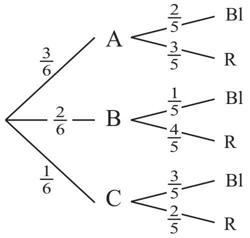
4





$$\begin{aligned}&\text{P (red)}\\&= \text{P (A and red or B and red)}\\&= \tfrac{1}{2}\times\tfrac{3}{5}\;+\;\tfrac{1}{2}\times\tfrac{1}{4}\\&= \tfrac{3}{10}+\tfrac{1}{8}\\&= \tfrac{17}{40}\end{aligned}$$

5



a $$\begin{aligned} \text{P (blue)} &= \text{P (A and Bl or B and Bl or C and Bl)} \\ &= \tfrac{3}{6} \times \tfrac{2}{5} + \tfrac{2}{6} \times \tfrac{1}{5} + \tfrac{1}{6} \times \tfrac{3}{5} \\ &= \tfrac{11}{30} \end{aligned}$$

b $$\begin{aligned} \text{P (red)} &= 1 - \text{P (blue)} \\ &= 1 - \tfrac{11}{30} \\ &= \tfrac{19}{30} \end{aligned}$$

EXERCISE 19G

1

a $$\begin{aligned} &\text{P (different colours)} \\ &= \text{P (PG or GP)} \\ &= \tfrac{2}{7} \times \tfrac{5}{7} + \tfrac{5}{7} \times \tfrac{2}{7} \\ &= \tfrac{20}{49} \end{aligned}$$

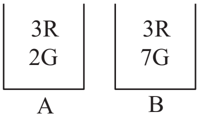
b $$\begin{aligned} &\text{P (different colours)} \\ &= \text{P (PG or GP)} \\ &= \tfrac{2}{7} \times \tfrac{5}{6} + \tfrac{5}{7} \times \tfrac{2}{6} \\ &= \tfrac{20}{42} \quad \text{i.e.,} \quad \tfrac{10}{21} \end{aligned}$$

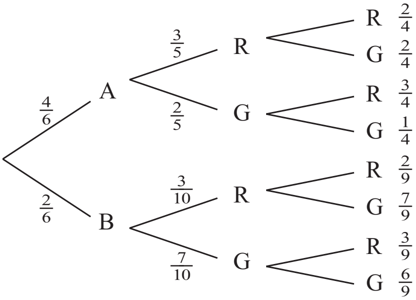
2 **a** $$\begin{aligned} &\text{P (both odd)} \\ &= \text{P (odd and odd)} \\ &= \tfrac{3}{5} \times \tfrac{2}{4} \\ &= \tfrac{3}{10} \end{aligned}$$

b $$\begin{aligned} &\text{P (both even)} \\ &= \text{P (even and even)} \\ &= \tfrac{2}{5} \times \tfrac{1}{4} \\ &= \tfrac{1}{10} \end{aligned}$$

c $$\begin{aligned} &\text{P (one odd and other even)} \\ &= 1 - \text{P (both odd)} - \text{P (both even)} \\ &= 1 - \tfrac{3}{10} - \tfrac{1}{10} \\ &= \tfrac{6}{10} \\ &= \tfrac{3}{5} \end{aligned}$$

3

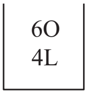


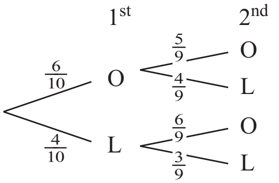


a $$\begin{aligned} &\text{P (both green)} \\ &= \text{P (AGG or BGG)} \\ &= \tfrac{4}{6} \times \tfrac{2}{5} \times \tfrac{1}{4} + \tfrac{2}{6} \times \tfrac{7}{10} \times \tfrac{6}{9} \\ &= \tfrac{1}{15} + \tfrac{7}{45} \\ &= \tfrac{10}{45} \\ &= \tfrac{2}{9} \end{aligned}$$

b $$\begin{aligned} &\text{P (different in colour)} \\ &= 1 - \text{P (both green)} - \text{P (both red)} \\ &= 1 - \tfrac{2}{9} - \text{P (ARR or BRR)} \\ &= \tfrac{7}{9} - \left(\tfrac{4}{6} \times \tfrac{3}{5} \times \tfrac{2}{4} + \tfrac{2}{6} \times \tfrac{3}{10} \times \tfrac{2}{9}\right) \\ &= \tfrac{7}{9} - \left(\tfrac{1}{5} + \tfrac{1}{45}\right) \\ &= \tfrac{5}{9} \end{aligned}$$

4





a $$\begin{aligned} &\text{P (both O)} \\ &= \tfrac{6}{10} \times \tfrac{5}{9} \\ &= \tfrac{1}{3} \end{aligned}$$

b $$\begin{aligned} &\text{P (both L)} \\ &= \tfrac{4}{10} \times \tfrac{3}{9} \\ &= \tfrac{2}{15} \end{aligned}$$

c $$\begin{aligned} &\text{P (OL)} \\ &= \tfrac{6}{10} \times \tfrac{4}{9} \\ &= \tfrac{4}{15} \end{aligned}$$

d $$\begin{aligned} &\text{P (LO)} \\ &= \tfrac{4}{10} \times \tfrac{6}{9} \\ &= \tfrac{4}{15} \end{aligned}$$

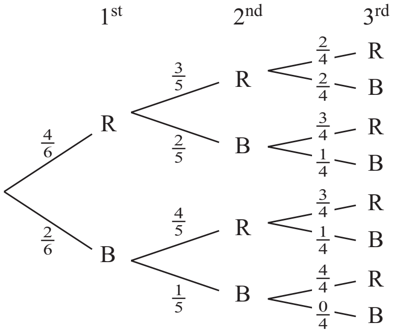
$\frac{1}{3}+\frac{2}{15}+\frac{4}{15}+\frac{4}{15}$

$=\frac{5}{15}+\frac{2}{15}+\frac{4}{15}+\frac{4}{15}$

$=\frac{15}{15}$ which is 1

The answer must be 1 as the four categories **a**, **b**, **c**, **d** are all the possibilities that could occur.

5



a P(all red)

$= \mathrm{P(RRR)}$

$=\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}$

$=\frac{1}{5}$

b P(only two are red)

$= \mathrm{P(RRB\ or\ RBR\ or\ BRR)}$

$=\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}+\frac{4}{6}\times\frac{2}{5}\times\frac{3}{4}+\frac{2}{6}\times\frac{4}{5}\times\frac{3}{4}$

$=3\times\left(\frac{24}{6\times5\times4}\right)$

$=\frac{3}{5}$

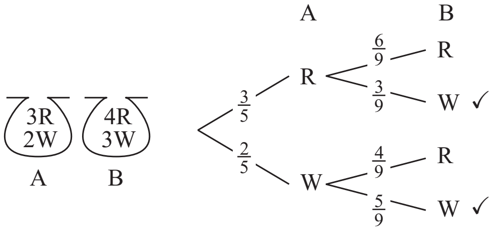
c P(at least two are red)

$= \mathrm{P}$(all red or only two are red)

$=\frac{1}{5}+\frac{3}{5}$ {from **a** and **b**}

$=\frac{4}{5}$

6

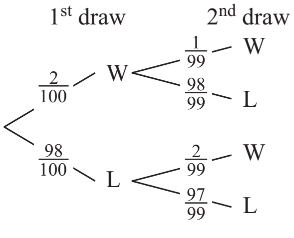


P(marble from B is W) $= \mathrm{P(RW\ or\ WW)}$ {paths ticked}

$=\frac{3}{5}\times\frac{3}{9}+\frac{2}{5}\times\frac{5}{9}$

$=\frac{19}{45}$

7



a P(wins both)

$= \mathrm{P(WW)}$

$=\frac{2}{100}\times\frac{1}{99}$

$\doteqdot 0.000\,202$

b P(wins neither)

$= \mathrm{P(LL)}$

$=\frac{98}{100}\times\frac{97}{99}$

$\doteqdot 0.960$

c P(wins at least one prize) $= 1-\mathrm{P}$(wins neither)

$=1-\frac{98}{100}\times\frac{97}{99}$

$\doteqdot 0.0398$

EXERCISE 19H

1 **a** $(p+q)^4=p^4+4p^3q+6p^2q^2+4pq^3+q^4$

b P(3 heads) $=4p^3q$

$=4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$ {as $p=q=\frac{1}{2}$}

$=\frac{1}{4}$

2 **a** $(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$

b P(4H and 1T)
$= 5p^4q$
$= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)$
$= \frac{5}{32}$

c P(2H and 3T)
$= 10p^2q^3$
$= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$
$= \frac{10}{32}$
$= \frac{5}{16}$

3 **a** $\left(\frac{2}{3}+\frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$

b $\mathrm{P(S)} = \frac{2}{3}$, $\mathrm{P(S')} = \frac{1}{3}$ S′ represents a non-strawberry cream (or an almond centre)

i P(all S)
$= \left(\frac{2}{3}\right)^4$
$= \frac{16}{81}$

ii P(two of each)
$= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$
$= \frac{8}{27}$

iii P(at least 2 strawberry creams)
= P(all S or 3S, 1T or 2S, 2T)
$= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$
$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81}$
$= \frac{72}{81}$
$= \frac{8}{9}$

4 **a** $\left(\frac{3}{4}+\frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$

b **i** P(2 ‘flat backs’)
= P(2Fs and 3F′s)
$= 10 \times \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$
$= \frac{135}{512}$

ii P(at least 3 are ‘flat backs’)
= P(3F, 2F′ or 4F, 1F′ or 5F)
$= 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$
$= \frac{53}{512}$ on simplifying

5 X is Bin (4, 0.8)

a P $(X = 2)$
= binompdf (4, 0.8, 2)
$\doteqdot$ 0.154

b $P(X \geqslant 2)$
$= 1 - P(X \leqslant 1)$
= 1 − binomcdf (4, 0.8, 1)
$\doteqdot$ 0.973

6 X is Bin (6, 0.05)

a P $(X = 2)$
= binompdf (6, 0.05, 2)
$\doteqdot$ 0.0305

b P $(X \geqslant 1)$
$= 1 - \mathrm{P}\ (X = 0)$
= 1 − binompdf (6, 0.05, 0)
= 0.265

7 X is Bin (10, 0.2) P (Raj passes) $= \mathrm{P}\ (X \geqslant 7)$
$= 1 - \mathrm{P}\ (X \leqslant 6)$
= 1 − binomcdf (10, 0.2, 6)
$\doteqdot$ 0.000 864 {i.e., about 9 in 10 000}

8 P(M wins a game against J) $= \frac{2}{3}$ i.e., P(M wins) $= \frac{2}{3}$ P(J wins) $= \frac{1}{3}$

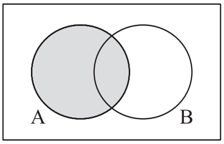
P(J wins a set 6 games to 4)

$$= \mathrm{P}(\underbrace{\text{J wins 5 of the first 9 games}}_{\text{this is Bin }(9,\ \frac{1}{3})}\ \textbf{and}\ \text{J wins the 10th game})$$

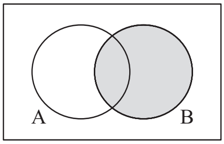
$= \mathrm{Binompdf}\ (9, \frac{1}{3}, 5) \times \frac{1}{3}$
$\doteqdot$ 0.0341

EXERCISE 19I

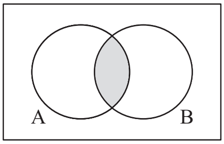
1 **a**



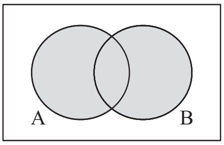
b



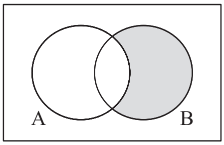
c



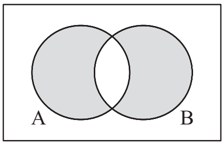
d



e



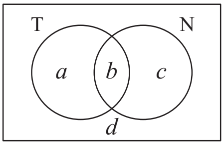
f



2 **a** Total number in the class $= 3+4+5+17 = 29$
b Number who study both $= 17$ {the intersection}
c Number who study at least one $= 5+17+4 = 26$ {the union}
d Number who study only Chem. $= 5$

3 **a** Total number in the survey $= 37+9+15+4 = 65$
b Number who liked both $= 9$ {the intersection}
c Number who liked neither $= 4$
d Number who liked exactly one $= 37+15 = 52$

4

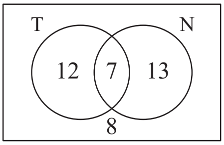


T represents those playing tennis
N represents those playing netball

$$\begin{aligned} \therefore \quad a+b+c+d &= 40 \\ a+b &= 19 \\ b+c &= 20 \\ d &= 8 \end{aligned}$$

$$\begin{aligned} \text{So,} \quad a+b+c &= 32 \\ \therefore \quad 19+c &= 32 \quad \text{and} \quad a+20 = 32 \\ \therefore \quad c &= 13 \quad \text{and} \quad a = 12 \end{aligned}$$

$$\begin{aligned} \text{Hence,} \quad 12+b+13+8 &= 40 \\ \therefore \quad b &= 7 \end{aligned}$$



a P (plays tennis)
$$= \frac{12+7}{40}$$
$$= \tfrac{19}{40}$$

b P (does not play netball)
$$= \frac{12+8}{40}$$
$$= \tfrac{1}{2}$$

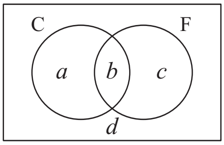
c P (plays at least one)
$$= \frac{12+7+13}{40}$$
$$= \tfrac{32}{40}$$
$$= \tfrac{4}{5}$$

d P (plays one and only one)
$$= \frac{12+13}{40}$$
$$= \tfrac{25}{40}$$
$$= \tfrac{5}{8}$$

e P (plays netball, but not tennis)
$$= \tfrac{13}{40}$$

f P (plays tennis given plays netball)
$$= \frac{7}{7+13}$$
$$= \tfrac{7}{20}$$

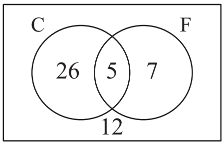
5



C represents men who gave chocolates
F represents men who gave flowers

$$\begin{aligned}\therefore\quad a+b+c+d &= 50\\ a+b &= 31\\ b+c &= 12\\ b &= 5\end{aligned}$$

Thus $c=7$, $a=26$ and $26+5+7+d=50$ $\therefore$ $d=12$



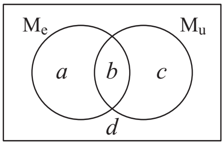
a P(C or F)
$= \frac{26+5+7}{50}$
$= \frac{38}{50}$ i.e., $\frac{19}{25}$

b P(C but not F)
$= \frac{26}{50}$
$= \frac{13}{25}$

c P(neither C nor F)
$= \frac{12}{50}$
$= \frac{6}{25}$

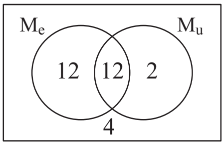
d P(F given that C$'$)
$= \frac{7}{7+12}$
$= \frac{7}{19}$

6



$$\begin{aligned}a+b+c+d &= 30\\ a+b &= 24\\ b &= 12\\ a+b+c &= 26\end{aligned}$$

$$\begin{aligned}\therefore\quad 26+d &= 30 \quad \text{i.e.,}\quad d=4\\ 24+c &= 26 \quad \text{i.e.,}\quad c=2\\ \text{and}\quad a+12+2 &= 26 \quad \therefore\quad a=12\end{aligned}$$



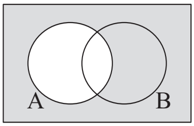
a P(Mu)
$= \frac{14}{30}$
$= \frac{7}{15}$

b P(Mu, but not Me)
$= \frac{2}{30}$
$= \frac{1}{15}$

c P(neither Mu nor Me)
$= \frac{4}{30}$
$= \frac{2}{15}$

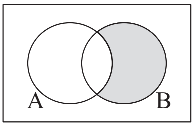
d P(Me given Mu)
$= \frac{12}{14}$
$= \frac{6}{7}$

7 **a**



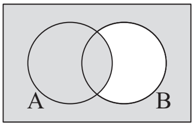
A'

b



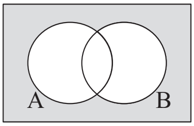
$A'\cap B$

c



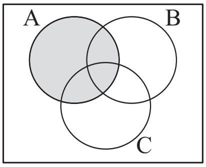
$A\cup B'$

d

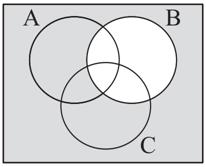


$A'\cap B'$

8 **a**



b



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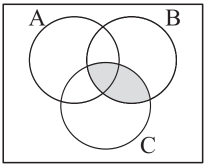

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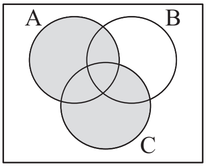

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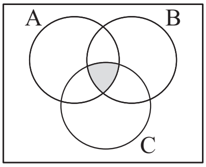

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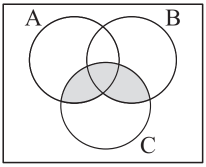
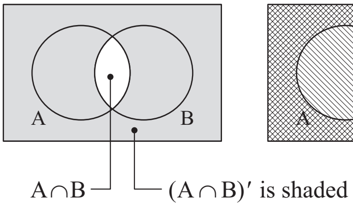
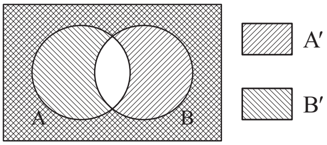


9 **a**

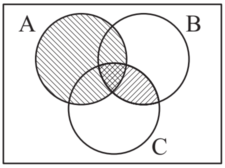


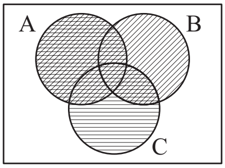


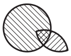
So $A' \cup B'$ is the region containing either type of shading.

Thus, as the regions are the same, $(A \cap B)' = A' \cup B'$ is verified.

b





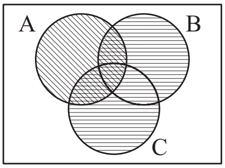
$A \cup (B \cap C)$ consists of the shaded region 

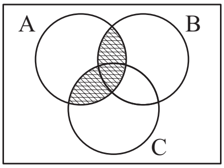
$(A \cup B) \cap (A \cup C)$ consists of the 'double shaded' region.

As the two regions are identical

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ is verified.

c





$A \cap (B \cup C)$ consists of the double shaded region 

$(A \cap B) \cup (A \cap C)$ consists of the region shaded. (all forms ▨ and ▨)

As the regions are identical, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is verified.

10 **a** $A = \{7, 14, 21, 28, 35, ..., 98\}$

$B = \{5, 10, 15, 20, 25, ..., 95\}$

i as $98 = 7 \times 14$, $n(A) = 14$ **ii** as $95 = 5 \times 19$, $n(B) = 19$

iii $A \cap B = \{35, 70\}$ $\therefore$ $n(A \cap B) = 2$

iv $A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50, 55, 56, 60, 63, 65, 70, 75, 77, 80, 84, 85, 90, 91, 95, 98\}$

$\therefore$ $n(A \cup B) = 31$

b
$$\begin{aligned} n(A) + n(B) - n(A \cap B) &= 14 + 19 - 2 \\ &= 31 \\ &= n(A \cup B) \quad \checkmark \end{aligned}$$

11 **a** **i**
$$\begin{aligned} &P(B) \\ &= \frac{n(B)}{n(S)} \\ &= \frac{b+c}{a+b+c+d} \end{aligned}$$

ii
$$\begin{aligned} &P(A \text{ and } B) \\ &= \frac{n(A \cap B)}{n(S)} \\ &= \frac{b}{a+b+c+d} \end{aligned}$$

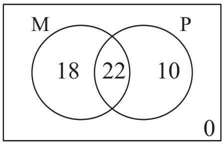
iii
$$\begin{aligned} &P(A \text{ or } B) \\ &= \frac{n(A \cup B)}{n(S)} \\ &= \frac{a+b+c}{a+b+c+d} \end{aligned}$$

iv
$$\begin{aligned} P(A) + P(B) - P(A \text{ and } B) &= \frac{a+b+b+c-b}{a+b+c+d} \\ &= \frac{a+b+c}{a+b+c+d} \end{aligned}$$

b P(A or B) = P(A) + P(B) − P(A and B) {using **iii** and **iv**}

EXERCISE 19J

1 **a**



So 22 study both.

b **i** P(M but not P)

$= \frac{18}{50}$

$= \frac{9}{25}$

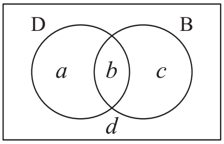
ii P(P given M)

$= \dfrac{22}{18+22}$

$= \frac{22}{40}$

$= \frac{11}{20}$

2



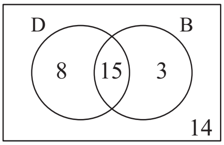
$$\begin{aligned} a+b+c+d &= 40 \quad .... (1)\\ a+b &= 23 \quad .... (2)\\ b+c &= 18 \quad .... (3)\\ a+b+c &= 26 \quad .... (4) \end{aligned}$$

$\therefore\ d = 14$ {using (1) and (4)}

$23 + c = 26$ and $a + 18 = 26$

$\therefore\ c = 3$ and $a = 8$

Thus $b = 18 - c = 15$



a P(D and B)

$= \frac{15}{40}$

$= \frac{3}{8}$

b P(neither D nor B)

$= \frac{14}{40}$

$= \frac{7}{20}$

c P(D, but not B)

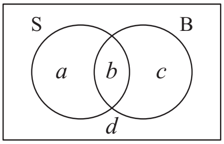
$= \frac{8}{40}$

$= \frac{1}{5}$

d P(B given D)

$= \frac{15}{23}$

3

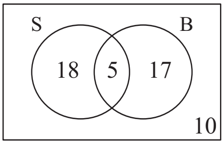


$$\begin{aligned} a+b+c+d &= 50\\ a+b &= 23\\ b+c &= 22\\ b &= 5 \end{aligned}$$

$\therefore\ c = 17,\ a = 18$

and $18 + 5 + 17 + d = 50$

$\therefore\ d = 10$



a P(not B)

$= \text{P}(\text{B}')$

$= \frac{28}{50}$

$= \frac{14}{25}$

b P(B or S)

$= \dfrac{18+5+17}{50}$

$= \frac{40}{50}$

$= \frac{4}{5}$

c P(neither B nor S)

$= \frac{10}{50}$

$= \frac{1}{5}$

d P(B, given S)

$= \dfrac{5}{18+5}$

$= \frac{5}{23}$

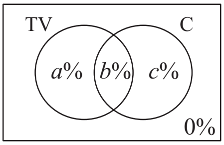
e P(S, given B$'$)

$= \dfrac{18}{18+10}$

$= \frac{18}{28}$

$= \frac{9}{14}$

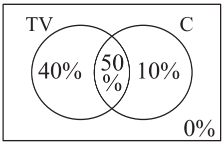
4



$$\begin{aligned} a+b+c &= 100\\ a+b &= 90\\ b+c &= 60 \end{aligned}$$

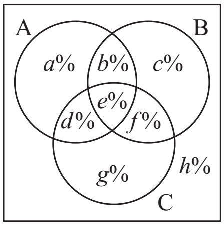
$\therefore\ c = 10$ and $a = 40$

$\therefore\ b = 50$



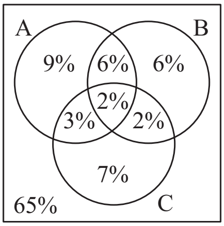
P(TV, given C) $= \dfrac{50}{50+10} = \frac{5}{6}$

5



$$\begin{aligned} a+b+c+d+e+f+g+h &= 100 \\ a+b+d+e &= 20 \\ b+c+e+f &= 16 \\ d+e+f+g &= 14 \\ b+e &= 8 \\ d+e &= 5 \\ e+f &= 4 \\ e &= 2 \end{aligned}$$

i.e., $e=2$, $f=2$, $d=3$, $b=6$, $\left\{\begin{aligned} a+6+3+2 &= 20 \\ 6+c+2+2 &= 16 \\ 3+2+2+g &= 14 \end{aligned}\right. \quad \therefore \quad \left\{\begin{aligned} a &= 9 \\ c &= 6 \\ g &= 7 \end{aligned}\right.$



a P (none)
$= \frac{65}{100}$
$= \frac{13}{20}$

b P (at least one)
$= 1 - \text{P (none)}$
$= 1 - \frac{13}{20}$
$= \frac{7}{20}$

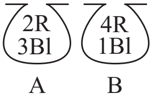
c P (exactly one)
$= \dfrac{9+6+7}{100}$
$= \frac{22}{100}$
$= \frac{11}{50}$

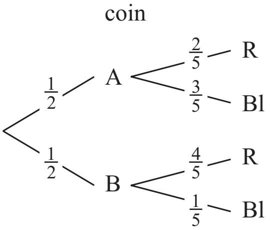
d P (A or B)
$= \frac{9+6+6+3+2+2}{100}$
$= \frac{28}{100}$
$= \frac{7}{25}$

e P (A, given at least one)
$= \frac{9+6+2+3}{35}$
$= \frac{20}{35}$
$= \frac{4}{7}$

f P (C, given A or B or both)
$= \frac{3+2+2}{9+6+6+3+2+2}$
$= \frac{7}{28}$
$= \frac{1}{4}$

6

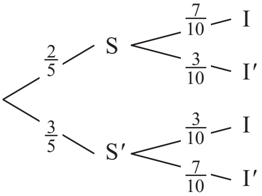




a $\text{P(R)} = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5}$
$= \frac{3}{5}$

b $\text{P(B|R)} = \dfrac{\text{P(B} \cap \text{R)}}{\text{P(R)}}$
$= \dfrac{\frac{1}{2} \times \frac{4}{5}}{\frac{3}{5}}$
$= \frac{2}{3}$

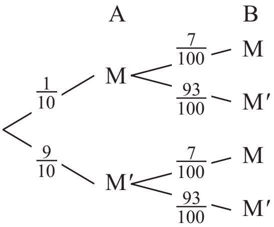
7



a $\text{P(I)} = \frac{2}{5} \times \frac{7}{10} + \frac{3}{5} \times \frac{3}{10}$
$= \frac{23}{50}$ (or 0.46)

b $\text{P(S|I)} = \dfrac{\text{P(S} \cap \text{I)}}{\text{P(I)}}$
$= \dfrac{\frac{2}{5} \times \frac{7}{10}}{\frac{23}{50}}$
$= \frac{14}{23}$

8



P(B | at least one malfunctions)

$= \dfrac{\text{P(B} \cap \text{at least one malfunctions)}}{\text{P(at least one malfunctions)}}$

$= \dfrac{\frac{1}{10} \times \frac{7}{100} + \frac{9}{10} \times \frac{7}{100}}{\frac{1}{10} \times \frac{7}{100} + \frac{1}{10} \times \frac{93}{100} + \frac{9}{10} \times \frac{7}{100}}$

$= \frac{7+63}{7+93+63}$

$= \frac{70}{163}$

9 $P(B) = 0.5,\quad P(G) = 0.6,\quad P(G\,|\,B) = 0.9$

a $P\,(\text{both eat})$

$= P(B \cap G)$

$= P(G\,|\,B) \times P(B) \qquad \left\{\text{as}\quad P(G\,|\,B) = \dfrac{P(G \cap B)}{P(B)}\right\}$

$= 0.9 \times 0.5$

$= 0.45$

b $P(B\,|\,G)$

$= \dfrac{P(B \cap G)}{P(G)}$

$= \dfrac{0.45}{0.6}$

$= 0.75$

c $P\,(\text{at least one eats})$

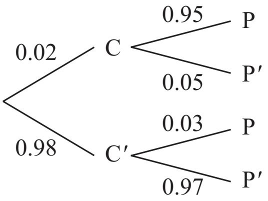
$= P(B \cup G)$

$= P(B) + P(G) - P(B \cap G)$

$= 0.5 + 0.6 - 0.45$

$= 0.65$

10



a $P(P)$

$= 0.02 \times 0.95 + 0.98 \times 0.03$

$= 0.0484$

b $P(C\,|\,P)$

$= \dfrac{P(C \cap P)}{P(P)}$

$= \dfrac{0.02 \times 0.95}{0.0484}$

$\doteqdot 0.3926$

11 The coins are H, H T, T and H, T.

Any one of these 6 faces could be seen uppermost, $\therefore\ P(\text{falls H}) = \frac{3}{6} = \frac{1}{2}$

Now $\displaystyle P(\text{HH coin}\,|\,\text{falls H}) = \frac{P(\text{HH coin} \cap \text{falls H})}{P(\text{falls H})}$

$\displaystyle = \frac{P(\text{HH})}{P(\text{falls H})}$

$= \dfrac{\frac{1}{3}}{\frac{1}{2}}$

$= \frac{2}{3}$

EXERCISE 19K

1 $P\,(R \cap S)$

$= P(R) + P(S) - P(R \cup S)$

$= 0.4 + 0.5 - 0.7$

$= 0.2$

Also, $P(R) \times P(S)$

$= 0.4 \times 0.5$

$= 0.2$

So, $P(R \cap S) = P(R) \times P(S) \;\Rightarrow\;$ R and S are independent events.

2 **a** $P(A \cap B)$

$= P(A) + P(B) - P(A \cup B)$

$= \frac{2}{5} + \frac{1}{3} - \frac{1}{2}$

$= \frac{7}{30}$

b $P(B|A)$

$= \dfrac{P(B \cap A)}{P(A)}$

$= \dfrac{\frac{7}{30}}{\frac{2}{5}}$

$= \frac{7}{12}$

c $P(A|B)$

$= \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{\frac{7}{30}}{\frac{1}{3}}$

$= \frac{7}{10}$

A and B are not independent as $P(A \cap B) = \frac{7}{30}$ whereas $P(A) \times P(B) = \frac{2}{15}$

i.e., $P(A \cap B)$

$\neq P(A) \times P(B)$

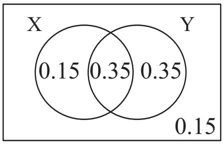
3 **a** As X and Y are independent

$$\begin{aligned} P(X \cap Y) &= P(X) \times P(Y) \\ &= 0.5 \times 0.7 \\ &= 0.35 \end{aligned}$$

i.e., P (both X and Y) $= 0.35$

b

$$\begin{aligned} &P(X \text{ or } Y) \\ &= P(X \cup Y) \\ &= P(X) + P(Y) - P(X \cap Y) \\ &= 0.5 + 0.7 - 0.35 \\ &= 0.85 \end{aligned}$$



c P (neither X nor Y)
$= 0.15$

d P (X but not Y)
$= 0.15$

e $P(X|Y) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{0.35}{0.70} = \frac{1}{2}$

4

$$\begin{aligned} &P\,(\text{at least one solves it}) \\ &= 1 - P\,(\text{no-one solves it}) \\ &= 1 - P\,(A' \text{ and } B' \text{ and } C') \\ &= 1 - \tfrac{2}{5} \times \tfrac{1}{3} \times \tfrac{1}{2} \\ &= 1 - \tfrac{1}{15} \\ &= \tfrac{14}{15} \end{aligned}$$

5 **a**

$$\begin{aligned} &P\,(\text{at least one } 6) \\ &= 1 - P(\text{no } 6\text{s}) \\ &= 1 - P(6' \text{ and } 6' \text{ and } 6') \\ &= 1 - \tfrac{5}{6} \times \tfrac{5}{6} \times \tfrac{5}{6} \\ &= 1 - \tfrac{125}{216} \\ &= \tfrac{91}{216} \end{aligned}$$

b

$$\begin{aligned} &P\,(\text{at least one } 6 \text{ in } n \text{ throws}) \\ &= 1 - (\tfrac{5}{6})^n \end{aligned}$$

So we want

$$\begin{aligned} 1 - (\tfrac{5}{6})^n &> 0.99 \\ \therefore\quad -(\tfrac{5}{6})^n &> -0.01 \\ \therefore\quad (\tfrac{5}{6})^n &< 0.01 \\ \therefore\quad n \log(\tfrac{5}{6}) &< \log(0.01) \\ \therefore\quad n &> \frac{\log(0.01)}{\log(\frac{5}{6})} \end{aligned}$$

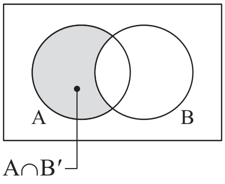
$\{\text{as } \log(\tfrac{5}{6}) < 0\}$

$$\begin{aligned} \therefore\quad n &> 25.2585.... \\ \text{i.e.,}\quad n &= 26 \end{aligned}$$

6 A and B are independent $\Rightarrow$ $P(A \cap B) = P(A)\,P(B)$ ...... (1)

Now

$$\begin{aligned} &P(A \cap B') \\ &= P(A \cup B) - P(B) \quad \{\text{see diagram}\} \\ &= P(A) + P(B) - P(A \cap B) - P(B) \\ &= P(A) - P(A \cap B) \\ &= P(A) - P(A)\,P(B) \quad \{\text{from (1)}\} \\ &= P(A)[1 - P(B)] \\ &= P(A) \times P(B') \end{aligned}$$



$\therefore$ A and B$'$ are also independent.

EXERCISE 19L

1 The total number of different committees is ${}_{11}C_4$. (**Note:** $C_4^{11} = {}_{11}C_4$)

The number of ways of X and Y being on a committee is ${}_2C_2 \times {}_9C_2$.

$$\begin{aligned} \therefore\quad P(\text{X and Y are on the committee}) &= \frac{{}_2C_2 \times {}_9C_2}{{}_{11}C_4} \\ &\doteqdot 0.109 \end{aligned}$$

2 AIDS and SAID are 2 of the 4! different orderings.

$\therefore$ P(AIDS or SAID) $= \dfrac{2}{4!} = \frac{1}{12}$

3 There are $_{12}C_7$ different teams that can be selected.

$\therefore$ P(captain and vice captain are chosen) $= \dfrac{_2C_2 \times {_{10}C_5}}{_{12}C_7} \doteqdot 0.318$

4 P(none of the golfers was killed) $= \dfrac{_3C_0 \times {_{19}C_4}}{_{22}C_4} \doteqdot 0.530$

5 $\boxed{5}\boxed{4}\boxed{3}\boxed{2}\boxed{1}$ i.e., there are 5! different possible seating arrangements.

a $\boxed{2}\boxed{4}\boxed{3}\boxed{2}\boxed{1}$ there are $2 \times 4!$ seating arrangements if K and U sit at the ends

$\therefore$ P(K and U sit at the ends) $= \dfrac{2 \times 4!}{5!} = \frac{2}{5}$

b K and U can sit together in 2! ways. They as a pair plus the other three people can then be ordered in 4! ways.

$\therefore$ P(sit together) $= \dfrac{2! \times 4!}{5!} = \frac{2}{5}$

6 There are $_{16}C_5$ different committees possible.

a P(all men) $= \dfrac{_9C_5 \times {_7C_0}}{_{16}C_5} = 0.0288$

b P(at least 3 men) $=$ P(3 men or 4 men or 5 men)

$= \dfrac{_9C_3 \times {_7C_2} + {_9C_4} \times {_7C_1} + {_9C_5} \times {_7C_0}}{_{16}C_5}$

$\doteqdot 0.635$

c P(at least one of each sex) $= 1 -$ P(no men or no women)

$= 1 - \dfrac{_9C_0 \times {_7C_5} + {_9C_5} \times {_7C_0}}{_{16}C_5}$

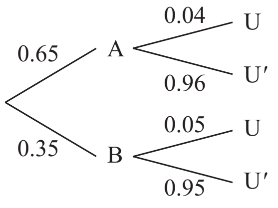
$\doteqdot 0.966$

7 If there are no restrictions there are 6! different orderings possible. A, B and C can be ordered in 3! ways. This triple together with the 3 others can be ordered in 4! ways.

$\therefore$ P(A, B, C together) $= \dfrac{3!4!}{6!} = \frac{1}{5}$

EXERCISE 19M

1



a P(underfilled)

$=$ P(A and U or B and U)

$= 0.65 \times 0.04 + 0.35 \times 0.05$

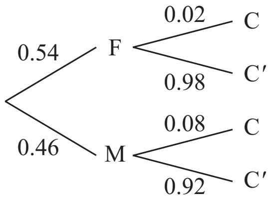
$= 0.0435$

b $\text{P}(\text{A}|\text{U}) = \dfrac{\text{P}(\text{A} \cap \text{U})}{\text{P}(\text{U})}$

$= \dfrac{0.65 \times 0.04}{0.0435}$

$\doteqdot 0.598$

2



a $\text{P}(\text{M}\,|\,\text{C}) = \dfrac{\text{P}(\text{C}\,|\,\text{M}) \times \text{P}(\text{M})}{\text{P}(\text{C}\,|\,\text{M}) \times \text{P}(\text{M}) + \text{P}(\text{C}\,|\,\text{F}) \times \text{P}(\text{F})}$

$= \dfrac{0.08 \times 0.46}{0.08 \times 0.46 + 0.02 \times 0.54}$

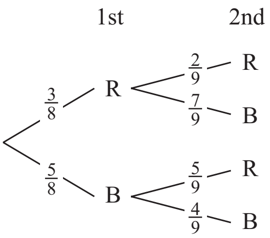
$\doteqdot 0.773$

b $\text{P}(\text{F}\,|\,\text{C}') = \dfrac{\text{P}(\text{C}'\,|\,\text{F}) \times \text{P}(\text{F})}{\text{P}(\text{C}'\,|\,\text{F}) \times \text{P}(\text{F}) + \text{P}(\text{C}'\,|\,\text{M}) \times \text{P}(\text{M})}$

$= \dfrac{0.98 \times 0.54}{0.98 \times 0.54 + 0.92 \times 0.46}$

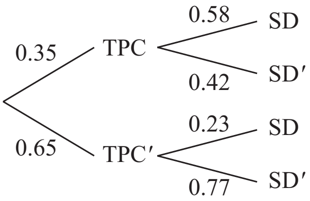
$\doteqdot 0.556$

3



$$\begin{aligned} &\text{P(BB} \mid \text{RR or BB)} \\ &= \frac{\text{P(BB} \cap \text{(RR or BB))}}{\text{P(RR or BB)}} \\ &= \frac{\text{P(BB)}}{\text{P(RR or BB)}} \\ &= \frac{\frac{5}{8} \times \frac{4}{9}}{\frac{3}{8} \times \frac{2}{9} + \frac{5}{8} \times \frac{4}{9}} \\ &= \tfrac{20}{26} \\ &= \tfrac{10}{13} \end{aligned}$$

4



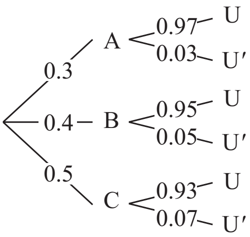
$$\begin{aligned} &\text{P(TPC}' \mid \text{SD)} \\ &= \frac{\text{P(TPC}' \cap \text{SD)}}{\text{P(SD)}} \\ &= \frac{0.65 \times 0.23}{0.35 \times 0.58 + 0.65 \times 0.23} \\ &\doteqdot 0.424 \end{aligned}$$

5 **a**

$$\begin{aligned} \text{P(A)} &= \text{P(A and in } C_1 \text{ or A and in } C_2 \text{ or A and in } C_3) \\ &= \text{P}((A \cap C_1) \cup (A \cap C_2) \cup (A \cap C_3)) \\ &= \text{P}(A \cap C_1) + \text{P}(A \cap C_2) + \text{P}(A \cap C_3) \\ &\qquad \text{as } A \cap C_1,\ A \cap C_2,\ A \cap C_3 \text{ are disjoint} \\ &= \text{P}(C_1)\,\text{P}(A|C_1) + \text{P}(C_2)\,\text{P}(A|C_2) + \text{P}(C_3)\,\text{P}(A|C_3) \end{aligned}$$

$$\left\{\text{as } \text{P}(X|Y) = \frac{\text{P}(X \cap Y)}{\text{P}(Y)}, \text{ then } \text{P}(X \cap Y) = \text{P}(Y)\,\text{P}(X|Y)\right\}$$

b

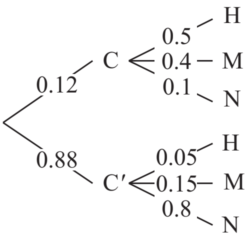


i $$\begin{aligned} \text{P(U)} &= \text{P}(A \cap U \text{ or } B \cap U \text{ or } C \cap U) \\ &= 0.3 \times 0.97 + 0.4 \times 0.95 + 0.3 \times 0.93 \\ &= 0.95 \end{aligned}$$

ii $$\text{P(A|U)} = \frac{\text{P}(A \cap U)}{\text{P(U)}} = \frac{0.3 \times 0.97}{0.95} \doteqdot 0.306$$

iii $$\begin{aligned} \text{P(A or C}|U') &= \frac{\text{P}((A \cup C) \cap U')}{\text{P}(U')} \\ &= \frac{\text{P}((A \cap U') \cup (C \cap U'))}{\text{P}(U')} \\ &= \frac{0.3 \times 0.03 + 0.3 \times 0.07}{0.05} \\ &= 0.6 \end{aligned}$$

6



a $$\begin{aligned} &\text{P(H)} \\ &= \text{P}(C \cap H \text{ or } C' \cap H) \\ &= 0.12 \times 0.5 + 0.88 \times 0.05 \\ &= 0.104 \end{aligned}$$

b $$\begin{aligned}P(C|M) &= \frac{P(C \cap M)}{P(M)}\\ &= \frac{0.12 \times 0.4}{0.12 \times 0.4 + 0.88 \times 0.15}\\ &\doteqdot 0.267\end{aligned}$$

c $$\begin{aligned}P(C|N) &= \frac{P(C \cap N)}{P(N)}\\ &= \frac{0.12 \times 0.1}{0.12 \times 0.1 + 0.88 \times 0.8}\\ &\doteqdot 0.0168\end{aligned}$$

REVIEW SET 19A

1 ABCD, ABDC, ACBD, ACDB, ADBC, ADCB,
BACD, BADC, BCAD, BCDA, BDAC, BDCA,
CABD, CADB, CBAD, CBDA, CDAB, CDBA,
DABC, DACB, DBAC, DBCA, DCAB, DCBA

a There are 24 possible orderings.

$\therefore$ P (A is next to C)

$= \frac{12}{24}$ {12 have A next to C}

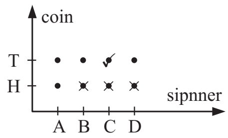
$= \frac{1}{2}$

b P (exactly one person between A and C)

$= \frac{8}{24}$ {8 have one person between A and C}

$= \frac{1}{3}$

2



a Consonants are B, C and D

$\therefore$ P(H and a consonant)

$= \frac{3}{8}$ {those with $\times$}

b P(T and C)

$= \frac{1}{8}$ {those with a ✓}

c P(T or vowel)

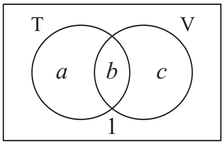
$=$ P(T or A)

$= \text{P(T)} + \text{P(A)} - \text{P(T and A)}$

$= \frac{4}{8} + \frac{2}{8} - \frac{1}{8}$

$= \frac{5}{8}$

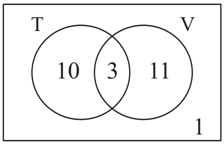
3



$$\begin{aligned}a+b+c &= 24\\ a+b &= 13\\ b+c &= 14\\ \text{Also} \quad b &= 13-a\\ &= 3\end{aligned}$$

$\therefore \quad 13 + c = 24 \quad \text{and} \quad a + 14 = 24$

$\therefore \quad c = 11 \quad \text{and} \quad a = 10$



a P(T and V)

$= \frac{3}{25}$

b P(at least one)

$= 1 - \text{P(neither)}$

$= 1 - \frac{1}{25}$

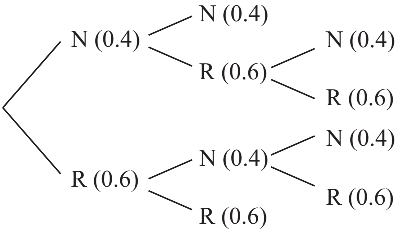
$= \frac{24}{25}$

c $\text{P}(V \mid T')$

$= \dfrac{11}{11+1}$

$= \frac{11}{12}$

4



P(Niklas wins)

$= (0.4)(0.4) + (0.4)(0.6)(0.4) + (0.6)(0.4)(0.4)$

$= 0.352$

5 $P(M) = \frac{3}{5}, \quad P(W) = \frac{2}{3}$

a $P(M \text{ and } W)$
$= \frac{3}{5} \times \frac{2}{3}$ {assuming independence}
$= \frac{2}{5}$

b $P(\text{at least one})$
$= P(M \text{ or } W)$
$= P(M) + P(W) - P(M \text{ and } W)$
$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$
$= \frac{13}{15}$

c $P(M' \text{ and } W)$
$= (1 - \frac{3}{5}) \times \frac{2}{3}$
$= \frac{2}{5} \times \frac{2}{3}$
$= \frac{4}{15}$

6 A repetition of 5 independent events $\therefore$ binomial model applies.
$P(M) = \frac{4}{5}, \quad P(M') = \frac{1}{5}$

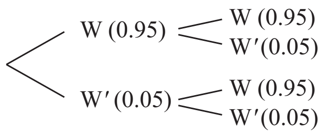
a $P(M \text{ wins 3 games})$
$= C_3^5 (\frac{4}{5})^3 (\frac{1}{5})^2$
$\doteqdot 0.205$

b $P(M \text{ wins 4 or 5 games})$
$= C_4^5 (\frac{4}{5})^4 (\frac{1}{5})^1 + C_5^5 (\frac{4}{5})^5$
$\doteqdot 0.737$

7 **a** $P(\text{wins first 3 prizes})$
$= P(WWW)$
$= \frac{4}{500} \times \frac{3}{499} \times \frac{2}{498}$
$\doteqdot 1.93 \times 10^{-7}$

b $P(\text{wins at least one of the 3 prizes})$
$= 1 - P(\text{wins none of them})$
$= 1 - P(W'W'W')$
$= 1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498}$
$\doteqdot 0.0239$

8



$P(\text{works on at least one day})$
$= 0.95 \times 0.95 + 0.95 \times 0.05 + 0.05 \times 0.95$
$= 0.9975$

9 **a** $P(\text{all doctors}) = \dfrac{{}_6C_5 \times {}_4C_0}{{}_{10}C_5} \doteqdot 0.0238$

b $P(\text{at least 2 doctors}) = 1 - P(1 \text{ doctor}) = 1 - \dfrac{{}_6C_1 \times {}_4C_4}{{}_{10}C_5} \doteqdot 0.976$

10 If there are no restrictions, there are 6! different orderings.

a

3	3	2	2	1	1
B	G	B	G	B	G

or

3	3	2	2	1	1
G	B	G	B	G	B

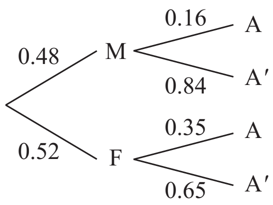
i.e., 36 + 36 consist of alternate sexes.

$\therefore \quad P(\text{alternate sexes}) = \frac{72}{6!} = \frac{1}{10}$

b The girls as a group can be ordered in 3! ways. This group plus the 3 boys can be ordered in 4! ways.

$\therefore \quad P(\text{girls are together}) = \dfrac{3! \times 4!}{6!} = \frac{1}{5}$

11



a $P(A) = P(M \cap A \text{ or } F \cap A)$
$= 0.48 \times 0.16 + 0.52 \times 0.35$
$= 0.2588 \quad (\doteqdot 0.259)$

b $P(F|A) = \dfrac{P(F \cap A)}{P(A)}$
$= \dfrac{0.52 \times 0.35}{0.2588}$
$\doteqdot 0.703$

REVIEW SET 19B

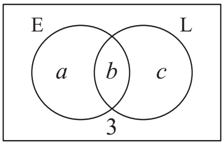
1 BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GGBB, GBBG, GBGB, BGGG, GBGG, GGBG, GGGB, GGGG

P(2B and 2G)

$= \frac{6}{16}$ ← 6 have 2B and 2G

$= \frac{3}{8}$

2



$$\begin{aligned} a+b+c &= 37 \\ a+b &= 22 \\ b+c &= 25 \end{aligned}$$

$\therefore\ 22+c = 37$ and $a+25 = 37$

$\therefore\ c = 15$ and $a = 12$

Hence, $b = 22 - a = 10$

a P(E and L)

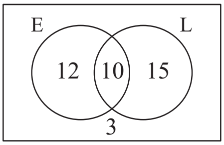
$= \frac{10}{40}$

$= \frac{1}{4}$

b P(at least one)

$= \frac{12+10+15}{40}$

$= \frac{37}{40}$



c $P(E\,|\,L) = \frac{10}{15+10} = \frac{10}{25} = \frac{2}{5}$

3 **a** P(both blue)

$= P(BB)$

$= \frac{5}{12} \times \frac{4}{11}$

$= \frac{5}{33}$

b P(both same colour)

$= P(BB \text{ or } RR \text{ or } YY)$

$= \frac{5}{12} \times \frac{4}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11}$

$= \frac{19}{66}$

c P(at least one R)

$= 1 - P(\text{no reds})$

$= 1 - P(R'R')$

$= 1 - \frac{9}{12} \times \frac{8}{11}$

$= 1 - \frac{6}{11}$

$= \frac{5}{11}$

d P(exactly one Y)

$= P(YY' \text{ or } Y'Y)$

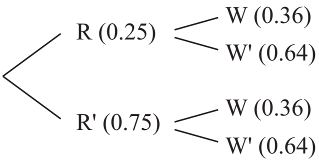
$= \frac{4}{12} \times \frac{8}{11} + \frac{8}{12} \times \frac{4}{11}$

$= \frac{16}{33}$

4 **a** Two events are independent if the occurrence of one does not influence the occurrence of the other. For A and B independent, P(A) × P(B) = P(A and B)

b Two events, A and B, are disjoint if they have no common outcomes, i.e., P(A and B) = 0 and so P(A or B) = P(A) + P(B)

5



a P(W and R)

$= 0.25 \times 0.36$

$= 0.09$

b P(W or R)

$= P(W) + P(R) - P(W \text{ and } R)$

$= 0.36 + 0.25 - 0.09$

$= 0.52$

or $P(W \text{ or } R) = 1 - P(W'R')$

$= 1 - 0.64 \times 0.75$

$= 0.52$

6 P(A) = 0.1, P(B) = 0.2, P(C) = 0.3 ∴ P(group solves it) = P(at least one solves it)

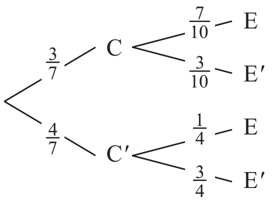
$= 1 - P(\text{no-one solves it})$

$= 1 - P(A' \text{ and } B' \text{ and } C')$

$= 1 - (0.9 \times 0.8 \times 0.7)$

$= 0.496$

7



a
$$\begin{aligned}\text{P(E)} &= \tfrac{3}{7}\times\tfrac{7}{10}+\tfrac{4}{7}\times\tfrac{1}{4}\\ &= \tfrac{3}{10}+\tfrac{1}{7}\\ &= \tfrac{31}{70}\end{aligned}$$

b
$$\begin{aligned}\text{P(C}\,|\,\text{E)} &= \frac{\text{P(C and E)}}{\text{P(E)}}\\ &= \frac{\frac{3}{7}\times\frac{7}{10}}{\frac{31}{70}}\\ &= \tfrac{21}{31}\end{aligned}$$

8 **a**
$$\left(\tfrac{3}{5}+\tfrac{2}{5}\right)^4 = \underbrace{\left(\tfrac{3}{5}\right)^4}_{4\text{B}} + \underbrace{4\left(\tfrac{3}{5}\right)^3\left(\tfrac{2}{5}\right)}_{\substack{3\text{B}\\ 1\text{B}'}} + \underbrace{6\left(\tfrac{3}{5}\right)^2\left(\tfrac{2}{5}\right)^2}_{\substack{2\text{B}\\ 2\text{B}'}} + \underbrace{4\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)^3}_{\substack{1\text{B}\\ 3\text{B}'}} + \underbrace{\left(\tfrac{2}{5}\right)^4}_{4\text{B}'}$$

$$\begin{aligned}\text{P(B)} &= \tfrac{12}{20}\\ &= \tfrac{3}{5}\\ \therefore\quad \text{P(B}') &= \tfrac{2}{5}\end{aligned}$$

b **i**
$$\begin{aligned}&\text{P(2 Blue inks)}\\ &= \text{P(2B and 2B}')\\ &= 6\left(\tfrac{3}{5}\right)^2\left(\tfrac{2}{5}\right)^2\\ &= \frac{6\times 9\times 4}{5^4}\\ &= \tfrac{144}{625}\end{aligned}$$

ii
$$\begin{aligned}&\text{P(at most 2 Blue inks)}\\ &= \text{P(2B and 2B}'\text{ or 1B and 3B}'\text{ or 4B}')\\ &= 6\left(\tfrac{3}{5}\right)^2\left(\tfrac{2}{5}\right)^2+4\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)^3+\left(\tfrac{2}{5}\right)^4\\ &= \frac{6\times 9\times 4+4\times 3\times 8+16}{625}\\ &= \tfrac{328}{625}\end{aligned}$$

9 **a** P(X wins) $= \frac{3}{5}$ P(Y wins) $= \frac{2}{5}$

Probability generator is $\left(\frac{3}{5}+\frac{2}{5}\right)^6$

$$= \underset{\substack{\uparrow\\ \text{X wins 6}}}{\left(\tfrac{3}{5}\right)^6} + \underset{\substack{\uparrow\\ \text{X wins 5}\\ \text{Y wins 1}}}{6\left(\tfrac{3}{5}\right)^5\left(\tfrac{2}{5}\right)} + \underset{\substack{\uparrow\\ \text{X wins 4}\\ \text{Y wins 2}}}{15\left(\tfrac{3}{5}\right)^4\left(\tfrac{2}{5}\right)^2} + \underset{\substack{\uparrow\\ \text{X wins 3}\\ \text{Y wins 3}}}{20\left(\tfrac{3}{5}\right)^3\left(\tfrac{2}{5}\right)^3} + \underset{\substack{\uparrow\\ \text{X wins 2}\\ \text{Y wins 4}}}{15\left(\tfrac{3}{5}\right)^2\left(\tfrac{2}{5}\right)^4}$$

$$+\ \underset{\substack{\uparrow\\ \text{X wins 1}\\ \text{Y wins 5}}}{6\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)^5} + \underset{\substack{\uparrow\\ \text{X wins 0}\\ \text{Y wins 6}}}{\left(\tfrac{3}{5}\right)^6}$$

b **i**
$$\begin{aligned}&\text{P(Y wins 3)}\\ &= 20(0.6)^3(0.4)^3\\ &\doteqdot 0.276\end{aligned}$$

ii
$$\begin{aligned}&\text{P(Y wins at least 5)}\\ &= 6(0.6)^1(0.4)^5+(0.4)^6\\ &\doteqdot 0.0410\end{aligned}$$

10 X = the number of goals scored X is Bin (5, 0.8)

a
P(3 goals **then** 2 misses)
= P(GGGG′G′)
$= (0.8)^3\times(0.2)^2$
$\doteqdot 0.0205$

b
P(3 goals and 2 misses)
$= \text{P}(X=3)$
= binompdf (5, 0.8, 3)
$\doteqdot 0.205$

Chapter 20

INTRODUCTION TO CALCULUS

EXERCISE 20A.1

1 **a** Average speed from Tailem Bend to Nhill

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(324 - 98)\text{ km}}{\left(\frac{204 - 63}{60}\right)\text{ h}}$$

$$= \frac{226 \times 60}{141}$$

$$= 96.2 \text{ kmph}$$

b Average speed from Horsham to Melbourne

$$= \frac{729 - 431 \text{ km}}{\left(\frac{534 - 261}{60}\right)\text{ h}}$$

$$= \frac{298 \times 60}{273}$$

$$= 65.5 \text{ kmph}$$

2 **a** 800 m to the newsagency

b $m = \frac{y\text{-step}}{x\text{-step}} = \frac{500 - 0}{4 - 0} = 125$

c Average speed $= \frac{(500 - 0)\text{ m}}{(4 - 0)\text{ min}} = 125$ m/minute

d The *slope* represents the *average walking speed.*

e Paul stayed between the 12th and 20th minutes i.e., 8 minutes.

f Average speed $= \frac{(800 - 0)\text{ m}}{(32 - 20)\text{ min}}$ {return journey is from (20, 800) to (32, 0)}

$= 66.7$ m/minute

g Total distance travelled $= 1600$ m $= 1.6$ km (to the shop and return)

3 Average water loss $= \frac{\text{amount of water lost}}{\text{time taken}}$

$$= \frac{53.8 - 48.2}{11}$$

$= 0.509$ million kL/day

$= 509\,000$ kL/day

4 **a** First quarter: Used 106.8 kL Number of days $= 31 + 28 + 31$

$$\therefore \text{ rate} = \frac{106.8}{90}$$

$= 1.19$ kL/day

b First six months: Used

$$\begin{array}{r} 106.8 \\ +\ \ 79.4 \\ \hline 186.2 \end{array} \text{ kL}$$

Number of days $= 90 + 30 + 31 + 30$

$= 181$ days

$$\therefore \text{ rate} = \frac{186.2}{181} = 1.03 \text{ kL/day}$$

c The whole year: Used $= 106.8 + 79.4 + 81.8 + 115.8$

$= 383.8$ kL

$$\therefore \text{ rate} = \frac{383.8}{365} \text{ kg}$$

$= 1.05$ kL/day

EXERCISE 20A.2

1 **a** In the first 4 seconds:

$$\begin{aligned}\text{average speed} &= \frac{\text{distance travelled}}{\text{time taken}}\\ &= \frac{(2.4-2)\text{ m}}{(4-0)\text{ sec}}\\ &= 0.1\text{ m/s}\end{aligned}$$

b In the last 4 seconds: $\text{average speed} = \dfrac{(6-2.4)\text{ m}}{(4-0)\text{ sec}} = 0.9\text{ m/s}$

c In the 8 second period: $\text{average speed} = \dfrac{(6-2)\text{ m}}{(8-0)\text{ sec}} = 0.5\text{ m/s}$

2 **a** **i** Initially there are 35 beetles. After a 10 g dose there are 3 beetles.

$$\begin{aligned}\text{rate of change} &= \frac{\text{change in population}}{\text{change in dose}}\\ &= \frac{(35-3)\text{ beetles}}{(0-10)\text{ gm}}\\ &= -3.2\text{ beetles/gm}\end{aligned}$$

i.e., rate of decrease = 3.2 beetles/gram

ii

$$\begin{aligned}\text{rate of change} &= \frac{(26-8)\text{ beetles}}{(4-8)\text{ beetles}}\\ &= -4.5\text{ beetles/gm}\end{aligned}$$

i.e., rate of decrease = 4.5 beetles/gram

b A dose of 0 to 1 g has little or no effect.
A dose of 1 to 8 g has good or considerable effect (rapid decrease).
A dose of 8 to 14 g has an effect, but less rapid than the 1 to 8 rate.

EXERCISE 20B.1

1 **a** The tangent passes through (2, 3) and (0, 1) $\therefore$ slope of tangent

$$\begin{aligned}&\doteqdot \frac{(3-1)\text{ m}}{(2-0)\text{ s}}\\ &\doteqdot \tfrac{2}{2}\text{ m/s}\\ &\doteqdot 1\text{ m/s}\end{aligned}$$

b The tangent passes through (3.5, 6) and (2.2, 2) $\therefore$ slope of tangent

$$\begin{aligned}&\doteqdot \frac{(6-2)\text{ km}}{(3-2.2)\text{ h}}\\ &\doteqdot \tfrac{4}{1.3}\text{ km/h}\\ &\doteqdot 3\text{ km/h}\end{aligned}$$

c The tangent passes through (20, 2700) and (40, 3700) $\therefore$ slope of tangent

$$\begin{aligned}&\doteqdot \frac{(3.7-2.7)\text{ thousands of \$}}{(40-20)\text{ items}}\\ &\doteqdot 0.05\text{ thousands of \$/item}\\ &\doteqdot \$50\text{/item}\end{aligned}$$

d The tangent passes through (0, 35) and (7, 0) $\therefore$ slope of tangent

$$\begin{aligned}&\doteqdot \frac{(35-0)\text{ bats}}{(0-7)\text{ weeks}}\\ &\doteqdot -5\ \ \text{bats/week}\end{aligned}$$

i.e., the population is decreasing at the rate of 5 bats/week after 5 weeks.

2 **a** Originally (when $x = 0$) the volume was 8000 L.

b After 1 hour $(x = 1)$ the volume was 3000 L.

c The tangent at (0, 8) passes through (1, 0.5)

$$\therefore \text{ slope of tangent } = \frac{(8 - 0.5)\text{ L}}{(0 - 1)\text{ h}}$$
$$= -7.5 \qquad \text{i.e., loses 7500 L/hour}$$

d After 1 hour, $x = 1$ and the tangent at (1, 3) passes through (0, 6)

$$\therefore \text{ slope of tangent } = \frac{(6 - 3)\text{ L}}{(0 - 1)\text{ h}}$$
$$= -3 \text{ L/h} \qquad \text{i.e., the rate of loss is 3000 L/hour}$$

EXERCISE 20B.2

1 Let M $(1.5 + h, (15 + h)^2)$ be a point on $y = x^2$ which is close to F $(1.5, 1.5^2)$.

$$\therefore \text{ slope of MF } = \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{(1.5 + h)^2 - (1.5)^2}{(1.5 + h) - 1.5}$$
$$= \frac{2.25 + h^2 + 3h - 2.25}{h}$$
$$= \frac{h^2 + 3h}{h}$$
$$= h + 3 \qquad \{\text{as } h \neq 0\}$$

As M approaches F, h approaches 0, $\therefore$ $h + 3$ approaches 3, $\therefore$ slope of the tangent = 3

2 **a**

$$(x + h)^3$$
$$= (x + h)^2(x + h)$$
$$= (x^2 + 2xh + h^2)(x + h)$$
$$= x^3 + 2x^2h + h^2x + hx^2 + 2xh^2 + h^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$

b

$$(1 + h)^3$$
$$= 1^3 + 3(1)^2h + 3(1)h^2 + h^3$$
$$= 1 + 3h + 3h^2 + h^3$$

c M $(1 + h, (1 + h)^3)$ is the point on $y = x^3$ which is close to F(1, 1).

d slope of chord MF

$$= \frac{y\text{-step}}{x\text{-step}}$$
$$= \frac{(1 + h)^3 - 1}{(1 + h) - 1}$$
$$= \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$$
$$= \frac{3h + 3h^2 + h^3}{h}$$
$$= 3 + 3h + h^2 \qquad \{\text{as } h \neq 0\}$$

e As M approaches F, h approaches 0

$\therefore$ slope MF approaches 3

$\therefore$ slope of tangent = 3

3 **b** from **2 a** $(2 + h)^3 = 2^3 + 3(2)^2h + 3(2)h^2 + h^3$

$$= 8 + 12h + 6h^2 + h^3$$

c M $(2 + h, (2 + h)^3)$ is the point on $y = x^3$ which is close to F(2, 8).

d slope of chord MF

$$= \frac{y\text{-step}}{x\text{-step}}$$

$$= \frac{(2+h)^3 - 2^3}{2+h-2}$$

$$= \frac{8+12h+6h^2+h^3-8}{h}$$

$$= \frac{12h+6h^2+h^3}{h}$$

$$= 12+6h+h^2 \qquad \{\text{as } h \neq 0\}$$

e as M approaches F,

h approaches 0,

slope approaches 12

$\therefore$ slope of tangent $= 12$

4 **a** $$\frac{1}{x+h} - \frac{1}{x} = \frac{x(1) - 1(x+h)}{(x+h)x}$$

$$= \frac{x-x-h}{x(x+h)}$$

$$= -\frac{h}{x(x+h)}$$

b M is $\left(2+h, \dfrac{1}{2+h}\right)$

$\therefore$ y-coordinate is $\dfrac{1}{2+h}$

$$\therefore \text{ slope of MF } = \frac{y\text{-step}}{x\text{-step}}$$

$$= \frac{\frac{1}{2+h} - \frac{1}{2}}{(2+h)-2}$$

$$= \frac{\frac{-h}{2(2+h)}}{h} \qquad \{\text{using } \textbf{a} \text{ with } x = 2\}$$

$$= \frac{-h}{2h(2+h)}$$

$$= \frac{-1}{2(2+h)} \qquad \{\text{as } h \neq 0\}$$

c F is the point where $x = 2$. Now as h approaches 0, M approaches F, and the slope MF approximates the slope of the tangent at M.

But as h approaches 0, the slope MF approaches $-\frac{1}{4}$, $\therefore$ slope of tangent $= -\frac{1}{4}$

d At the point where $x = 3$, $y = \frac{1}{3}$.

Also, the point with x-coordinate $3+h$ has y-coordinate $\dfrac{1}{3+h}$.

Using the method in **b**, the slope of the line between these points is $-\dfrac{1}{3(3+h)}$, $h \neq 0$.

As h approaches 0, the slope of the line approximates the slope of the tangent when $x = 3$, and this is $-\frac{1}{9}$.

5 **a** $$\frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2 - (x+h)^2}{x^2(x+h)^2}$$

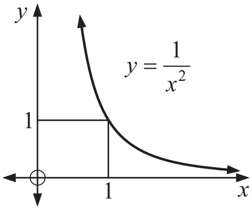
$$= \frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2}$$

$$= \frac{-2xh - h^2}{x^2(x+h)^2}$$

b When $x = 2$,

$$\frac{1}{(2+h)^2} - \frac{1}{4} = \frac{-4h-h^2}{4(2+h)^2}$$

c



d Let M be close to F$(2, \frac{1}{2^2})$ i.e., $\left(2+h, \frac{1}{(2+h)^2}\right)$

$$\therefore \text{ slope MF } = \frac{y\text{-step}}{x\text{-step}}$$

$$= \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{(2+h)-2}$$

$$= \frac{\frac{-4h-h^2}{4(2+h)^2}}{h} \quad \{\text{using } \mathbf{b}\}$$

$$= \frac{-4h-h^2}{4h(2+h)^2}$$

$$= \frac{-4-h}{4(2+h)^2} \quad \{\text{as } h \neq 0\}$$

as h approaches 0, slope MF approaches $\frac{-4}{16}$ i.e., $-\frac{1}{4}$ $\therefore$ slope of tangent $= -\frac{1}{4}$

6 **a**

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

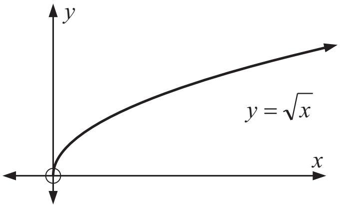
$$= \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}} \qquad h \neq 0$$

b When $x = 9$,

$$\frac{\sqrt{9+h}-3}{h} = \frac{1}{\sqrt{9+h}+3} \quad \text{...... } (*)$$

c



d Slope MF $= \frac{y\text{-step}}{x\text{-step}}$

$$= \frac{\sqrt{9+h}-\sqrt{9}}{(9+h)-9}$$

$$= \frac{\sqrt{9+h}-3}{h}$$

$$= \frac{1}{\sqrt{9+h}+3} \qquad \{\text{using } *\}$$

$\therefore$ as h approaches 0, slope MF approaches $\frac{1}{6}$, $\therefore$ slope of tangent $= \frac{1}{6}$

e $N = \sqrt{t}$ for $t \geqslant 4$ is the same graph as $y = \sqrt{x}$ for $x \geqslant 4$,

$\therefore$ rate after 9 days is the same as the slope of the tangent in **d**, i.e., $\frac{1}{6}$

$\therefore$ the population is increasing at a rate of $\frac{1}{6}$ (thousand insects/day) = 167 insects/day

REVIEW SET 20

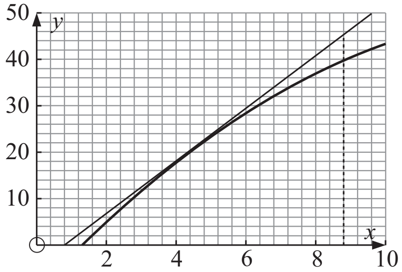
1 **a** When $t = 1$, $d = 40$ and when $t = 4$, $d = 70$

$$\therefore \text{ average speed on } 1 \leqslant t \leqslant 4 \text{ is } \frac{\text{distance travelled}}{\text{time taken}} = \frac{70-40}{4-1} = \frac{30}{3} = 10 \text{ m/s}$$

b When $t = 1$, $d = 40$ and when $t = 10$, $d \doteqdot 81$

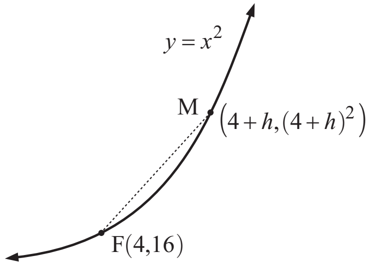
$$\therefore \text{ average speed } = \frac{81-40}{10-1} = \frac{41}{9} \doteqdot 4.6 \text{ m/s}$$

2 **a**



b slope $= \dfrac{y\text{-step}}{x\text{-step}} \doteqdot \dfrac{46-0}{8.8-0.8} \doteqdot \dfrac{46}{8} \doteqdot 5.8$

3



Let M $(4+h, (4+h)^2)$ be a point close to F(4, 16).

$$\therefore \text{ slope of MF } = \frac{(4+h)^2 - 16}{4+h-4} = \frac{16 + 8h + h^2 - 16}{h} = \frac{8h + h^2}{h} = 8 + h \quad (\text{as } h \neq 0)$$

Now as M approaches F, $h \to 0$ and $8 + h$ approaches 8. $\therefore$ the tangent at F has slope 8.

4 **a**
$$\begin{aligned} & f(x+h) \\ &= (2(x+h)+3)^2 \\ &= ((2x+2h)+3)^2 \\ &= (2x+2h)^2 + 2(2x+2h)3 + 3^2 \\ &= 4x^2 + 8xh + 4h^2 + 12x + 12h + 9 \end{aligned}$$

b
$$\begin{aligned} & \frac{f(x+h)-f(x)}{h} \\ &= \frac{\cancel{4x^2} + 8xh + 4h^2 + \cancel{12x} + 12h + \cancel{9} - \cancel{4x^2} - \cancel{12x} - \cancel{9}}{h} \\ &= \frac{8xh + 4h^2 + 12h}{h} \\ &= 8x + 4h + 12 \qquad \{\text{as } h \neq 0\} \end{aligned}$$

c $\dfrac{f(x+h)-f(x)}{h}$ is the slope of the secant (chord) AB.

d **i** As $h \to 0$, $\dfrac{f(x+h)-f(x)}{h} = 8x + 4h + 12$ approaches $8x + 12$.

ii As $h \to 0$, $\dfrac{f(1+h)-f(1)}{h} = 8 + 4h + 12$ approaches 20.

e **i** $8x + 12$ gives us the slope of the tangent at a point with x-coordinate x.

ii 20 is the slope of the tangent at $x = 1$.

5 **a** Average speed on $2 \leqslant t \leqslant 5$ is

$$\begin{aligned} &\frac{s(5)-s(2)}{5-2} \text{ m/s} \\ &= \frac{(25+20)-(4+8)}{3} \text{ m/s} \\ &= \frac{33}{3} \text{ m/s} \\ &= 11 \text{ m/s} \end{aligned}$$

b Average speed on $2 \leqslant t \leqslant 2+h$ is

$$\begin{aligned} &\frac{s(2+h)-s(2)}{2+h-2} \text{ m/s} \\ &= \frac{(2+h)^2+4(2+h)-12}{h} \text{ m/s} \\ &= \frac{4+4h+h^2+8+4h-12}{h} \text{ m/s} \\ &= \frac{8h+h^2}{h} \text{ m/s} \\ &= (8+h) \text{ m/s} \end{aligned}$$

c as $h \to 0$, $8+h \to 8$

$$\therefore \quad \frac{s(2+h)-s(2)}{h} \to 8 \text{ m/s}$$

8 m/s is the instantaneous velocity at $t = 2$ seconds.

Chapter 21

DIFFERENTIAL CALCULUS

EXERCISE 21A

1 **a** Slope AB $= \dfrac{x^2-9}{x-3}$

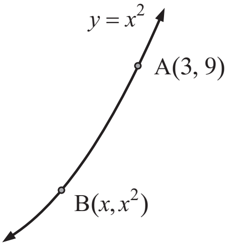
$= \dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$

$= x+3 \qquad$ (provided $x \neq 3$)

$\therefore \displaystyle\lim_{x\to 3}(\text{slope AB}) = 6 \quad \text{......} \ (1)$

But, as B → A, i.e., $x \to 3$ and $\displaystyle\lim_{x\to 3}(\text{slope AB}) =$ slope of tangent at A

$\therefore$ slope of tangent $= 6$



b Slope AB $= \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$

$= \dfrac{2-x}{2x(x-2)}$

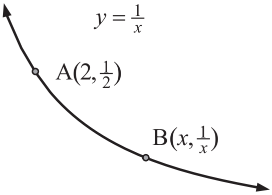
$= \dfrac{-1\cancel{(x-2)}}{2x\cancel{(x-2)}}$

$= -\dfrac{1}{2x} \quad$ (provided $x \neq 2$)

$\therefore \displaystyle\lim_{x\to 2}(\text{slope AB}) = -\tfrac{1}{4}$

But, as B → A, i.e., $x \to 2$ and $\displaystyle\lim_{x\to 2}(\text{slope AB}) =$ slope of tangent at A

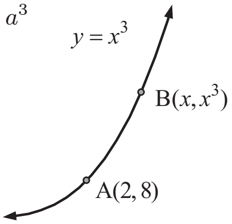
$\therefore$ slope of tangent $= -\tfrac{1}{4}$



2 **a** $(x-a)(x^2+ax+a^2)$

$= x^3 + ax^2 + a^2x - ax^2$
$\quad -a^2x - a^3$

$= x^3 - a^3$



b Slope AB $= \dfrac{x^3-8}{x-2}$

$= \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$

$= x^2+2x+4 \quad$ (provided $x \neq 2$)

$\therefore \displaystyle\lim_{x\to 2}(\text{slope of AB}) = 2^2+4+4 = 12$

But, as B → A, i.e., $x \to 2$,

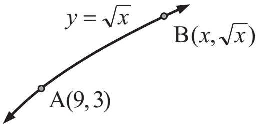
$\displaystyle\lim_{x\to 2}(\text{slope AB}) =$ slope of tangent at A

$\therefore$ slope of tangent $= 12$

3 **a** $\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$

$= \dfrac{\cancel{\sqrt{x}-\sqrt{a}}^{\,1}}{(\sqrt{x}+\sqrt{a})\cancel{(\sqrt{x}-\sqrt{a})}_{1}}$

$= \dfrac{1}{\sqrt{x}+\sqrt{a}}$



b Slope AB $= \dfrac{\sqrt{x}-3}{x-9} = \dfrac{1}{\sqrt{x}+3}$

{from **a**, provided $x \neq 9$}

$\therefore \displaystyle\lim_{x\to 9}(\text{slope AB}) = \tfrac{1}{6}$

But, as B → A, i.e., $x \to 9$,

$\displaystyle\lim_{x\to 9}(\text{slope AB}) =$ slope of tangent at A

$\therefore$ slope of tangent $= \tfrac{1}{6}$

EXERCISE 21B

1 **a**

$$f(x) = 1 - x^2$$
$$\therefore \quad f(2) = 1 - 2^2 = -3$$
$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{(1 - x^2) - (-3)}{x - 2} \\ &= \lim_{x\to 2} \frac{4 - x^2}{x - 2} \\ &= \lim_{x\to 2} \frac{-(x+2)\cancel{(x-2)}}{\cancel{x-2}} \\ &= \lim_{x\to 2} -(x+2) \qquad \{\text{as } x \neq 2\} \\ &= -4 \end{aligned}$$

b

$$f(x) = 2x^2 + 5x \quad \text{at} \quad x = -1$$
$$f(-1) = 2(-1)^2 + 5(-1) = -3$$
$$\begin{aligned} f'(-1) &= \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x\to -1} \frac{(2x^2 + 5x) - (-3)}{x + 1} \\ &= \lim_{x\to -1} \frac{2x^2 + 5x + 3}{x + 1} \\ &= \lim_{x\to -1} \frac{(2x+3)\cancel{(x+1)}}{\cancel{x+1}} \\ &= \lim_{x\to -1} 2x + 3 \qquad \{\text{as } x \neq -1\} \\ &= 1 \end{aligned}$$

c

$$f(x) = 5 - 2x^2 \quad \text{at} \quad x = 3$$
$$\therefore \quad f(3) = 5 - 2(3)^2 = -13$$
$$\begin{aligned} f'(3) &= \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x\to 3} \frac{(5 - 2x^2) - (-13)}{x - 3} \\ &= \lim_{x\to 3} \frac{18 - 2x^2}{x - 3} \\ &= \lim_{x\to 3} \frac{-2(x^2 - 9)}{x - 3} \\ &= \lim_{x\to 3} \frac{-2(x+3)\cancel{(x-3)}}{\cancel{x-3}} \\ &= \lim_{x\to 3} -2(x+3) \qquad \{\text{as } x \neq 3\} \\ &= -2(6) \\ &= -12 \end{aligned}$$

d

$$f(x) = 3x + 5 \quad \text{at} \quad x = -2$$
$$\therefore \quad f(-2) = 3(-2) + 5 = -1$$
$$\begin{aligned} f'(-2) &= \lim_{x\to -2} \frac{f(x) - f(-2)}{x + 2} \quad \text{where} \\ &= \lim_{x\to -2} \frac{(3x + 5) - (-1)}{x + 2} \\ &= \lim_{x\to -2} \frac{3x + 6}{x + 2} \\ &= \lim_{x\to -2} \frac{3\cancel{(x+2)}}{\cancel{x+2}} \\ &= \lim_{x\to -2} 3 \qquad \{\text{as } x \neq -2\} \\ &= 3 \end{aligned}$$

2 **a**

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{\frac{4}{x} - \frac{4}{2}}{x - 2} \\ &= \lim_{x\to 2} \frac{8 - 4x}{2x(x - 2)} \\ &= \lim_{x\to 2} \frac{-4\cancel{(x-2)}}{2x\cancel{(x-2)}} \\ &= \lim_{x\to 2} \frac{-4}{2x} \qquad \{\text{as } x \neq 2\} \\ &= -\tfrac{4}{4} \\ &= -1 \end{aligned}$$

b

$$\begin{aligned} f'(-2) &= \lim_{x\to -2} \frac{f(x) - f(-2)}{x - (-2)} \\ &= \lim_{x\to -2} \frac{-\frac{3}{x} - \frac{3}{2}}{(x + 2)} \\ &= \lim_{x\to -2} \frac{-6 - 3x}{2x(x + 2)} \\ &= \lim_{x\to -2} \frac{-3\cancel{(x+2)}}{2x\cancel{(x+2)}} \\ &= \lim_{x\to -2} \frac{-3}{2x} \qquad \{\text{as } x \neq -2\} \\ &= \tfrac{-3}{-4} \\ &= \tfrac{3}{4} \end{aligned}$$

c
$$\begin{aligned} f'(4) &= \lim_{x\to4}\frac{f(x)-f(4)}{x-4} \\ &= \lim_{x\to4}\frac{\dfrac{1}{x^2}-\dfrac{1}{16}}{x-4} \\ &= \lim_{x\to4}\frac{16-x^2}{16x^2(x-4)} \\ &= \lim_{x\to4}\frac{-(x+4)\cancel{(x-4)}}{16x^2\cancel{(x-4)}} \\ &= \lim_{x\to4}\frac{-(x+4)}{16x^2} \qquad \{\text{as } \ x\neq 4\} \\ &= \tfrac{-8}{256} \\ &= -\tfrac{1}{32} \end{aligned}$$

d
$$\begin{aligned} f'(2) &= \lim_{x\to2}\frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to2}\frac{\dfrac{4x}{x-3}-\left(-\dfrac{8}{1}\right)}{x-2} \\ &= \lim_{x\to2}\frac{4x+8(x-3)}{1(x-2)(x-3)} \\ &= \lim_{x\to2}\frac{12\cancel{(x-2)}}{\cancel{(x-2)}(x-3)} \\ &= \lim_{x\to2}\frac{12}{x-3} \qquad \{\text{as } \ x\neq 2\} \\ &= \tfrac{12}{-1} \\ &= -12 \end{aligned}$$

e
$$\begin{aligned} f'(5) &= \lim_{x\to5}\frac{f(x)-f(5)}{(x-5)} \\ &= \lim_{x\to5}\frac{\dfrac{4x+1}{x-2}-\dfrac{7}{1}}{x-5} \\ &= \lim_{x\to5}\frac{4x+1-7(x-2)}{(x-2)(x-5)} \\ &= \lim_{x\to5}\frac{4x+1-7x+14}{(x-2)(x-5)} \\ &= \lim_{x\to5}\frac{-3x+15}{(x-2)(x-5)} \\ &= \lim_{x\to5}\frac{-3\cancel{(x-5)}}{(x-2)\cancel{(x-5)}} \\ &= \lim_{x\to5}\frac{-3}{x-2} \qquad \{\text{as } \ x\neq 5\} \\ &= -\tfrac{3}{3} \\ &= -1 \end{aligned}$$

f
$$\begin{aligned} f'(-4) &= \lim_{x\to-4}\frac{f(x)-f(-4)}{x-(-4)} \\ &= \lim_{x\to-4}\frac{\dfrac{3x}{x^2+1}-\left(-\dfrac{12}{17}\right)}{x+4} \\ &= \lim_{x\to-4}\frac{51x+12(x^2+1)}{17(x^2+1)(x+4)} \\ &= \lim_{x\to-4}\frac{12x^2+51x+12}{17(x+4)(x^2+1)} \\ &= \lim_{x\to-4}\frac{\cancel{(x+4)}(12x+3)}{17(x^2+1)\cancel{(x+4)}} \\ &= \lim_{x\to-4}\frac{12x+3}{17(x^2+1)} \qquad \{x\neq -4\} \\ &= -\frac{45}{17\times17} \\ &= -\tfrac{45}{289} \end{aligned}$$

3 **a** $f(x)=\sqrt{x}$ and $f(4)=\sqrt{4}=2$

$$\begin{aligned} f'(4) &= \lim_{x\to4}\frac{f(x)-f(4)}{x-4} \\ &= \lim_{x\to4}\frac{\sqrt{x}-2}{x-4} \\ &= \lim_{x\to4}\frac{\cancel{\sqrt{x}-2}}{(\sqrt{x}+2)\cancel{(\sqrt{x}-2)}} \\ &= \lim_{x\to4}\frac{1}{(\sqrt{x}+2)} \qquad \{\text{as } \ x\neq 4\} \\ &= \frac{1}{2+2} \\ &= \tfrac{1}{4} \end{aligned}$$

b $f(x)=\sqrt{x}$ and $f\left(\frac{1}{4}\right)=\sqrt{\frac{1}{4}}=\frac{1}{2}$

$$\begin{aligned} f'(\tfrac{1}{4}) &= \lim_{x\to\frac{1}{4}}\frac{f(x)-f(\frac{1}{4})}{x-\frac{1}{4}} \\ &= \lim_{x\to\frac{1}{4}}\frac{\sqrt{x}-\frac{1}{2}}{x-\frac{1}{4}} \\ &= \lim_{x\to\frac{1}{4}}\frac{\cancel{(\sqrt{x}-\frac{1}{2})}}{(\sqrt{x}+\frac{1}{2})\cancel{(\sqrt{x}-\frac{1}{2})}} \\ &= \lim_{x\to\frac{1}{4}}\frac{1}{\sqrt{x}+\frac{1}{2}} \qquad \{\text{as } \ x\neq \tfrac{1}{4}\} \\ &= \frac{1}{\frac{1}{2}+\frac{1}{2}} \\ &= 1 \end{aligned}$$

c $f(x) = \dfrac{2}{\sqrt{x}}$ and $f(9) = \frac{2}{3}$

$$\begin{aligned}
f'(9) &= \lim_{x\to 9} \frac{f(x) - f(9)}{x - 9}\\
&= \lim_{x\to 9} \frac{\frac{2}{\sqrt{x}} - \frac{2}{3}}{x - 9}\\
&= \lim_{x\to 9} \frac{2(3 - \sqrt{x})}{3\sqrt{x}(x - 9)}\\
&= \lim_{x\to 9} \frac{-2\cancel{(\sqrt{x} - 3)}}{3\sqrt{x}(\sqrt{x} + 3)\cancel{(\sqrt{x} - 3)}}\\
&= \lim_{x\to 9} \frac{-2}{3\sqrt{x}(\sqrt{x} + 3)} \quad \{x \neq 9\}\\
&= \frac{-2}{3(3)(6)}\\
&= -\tfrac{1}{27}
\end{aligned}$$

d $f(x) = \sqrt{x - 6}$ and $f(10) = 2$

$$\begin{aligned}
f'(10) &= \lim_{x\to 10} \frac{f(x) - f(10)}{x - 10}\\
&= \lim_{x\to 10} \frac{\sqrt{x - 6} - 2}{x - 10}\\
&= \lim_{x\to 10} \frac{(\sqrt{x - 6} - 2)(\sqrt{x - 6} + 2)}{(x - 10)(\sqrt{x - 6} + 2)}\\
&= \lim_{x\to 10} \frac{x - 6 - 4}{(x - 10)(\sqrt{x - 6} + 2)}\\
&= \lim_{x\to 10} \frac{{}^{1}\cancel{x - 10}}{\cancel{(x - 10)}(\sqrt{x - 6} + 2)}\\
&= \lim_{x\to 10} \frac{1}{\sqrt{x - 6} + 2} \quad \{\text{as } x \neq 10\}\\
&= \frac{1}{2 + 2}\\
&= \tfrac{1}{4}
\end{aligned}$$

4 **a** $f(x) = x^2 + 3x - 4$ at $x = 3$

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3 + h) - f(3)}{h} \quad \text{where} \quad f(3) = 3^2 + 3(3) - 4 = 14\\
&= \lim_{h\to 0} \frac{\left[(3 + h)^2 + 3(3 + h) - 4\right] - 14}{h}\\
&= \lim_{h\to 0} \frac{9 + 6h + h^2 + 9 + 3h - 4 - 14}{h}\\
&= \lim_{h\to 0} \frac{9h + h^2}{h}\\
&= \lim_{h\to 0} 9 + h \quad \{\text{as } h \neq 0\}\\
&= 9
\end{aligned}$$

b $f(x) = 5 - 2x - 3x^2$ at $x = -2$

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2 + h) - f(-2)}{h} \quad \text{where} \quad f(-2) = 5 - 2(-2) - 3(4) = -3\\
&= \lim_{h\to 0} \frac{\left[5 - 2(-2 + h) - 3(-2 + h)^2\right] - (-3)}{h}\\
&= \lim_{h\to 0} \frac{5 + 4 - 2h - 12 + 12h - 3h^2 + 3}{h}\\
&= \lim_{h\to 0} \frac{10h - 3h^2}{h}\\
&= \lim_{h\to 0} 10 - 3h \quad \{\text{as } h \neq 0\}\\
&= 10
\end{aligned}$$

c

$$f(x) = \frac{1}{2x-1}$$

$$\therefore \quad f(-2) = \frac{1}{2(-2)-1} = -\frac{1}{5}$$

$$\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{2(-2+h)-5} - \left(-\dfrac{1}{5}\right)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{2h-5} + \dfrac{1}{5}}{h} \\
&= \lim_{h\to 0} \frac{5+1\cancel{(2h-5)}}{5h\cancel{(2h-5)}} \\
&= \lim_{h\to 0} \frac{2h}{5h(2h-5)} \\
&= \lim_{h\to 0} \frac{2}{5(2h-5)} \qquad \{\text{as} \quad h \neq 0\} \\
&= -\tfrac{2}{25}
\end{aligned}$$

d

$$f(x) = \frac{1}{x^2}$$

$$\therefore \quad f(3) = \frac{1}{3^2} = \frac{1}{9}$$

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(3+h)^2} - \dfrac{1}{9}}{h} \\
&= \lim_{h\to 0} \frac{9-(3+h)^2}{9h(3+h)^2} \\
&= \lim_{h\to 0} \frac{9-9-6h-h^2}{9h(3+h)^2} \\
&= \lim_{h\to 0} \frac{-\cancel{h}(6+h)}{9\cancel{h}(3+h)^2} \\
&= \lim_{h\to 0} \frac{-(6+h)}{9(3+h)^2} \qquad \{\text{as} \quad h \neq 0\} \\
&= \tfrac{-6}{81} \\
&= -\tfrac{2}{27}
\end{aligned}$$

e

$$f(x) = \sqrt{x}$$

$$\therefore \quad f(4) = \sqrt{4} = 2$$

$$\begin{aligned}
f'(4) &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) \\
&= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{{}^{1}\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} \qquad \{\text{as} \quad h \neq 0\} \\
&= \tfrac{1}{4}
\end{aligned}$$

f

$$f(x) = \frac{1}{\sqrt{x}}$$

$$\therefore \quad f(1) = \tfrac{1}{\sqrt{1}} = 1$$

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+h}} - \dfrac{1}{1}}{h} \\
&= \lim_{h\to 0} \frac{1-\sqrt{1+h}}{h\sqrt{1+h}} \\
&= \lim_{h\to 0} \frac{(1-\sqrt{1+h})}{h\sqrt{1+h}} \left(\frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}\right) \\
&= \lim_{h\to 0} \frac{1-(1+h)}{h(\sqrt{1+h})(1+\sqrt{1+h})} \\
&= \lim_{h\to 0} \frac{-\cancel{h}^{\,1}}{\cancel{h}\sqrt{1+h}(1+\sqrt{1+h})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{1+h}(1+\sqrt{1+h})} \qquad \{h \neq 0\} \\
&= \frac{-1}{1(1+1)} \\
&= -\tfrac{1}{2}
\end{aligned}$$

5 **a** $f(x) = x^3$, $\quad \therefore \quad f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$ where $f(2) = 2^3 = 8$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(2+h)^3 - 8}{h} \\
&= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
&= \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} \\
&= \lim_{h\to 0} 12 + 6h + h^2 \quad \{\text{as } h \neq 0\} \\
&= 12
\end{aligned}$$

b $f(x) = x^4$, $\quad \therefore \quad f'(3) = \lim_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$ where $f(3) = 3^4 = 81$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(3+h)^4 - 3^4}{h} \\
&= \lim_{h\to 0} \frac{81 + 108h + 54h^2 + 12h^3 + h^4 - 81}{h} \\
&= \lim_{h\to 0} \frac{108h + 54h^2 + 12h^3 + h^4}{h} \\
&= \lim_{h\to 0} 108 + 54h + 12h^2 + h^3 \quad \{\text{as } h \neq 0\} \\
&= 108
\end{aligned}$$

EXERCISE 21C

1 **a**

$$\begin{aligned}
f(x) &= x \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= \lim_{h\to 0} 1 \quad \{\text{as } h \neq 0\} \\
&= 1
\end{aligned}$$

b

$$\begin{aligned}
f(x) &= 5 \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{5 - 5}{h} \\
&= \lim_{h\to 0} \frac{0}{h} \\
&= \lim_{h\to 0} 0 \quad \{\text{as } h \neq 0\} \\
&= 0
\end{aligned}$$

c $f(x) = x^3$, $\quad \therefore \quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\begin{aligned}
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 \quad \{\text{as } h \neq 0\} \\
&= 3x^2
\end{aligned}$$

d
$$\begin{aligned}
f(x) &= x^4 \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\
&= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \quad (\text{as } h \neq 0) \\
&= 4x^3
\end{aligned}$$

2 **a**
$$\begin{aligned}
&f(x) = 2x + 5 \\
&\therefore\ f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(2(x+h)+5) - (2x+5)}{h} \\
&= \lim_{h\to 0} \frac{2x + 2h + 5 - 2x - 5}{h} \\
&= \lim_{h\to 0} \frac{2h}{h} \\
&= \lim_{h\to 0} 2 \qquad \{\text{as } h \neq 0\} \\
&= 2
\end{aligned}$$

b
$$\begin{aligned}
&f(x) = x^2 - 3x \\
&\therefore\ f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\
&= \lim_{h\to 0} 2x + h - 3 \quad \{\text{as } h \neq 0\} \\
&= 2x - 3
\end{aligned}$$

c
$$\begin{aligned}
f(x) &= x^3 - 2x^2 + 3 \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 + 3) - (x^3 - 2x^2 + 3)}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 4x - 2h \quad \{\text{as } h \neq 0\} \\
&= 3x^2 - 4x
\end{aligned}$$

3 **a** $f(x) = \dfrac{1}{x+2},$

$\therefore \quad f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)+2} - \dfrac{1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{h(x+h+2)(x+2)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}^{\,1}}{\cancel{h}(x+2)(x+h+2)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+2)(x+h+2)} \quad \{\text{as } h \neq 0\}$$

$$= \frac{-1}{(x+2)^2}$$

b $f(x) = \dfrac{1}{2x-1},$

$\therefore \quad f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{2(x+h)-1} - \dfrac{1}{2x-1}}{h}$$

$$= \lim_{h\to 0} \frac{(2x-1)-[2(x+h)-1]}{h[2(x+h)-1](2x-1)}$$

$$= \lim_{h\to 0} \frac{2x-1-2x-2h+1}{h[2(x+h)-1](2x-1)}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}[2(x+h)-1](2x-1)}$$

$$= \lim_{h\to 0} \frac{-2}{[2(x+h)-1](2x-1)} \quad \{\text{as } h \neq 0\}$$

$$= \frac{-2}{(2x-1)^2}$$

c $f(x) = \dfrac{1}{x^2},$

$\therefore \quad f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{[x+(x+h)][x-(x+h)]}{hx^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{(2x+h)(-\overset{1}{\cancel{h}})}{\cancel{h}x^2(x+h)^2}$$

$$= \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2} \quad \{\text{as } h \neq 0\}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

d $f(x) = \dfrac{1}{x^3},$

$\therefore \quad f'(x)$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^3} - \dfrac{1}{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{x^3-(x+h)^3}{hx^3(x+h)^3}$$

$$= \lim_{h\to 0} \frac{x^3-x^3-3x^2h-3xh^2-h^3}{hx^3(x+h)^3}$$

$$= \lim_{h\to 0} \frac{-3x^2h-3xh^2-h^3}{hx^3(x+h)^3}$$

$$= \lim_{h\to 0} \frac{-3x^2-3xh-h^2}{x^3(x+h)^3} \quad \{\text{as } h \neq 0\}$$

$$= \frac{-3x^2}{x^3 \times x^3}$$

$$= -\frac{3}{x^4}$$

4 **a**

$$
\begin{aligned}
f(x) &= \sqrt{x+2}, \\
\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{x+2+h}-\sqrt{x+2})(\sqrt{x+2+h}+\sqrt{x+2})}{h(\sqrt{x+2+h}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{(x+2+h)-(x+2)}{h(\sqrt{x+2+h}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{\cancel{h}^{1}}{\cancel{h}(\sqrt{x+2+h}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+2+h}+\sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+2}+\sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}
$$

b

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{x}}, \\
\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\times\sqrt{x}} \\
&= \lim_{h\to 0} \frac{(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x+h}\times\sqrt{x}}\left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right) \\
&= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\times\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-\cancel{h}^{1}}{\cancel{h}\sqrt{x+h}\times\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\times\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\
&= \frac{-1}{\sqrt{x}\times\sqrt{x}\times 2\sqrt{x}} \\
&= \frac{-1}{2x\sqrt{x}}
\end{aligned}
$$

b

$$
\begin{aligned}
f(x) &= \sqrt{2x+1} \\
\therefore\ f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\left(\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}\right) \\
&= \lim_{h\to 0} \frac{[2(x+h)+1]-[2x+1]}{h(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\
&= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \qquad \{\text{as} \quad h \neq 0\} \\
&= \lim_{h\to 0} \frac{2}{(\sqrt{2(x+h)+1}+\sqrt{2x+1}} \\
&= \frac{2}{2\sqrt{2x+1}} \\
&= \frac{1}{\sqrt{2x+1}}
\end{aligned}
$$

5

Function	*Derivative*
x^1	$1x^0 = 1$
x^2	$2x^1 = 2x$
x^3	$3x^2$
x^4	$4x^3$

Function	*Derivative*
x^{-1}	$-1x^{-2}$
x^{-2}	$-2x^{-3}$
x^{-3}	$-3x^{-4}$

Function	*Derivative*
$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$
$x^{-\frac{1}{2}}$	$-\frac{1}{2}x^{-\frac{3}{2}}$

If $f(x) = x^n, \quad f'(x) = nx^{n-1}$

6 In order to prove this rule, we need the *binomial expansion* discussed in **Chapter 9**.

$$\begin{aligned}
\text{Now if} \quad f(x) &= x^n, \\
\text{then} \quad f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ... + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n\right] - x^n}{h} \\
&\qquad \{\text{for } n \in Z^+ \text{ using the Binomial expansion}\} \\
&= \lim_{h\to 0} \frac{\left[x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ... + \binom{n}{n-1}xh^{n-1} + h^n\right] - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ... + \binom{n}{n-1}xh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} \left[nx^{n-1} + \binom{n}{2}x^{n-2}h + ... + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right] \\
&= nx^{n-1} \quad \text{for } n \in Z^+ \text{ as required}
\end{aligned}$$

EXERCISE 21D

1 **a** $f(x) = x^3,$

$\therefore \; f'(x) = 3x^2$

b $f(x) = 2x^3,$

$$\begin{aligned}\therefore \; f'(x) &= 3 \times 2x^2 \\ &= 6x^2\end{aligned}$$

c $f(x) = 7x^2,$

$$\begin{aligned}\therefore \; f'(x) &= 2 \times 7x \\ &= 14x\end{aligned}$$

d $f(x) = x^2 + x,$

$\therefore \; f'(x) = 2x + 1$

e $f(x) = 4 - 2x^2,$

$$\begin{aligned}\therefore \; f'(x) &= 0 - 2 \times 2x \\ &= -4x\end{aligned}$$

f $f(x) = x^2 + 3x - 5,$

$$\begin{aligned}\therefore \; f'(x) &= 2x + 3 - 0 \\ &= 2x + 3\end{aligned}$$

g

$$\begin{aligned}f(x) &= x^3 + 3x^2 + 4x - 1 \\ \therefore \; f'(x) &= 3x^2 + 2(3x) + 4 - 0 \\ &= 3x^2 + 6x + 4\end{aligned}$$

h

$$\begin{aligned}f(x) &= 5x^4 - 6x^2 \\ \therefore \; f'(x) &= 4(5x^3) - 2(6x) \\ &= 20x^3 - 12x\end{aligned}$$

i

$$\begin{aligned}f(x) &= \frac{3x-6}{x} = 3 - 6x^{-1} \\ \therefore \; f'(x) &= 0 - (-1) \times 6x^{-2} \\ &= \frac{6}{x^2}\end{aligned}$$

j

$$\begin{aligned}f(x) &= \frac{2x-3}{x^2} = \frac{2x}{x^2} - \frac{3}{x^2} \\ &\qquad = 2x^{-1} - 3x^2 \\ \therefore \; f'(x) &= -2x^{-2} + 6x^{-3} = \frac{-2}{x^2} + \frac{6}{x^3}\end{aligned}$$

k

$$\begin{aligned}f(x) &= \frac{x^3+5}{x} = x^2 + 5x^{-1} \\ \therefore \; f'(x) &= 2x - 5x^{-2} \\ &= 2x - \frac{5}{x^2}\end{aligned}$$

l

$$\begin{aligned}f(x) &= \frac{x^3+x-3}{x} = x^2 + 1 - 3x^{-1} \\ \therefore \; f'(x) &= 2x + 0 + 3x^{-2} \\ &= 2x + \frac{3}{x^2}\end{aligned}$$

m $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\therefore\ f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = \frac{-1}{2x\sqrt{x}}$

n $f(x) = (2x-1)^2 = 4x^2 - 4x + 1$

$\therefore\ f'(x) = 8x - 4$

o $f(x) = (x+2)^3$

$= x^3 + 3x^2(2) + 3x(2^2) + 2^3$

$= x^3 + 6x^2 + 12x + 8 \qquad \therefore\ f'(x) = 3x^2 + 12x + 12$

2 **a** $y = 2x^3 - 7x^2 - 1$

$\therefore\ \frac{dy}{dx} = 6x^2 - 14x$

b $y = \pi x^2$

$\therefore\ \frac{dy}{dx} = 2\pi x$

c $y = \frac{1}{5x^2} = \frac{1}{5}x^{-2}$

$\therefore\ \frac{dy}{dx} = -\frac{2}{5}x^{-3} = \frac{-2}{5x^3}$

d $y = 100x$

$\therefore\ \frac{dy}{dx} = 100$

e $y = 10(x+1)$

$= 10x + 10$

$\therefore\ \frac{dy}{dx} = 10$

f $y = 4\pi x^3$

$\therefore\ \frac{dy}{dx} = 12\pi x^2$

3 **a** $\frac{d}{dx}(6x+2)$

$= 6$

b $\frac{d}{dx}(x\sqrt{x})$

$= \frac{d}{dx}(x^{\frac{3}{2}})$

$= \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x}$

c $\frac{d}{dx}(5-x)^2$

$= \frac{d}{dx}(25 - 10x + x^2)$

$= -10 + 2x$

$= 2x - 10$

d $\frac{d}{dx}\left(\frac{6x^2 - 9x^4}{3x}\right)$

$= \frac{d}{dx}(2x - 3x^3)$

$= 2 - 9x^2$

e $\frac{d}{dx}\left(4x - \frac{1}{4x}\right)$

$= \frac{d}{dx}\left(4x - \frac{1}{4}x^{-1}\right)$

$= 4 + \frac{1}{4}x^{-2}$

$= 4 + \frac{1}{4x^2}$

f $\frac{d}{dx}\left(x(x+1)(2x-5)\right)$

$= \frac{d}{dx}\left(x(2x^2 - 3x - 5)\right)$

$= \frac{d}{dx}\left(2x^3 - 3x^2 - 5x\right)$

$= 6x^2 - 6x - 5$

4 **a** $y = x^2$ at $x = 2$

$f(x) = x^2$

$\therefore\ f'(x) = 2x$

and so $f'(2) = 2(2)$

$\therefore$ tangent has slope of 4

b $y = \frac{8}{x^2}$ at $x = 9$

$f(x) = 8x^{-2}$

$\therefore\ f'(x) = -16x^3 = -\frac{16}{x^3}$

and so $f'(9) = -\frac{16}{729}$

$\therefore$ tangent has slope of $-\frac{16}{729}$

c $y = 2x^2 - 3x + 7$ at $x = -1$

$f(x) = 2x^2 - 3x + 7$

$\therefore\ f'(x) = 4x - 3 + 0$

and so $f'(-1) = 4(-1) - 3$

$= -7$

$\therefore$ tangent has slope of -7

d $y = \frac{2x^2 - 5}{x}$ at $x = 2$

$f(x) = 2x - 5x^{-1}$

$\therefore\ f'(x) = 2 + 5x^{-2}$

$= 2 + \frac{5}{x^2}$

and so $f'(2) = 2 + \frac{5}{4} = \frac{13}{4}$

$\therefore$ tangent has slope of $\frac{13}{4}$

e $y = \dfrac{x^2 - 4}{x^2}$ at $x = 4$

$$\begin{aligned} f(x) &= 1 - 4x^{-2} \\ \therefore\quad f'(x) &= 0 + 8x^{-3} \\ &= \frac{8}{x^3} \end{aligned}$$

and so $f'(4) = \dfrac{8}{4^3} = \frac{1}{8}$

$\therefore$ tangent has slope of $\frac{1}{8}$

f $y = \dfrac{x^3 - 4x - 8}{x^2}$ at $x = -1$

$$\begin{aligned} f(x) &= x - 4x^{-1} - 8x^{-2} \\ \therefore\quad f'(x) &= 1 + 4x^{-2} + 16x^{-3} \\ &= 1 + \frac{4}{x^2} + \frac{16}{x^3} \end{aligned}$$

$$\begin{aligned} \text{and so}\quad f'(-1) &= 1 + 4 - 16 \\ &= -11 \end{aligned}$$

$\therefore$ tangent has slope of -11

5 **a**

$$\begin{aligned} f(x) &= 4\sqrt{x} + x = 4x^{\frac{1}{2}} + x \\ \therefore\quad f'(x) &= 4\left(\tfrac{1}{2}\right)x^{-\frac{1}{2}} + 1 \\ &= \frac{2}{\sqrt{x}} + 1 \end{aligned}$$

b

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{\frac{1}{3}} \\ \therefore\quad f'(x) &= \tfrac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}} \end{aligned}$$

c

$$\begin{aligned} f(x) &= -\frac{2}{\sqrt{x}} = -2x^{-\frac{1}{2}} \\ \therefore\quad f'(x) &= -2(-\tfrac{1}{2})\,x^{-\frac{3}{2}} \\ &= x^{-\frac{3}{2}} \\ &= \frac{1}{x\sqrt{x}} \end{aligned}$$

d

$$\begin{aligned} f(x) &= 2x - \sqrt{x} = 2x - x^{\frac{1}{2}} \\ \therefore\quad f'(x) &= 2 - \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= 2 - \frac{1}{2\sqrt{x}} \end{aligned}$$

e

$$\begin{aligned} f(x) &= \frac{4}{\sqrt{x}} - 5 = 4x^{-\frac{1}{2}} - 5 \\ \therefore\quad f'(x) &= 4(-\tfrac{1}{2})\,x^{-\frac{3}{2}} - 5 \\ &= -2x^{-\frac{3}{2}} \quad \text{or} \quad \frac{-2}{x\sqrt{x}} \end{aligned}$$

f

$$\begin{aligned} f(x) &= 3x^2 - x\sqrt{x} = 3x^2 - x^{\frac{3}{2}} \\ \therefore\quad f'(x) &= 6x - \tfrac{3}{2}x^{\frac{1}{2}} \\ &= 6x - \tfrac{3}{2}\sqrt{x} \end{aligned}$$

g

$$\begin{aligned} f(x) &= \frac{5}{x^2\sqrt{x}} = 5x^{-\frac{5}{2}} \\ \therefore\quad f'(x) &= 5(-\tfrac{5}{2})x^{-\frac{7}{2}} \\ &= -\tfrac{25}{2}x^{-\frac{7}{2}} \\ &= \frac{-25}{2x^3\sqrt{x}} \end{aligned}$$

h

$$\begin{aligned} f(x) &= 2x - \frac{3}{x\sqrt{x}} = 2x - 3x^{-\frac{3}{2}} \\ \therefore\quad f'(x) &= 2 - 3(-\tfrac{3}{2})x^{-\frac{5}{2}} \\ &= 2 + \tfrac{9}{2}x^{-\frac{5}{2}} \\ &= 2 + \frac{9}{2x^2\sqrt{x}} \end{aligned}$$

6 **a** $y = 4x - \dfrac{3}{x} = 4x - 3x^{-1} \quad \therefore \quad \dfrac{dy}{dx} = 4 + 3x^{-2} = 4 + \dfrac{3}{x^2}$

$\dfrac{dy}{dx}$ is the slope function of $y = 4x - \dfrac{3}{x}$ from which the slope at any point can be found.

b $S = 2t^2 + 4t$ m $\quad \therefore \quad \dfrac{dS}{dt} = 4t + 4$ ms^{-1}

$\dfrac{dS}{dt}$ is the instantaneous rate of change in position at time t, i.e., it is the velocity function.

c $C = 1785 + 3x + 0.002x^2$ dollars.

$\dfrac{dC}{dx} = 3 + 2(0.002)x = 3 + 0.004x$ dollars/toaster

$\dfrac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 21E.1

1 **a** $f(x)=x^2,\quad g(x)=2x+7,$

$\therefore\quad f(g(x))=f(2x+7)=(2x+7)^2$

b $f(x)=2x+7,\quad g(x)=x^2,$

$f(g(x))=f(x^2)=2x^2+7$

c $f(x)=\sqrt{x},\quad g(x)=3-4x,$

$f(g(x))=f(3-4x)=\sqrt{3-4x}$

d $f(x)=3-4x,\quad g(x)=\sqrt{x},$

$f(g(x))=f(\sqrt{x})=3-4\sqrt{x}$

e $f(x)=\dfrac{2}{x},\quad g(x)=x^2+3,$

$f(g(x))=f(x^2+3)=\dfrac{2}{x^2+3}$

f $f(x)=x^2+3,\quad g(x)=\dfrac{2}{x},$

$f(g(x))=f(\frac{2}{x})=\left(\dfrac{2}{x}\right)^2+3=\dfrac{4}{x^2}+3$

2 **a** $f(g(x))=(3x+10)^3\quad\therefore\quad f(x)=x^3,\quad g(x)=3x+10$

b $f(g(x))=\dfrac{1}{2x+4}\quad\therefore\quad f(x)=\dfrac{1}{x},\quad g(x)=2x+4$

c $f(g(x))=\sqrt{x^2-3x}\quad\therefore\quad f(x)=\sqrt{x},\quad g(x)=x^2-3x$

d $f(g(x))=\dfrac{10}{(3x-x^2)^3}\quad\therefore\quad f(x)=\dfrac{10}{x^3},\quad g(x)=3x-x^2$

EXERCISE 21E.2

1 **a** $\dfrac{1}{(2x-1)^2}$

$=(2x-1)^{-2}$

$=u^{-2},$

where $u=2x-1$

b $\sqrt{x^2-3x}$

$=(x^2-3x)^{\frac{1}{2}}$

$=u^{\frac{1}{2}},$

where $u=x^2-3x$

c $\dfrac{2}{\sqrt{2-x^2}}$

$=2(2-x^2)^{-\frac{1}{2}}$

$=2u^{-\frac{1}{2}},$

where $u=2-x^2$

d $\sqrt[3]{x^3-x^2}$

$=(x^3-x^2)^{\frac{1}{3}}$

$=u^{\frac{1}{3}},$

where $u=x^3-x^2$

e $\dfrac{4}{(3-x)^3}$

$=4(3-x)^{-3}$

$=4u^{-3},$

where $u=3-x$

f $\dfrac{10}{x^2-3}$

$=10(x^2-3)^{-1}$

$=10u^{-1},$

where $u=x^2-3$

2 **a** $y=(4x-5)^2$

$\therefore\quad y=u^2$ where $u=4x-5$

Now $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=2u(4)$

$=8u$

$=8(4x-5)$

b $y=\dfrac{1}{5-2x}=u^{-1}$ where $u=5-2x$

Now $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=-u^{-2}(-2)$

$=\dfrac{2}{u^2}$

$=\dfrac{2}{(5-2x)^2}$

c $y=\sqrt{3x-x^2}$

$\therefore\quad y=u^{\frac{1}{2}}$ where $u=3x-x^2$

Now $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}(3-2x)$

$=\dfrac{(3-2x)}{2\sqrt{u}}$

$=\dfrac{3-2x}{2\sqrt{3x-x^2}}$

d $y=(1-3x)^4$

$\therefore\quad y=u^4$ where $u=1-3x$

Now $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=4u^3(-3)$

$=-12u^3$

$=-12(1-3x)^3$

e

$$y = 6(5-x)^3$$
$$\therefore \quad y = 6u^3 \quad \text{where} \quad u = 5-x$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 18u^2(-1)$$
$$= -18u^2$$
$$= -18(5-x)^2$$

f

$$y = \sqrt[3]{2x^3 - x^2}$$
$$\therefore \quad y = u^{\frac{1}{3}} \quad \text{where} \quad u = 2x^3 - x^2$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \tfrac{1}{3}u^{-\frac{2}{3}}(6x^2 - 3x)$$
$$= \frac{6x^2 - 2x}{3\sqrt[3]{(2x^3 - x^2)^2}}$$

g

$$y = \frac{6}{(5x-4)^2}$$
$$\therefore \quad y = 6u^{-2} \quad \text{where} \quad u = 5x-4$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -12u^{-3}(5)$$
$$= -\frac{60}{u^3}$$
$$= \frac{-60}{(5x-4)^3}$$

h

$$y = \frac{4}{3x - x^2}$$
$$\therefore \quad y = 4u^{-1} \quad \text{where} \quad u = 3x - x^2$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -4u^{-2}(3-2x)$$
$$= \frac{-4(3-2x)}{u^2}$$
$$= \frac{-4(3-2x)}{(3x-x^2)^2}$$

i

$$y = 2\left(x^2 - \frac{2}{x}\right)^3$$
$$\therefore \quad y = 2u^3 \quad \text{where} \quad u = x^2 - 2x^{-1}$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6u^2(2x + 2x^{-2})$$
$$= 6\left(2x + \frac{2}{x^2}\right)\left(x^2 - \frac{2}{x}\right)^2$$

3 **a** $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

$$\therefore \quad y = \sqrt{u} \quad \text{where} \quad u = 1 - x^2$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \tfrac{1}{2}u^{-\frac{1}{2}}(-2x)$$
$$= \frac{-x}{\sqrt{u}}$$
$$= \frac{-x}{\sqrt{1-x^2}}$$
$$\text{at} \quad x = \tfrac{1}{2}, \quad \frac{dy}{dx} = \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = -\tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)$$

$\therefore$ slope of tangent $= -\frac{1}{\sqrt{3}}$

b $y = (3x+2)^6$ at $x = -1$

$$\therefore \quad y = u^6 \quad \text{where} \quad u = 3x+2$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6u^5(3)$$
$$= 18u^5$$
$$= 18(3x+2)^5$$
$$\text{at} \quad x = -1, \quad \frac{dy}{dx} = 18(-1)^5$$

$\therefore$ slope of tangent $= -18$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

$$\therefore \quad y = u^{-4} \quad \text{where} \quad u = 2x-1$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -4u^{-5}(2)$$
$$= \frac{-8}{u^5}$$
$$= \frac{-8}{(2x-1)^5}$$
$$\text{at} \quad x = 1, \quad \frac{dy}{dx} = \frac{-8}{1^5}$$

$\therefore$ slope of tangent $= -8$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

$$\therefore \quad y = 6u^{\frac{1}{3}} \quad \text{where} \quad u = 1-2x$$
$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6(\tfrac{1}{3})u^{-\frac{2}{3}}(-2)$$
$$= 2u^{-\frac{2}{3}}(-2)$$
$$= \frac{-4}{\sqrt[3]{u^2}}$$
$$= \frac{-4}{\sqrt[3]{(1-2x)^2}}$$
$$\text{at} \quad x = 0, \quad \frac{dy}{dx} = \frac{-4}{\sqrt[3]{1^2}}$$

$\therefore$ slope of tangent $= -4$

e $y = \dfrac{4}{x+2\sqrt{x}}$ at $x = 4$

$$\therefore \quad y = 4u^{-1} \quad \text{where} \quad u = x + 2x^{\frac{1}{2}}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= -4u^{-2}(1 + x^{-\frac{1}{2}}) \\ &= -\frac{4}{u^2}\left(1 + \frac{1}{\sqrt{x}}\right) \\ &= \frac{-4}{(x+2\sqrt{x})^2}\left(1 + \frac{1}{\sqrt{x}}\right) \end{aligned}$$

$$\begin{aligned} \text{at } x = 4, \quad \frac{dy}{dx} &= \frac{-4}{(4+4)^2}(1 + \tfrac{1}{2}) \\ &= -\tfrac{6}{64} \end{aligned}$$

$\therefore$ slope of tangent $= -\tfrac{3}{32}$

f $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$

$$\therefore \quad y = u^3 \quad \text{where} \quad u = x + x^{-1}$$

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 3u^2(1 - u^{-2}) \\ &= 3\left(x + \frac{1}{x}\right)^2\left(1 - \frac{1}{x^2}\right) \end{aligned}$$

$$\text{at} \quad x = 1, \quad \frac{dy}{dx} = 3(1+1)^2(1-1)$$

$\therefore$ slope of tangent $= 0$

4 **a**

$$y = x^3 \quad \therefore \quad \frac{dy}{dx} = 3x^2$$

$$x = y^{\frac{1}{3}} \quad \therefore \quad \frac{dx}{dy} = \tfrac{1}{3}y^{-\frac{2}{3}}$$

$$\begin{aligned} \frac{dy}{dx}\frac{dx}{dy} &= 3x^2\left(\tfrac{1}{3}\right)y^{-\frac{2}{3}} \\ &= x^2(y)^{-\frac{2}{3}} \\ &= x^2(x^3)^{-\frac{2}{3}} \quad \{\text{substituting} \quad y = x^3\} \\ &= x^2(x^{-2}) \\ &= x^0 \\ &= 1 \quad \text{as required.} \end{aligned}$$

b We know that $\dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{dy}{dx}$ {chain rule}

$$\begin{aligned} \therefore \quad \frac{dy}{dx}\frac{dx}{dy} &= \frac{dy}{dy} \\ &= 1 \end{aligned}$$

EXERCISE 21F.1

1 **a** $y = x^2(2x-1)$ is the product of $u = x^2$ and $v = 2x - 1$

$\therefore$ $u' = 2x$ and $v' = 2$

$$\text{Now} \quad \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 2x(2x-1) + x^2(2) \\ &= 4x^2 - 2x + 2x^2 \\ &= 6x^2 - 2x \end{aligned}$$

b $y = 4x(2x+1)^3$ is the product of $u = 4x$ and $v = (2x+1)^3$

$\therefore$ $u' = 4$ and $v' = 6(2x+1)^2$

$$\text{Now} \quad \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 4(2x+1)^3 + 24x(2x+1)^2 \\ &= [4(2x+1) + 24x](2x+1)^2 \\ &= [32x+4](2x+1)^2 \end{aligned}$$

c $y = x^2\sqrt{3-x}$ is the product of $u = x^2$ and $v = (3-x)^{\frac{1}{2}}$

$\therefore\ u' = 2x$ and $v' = -\frac{1}{2}(3-x)^{-\frac{1}{2}}$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx} = 2x(3-x)^{\frac{1}{2}} + x^2\left[-\tfrac{1}{2}(3-x)^{-\frac{1}{2}}\right]$$

$$= 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}}$$

d $y = \sqrt{x}(x-3)^2$ is the product of $y = x^{\frac{1}{2}}$ and $v = (x-3)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x-3)^1$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$$

e $y = 5x^2(3x^2-1)^2$ is the product of $u = 5x^2$ and $v = (3x^2-1)^2$

$\therefore\ u' = 10x$ and $v' = 12x(3x^2-1)$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx} = 10x(3x^2-1)^2 + 5x^2(12x)(3x^2-1)$$

$$= 10x(3x^2-1)^2 + 60x^3(3x^2-1)$$

f $y = \sqrt{x}(x-x^2)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-x^2)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(1-2x)(x-x^2)^2$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(1-2x)(x-x^2)^2$$

2 **a** $y = x^4(1-2x)^2$ is the product of $u = x^4$ and $v = (1-2x)^2$

$u' = 4x^3$ and $v' = 2(1-2x)^1(-2)$

$= -4(1-2x)$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx} = 4x^3(1-2x)^2 - 4x^4(1-2x)$$

at $x = -1$, $\frac{dy}{dx} = -4(3^2) - 4(3) = -48$ $\therefore$ slope of tangent $= -48$

b $y = \sqrt{x}(x^2-x+1)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x^2-x+1)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x^2-x+1)(2x-1)$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore\ \frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + 2\sqrt{x}(2x-1)(x^2-x+1)$$

at $x = 4$, $\frac{dy}{dx} = \frac{1}{4}(13)^2 + 4(7)(13) = 406\frac{1}{4}$

$\therefore$ slope of tangent $= 406\frac{1}{4}$

c $y = x\sqrt{1-2x}$ is the product of $u = x$ and $v = (1-2x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)$

$= -(1-2x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore\ \dfrac{dy}{dx} = \sqrt{1-2x} - \dfrac{x}{\sqrt{1-2x}}$

at $x = -4$, $\dfrac{dy}{dx} = \sqrt{9} - \dfrac{(-4)}{\sqrt{9}} = 3 + \frac{4}{3} = \frac{13}{3}$

$\therefore$ slope of tangent $= \frac{13}{3}$

d $y = x^3\sqrt{5-x^2}$ is the product of $u = x^3$ and $v = (5-x^2)^{\frac{1}{2}}$

$\therefore\ u' = 3x^2$ and $v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x)$

$= -x(5-x^2)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$\therefore\ \dfrac{dy}{dx} = 3x^2\sqrt{5-x^2} - \dfrac{x}{\sqrt{5-x^2}}$

at $x = 1$, $\dfrac{dy}{dx} = 3(1)^2\sqrt{4} - \dfrac{1}{\sqrt{4}} = 6 - \frac{1}{2} = \frac{11}{2}$ $\therefore$ slope of tangent $= \frac{11}{2}$

3 $y = \sqrt{x}(3-x)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (3-x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(3-x)^1(-1) = -2(3-x)$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}(3-x)^2 - 2\sqrt{x}(3-x)$$

$$= \frac{(3-x)^2 - 4x(3-x)}{2\sqrt{x}}$$

$$= \frac{(3-x)\left[(3-x) - 4x\right]}{2\sqrt{x}}$$

$$= \frac{(3-x)(3-5x)}{2\sqrt{x}} \quad \text{as required}$$

Tangents are horizontal when their slopes are 0.

$\therefore\ \dfrac{dy}{dx} = 0$ when $(3-x)(3-5x) = 0$

i.e., $3 - x = 0$ or $3 - 5x = 0$

i.e., $x = 3$ or $x = \frac{3}{5}$

EXERCISE 21F.2

1 **a** $y = \dfrac{1+3x}{2-x}$ is a quotient where $u = 1 + 3x$ and $v = 2 - x$

$\therefore\ u' = 3$ and $v' = -1$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{3(2-x) - (-1)(1+3x)}{(2-x)^2} = \frac{7}{(2-x)^2}$$

b $y = \dfrac{x^2}{2x+1}$ is a quotient where $u = x^2$ and $v = 2x+1$

$\therefore\ u' = 2x$ and $v' = 2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{2x(2x+1) - x^2(2)}{(2x+1)^2} = \frac{2x^2+2x}{(2x+1)^2}$$

c $y = \dfrac{x}{x^2-3}$ is a quotient where $u = x$ and $v = x^2 - 3$

$\therefore\ u' = 1$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{1(x^2-3) - x(2x)}{(x^2-3)^2} = \frac{-3-x^2}{(x^2-3)^2}$$

d $y = \dfrac{\sqrt{x}}{1-2x}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 1-2x$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = -2$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\dfrac{1-2x}{2\sqrt{x}} - (-2)\sqrt{x}}{(1-2x)^2} = \frac{1-2x+4x}{2\sqrt{x}(1-2x)^2} = \frac{1+2x}{2\sqrt{x}(1-2x)^2}$$

e $y = \dfrac{x^2-3}{3x-x^2}$ is a quotient where $u = x^2-3$ and $v = 3x - x^2$

$\therefore\ u' = 2x$ and $v' = 3-2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2} = \frac{6x^2-2x^3-3x^2+2x^3+9-6x}{(3x-x^2)^2}$$

$$= \frac{3x^2-6x+9}{(3x-x^2)^2}$$

f $y = \dfrac{x}{\sqrt{1-3x}}$ is a quotient where $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$\therefore\ u' = 1$ and $v' = -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{\sqrt{1-3x} - x\left(\dfrac{-3}{2\sqrt{1-3x}}\right)}{(1-3x)} = \frac{2(1-3x)+3x}{2(1-3x)^{\frac{3}{2}}} = \frac{2-3x}{2(1-3x)^{\frac{3}{2}}}$$

2 **a** $y = \dfrac{x}{1-2x}$ is a quotient where $u = x$ and $v = 1-2x$

$\therefore\ u' = 1$ and $v' = -2$ Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx} = \frac{1(1-2x) - x(-2)}{(1-2x)^2} = \frac{1}{(1-2x)^2}$$

at $x = 1$ $\dfrac{dy}{dx} = \dfrac{1}{(1-2)^2} = \dfrac{1}{(-1)^2} = 1$ $\therefore$ slope of tangent $= 1$

b $y=\dfrac{x^3}{x^2+1}$ is a quotient where $u=x^3$ and $v=x^2+1$

$\therefore\ u'=3x^2$ and $v'=2x$

Now $\dfrac{dy}{dx}=\dfrac{u'v-uv'}{v^2}=\dfrac{3x^2(x^2+1)-x^3(2x)}{(x^2+1)^2}=\dfrac{x^4+3x^2}{(x^2+1)^2}$

$\therefore$ at $x=-1$ $\dfrac{dy}{dx}=\dfrac{1+3}{(1+1)^2}=\dfrac{4}{4}=1$

and so, the slope of tangent $=1$

c $y=\dfrac{\sqrt{x}}{2x+1}$ is a quotient where $u=x^{\frac{1}{2}}$ and $v=2x+1$

$u'=\frac{1}{2}x^{-\frac{1}{2}}$ and $v'=2$

Now $\dfrac{dy}{dx}=\dfrac{u'v-uv'}{v^2}=\dfrac{\dfrac{1}{2\sqrt{x}}(2x+1)-\sqrt{x}(2)}{(2x+1)^2}$

at $x=4$ $\dfrac{dy}{dx}=\dfrac{\frac{9}{4}-4}{81}=\dfrac{\left(\frac{9}{4}-4\right)}{81}\times\dfrac{4}{4}=\dfrac{9-16}{324}$

$\therefore$ slope of tangent $=-\frac{7}{324}$

d $y=\dfrac{x^2}{\sqrt{x^2+5}}$ is a quotient where $u=x^2$ and $v=(x^2+5)^{\frac{1}{2}}$

$$\therefore\ u'=2x \text{ and } v'=\tfrac{1}{2}(x^2+5)^{-\frac{1}{2}}(2x)$$
$$=x(x^2+5)^{-\frac{1}{2}}$$

Now $\dfrac{dy}{dx}=\dfrac{u'v-uv'}{v^2}=\dfrac{2x\sqrt{x^2+5}-x^2\left(\dfrac{x}{\sqrt{x^2+5}}\right)}{(x^2+5)}$

at $x=-2$ $\dfrac{dy}{dx}=\dfrac{-4(3)-\left(\frac{-8}{3}\right)}{9}=\dfrac{\left(-12+\frac{8}{3}\right)}{9}\times\dfrac{3}{3}=\dfrac{-36+8}{27}$

$\therefore$ slope of tangent $=-\frac{28}{27}$

3 **a** $y=\dfrac{2\sqrt{x}}{1-x}$ is a quotient where $u=2x^{\frac{1}{2}}$ and $v=1-x$

$\therefore\ u'=2x^{\frac{1}{2}}$ and $v'=-1$

Now $\dfrac{dy}{dx}=\dfrac{u'v-uv'}{v^2}$ {quotient rule}

$$\therefore\ \frac{dy}{dx}=\left(\frac{\dfrac{1}{\sqrt{x}}(1-x)-2\sqrt{x}(-1)}{(1-x)^2}\right)\frac{\sqrt{x}}{\sqrt{x}}=\frac{(1-x)+2x}{\sqrt{x}(1-x)^2}=\frac{x+1}{\sqrt{x}(1-x)^2}\quad\text{as required.}$$

i $\dfrac{dy}{dx}=0$ when $x+1=0$ i.e., $x=-1$. However $\dfrac{dy}{dx}$ is not defined for $x\leqslant 0$ because of the $\sqrt{x}$ term. Hence $\dfrac{dy}{dx}$ never equals 0.

ii $\dfrac{dy}{dx}$ is undefined when $x\leqslant 0$ and when $x=1$

b $y = \dfrac{x^2-3x+1}{x+2}$ is a quotient where $y = x^2-3x+1$ and $v = x+2$

$\therefore \quad u' = 2x-3$ and $v' = 1$

Now $\dfrac{dy}{dx} = \dfrac{u'v-uv'}{v^2}$ {quotient rule}

$$\therefore \quad \frac{dy}{dx} = \frac{(2x-3)(x+2)-(x^2-3x+1)(1)}{(x+2)^2} = \frac{2x^2+4x-3x-6-x^2+3x-1}{(x+2)^2}$$

$$= \frac{x^2+4x-7}{(x+2)^2} \quad \text{as required.}$$

i $\dfrac{dy}{dx} = 0$ when $x^2+4x-7 = 0$ i.e., $x = \dfrac{-4 \pm \sqrt{44}}{2} = -2 \pm \sqrt{11}$

ii $\dfrac{dy}{dx}$ is undefined when $(x+2)^2 = 0$ i.e., $x = -2$

EXERCISE 21G

1 **a** $y = x - 2x^2 + 3$ at $x = 2$

Since when $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$, the point of contact is $(2, -3)$.

Now $\dfrac{dy}{dx} = 1 - 4x$, $\therefore$ at $x = 2$, $\dfrac{dy}{dx} = 1 - 8 = -7$

$\therefore$ the tangent has equation $\dfrac{y-(-3)}{x-2} = -7$ i.e., $y + 3 = -7(x-2)$

i.e., $y = -7x + 14 - 3$

i.e., $y = -7x + 11$

b $y = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$ at $x = 4$

Since when $x = 4$, $y = \sqrt{4} + 1 = 3$, the point of contact is $(4, 3)$.

Now $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, $\therefore$ at $x = 4$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore$ the tangent has equation $\dfrac{y-3}{x-4} = \frac{1}{4}$ i.e., $4y - 12 = x - 4$

i.e., $4y = x + 8$

c $y = x^3 - 5x$ at $x = 1$

Since when $x = 1$, $y = 1^3 - 5(1) = -4$ the point of contact is $(1, -4)$.

Now $\dfrac{dy}{dx} = 3x^2 - 5$ $\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 3 - 5 = -2$

$\therefore$ tangent has equation $\dfrac{y-(-4)}{x-1} = -2$ i.e., $y + 4 = -2x + 2$

i.e., $y = -2x - 2$

d $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$. Now $y = \dfrac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$$\therefore \quad \frac{dy}{dx} = -2x^{-\frac{3}{2}}$$

$$\text{at } x = 1, \quad \frac{dy}{dx} = -2\left(1^{-\frac{3}{2}}\right) = -2$$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-1} = -2$ i.e., $y - 4 = -2x + 2$

i.e., $y = -2x + 6$

2 **a** $y = x^2$ at $(3, 9)$

$\dfrac{dy}{dx} = 2x \quad \therefore \quad$ at $x = 3$, $\dfrac{dy}{dx} = 2(3) = 6 = \frac{6}{1}$

$\therefore$ the normal at (3, 9) has slope $-\frac{1}{6}$, so the equation of the normal is

$\dfrac{y-9}{x-3} = -\frac{1}{6}$ i.e., $6y - 54 = -x + 3$ i.e., $6y = -x + 57$

b $y = x^3 - 5x + 2$ at $x = -2$

Since when $x = -2$, $y = (-2)^3 - 5(-2) + 2 = 4$

and so the point of contact is $(-2, 4)$

Now $\dfrac{dy}{dx} = 3x^2 - 5 \quad \therefore$ at $x = -2$, $\dfrac{dy}{dx} = 3(-2)^2 - 5 = 7$

$\therefore$ the normal at $(-2, 4)$ has slope $-\frac{1}{7}$, so the equation of the normal is

$\dfrac{y-4}{x-(-2)} = -\frac{1}{7}$ i.e., $7y - 28 = -(x+2)$ i.e., $7y = -x + 26$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$

Now $y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}} \quad \therefore \quad \dfrac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = -\frac{5}{2}\left(1^{-\frac{3}{2}}\right) - \frac{1}{2}\left(1^{-\frac{1}{2}}\right) = -\frac{5}{2} - \frac{1}{2} = -3$

$\therefore$ the normal at (1, 4) has slope $\frac{1}{3}$, so the equation of the normal is

$\dfrac{y-4}{x-1} = \frac{1}{3}$ i.e., $3y - 12 = x - 1$
i.e., $3y = x + 11$

d $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$

Since when $x = 1$, $y = 8\sqrt{1} - \dfrac{1}{1^2} = 7$ the point of contact is (1, 7)

Now $y = 8\sqrt{x} - \dfrac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2}$

$\therefore \quad \dfrac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 4 + 2 = 6$

$\therefore$ the normal at (1, 7) has slope $-\frac{1}{6}$, so the equation of the normal is

$\dfrac{y-7}{x-1} = -\frac{1}{6}$ i.e., $6y - 42 = -x + 1$
i.e., $6y = -x + 43$

3 **a** $y = 2x^3 + 3x^2 - 12x + 1 \quad \therefore \quad \dfrac{dy}{dx} = 6x^2 + 6x - 12$

Horizontal tangents have slope $= 0$ so

$$\begin{aligned} 6x^2 + 6x - 12 &= 0 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ \therefore \quad x = -2 \quad \text{or} \quad x &= 1 \end{aligned}$$

Now at $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$

and at $x = 1$, $y = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$

i.e., points of contact are $(-2, 21)$ and $(1, -6)$

$\therefore$ tangents are $y = -6$ and $y = 21$

b Now $y = \dfrac{2x+1}{\sqrt{x}} = 2\sqrt{x} + \dfrac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \dfrac{1}{\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}$

Horizontal tangents have slope $= 0$

$$\begin{aligned} \therefore \quad \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}} &= 0 \\ \therefore \quad \frac{2x-1}{2x\sqrt{x}} &= 0 \\ \therefore \quad 2x - 1 &= 0 \\ \therefore \quad x &= \tfrac{1}{2} \end{aligned}$$

Now at $x = \frac{1}{2}$, $y = \dfrac{2\left(\frac{1}{2}\right)+1}{\sqrt{\frac{1}{2}}} = \dfrac{2}{\frac{1}{2}\sqrt{2}} = 2\sqrt{2}$

$\therefore$ the only horizontal tangent touches at $(\frac{1}{2}, 2\sqrt{2})$

c Now $y = 2x^3 + kx^2 - 3 \quad \therefore \quad \dfrac{dy}{dx} = 6x^2 + 2kx$

and so when $x = 2$, $\dfrac{dy}{dx} = 4$

$$\begin{aligned} \therefore \quad 6(2)^2 + 2k(2) &= 4 \\ \therefore \quad 24 + 4k &= 4 \\ \therefore \quad 4k &= -20 \\ \therefore \quad k &= -5 \end{aligned}$$

d Now $y = 1 - 3x + 12x^2 - 8x^3 \quad \therefore \quad \dfrac{dy}{dx} = -3 + 24x - 24x^2$

when $x = 1$, $\dfrac{dy}{dx} = -3 + 24 - 24 = -3$

i.e., the tangent at (1, 2) has slope -3

$\therefore \quad -3 + 24x - 24x^2 = -3$ for all points x where the tangent has slope $= -3$

$\therefore \quad 24x^2 - 24x = 0$

$\therefore \quad 24x(x-1) = 0$

i.e., when $x = 0$ or $x = 1$

$\therefore$ the other x-value for which the tangent ot the curve has slope -3 is $x = 0$ and when $x = 0$, $y = 1 - 0 + 0 - 0 = 1$

i.e., the tangent to the curve at (0, 1) is parallel to the tangent at (1, 2)

This tangent has equation $\dfrac{y-1}{x-0} = -3$ i.e., $y = -3x + 1$

4 **a** Now $y = x^2 + ax + b \quad \therefore \quad \dfrac{dy}{dx} = 2x + a$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 2 + a$

$\therefore$ the slope of the tangent to the curve at $x = 1$ will be $2 + a$

However the equation of the tangent is $y + 2x = 6$ i.e., $y = -2x + 6$

and so the slope of the tangent is -2.

$$\begin{aligned} \therefore \quad 2 + a &= -2 \\ a &= -4 \\ \therefore \quad y &= x^2 - 4x + b \end{aligned}$$

Also, at $x = 1$, the tangent line contacts the curve

$$\begin{aligned} \text{i.e.,} \quad 1^2 - 4(1) + b &= -2(1) + 6 \\ 1 - 4 + b &= 4 \\ b &= 7 \end{aligned}$$

$\therefore \quad a = -4, \quad b = 7$

b Now $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$$\therefore \quad \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

$$\therefore \text{ at } x = 4, \quad \frac{dy}{dx} = \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right) = \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right) = \frac{a}{4} - \frac{b}{16}$$

$\therefore$ the slope of the tangent to the curve at $x = 4$ will be $\dfrac{a}{4} - \dfrac{b}{16} = \dfrac{4a-b}{16}$

However the equation of the *normal* is $4x + y = 22$, i.e., $y = -4x + 22$

$\therefore$ the normal has slope -4

$\therefore$ the tangent has slope $\frac{1}{4}$ and so, $\dfrac{4a-b}{16} = \frac{1}{4}$ $\quad\therefore\quad 4a - b = 4$

i.e., $b = 4a - 4$ ...... (1)

Also, at $x = 4$ the normal line intersects the curve.

$$\text{i.e., } a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22$$

$$\therefore \quad 2a + \frac{b}{2} = 6$$

Consequently, $2a + \dfrac{4a-4}{2} = 6$ using (1)

$$\therefore \quad 2a + 2a - 2 = 6$$

$$\therefore \quad 4a = 8$$

$\therefore \quad a = 2$ and so $b = 4(2) - 4 = 4$ from (1)

5 **a** $y = \sqrt{2x+1}$ when $x = 4$, $y = \sqrt{2(4)+1} = 3$ $\quad\therefore$ the point of contact is $(4, 3)$

Since $y = (2x+1)^{\frac{1}{2}}$ then $\dfrac{dy}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) = \dfrac{1}{\sqrt{2x+1}}$

$$\therefore \text{ at } x = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{2(4)+1}} = \tfrac{1}{3}$$

so the tangent has equation $\dfrac{y-3}{x-4} = \frac{1}{3}$ i.e., $3y = x + 5$

b $y = \dfrac{1}{2-x} = (2-x)^{-1}$ at $x = -1$, $y = \dfrac{1}{2-(-1)} = \frac{1}{3}$

$\therefore$ the point of contact is $(-1, \frac{1}{3})$

$$\text{Now } \frac{dy}{dx} = -1(2-x)^{-2}(-1) = \frac{1}{(2-x)^2}$$

$$\therefore \text{ at } x = -1, \quad \frac{dy}{dx} = \frac{1}{(2-(-1))^2} = \tfrac{1}{9}$$

$\therefore$ the tangent has equation $\dfrac{y - \frac{1}{3}}{x-(-1)} = \frac{1}{9}$ i.e., $9y - 3 = x + 1$ i.e., $9y = x + 4$

c $f(x) = \dfrac{x}{1-3x}$ at $(-1, -\frac{1}{4})$

$f(x)$ is a quotient where $u = x$ and $v = 1 - 3x$ $\quad\therefore\quad u' = 1$ and $v' = -3$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \quad f'(x) = \dfrac{1(1-3x) - x(-3)}{(1-3x)^2} = \dfrac{1}{(1-3x)^2}$

$\therefore \quad f(-1) = \dfrac{1}{(1-3(-1))^2} = \frac{1}{16}$

So the tangent has equation $\dfrac{y - \left(-\frac{1}{4}\right)}{x - (-1)} = \frac{1}{16}$ i.e., $16y + 4 = x + 1$
i.e., $16y = x - 3$

d $f(x) = \dfrac{x^2}{1-x}$ at $(2, -4)$

$f(x)$ is a quotient where $u = x^2$ and $v = 1 - x$
$u' = 2x$ and $v' = -1$

Now $f'(x) = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$\therefore \quad f'(x) = \dfrac{2x(1-x) - x^2(-1)}{(1-x)^2} = \dfrac{2x - 2x^2 + x^2}{(1-x)^2} = \dfrac{2x - x^2}{(1-x)^2}$

$\therefore \quad f'(2) = \dfrac{2(2) - 2^2}{(1-2)^2} = \dfrac{4-4}{1} = 0$

As the tangent has slope 0, i.e., is horizontal, it has equation $y = c$ and as the contact point is $(2, -4)$, the tangent has equation $y = -4$

6 **a** $y = \dfrac{1}{(x^2+1)^2}$ at $(1, \frac{1}{4})$

As $y = (x^2+1)^{-2}$,

$\dfrac{dy}{dx} = -2(x^2+1)^{-3}(2x) = \dfrac{-4x}{(x^2+1)^3}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{-4}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2}$

$\therefore$ normal at $(1, \frac{1}{4})$ has slope 2.

So the equation of the normal is $\dfrac{y - \frac{1}{4}}{x - 1} = 2$ i.e., $y - \frac{1}{4} = 2x - 2$
$2x - y = \frac{7}{4}$
i.e., $y = 2x - \frac{7}{4}$

b $y = \dfrac{1}{\sqrt{3-2x}}$ at $x = -3$, $y = \dfrac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$

$\therefore$ the point of contact is $(-3, \frac{1}{3})$

Now $y = (3-2x)^{-\frac{1}{2}}$

$\therefore \quad \dfrac{dy}{dx} = -\frac{1}{2}(3-2x)^{-\frac{3}{2}}(-2) = (3-2x)^{-\frac{3}{2}}$

$\therefore$ at $x = -3$, $\dfrac{dy}{dx} = (3-2(-3))^{-\frac{3}{2}} = 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$

$\therefore$ normal at $(-3, \frac{1}{3})$ has slope -27,

So the equation of the normal is $\dfrac{y - \frac{1}{3}}{x - (-3)} = -27$ i.e., $y - \frac{1}{3} = -27(x+3)$
i.e., $y = -27x - \frac{242}{3}$

c $f(x) = \sqrt{x}(1-x)^2$

since $f(4) = \sqrt{4}(1-4)^2 = 18$, the point of contact is $(4, 18)$

Now $f(x)$ is a product where $u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$\therefore$ $u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-x)(-1) = -2(1-x)$

Now $f'(x) = u'v + uv'$ {product rule}

$\therefore$ $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^2 - x^{\frac{1}{2}}2(1-x)$

$\therefore$ $f'(4) = \frac{1}{2\sqrt{4}}(1-4)^2 - \sqrt{4}(2)(1-4) = \frac{1}{4}(9) - 2(2)(-3) = \frac{57}{4}$

$\therefore$ the normal at (4, 18) has slope $-\frac{4}{57}$.

So the equation of the normal is $\dfrac{y-18}{x-4} = -\frac{4}{57}$ i.e., $57(y-18) = -4(x-4)$

i.e., $57y = -4x + 1042$

d $f(x) = \dfrac{x^2-1}{2x+3}$

Since $f(-1) = \dfrac{(-1)^2-1}{2(-1)+3} = \dfrac{0}{1} = 0$ the point of contact is $(-1, 0)$

Now $f(x)$ is a quotient where $u = x^2 - 1$ and $v = 2x + 3$

$\therefore$ $u' = 2x$ and $v' = 2$

Now $f'(x) = \dfrac{u'v - uv'}{v^2} = \dfrac{2x(2x+3) - (x^2-1)(2)}{(2x+3)^2}$

$\therefore$ $f'(-1) = \dfrac{2(-1)(-2+3) - ((-1)^2-1)(2)}{(2(-1)+3)^2} = \dfrac{-2(-1) - (0)(2)}{(1)^2} = -2$

$\therefore$ the normal at $(-1, 0)$ has slope $\frac{1}{2}$.

So the equation of the normal is $\dfrac{y-0}{x-(-1)} = \frac{1}{2}$ i.e., $2y = x + 1$

7 The tangent has equation $3x + y = 5$, i.e., $y = -3x + 5$

$\therefore$ tangent has slope -3 ...... (1)

Also, at $x = -1$, $y = -3(-1) + 5 = 8$

$\therefore$ the tangent contacts the curve at $(-1, 8)$ ...... (2)

Now $y = a(1-bx)^{\frac{1}{2}}$, $\therefore$ $\dfrac{dy}{dx} = \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b)$

$\therefore$ $-3 = \frac{1}{2}a(1+b)^{-\frac{1}{2}}(-b)$ using (1)

$6 = \dfrac{ab}{\sqrt{1+b}}$ ...... (3)

and $(-1, 8)$ must lie on the curve $y = a\sqrt{1-bx}$

and so $8 = a\sqrt{1+b}$ ...... (4)

$\therefore$ $\dfrac{6\sqrt{1+b}}{b} = \dfrac{8}{\sqrt{1+b}}$ {equating a's in (3) and (4)}

$\therefore$ $6(1+b) = 8b$

$\therefore$ $6 + 6b = 8b$

$\therefore$ $6 = 2b$

$\therefore$ $b = 3$ and $a = \dfrac{8}{\sqrt{4}} = 4$

8 **a** $y = x^3$ at $x = 2$

Since when $x = 2$, $y = 2^3 = 8$ the point of contact is $(2, 8)$

Now $\dfrac{dy}{dx} = 3x^2$ and so at $x = 2$, $\dfrac{dy}{dx} = 3(2)^2 = 12$

$\therefore$ tangent at (2, 8) has slope 12 and its equation is $\dfrac{y-8}{x-2} = 12$ i.e., $y - 8 = 12x - 24$

i.e., $y = 12x - 16$

$\therefore$ the tangent meets the curve where $12x - 16 = x^3$

i.e., $x^3 - 12x + 16 = 0$

Because the tangent touches the curve at $x = 2$, there must be a repeated solution at this point. $\therefore$ $(x-2)^2$ must be a factor of this cubic

$$\therefore \quad (x-2)^2(x+4) = 0$$

$\therefore$ tangent meets curve again when $x = -4$

and when $x = -4$, $y = (-4)^3 = -64$

$\therefore$ tangent meets curve again at $(-4, -64)$

b $y = -x^3 + 2x^2 + 1$ at $x = -1$

Since when $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1 = 4$

and so the point of contact is $(-1, 4)$

Now $\dfrac{dy}{dx} = -3x^2 + 4x$ $\therefore$ at $x = -1$, $\dfrac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$

$\therefore$ tangent at $(-1, 4)$ has slope -7

$\therefore$ its equation is $\dfrac{y-4}{x-(-1)} = -7$ $\therefore$ $y - 4 = -7(x+1)$

i.e., $y = -7x - 3$

Now the tangent meets the curve where $-7x - 3 = -x^3 + 2x^2 + 1$

i.e., $x^3 - 2x^2 - 7x - 4 = 0$

Because the tangent touches the curve at $x = -1$, there must be a repeated solution at this point. $\therefore$ $(x+1)^2$ must be a factor of this cubic

$$\therefore \quad (x+1)^2(x-4) = 0$$

$\therefore$ tangent meets curve again when $x = 4$

and when $x = 4$, $y = -(4)^3 + 2(4)^2 + 1 = -64 + 32 + 1 = -31$

$\therefore$ tangent meets curve again at $(4, -31)$

c $y = x^3 + \dfrac{4}{x}$ at $x = 1$

Since when $x = 1$, $y = 1^3 + \frac{4}{1} = 5$ and so the point of contact is $(1, 5)$

Now $\dfrac{dy}{dx} = 3x^2 - \dfrac{4}{x^2}$ and at $x = 1$, $\dfrac{dy}{dx} = 3 - 4 = -1$

$\therefore$ tangent at (1, 5) has slope -1 and therefore its equation is

$\dfrac{y-5}{x-1} = -1$ i.e., $y - 5 = -x + 1$

i.e., $y = -x + 6$

Now the tangent meets the curve where $-x + 6 = x^3 + \dfrac{4}{x}$

$$\therefore \quad x^3 + x - 6 + \frac{4}{x} = 0$$

$$\therefore \quad x^4 + x^2 - 6x + 4 = 0$$

Using a graphics calculator, this quartic has a graph which touches the x-axis at $x = 1$ and has no other x-intercepts. i.e., the tangent *never* meets the curve again.

9 **a** $y = x^2 - x + 9$ at $x = a$

Since when $x = a$, $y = a^2 - a + 9$ the point of contact is $(a, a^2 - a + 9)$

Now $\dfrac{dy}{dx} = 2x - 1$ $\therefore$ at $x = a$, $\dfrac{dy}{dx} = 2a - 1$

$\therefore$ the slope of the tangent at $(a, a^2 - a + 9)$ is $2a - 1$

$\therefore$ the equation of the tangent is $\dfrac{y - (a^2 - a + 9)}{x - a} = 2a - 1$

$$\begin{aligned} \text{i.e., } y - (a^2 - a + 9) &= (2a - 1)(x - a) \\ \text{i.e., } y &= (2a - 1)x - 2a^2 + a + a^2 - a + 9 \\ \text{i.e., } y &= (2a - 1)x - a^2 + 9 \end{aligned}$$

But this tangent passes through (0, 0)
$$\begin{aligned} \therefore\ 0 &= a^2 - 9 \\ \therefore\ (a + 3)(a - 3) &= 0 \\ \therefore\ a &= \pm 3 \end{aligned}$$

$\therefore$ tangents are: at $a = 3$: $y - (9 - 3 + 9) = 5(x - 3)$

$\therefore$ $y = 5x$, with contact at (3, 15)

at $a = -3$: $y - (9 + 3 + 9) = -7(x + 3)$

$\therefore$ $y = -7x$, with contact at (–3, 21)

points of contact are (3, 15) and (–3, 21)

b Let (a, a^3) lie on $y = x^3$

Now $\dfrac{dy}{dx} = 3x^2$, $\therefore$ at $x = a$, $\dfrac{dy}{dx} = 3a^2$

$\therefore$ the slope of the tangent at (a, a^3) is $3a^2$

$\therefore$ the equation of the tangent is $\dfrac{y - a^3}{x - a} = 3a^2$ i.e., $y - a^3 = (3a^2)(x - a)$

But this tangent passes through (–2, 0)
$$\begin{aligned} \therefore\ a - a^3 &= 3a^2(-2 - a) \\ \therefore\ -a^3 &= -6a^2 - 3a^3 \\ \therefore\ 2a^3 + 6a^2 &= 0 \\ \therefore\ 2a^2(a + 3) &= 0 \\ \therefore\ a &= 0 \text{ or } -3 \end{aligned}$$

If $a = 0$, tangent equation is $y = 0$, with contact point (0, 0)

If $a = -3$, tangent equation is $y - (-27) = 27(x + 3)$

i.e., $y = 27x + 54$, with contact point (–3, –27)

c Let $(a, \sqrt{a})$ lie on $y = \sqrt{x}$.

Now $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ $\therefore$ at $x = a$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{a}}$

$\therefore$ the slope of the tangent at $(a, \sqrt{a})$ is $\dfrac{1}{2\sqrt{a}}$ and so the slope of the normal is $-2\sqrt{a}$

$\therefore$ the normal has equation $\dfrac{y - \sqrt{a}}{x - a} = -2\sqrt{a}$ i.e., $y - \sqrt{a} = -2\sqrt{a}(x - a)$

But this normal passes through (4, 0)
$$\begin{aligned} \therefore\ 0 - \sqrt{a} &= -2\sqrt{a}(4 - a) \\ \therefore\ 2\sqrt{a}(4 - a) - \sqrt{a} &= 0 \\ \therefore\ \sqrt{a}(8 - 2a - 1) &= 0 \\ \therefore\ \sqrt{a}(7 - 2a) &= 0 \\ \therefore\ a &= 0 \text{ or } \tfrac{7}{2} \end{aligned}$$

When $a = 0$, normal equation is $y = 0$, and contact point is (0, 0).

When $a = \frac{7}{2}$,
$$\begin{aligned} y - \sqrt{\tfrac{7}{2}} &= -2\sqrt{\tfrac{7}{2}}\left(x - \tfrac{7}{2}\right) \\ \text{i.e., } \sqrt{2}y - \sqrt{7} &= -2\sqrt{7}\left(x - \tfrac{7}{2}\right) \\ \text{i.e., } \sqrt{2}y + 2\sqrt{7}x &= 7\sqrt{7} + \sqrt{7} \\ \text{i.e., } \sqrt{2}y + 2\sqrt{7}x &= 8\sqrt{7} \\ \text{i.e., } y &= -\sqrt{14}x + 4\sqrt{14} \quad \text{with contact point } \left(\tfrac{7}{2}, \sqrt{\tfrac{7}{2}}\right) \end{aligned}$$

EXERCISE 21H

1 **a**
$$\begin{aligned} f(x) &= 3x^2 - 6x + 2 \\ \therefore\ f'(x) &= 6x - 6 \\ \therefore\ f''(x) &= 6 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - x + 5 \\ \therefore\ f'(x) &= 6x^2 - 6x - 1 + 0 \\ \therefore\ f''(x) &= 12x - 6 \end{aligned}$$

c
$$\begin{aligned} f(x) &= \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1 \\ \therefore\ f'(x) &= -x^{-\frac{3}{2}} \\ \therefore\ f''(x) &= \tfrac{3}{2}x^{-\frac{5}{2}} \end{aligned}$$

d
$$\begin{aligned} f(x) &= \frac{2 - 3x}{x^2} = 2x^{-2} - 3x^{-1} \\ \therefore\ f'(x) &= -4x^{-3} + 3x^{-2} \\ \therefore\ f''(x) &= 12x^{-4} - 6x^{-3} \\ &= \frac{12 - 6x}{x^4} \end{aligned}$$

e
$$\begin{aligned} f(x) &= (1 - 2x)^3 \\ \therefore\ f'(x) &= 3(1 - 2x)^2(-2) \\ &= -6(1 - 2x)^2 \qquad \therefore\ f''(x) = -12(1 - 2x)^1(-2) = 24(1 - 2x) \end{aligned}$$

f $f(x) = \dfrac{x + 2}{2x - 1}$ is a quotient where $u = x + 2$ and $v = 2x - 1$

$\therefore\ u' = 1$ and $v' = 2$

$$\begin{aligned} \therefore\ f'(x) &= \frac{1(2x - 1) - 2(x + 2)}{(2x - 1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{-5}{(2x - 1)^2} \\ &= -5(2x - 1)^{-2} \end{aligned}$$

and $f''(x) = 10(2x - 1)^{-3}(2) = \dfrac{20}{(2x - 1)^3}$

2 **a**
$$\begin{aligned} y &= x - x^3 \\ \therefore\ \frac{dy}{dx} &= 1 - 3x^2 \\ \therefore\ \frac{d^2y}{dx^2} &= -6x \end{aligned}$$

b
$$\begin{aligned} y &= x^2 - \frac{5}{x^2} \\ &= x^2 - 5x^{-2} \\ \therefore\ \frac{dy}{dx} &= 2x + 10x^{-3} \\ \therefore\ \frac{d^2y}{dx^2} &= 2 - 30x^{-4} = 2 - \frac{30}{x^4} \end{aligned}$$

c
$$\begin{aligned} y &= 2 - \frac{3}{\sqrt{x}} \\ &= 2 - 3x^{-\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{3}{2}x^{-\frac{3}{2}} \\ \therefore\ \frac{d^2y}{dx^2} &= -\tfrac{9}{4}x^{-\frac{5}{2}} \end{aligned}$$

d
$$\begin{aligned} y &= \frac{4 - x}{x} = 4x^{-1} - 1 \\ \therefore\ \frac{dy}{dx} &= -4x^{-2} \\ \therefore\ \frac{d^2y}{dx^2} &= 8x^{-3} = \frac{8}{x^3} \end{aligned}$$

e
$$\begin{aligned} y &= (x^2 - 3x)^3 \\ \therefore\ \frac{dy}{dx} &= 3(x^2 - 3x)^2(2x - 3) = (6x - 9)(x^2 - 3x)^2 \end{aligned}$$

which is a product where $u = 6x - 9$ and $v = (x^2 - 3x)^2$

$\therefore\ u' = 6$ and $v' = 2(x^2 - 3x)^1(2x - 3)$

$$\begin{aligned} \therefore\ \frac{d^2y}{dx^2} &= 6(x^2 - 3x)^2 + (6x - 9)(2)(x^2 - 3x)(2x - 3) \\ &= 6(x^2 - 3x)\left[(x^2 - 3x) + (2x - 3)^2\right] \\ &= 6(x^2 - 3x)(x^2 - 3x + 4x^2 - 12x + 9) \\ &= 6(x^2 - 3x)(5x^2 - 15x + 9) \end{aligned}$$

f
$$y = x^2 - x + \frac{1}{1-x} = x^2 - x + (1-x)^{-1}$$
$$\therefore \quad \frac{dy}{dx} = 2x - 1 + (-1)(1-x)^{-2}(-1) = 2x - 1 + (1-x)^{-2}$$
$$\therefore \quad \frac{d^2y}{dx^2} = 2 - 2(1-x)^{-3}(-1) = 2 + \frac{2}{(1-x)^3}$$

3 **a**
$$f(x) = 2x^3 - 6x^2 + 5x + 1$$
$$\therefore \quad f'(x) = 6x^2 - 12x + 5$$
$$\therefore \quad f''(x) = 12x - 12$$ and so $f''(x) = 0$ when $12x - 12 = 0$
$$\therefore \quad 12x = 12$$
i.e., when $x = 1$

b $f(x) = \dfrac{x}{x^2+2}$ is a quotient where $u = x$ and $v = x^2 + 2$

$\therefore \quad u' = 1, \quad v' = 2x$

$$\therefore \quad f'(x) = \frac{1(x^2+2) - 2x^2}{(x^2+2)^2} \quad \{\text{using the quotient rule}\}$$
$$= \frac{2 - x^2}{(x^2+2)^2}$$ which is another quotient

with $u = 2 - x^2$ and $v = (x^2+2)^2$

$\therefore \quad u' = -2x, \quad v' = 2(x^2+2)(2x)$

$$f''(x) = \frac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4}$$
$$= \frac{-2x(x^2+2)[x^2+2+2(2-x^2)]}{(x^2+2)^4}$$
$$= \frac{-2x[-x^2+6]}{(x^2+2)^3}$$
$$= \frac{2x[x^2-6]}{(x^2+2)^3}$$ and so $f''(x) = 0$ when $2x[x^2-6] = 0$

i.e., $x = 0$ or $x^2 - 6 = 0$

i.e., when $x = 0$ or $x = \pm\sqrt{6}$

REVIEW SET 21A

1 $y = -2x^2$. Since when $x = -1$, $y = -2(-1)^2 = -2$, the point of contact is $(-1, -2)$.

$$\text{Now} \quad \frac{dy}{dx} = -4x$$
$$\therefore \quad \text{at} \quad x = -1, \quad \frac{dy}{dx} = -4(-1) = 4$$
$$\therefore \quad \text{tangent has equation} \quad \frac{y-(-2)}{x+1} = 4$$ i.e., $y + 2 = 4x + 4$

i.e., $y = 4x + 2$

2 **a**
$$y = 3x^2 - x^4$$
$$\therefore \quad \frac{dy}{dx} = 6x - 4x^3$$

b
$$y = \frac{x^3 - x}{x^2} = x - x^{-1}$$
$$\therefore \quad \frac{dy}{dx} = 1 + x^{-2} = 1 + \frac{1}{x^2}$$

3 $f(x) = x^2 + 2x, \quad \therefore \quad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h}$$

$$= \lim_{h \to 0} 2x + 2 + h \quad \{\text{as} \quad h \neq 0\}$$

$$= 2x + 2 \qquad \textit{Checking:} \quad f(x) = x^2 + 2x, \quad \therefore \quad f'(x) = 2x + 2 \ \checkmark$$

4 $y = \dfrac{1 - 2x}{x^2}$. Since when $x = 1$, $y = \dfrac{1 - 2(1)}{1^2} = -1$ the point of contact is (1, -1)

Since $y = \dfrac{1}{x^2} - \dfrac{2}{x}$ then $\dfrac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\dfrac{2}{x^3} + \dfrac{2}{x^2}$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = -2 + 2 = 0$

i.e., the tangent is a horizontal line. $\therefore$ the normal is a vertical line, of the form $x = k$.

As the normal passes through (1, -1) its equation must be $x = 1$

5 $y = 2x^3 + 4x - 1$ at (1, 5)

Now $\dfrac{dy}{dx} = 6x^2 + 4 \qquad \therefore$ at $x = 1$, $\dfrac{dy}{dx} = 6(1)^2 + 4 = 10$

$\therefore$ the tangent has equation $\dfrac{y - 5}{x - 1} = 10$ i.e., $y = 10x - 5$

Now the tangent meets the curve again where $10x - 5 = 2x^3 + 4x - 1$

$$2x^3 - 6x + 4 = 0$$
$$x^3 - 3x + 2 = 0$$

We know that $(x - 1)^2$ is a factor since the line is tangent to the curve (i.e., touches) at $x = 1$

Consequently, $x^3 - 3x + 2 = (x - 1)^2(x + 2) = 0 \qquad$ {since the constant term is 2}

Thus $x = -2$ is the other solution and when $x = -2$, $y = 2(-2)^3 + 4(-2) - 1 = -25$

$\therefore$ tangent meets curve again at (-2, -25)

6 $y = \dfrac{ax + b}{\sqrt{x}} = a\sqrt{x} + \dfrac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$$\therefore \quad \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} = \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}$$

The equation of the tangent at $x = 1$ is $2x - y = 1$

i.e., $y = 2x - 1 \quad \therefore$ the slope is 2

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{a}{2} - \dfrac{b}{2} = 2 \quad \therefore \quad a - b = 4$

$\therefore \quad a = b + 4$ ...... (1)

Also, at $x = 1$ the tangent touches the curve i.e., $\dfrac{a(1) + b}{\sqrt{1}} = 2(1) - 1$

i.e., $a + b = 1$

$\therefore \quad b + 4 + b = 1$ {using (1)}

$\therefore \quad 2b = -3$

$\therefore \quad b = -\frac{3}{2}$

$\therefore \quad a = 4 - \frac{3}{2} = \frac{5}{2}$

i.e., $a = \frac{5}{2}$, $b = -\frac{3}{2}$

7 $y = 4(ax+1)^{-2}$

Since when $x = 0$, $y = 4(0+1)^{-2} = 4$, the point of contact is (0, 4)

Now $\dfrac{dy}{dx} = -8(ax+1)^{-3}(a) = \dfrac{-8a}{(ax+1)^3}$ $\quad\therefore$ at $x = 0$, $\dfrac{dy}{dx} = -8a$

$\therefore$ the tangent has equation $\dfrac{y-4}{x-0} = -8a$ i.e., $y - 4 = -8ax$

and this passes through (1, 0) $\quad\therefore\quad 0 - 4 = -8a(1) \quad\therefore\quad a = \frac{1}{2}$

8 $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$

Since when $x = 4$ $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$ the point of contact is $(4, \frac{1}{2})$

Now $\dfrac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ $\quad\therefore$ at $x = 4$, $\dfrac{dy}{dx} = -\frac{1}{2}\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16}$

$\therefore$ the normal at $(4, \frac{1}{2})$ has slope 16.

So the equation is $\dfrac{y-\frac{1}{2}}{x-4} = 16$ i.e., $y - \frac{1}{2} = 16x - 64$ i.e., $y = 16x - \frac{127}{2}$

9 **a**

$$M = (t^2+3)^4$$
$$\frac{dM}{dt} = 4(t^2+3)^3(2t)$$
$$\frac{dM}{dt} = 8t(t^2+3)^3$$

b $A = \dfrac{\sqrt{t+5}}{t^2}$ is a quotient with

$u = (t+5)^{\frac{1}{2}}$ and $v = t^2$

$\therefore\quad u' = \frac{1}{2}(t+5)^{-\frac{1}{2}}$, $v' = 2t$

$$\frac{dA}{dt} = \frac{\left(\dfrac{1}{2\sqrt{t+5}}\right)t^2 - \sqrt{t+5}(2t)}{t^4}$$
$$= \frac{t^2 - 2(2t)(t+5)}{2t^4\sqrt{t+5}}$$
$$= \frac{t - 4(t+5)}{2t^3\sqrt{t+5}}$$
$$= \frac{-3t-20}{2t^3\sqrt{t+5}}$$

10 **a**

$$y = \frac{4}{\sqrt{x}} - 3x = 4x^{-\frac{1}{2}} - 3x$$
$$\therefore\quad \frac{dy}{dt} = -2x^{-\frac{3}{2}} - 3 = \frac{-2}{x\sqrt{x}} - 3$$

b

$$y = \left(x - \frac{1}{x}\right)^4 = (x - x^{-1})^4$$
$$\therefore\quad \frac{dy}{dx} = 4(x - x^{-1})^3(1 + x^{-2})$$
$$= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)$$

c

$$y = \sqrt{x^2-3x} = (x^2-3x)^{\frac{1}{2}}$$
$$\therefore\quad \frac{dy}{dx} = \tfrac{1}{2}(x^2-3x)^{-\frac{1}{2}}(2x-3)$$
$$= \frac{2x-3}{2\sqrt{x^2-3x}}$$

REVIEW SET 21B

1 **a**

$$y = 5x - 3x^{-1}$$
$$\frac{dy}{dx} = 5 + 3x^{-2} = 5 + \frac{3}{x^2}$$

b

$$y = (3x^2+x)^4$$
$$\frac{dy}{dx} = 4(3x^2+x)^3(6x+1)$$

c $y = (x^2+1)(1-x^2)^3$ is a product with $u = x^2+1$ and $v = (1-x^2)^3$

$\therefore\quad u' = 2x$ and $v' = 3(1-x^2)^2(-2x)$

$\dfrac{dy}{dx} = 2x(1-x^2)^3 - 6x(x^2+1)(1-x^2)^2$ {using the product rule}

2 $y = x^3 - 3x^2 - 9x + 2, \quad \therefore \quad \dfrac{dy}{dx} = 3x^2 - 6x - 9$

horizontal tangents occur when $\dfrac{dy}{dx} = 0$ i.e., when

$$\begin{aligned} 3x^2 - 6x - 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ \therefore \quad x &= 3 \quad \text{or} \quad x = -1 \end{aligned}$$

when $x = 3$, horizontal tangent has equation $y = -25$

when $x = -1$, horizontal tangent has equation $y = 7$

3 $y = \dfrac{x+1}{x^2-2}$ and since when $x = 1$, $y = \dfrac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

Now $y = \dfrac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2 - 2$

$\therefore \quad u' = 1$ and $v' = 2x$

$$\frac{dy}{dx} = \frac{1(x^2-2) - (x+1)(2x)}{(x^2-2)^2} \qquad \{\text{using the quotient rule}\}$$

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{(1^2-2) - (1+1)2(1)}{(1^2-2)^2} = \dfrac{(-1)-4}{1} = -5$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$

$\therefore$ the equation of the normal is $\dfrac{y-(-2)}{x-1} = \frac{1}{5}$ i.e., $5y = x - 11$

4 **a** $f(x) = \dfrac{(x+3)^3}{\sqrt{x}} = (x+3)^3 x^{-\frac{1}{2}}$ which is a product with $u = (x+3)^3$ and $v = x^{-\frac{1}{2}}$

$\therefore \quad u' = 3(x+3)^2$ and $v' = -\frac{1}{2}x^{-\frac{3}{2}}$

$$\therefore \quad f'(x) = \frac{3(x+3)^2}{\sqrt{x}} - \frac{(x+3)^3}{2x\sqrt{x}} \qquad \{\text{using the product rule}\}$$

b $f(x) = x^4\sqrt{x^2+3}$ which is a product with $u = x^4$ and $v = (x^2+3)^{\frac{1}{2}}$

$\therefore \quad u' = 4x^3$ and $v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x) = x(x^2+3)^{-\frac{1}{2}}$

$$\therefore \quad f'(x) = 4x^3\sqrt{x^2+3} + \frac{x^5}{\sqrt{x^2+3}} \qquad \{\text{using the product rule}\}$$

5 **a**

$$\begin{aligned} f(x) &= 3x^2 - \frac{1}{x} = 3x^2 - x^{-1} \\ \therefore \quad f'(x) &= 6x + x^{-2} \\ \text{and} \quad f''(x) &= 6 - 2x^{-3} = 6 - \frac{2}{x^3} \end{aligned}$$

b

$$\begin{aligned} f(x) &= \sqrt{x} = x^{\frac{1}{2}} \\ \therefore \quad f'(x) &= \tfrac{1}{2}x^{-\frac{1}{2}} \\ \therefore \quad f''(x) &= -\tfrac{1}{4}x^{-\frac{3}{2}} \end{aligned}$$

6 $y = x^2\sqrt{1-x}$ at $x = -3$

Since when $x = -3$, $y = (-3)^2\sqrt{1-(-3)} = 9\sqrt{4} = 18$,

the point of contact is $(-3, 18)$

Also, $y = x^2\sqrt{1-x}$ is a product, with $u = x^2$ and $v = (1-x)^{\frac{1}{2}}$

$\therefore \quad u' = 2x$ and $v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$$\therefore \quad \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2\left(\tfrac{1}{2}(1-x)^{-\frac{1}{2}}\right)$$

$\therefore$ at $x=-3$, $\dfrac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2(\tfrac{1}{2})(1-(-3))^{-\frac{1}{2}} = -6(2) - 9(\tfrac{1}{2})^2 = -\tfrac{57}{4}$

$\therefore$ the tangent at (−3, 18) has equation $\dfrac{y-18}{x-(-3)} = -\tfrac{57}{4}$ i.e., $4y - 72 = -57x - 171$

i.e., $4y = -57x - 99$

Now when $x=0$, $y = -\tfrac{99}{4}$ and when $y=0$, $x = -\tfrac{99}{57}$

$\therefore$ area of ΔOAB $= \tfrac{1}{2}\left(\tfrac{99}{4}\right)\left(\tfrac{99}{57}\right) \doteqdot 21.5$ units^2

7 $y = x^3 + ax + b \quad \therefore \quad \dfrac{dy}{dx} = 3x^2 + a$ and so at $x=1$, $\dfrac{dy}{dx} = 3 + a$

Now the equation of the tangent at $x=1$ is $y = 2x$

$\therefore$ the slope is 2 $\quad \therefore \quad 3 + a = 2$ and so $a = -1$

Also, at $x=1$ the tangent touches the curve.

i.e., $x^3 + ax + b = 2x$ when $x = 1$

$\therefore \quad (1)^3 + (-1)(1) + b = 2(1)$

$\therefore \quad 1 - 1 + b = 2$

$\therefore \quad b = 2$ and so $a = -1$.

8 $y = x^3 + ax^2 - 4x + 3 \quad \therefore \quad \dfrac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x=1$ is parallel to $y = 3x$, so at $x=1$, $\dfrac{dy}{dx} = 3$

i.e., $3 = 3(1)^2 + 2a(1) - 4$

$\therefore \quad 2a = 4$

$\therefore \quad a = 2$

Since when $x=1$, $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

The contact point is (1, 2) and since the slope is 3, the tangent at (1, 2) has equation

$\dfrac{y-2}{x-1} = 3$ i.e., $y - 2 = 3x - 3$

i.e., $y = 3x - 1$

The tangent meets the curve where $x^3 + 2x^2 - 4x + 3 = 3x - 1$

i.e., $x^3 + 2x^2 - 7x + 4 = 0$

Since the line touches the curve (i.e., is tangent to it) at $x=1$, $(x-1)^2$ must be a factor.

Consequently, $x^3 + 2x^2 - 7x + 4 = (x-1)^2(x+4) = 0$ {since the constant term is 4}

$\therefore$ the curve cuts the tangent at $x = -4$, $y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$

i.e., at (−4, −13)

9 $f(x) = 2x^3 + Ax + B \quad \therefore \quad f'(x) = 6x^2 + A$

Now as the slope at (−2, 33) is 10,

$\therefore \quad f'(-2) = 10$

$\therefore \quad 10 = 6(-2)^2 + A$

$\therefore \quad A = -14$

$\therefore \quad f(x) = 2x^3 - 14x + B$

and as (−2, 33) lies on the curve

$f(-2) = 33$

$\therefore \quad 2(-2)^3 - 14(-2) + B = 33$

$\therefore \quad -16 + 28 + B = 33$

$\therefore \quad B = 49 - 28$

$\therefore \quad B = 21$

10 So, P_n is: "If $y = x^n$ where $n \in Z^+$, then $\dfrac{dy}{dx} = nx^{n-1}$."

Proof: (By the Principle of Mathematical Induction)

(1) If $p = 1$, then $y = x$. This has slope 1, so $\dfrac{dy}{dx} = 1 = 1x^0$. $\therefore$ P_1 is true.

(2) If P_k is true, then $y = x^k$ implies that $\dfrac{dy}{dx} = kx^{k-1}$

$$\begin{aligned} \text{If } \quad y &= x^{k+1} = x^k x, \\ \text{then } \quad \frac{dy}{dx} &= \frac{d}{dx}\left(x^k\right)x + x^k\frac{d}{dx}(x) \qquad \{\text{Product Rule}\} \\ &= kx^{k-1}x + x^k \times 1 \\ &= kx^k + x^k \\ &= (k+1)x^k \end{aligned}$$

Hence P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for $n \in Z^+$. {Principle of Mathematical Induction}

REVIEW SET 21C

1 **a** $y = x^3\sqrt{1-x^2}$ is a product where $y = x^3$ and $v = (1-x^2)^{\frac{1}{2}}$

$\therefore$ $u' = 3x^2$ and $v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -x(1-x^2)^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = 3x^2\sqrt{1-x^2} - \dfrac{x^4}{\sqrt{1-x^2}}$ {using the product rule}

b $y = \dfrac{x^2-3x}{\sqrt{x+1}}$ is a quotient where $u = x^2 - 3x$ and $v = (x+1)^{\frac{1}{2}}$

$u' = 2x - 3$ and $v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$ {using the quotient rule}

2 $y = \dfrac{x+1}{x^2-2}$. Since when $x = 1$, $y = \dfrac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

$y = \dfrac{x+1}{x^2-2}$ is a quotient with $u = x + 1$ and $v = x^2 - 2$

$\therefore$ $u' = 1$ and $v' = 2x$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1(x^2-2)-(x+1)2x}{(x^2-2)^2}$ {using the quotient rule}

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = \dfrac{(1-2)-2(1+1)}{(1-2)^2} = \dfrac{-1-4}{1} = -5$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$.

So the normal has equation: $\dfrac{y-(-2)}{x-1} = \frac{1}{5}$ i.e., $5y + 10 = x - 1$

i.e., $5y = x - 11$

3

$$\begin{aligned} f(x) &= 2x^4 - 4x^3 - 9x^2 + 4x + 7 \\ \therefore \quad f'(x) &= 8x^3 - 12x^2 - 18x + 4 \\ \therefore \quad f''(x) &= 24x^2 - 24x - 18, \quad \therefore \quad f''(x) = 0 \text{ where } \quad 24x^2 - 24x - 18 = 0 \end{aligned}$$

$$\begin{aligned} 4x^2 - 4x - 3 &= 0 \\ (2x+1)(2x-3) &= 0 \\ \therefore \quad x &= -\tfrac{1}{2} \text{ or } x = \tfrac{3}{2} \end{aligned}$$

4 $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$

$f(x)$ is the product of u and v where $u = 3x$ and $v = 1 + x$ $\therefore$ $u' = 3$ and $v' = 1$

$\therefore$ by the quotient rule $f'(x) = \dfrac{3(1+x) - 1(3x)}{(1+x)^2} = \dfrac{3}{(1+x)^2}$ $\therefore$ $f'(2) = \frac{3}{9} = \frac{1}{3}$

$\therefore$ the normal at (2, 2) has slope -3

So the equation of the normal is $\dfrac{y-2}{x-2} = -3$ $\therefore$ $y - 2 = -3(x-2)$

i.e., $y - 2 = -3x + 6$

i.e., $y = -3x + 8$

$\therefore$ when $x = 0$, $y = 8$ and when $y = 0$, $x = \frac{8}{3}$

$\therefore$ B and C are at $(0, 8)$ and $(\frac{8}{3}, 0)$, and the distance BC

$$= \sqrt{\left(0 - \tfrac{8}{3}\right)^2 + (8-0)^2} = \sqrt{\tfrac{64}{9} + 64} = \sqrt{\tfrac{640}{9}} = \tfrac{8\sqrt{10}}{3} \text{ units}$$

5 **a** $y = 3x^4 - \dfrac{2}{x} = 3x^4 - 2x^{-1}$

$\therefore$ $\dfrac{dy}{dx} = 12x^3 + 2x^{-2}$

$\therefore$ $\dfrac{d^2y}{dx^2} = 36x^2 - 4x^{-3} = 36x^2 - \dfrac{4}{x^3}$

b $y = x^3 - x + \dfrac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$

$\therefore$ $\dfrac{dy}{dx} = 3x^2 - 1 - \frac{1}{2}x^{-\frac{3}{2}}$

$\therefore$ $\dfrac{d^2y}{dx^2} = 6x + \frac{3}{4}x^{-\frac{5}{2}}$

6 $y = \dfrac{x}{\sqrt{1-x}}$ at $x = -3$.

Since when $x = -3$, $y = \dfrac{-3}{\sqrt{1-(-3)}} = -\frac{3}{2}$ the point of contact is $(-3, -\frac{3}{2})$

Now $y = \dfrac{x}{\sqrt{1-x}}$ is a quotient with $u = x$ and $v = (1-x)^{\frac{1}{2}}$

$\therefore$ $u' = 1$, $v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$

$$\therefore \quad \frac{dy}{dx} = \frac{(1)\sqrt{1-x} - x(\frac{1}{2})(1-x)^{-\frac{1}{2}}(-1)}{1-x} \quad \{\text{quotient rule}\}$$

at $x = -3$, $\dfrac{dy}{dx} = \dfrac{\sqrt{4} - (-3)(\frac{1}{2})(4)^{-\frac{1}{2}}(-1)}{4} = \dfrac{2 - 3(\frac{1}{2})(\frac{1}{2})}{4} = \dfrac{2 - \frac{3}{4}}{4} = \frac{5}{16}$

$\therefore$ the tangent at $(-3, -\frac{3}{2})$ has equation $\dfrac{y - (-\frac{3}{2})}{x - (-3)} = \frac{5}{16}$ i.e., $16(y + \frac{3}{2}) = 5(x+3)$

$16y + 24 = 5x + 15$

i.e., $5x - 16y = 9$

$\therefore$ $b = -16$ and $a = 9$

7 $f(x) = 3x^3 + Ax^2 + B$

As the point $(-2, 14)$ lies on the curve, $14 = -24 + 4A + B$

$4A + B = 38$ ...... (1)

Now $f'(x) = 9x^2 + 2Ax$ and as $f'(-2) = 0$

$36 - 4A = 0$

$\therefore$ $A = 9$

$\therefore$ $4(9) + B = 38$ {using (1)}

$\therefore$ $B = 38 - 36 = 2$

That is, $A = 9$ and $B = 2$

$\therefore$ $f'(x) = 9x^2 + 18x$

$\therefore$ $f''(x) = 18x + 18$ and so $f''(-2) = -36 + 18 = -18$

8 $y = \dfrac{a}{(x+2)^2} = a(x+2)^{-2}$

The slope of the line AB is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-4}{0-2} = \dfrac{4}{-2} = -2$

$\therefore$ the equation of the tangent is $\dfrac{y-8}{x-0} = -2$ i.e., $y = -2x + 8$

$$\frac{dy}{dx} = -2(a)(x+2)^{-3} = -2$$

$$\therefore \quad \frac{a}{(x+2)^3} = 1 \quad \therefore \quad a = (x+2)^3 \quad \text{...... (1)}$$

The line AB meets the curve where

$$\begin{aligned} -2x + 8 &= \frac{a}{(x+2)^2} \\ \therefore \quad -2x+8 &= \frac{(x+2)^3}{(x+2)^2} \quad \{\text{using (1)}\} \\ \therefore \quad -2x+8 &= x+2 \\ \therefore \quad -3x &= -6 \\ \therefore \quad x &= 2 \\ \text{and so} \quad a &= (2+2)^3 = 64 \end{aligned}$$

9 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ meet when $\sqrt{(3x+1)} = \sqrt{5x - x^2}$

Squaring both sides, $3x + 1 = 5x - x^2$

$$\begin{aligned} \therefore \quad x^2 - 2x + 1 &= 0 \\ \therefore \quad (x-1)^2 &= 0 \\ \therefore \quad x &= 1 \end{aligned}$$

and when $x = 1$, $y = \sqrt{3+1} = 2$ $\quad\therefore$ the curves meet at (1, 2)

Now for $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \tfrac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$$

$$\therefore \text{ at } (1, 2) \quad \frac{dy}{dx} = \frac{3}{2(3+1)^{\frac{1}{2}}} = \tfrac{3}{4}$$

and for $y = \sqrt{5x - x^2} = (5x - x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \tfrac{1}{2}(5x - x^2)^{-\frac{1}{2}}(5 - 2x) = \frac{5 - 2x}{2\sqrt{5x - x^2}}$$

$$\therefore \text{ at } (1, 2) \quad \frac{dy}{dx} = \frac{5-2}{2\sqrt{5-1}} = \tfrac{3}{4}$$

i.e., the curves have the same slope of $\frac{3}{4}$ at their point of intersection.

Now the equation of the common tangent at (1, 2) is $\dfrac{y-2}{x-1} = \dfrac{3}{4}$ i.e., $4(y-2) = 3(x-1)$

$$\begin{aligned} 4y - 8 &= 3x - 3 \\ \text{i.e.,} \quad 4y &= 3x + 5 \end{aligned}$$

10 **a**

$$y = \frac{1}{1-x} = (1-x)^{-1}$$

$$\therefore \quad \frac{dy}{dx} = -(-1)\,(1-x)^{-2} = \frac{1}{(1-x)^2}$$

b P_n is: "If $y = \dfrac{1}{1-x}$ then $\dfrac{d^n y}{dx^n} = \dfrac{n!}{(1-x)^{n-1}}$ for $n \in Z^+$."

Proof: (By the Principle of Mathematical Induction)

(1) Using **a**, $\dfrac{d^1 y}{dx^1} = \dfrac{dy}{dx} = \dfrac{1}{(1-x)^2} = \dfrac{1!}{(1-x)^{(1+1)}}$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$$\begin{aligned}\frac{d^k y}{dx^k} &= \frac{k!}{(1-x)^{k+1}}\\ &= k!\,(1-x)^{-(k+1)}\\ \therefore \quad \frac{d^{k+1}y}{dx^{k+1}} &= -k!(k+1)(-1)\,(1-x)^{-(k+1)-1}\\ &= \frac{(k+1)!}{(1-x)^{(k+1)+1}}\end{aligned}$$

Hence P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for $n \in Z^+$. {Principle of Mathematical Induction}

Chapter 22

APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 22A

1 $P(t) = 2t^2 - 12t + 118$ thousand dollars, $t \geqslant 0$

a $P(0) = \$118\,000$ is the current annual profit

b $\dfrac{dP}{dt} = 4t - 12$ thousand dollars/year

c $\dfrac{dP}{dt}$ is the rate of change in profit with time

d **i** Profit decreases when $\dfrac{dP}{dt} \leqslant 0$ i.e., $4t - 12 \leqslant 0$

$\therefore\ 4t \leqslant 12$

i.e., $t \leqslant 3$

But $t \geqslant 0$ $\therefore$ $0 \leqslant t \leqslant 3$ years

ii Profit increases when $\dfrac{dP}{dt} \geqslant 0$ i.e., $t \geqslant 3$ years

e The profit function is a quadratic with $a > 0$ $\therefore$ shape is

So, a minimum profit occurs when $\dfrac{dP}{dt} = 0$ i.e., at $t = 3$ years

and $P(3) = 18 - 36 + 118 = 100$ thousand dollars i.e., $\$100\,000$.

f **i** When $t = 4$, $\dfrac{dP}{dt} = 4$ thousand dollars per year.

So, the profit is increasing at \$4000/year after 4 years.

ii When $t = 10$, $\dfrac{dP}{dt} = 28$ thousand dollars per year.

So, the profit is increasing at \$28 000/year after 10 years.

iii When $t = 25$, $\dfrac{dP}{dt} = 88$ thousand dollars per year.

So, the profit is increasing at \$88 000/year after 25 years.

2 $V = 200(50 - t)^2$ m^3

a average rate on $0 \leqslant t \leqslant 5$

$$= \frac{V(5) - V(0)}{5 - 0}$$

$$= \frac{200(45)^2 - 200(50)^2}{5}$$

$= -19\,000$ m^3/min

i.e., leaving at 19 000 m^3/min

b
$$\begin{aligned} V'(t) &= 400(50 - t)^1 \times (-1) \\ \therefore\ V'(5) &= 400 \times 45 \times -1 \\ &= -18\,000 \text{ m}^3\text{/min} \end{aligned}$$

i.e., leaving at 18 000 m^3/min

3 $s(t) = 1.2 + 28.1t - 4.9t^2$ metres

a When released, $t = 0$ and $s(0) = 1.2$ m $\therefore$ it is released 1.2 m above the ground.

b $s'(t) = 28.1 - 9.8t$ m/s is the instantaneous velocity of the ball at the time t seconds after release.

c When $s'(t) = 0$, $28.1 - 9.8t = 0$ $\therefore$ $t = \dfrac{28.1}{9.8} \doteqdot 2.87$ sec.

So, after 2.87 sec the ball has reached its maximum height.

d $s(2.867) = 1.2 + 28.1 \times 2.867 - 4.9 \times 2.867^2 \doteqdot 41.5$ m

So, the maximum height reached is about 41.5 m.

e **i** $s'(0) = 28.1$ m/s **ii** $s'(2) = 28.1 - 19.6 = 8.5$ m/s **iii** $s'(5) = 28.1 - 49 = -20.9$ m/s

If $s'(t) \geqslant 0$, the ball is travelling upwards.
If $s'(t) \leqslant 0$, the ball is travelling downwards.

f $s(t) = 0$ when

$$\begin{aligned} 1.2 + 28.1t - 4.9t^2 &= 0 \\ \text{i.e.,}\quad 4.9t^2 - 28.1t - 1.2 &= 0 \\ \therefore\quad t &= \frac{28.1 \pm \sqrt{28.1^2 - 4(4.9)(-1.2)}}{9.8} \\ &\doteqdot -0.0424 \quad \text{or} \quad 5.777 \end{aligned}$$

$\therefore$ hits the ground after 5.78 sec.

g $\frac{d^2s}{dt^2} = -9.8$ m/s/s and is the constant rate of change in $\frac{ds}{dt}$

i.e., the instantaneous acceleration is constant at -9.8 m/s^2 for the entire motion.

4 **a**

$$\begin{aligned} s(t) &= bt - 4.9t^2 \\ s'(t) &= b - 9.8t \\ \therefore\quad s'(0) &= b \end{aligned}$$

i.e., the initial velocity is b m/s

b

$$\begin{aligned} \text{since}\quad s(14.2) &= 0 \\ b(14.2) - 4.9(14.2)^2 &= 0 \\ \therefore\quad 14.2\,[b - 4.9 \times 14.2] &= 0 \\ \therefore\quad b &= 4.9 \times 14.2 \\ \therefore\quad b &\doteqdot 69.6 \end{aligned}$$

$\therefore$ the initial velocity is 69.6 m/s

EXERCISE 22B

1 $Q = 100 - 10\sqrt{t}$, $t \geqslant 0$

a **i** At $t = 0$, $Q = 100$ units
ii At $t = 25$, $Q = 50$ units
iii At $t = 100$, $Q = 0$ units

b $\frac{dQ}{dt} = -5t^{-\frac{1}{2}} = -\frac{5}{\sqrt{t}}$

i At $t = 25$, $\frac{dQ}{dt} = -1$ units/year

i.e., decreasing at 1 unit/year

ii At $t = 50$, $\frac{dQ}{dt} = -\frac{5}{\sqrt{50}} = -\frac{1}{\sqrt{2}}$ units/year

i.e., decreasing at $\frac{1}{\sqrt{2}}$ units/year

c $\frac{dQ}{dt} = -\frac{5}{\sqrt{t}}$

$\therefore\ \frac{dQ}{dt} < 0$, for all $t > 0$, $\therefore\ \frac{dQ}{dt}$ is decreasing for all $t > 0$

2 $H = 20 - \frac{9}{t+5}$ m, $t \geqslant 0$

a At planting, $t = 0$ $\therefore$ $H(0) = 20 - \frac{9}{0+5} = 18.2$ m

b **i** At $t = 4$, $H(4) = 20 - \frac{9}{4+5} = 19$ m **ii** At $t = 8$, $H(8) = 20 - \frac{9}{8+5} \doteqdot 19.3$ m **iii** At $t = 12$, $H(12) = 20 - \frac{9}{12+5} \doteqdot 19.5$ m

c Now $\frac{dH}{dt} = 9(t+5)^{-2} = \frac{9}{(t+5)^2}$

i When $t = 0$, $\frac{dH}{dt} = \frac{9}{25} = 0.36$ m/yr **ii** When $t = 5$, $\frac{dH}{dt} = \frac{9}{100} = 0.09$ m/yr **iii** When $t = 10$, $\frac{dH}{dt} = \frac{9}{225} = 0.04$ m/yr

d Now $\dfrac{dH}{dt}=\dfrac{9}{(t+5)^2}$, $\therefore$ as $(t+5)^2>0$ for all $t\geqslant 0$, $\dfrac{dH}{dt}>0$ for all $t\geqslant 0$

i.e., the height of the tree is always increasing, which means that the tree is always growing.

3 **a** $C(v)=200v+10\,000v^{-1}$ dollars **i** At $v=20$ kmph, $C=\$4500$
ii At $v=40$ kmph, $C=\$8250$

b $\dfrac{dC}{dv}=200-10\,000v^{-2}=200-\dfrac{10\,000}{v^2}$

i At $v=10$ kmph, $\dfrac{dC}{dv}=\$100$ per kmph

ii At $v=30$ kmph, $\dfrac{dC}{dv}=\$188.89$ per kmph

c Cost is a minimum when $\dfrac{dC}{dv}=0$, i.e., $200-\dfrac{10\,000}{v^2}=0$

$$\therefore\ 200=\frac{10\,000}{v^2}$$
$$\therefore\ v^2=50$$
$$\therefore\ v=\pm\sqrt{50}$$
i.e., $v=\sqrt{50}$ kmph

4 $y=\frac{1}{10}x(x-2)(x-3)=\frac{1}{10}(x^3-5x^2+6x)$

a When $y=0$, $x=0$, 2 or 3

$\therefore$ the lake is between 2 and 3 km from the shoreline.

b $\dfrac{dy}{dx}=\frac{1}{10}(3x^2-10x+6)$ When $x=\frac{1}{2}$, $\dfrac{dy}{dx}=\frac{7}{40}$ $\therefore$ land is sloping upwards.

When $x=\frac{3}{2}$, $\dfrac{dy}{dx}=-\frac{9}{40}$ $\therefore$ land is sloping downwards.

c The deepest point of the lake occurs when the slope of the land is 0, i.e., $\dfrac{dy}{dx}=0$

$\therefore$ $\frac{1}{10}(3x^2-10x+6)=0$

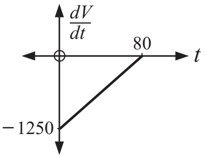
i.e., $3x^2-10x+6=0$ $\therefore$ $x=\dfrac{10\pm\sqrt{100-72}}{6}=\dfrac{5\pm\sqrt{7}}{3}$

but it must be the value between 2 and 3 km, i.e., $x=\dfrac{5+\sqrt{7}}{3}\doteqdot 2.549$ km

$\therefore$ the depth at this point is $y(2.549)\doteqdot\frac{1}{10}(2.549)(0.549)(-0.451)$
$\doteqdot -0.06311$ km
i.e., $\doteqdot$ 63.11 m below sea level.

5 **a** $V=50\,000\left(1-\frac{t}{80}\right)^2$, $0\leqslant t\leqslant 80$

$$\frac{dV}{dt}=2\times 50\,000\left(1-\frac{t}{80}\right)^1\times\left(-\tfrac{1}{80}\right)=-1250\left(1-\frac{t}{80}\right)$$



b Outflow was fastest when $t=0$ (when the tap was first opened).

c As $\dfrac{dV}{dt}=-1250+\frac{1250}{80}t$, then $\dfrac{d^2V}{dt^2}=\frac{1250}{80}=\frac{125}{8}$

and since $\dfrac{d^2V}{dt^2}$ is constant and positive, it shows that $\dfrac{dV}{dt}$ is constantly increasing

i.e., the outflow is decreasing at a constant rate.

6 **a** $\frac{dP}{dt} = aP\left(1-\frac{P}{b}\right) - \left(\frac{c}{100}\right)P$ and when $\frac{dP}{dt} = 0$, the rate of change of population is zero, $\therefore$ the population is not changing and is stable.

b If $a = 0.06$, $b = 24\,000$, $c = 5$ then

$$\begin{aligned}\frac{dP}{dt} &= 0.06P\left(1-\frac{P}{24\,000}\right) - \tfrac{5}{100}P\\ &= 0.06P - 0.05P - \frac{0.06P^2}{24\,000}\\ &= P\left(0.01 - \frac{P}{400\,000}\right)\end{aligned}$$

Now for a stable population

$$\frac{dP}{dt} = 0$$

$$\therefore \quad P = 0 \quad \text{or} \quad \frac{P}{400\,000} = 0.01$$

i.e., $P = 0$ or 4000

i.e, the stable population is 4000 fish

c If the harvest rate is 4%, then

$$\begin{aligned}\frac{dP}{dt} &= 0.06P\left(1-\frac{P}{24\,000}\right) - \tfrac{4}{100}P\\ &= P\left(0.02 - \frac{0.06P}{24\,000}\right)\end{aligned}$$

For stable population $\frac{dP}{dt} = 0$ $\quad \therefore \quad 0 = P\left(0.02 - \frac{0.06P}{24\,000}\right)$

$$\therefore \quad P = 0 \quad \text{or} \quad \frac{0.06P}{24\,000} = 0.02$$

$$\therefore \quad P = 0 \quad \text{or} \quad \frac{0.02 \times 24\,000}{0.06}$$

$$\therefore \quad P = 0 \quad \text{or} \quad 8000$$

i.e., the stable population is 8000 fish

7 **a**

$$\begin{aligned} C(x) &= 0.0003x^3 + 0.02x^2 + 4x + 2250\\ \therefore \quad C'(x) &= 0.0009x^2 + 0.04x + 4 \quad \text{(dollars/pair)}\end{aligned}$$

b $C'(220) = 0.0009(220)^2 + 0.04(220) + 4 = \$56.36$ per pair

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) = \$7348.98 - \$7292.40 = \$56.58$

This is the actual cost to make the extra pair of jeans (221 instead of 220).

d $C''(x) = 0.0018x + 0.04$

$C''(x) = 0$ when $0.0018x + 0.04 = 0$ i.e., $x = -\frac{0.04}{0.0018} \doteqdot -22.2$ but $x \geqslant 0$

$\therefore$ when the rate of change is a minimum it is out of the bounds of our model (we cannot make < 0 jeans!)

EXERCISE 22C.1

1 **a** $s(t) = t^2 + 3t - 2 \quad t \geqslant 0$

Average velocity

$$\begin{aligned} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1}\\ &= \frac{s(3) - s(1)}{3-2}\\ &= \frac{16-2}{2}\\ &= 7 \text{ ms}^{-1}\end{aligned}$$

b Average velocity

$$\begin{aligned} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1}\\ &= \frac{s(1+h) - s(1)}{(1+h) - 1}\\ &= \frac{(1+h)^2 + 3(1+h) - 2 - 2}{h}\\ &= \frac{2h + h^2 + 3h}{h}\\ &= (5+h) \text{ ms}^{-1}\end{aligned}$$

c $\lim_{h\to 0} \dfrac{s(1+h)-s(1)}{h}$

$= \lim_{h\to 0} 5+h$

$= 5 \text{ ms}^{-1}$

This is the *instantaneous velocity* at $t = 1$ second.

d Average velocity

$$= \frac{s(t_2)-s(t_1)}{t_2-t_1}$$

$$= \frac{s(t+h)-s(t)}{(t+h)-t}$$

$$= \frac{\left[(t+h)^2+3(t+h)-2\right]-\left[t^2+3t-2\right]}{h}$$

$$= \frac{2ht+h^2+3h}{h}$$

$$= (2t+3+h) \text{ ms}^{-1}$$

Now $\lim_{h\to 0} \dfrac{s(t+h)-s(t)}{h} = \lim_{h\to 0} (2t+3+h)$

$= (2t+3) \text{ ms}^{-1}$

This is the *instantaneous velocity* at t seconds.

2 **a** $s(t) = 5-2t^2$ cm

Average velocity $= \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$$= \frac{s(5)-s(2)}{5-2}$$

$$= \frac{(-45)-(-3)}{3}$$

$$= -14 \text{ cm s}^{-1}$$

b Average velocity $= \dfrac{s(t_2)-s(t_1)}{t_2-t_1}$

$$= \frac{s(2+h)-s(2)}{(2+h)-2}$$

$$= \frac{5-2(2+h)^2+3}{h}$$

$$= \frac{-8h-2h^2}{h}$$

$$= (-8-2h) \text{ cm s}^{-1}$$

c $\lim_{h\to 0} \dfrac{s(2+h)-s(2)}{h} = \lim_{h\to 0} (-8-2h)$

$= -8 \text{ cm s}^{-1}$

This is the *instantaneous velocity* when $t = 2$ seconds.

d $\lim_{h\to 0} \dfrac{s(t+h)-s(t)}{h}$

$$= \lim_{h\to 0} \frac{\left[5-2(t+h)^2\right]-\left[5-2t^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{-4th-2h^2}{h}$$

$$= \lim_{h\to 0} (-4t-2h)$$

$$= -4t \text{ cm s}^{-1}$$

$-4t \text{ cm s}^{-1}$ is the *instantaneous velocity* at t seconds.

3 $v(t) = 2\sqrt{t}+3 \text{ cm s}^{-1}, \quad t \geqslant 0$

a Average acceleration

$$= \frac{v(t_2)-v(t_1)}{t_2-t_1}$$

$$= \frac{v(4)-v(1)}{4-1}$$

$$= \frac{7-5}{3}$$

$$= \tfrac{2}{3} \text{ cm s}^{-2}$$

b Average acceleration

$$= \frac{v(t_2)-v(t_1)}{t_2-t_1}$$

$$= \frac{v(1+h)-v(1)}{(1+h)-1}$$

$$= \frac{[2\sqrt{1+h}+3]-[2\sqrt{1}+3]}{h}$$

$$= \frac{2\sqrt{1+h}-2}{h} \text{ cm s}^{-2}$$

c $$\lim_{h\to 0}\frac{v(1+h)-v(1)}{(1+h)-1}$$
$$=\lim_{h\to 0}\frac{[2\sqrt{1+h}+3]-[2+3]}{h}$$
$$=\lim_{h\to 0}\frac{2[\sqrt{1+h}-1]}{h}\times\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$
$$=\lim_{h\to 0}\frac{2h}{h(\sqrt{1+h}+1)}$$
$$=\tfrac{2}{2}$$
$=1$ cm s^{-2} This is the *instantaneous acceleration* when $t=1$ second.

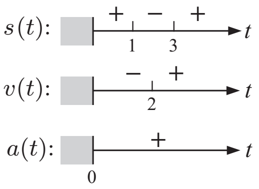
d $$\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$$
$$=\lim_{h\to 0}\frac{2\sqrt{t+h}-2\sqrt{t}}{h}$$
$$=\lim_{h\to 0}\frac{2(\sqrt{t+h}-\sqrt{t})}{h}\times\frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+t}$$
$$=\lim_{h\to 0}\frac{2h}{h(\sqrt{t+h}+\sqrt{t})}$$
$$=\frac{2}{2\sqrt{t}}$$
$=\dfrac{1}{\sqrt{t}}$ cm s^{-2} This is the *instantaneous acceleration* at t seconds.

4 **a** This is the *instantaneous velocity* at $t=4$ seconds.

b This is the *instantaneous acceleration* at $t=4$ seconds.

EXERCISE 22C.2

1 **a** $s(t)=t^2-4t+3$ cm, $t\geqslant 0$ $\therefore$ $v(t)=2t-4$ cm s^{-1}
and $a(t)=2$ cm s^{-2}

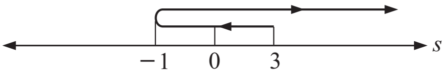


b When $t=0$, $s(0)=3$ cm
$v(0)=-4$ cm s^{-1}
$a(0)=2$ cm s^{-2}
$\therefore$ the object is 3 cm right of O and is moving to the left with a velocity of 4 cm s^{-1} and slowing down, its acceleration being 2 cm s^{-2} to the right.

c When $t=2$, $s(2)=-1$ cm
$v(2)=0$ cm s^{-1}
$a(2)=2$ cm s^{-2}
$\therefore$ the object is 1 cm left of O, momentarily at rest, but with acceleration 2 cm s^{-2} to the right.

d The object reverses direction when $v(t)=0$ i.e., at $t=2$ seconds.
At $t=2$, the particle is 1 cm left of O.

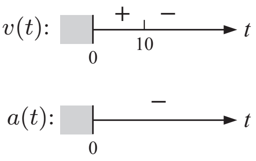
e



f Speed decreases when $v(t)$ and $a(t)$ have opposite signs, i.e., when $0\leqslant t\leqslant 2$.

2 $s(t)=98t-4.9t^2$ m $t\geqslant 0$

a $v(t)=98-9.8t$ ms^{-1}
$a(t)=-9.8$ ms^{-2}



b When $t=0$, $s(0)=0$ m, $v(0)=98$ ms^{-1}

c When $t=5$, $s(5)=367.5$ m
$v(5)=49$ ms^{-1}
$a(5)=-9.8$ ms^{-2}
The stone is 367.5 m above the ground, travelling upwards at 49 ms^{-1} and slowing down.

When $t=12$, $s(5)=470.4$ m
$v(5)=-19.6$ ms^{-1}
$a(5)=-9.8$ ms^{-2}
The stone is 470.4 m above the ground and travelling downwards at 19.6 ms^{-1}, increasing in speed.

d Maximum height is reached when $v(t) = 0 \text{ ms}^{-1}$

$$\begin{aligned} \therefore \quad 98 - 9.8t &= 0 \\ 9.8t &= 98 \\ t &= 10 \text{ seconds} \end{aligned}$$

$\therefore$ the maximum height is

$$\begin{aligned} s(10) &= 9.8(10) - 4.9\,(100) \\ &= 980 - 490 \\ &= 490 \text{ m} \end{aligned}$$

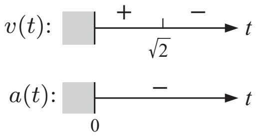
e The stone is at ground level when $s(t) = 0$ i.e., when

$$\begin{aligned} 98t - 4.9t^2 &= 0 \\ \therefore \quad 4.9t(20 - t) &= 0 \\ \therefore \quad t &= 0 \text{ or } 20 \text{ seconds} \end{aligned}$$

i.e., it hits the ground after 20 seconds.

3 **a**

$$\begin{aligned} s(t) &= 12t - 2t^3 - 1 \text{ cm}, \quad t \geqslant 0 \\ \therefore \quad v(t) &= 12 - 6t^2 \quad \text{cm s}^{-1} \\ \text{and} \quad a(t) &= -12t \quad \text{cm s}^{-2} \end{aligned}$$



b When $t = 0$,

$$\begin{aligned} s(0) &= -1 \text{ cm} \\ v(0) &= 12 \text{ cm s}^{-1} \\ a(0) &= 0 \text{ cm s}^{-2} \end{aligned}$$

The particle is 1 cm left of O, moving right at 12 cm s^{-1} with constant speed.

c The particle reverses direction when $v(t) = 0$ i.e., at $t = \sqrt{2}$ seconds.

When $t = \sqrt{2}$,

$$\begin{aligned} s(\sqrt{2}) &= 12\sqrt{2} - 2(2\sqrt{2}) - 1 \\ &= 8\sqrt{2} - 1 \end{aligned}$$

i.e., the particle is $(8\sqrt{2} - 1)$ cm to the right of O.

d **i** From the sign diagrams in **a**, the speed increases for $t \geqslant \sqrt{2}$ seconds.

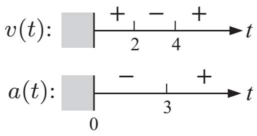
ii The velocity of the particle never increases $\{a(t) \leqslant 0\}$.

4 **a**

$$\begin{aligned} x(t) &= t^3 - 9t^2 + 24t \text{ m}, \quad t \geqslant 0 \\ v(t) &= 3t^2 - 18t + 24 \\ &= 3(t^2 - 6t + 8) \\ &= 3(t - 4)(t - 2) \text{ ms}^{-1} \end{aligned}$$

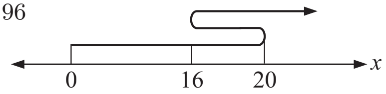
and

$$\begin{aligned} a(t) &= 6t - 18 \\ &= 6(t - 3) \text{ ms}^{-2} \end{aligned}$$



b Reverses direction when $v(t) = 0$, i.e., at $t = 2$ seconds and $t = 4$ seconds.

$$\begin{aligned} x(2) &= 8 - 36 + 48 \text{ m} \\ &= 20 \text{ m} \end{aligned} \qquad \text{and} \qquad \begin{aligned} x(4) &= 64 - 144 + 96 \\ &= 16 \text{ m} \end{aligned}$$



c **i** The speed decreases when $v(t)$ and $a(t)$ have the same sign, i.e., when $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$.

ii The velocity decreases when $a(t) < 0$, i.e., when $0 \leqslant t \leqslant 3$.

d When $t = 5$,

$$\begin{aligned} s(5) &= 5^3 - 9.5^2 + 24.5 \\ &= 125 - 225 + 120 \\ &= 20 \text{ m} \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{distance travelled} &= 20 + 4 + 4 \text{ m} \\ &= 28 \text{ m} \end{aligned}$$

5 **a** Let the equation be

$$\begin{aligned} s(t) &= at^2 + bt + c \\ \therefore \quad v(t) &= 2at + b \\ \text{and} \quad a(t) &= 2a = g \quad \{\text{gravitational acceleration}\} \\ \therefore \quad a &= \tfrac{1}{2}g \\ \text{Also} \quad v(t) &= gt + b \end{aligned}$$

But, when $t = 0$,

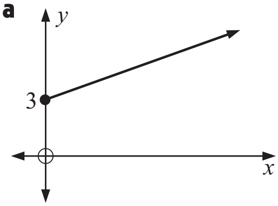
$$\begin{aligned} v(0) &= g \times 0 \; + \; b \\ \therefore \quad v(0) &= b \quad \text{i.e., initial velocity is } b \\ \therefore \quad v(t) &= v(0) + gt \quad \text{as required} \end{aligned}$$

b Now when $t = 0$, $s(0) = 0$

$$\therefore \quad a \times 0^2 + b \times 0 + c = 0$$
$$\therefore \quad c = 0$$
$$\text{and so} \quad s(t) = \left(\tfrac{1}{2}g\right)t^2 + v(0)t$$
$$\text{i.e.,} \quad s(t) = v(0) \times t + \tfrac{1}{2}gt^2 \quad \text{as required}$$

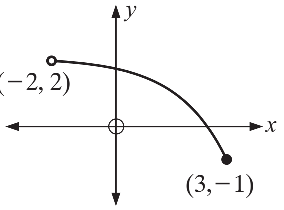
EXERCISE 22D.1

1 **a**



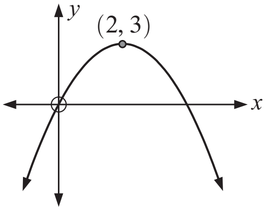
i $x \geqslant 0$
ii never

b



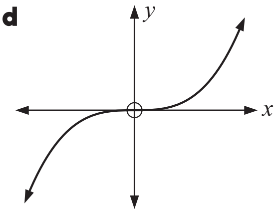
i never
ii $-2 < x \leqslant 3$

c



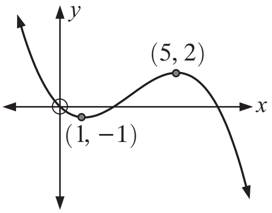
i $x \leqslant 2$
ii $x \geqslant 2$

d



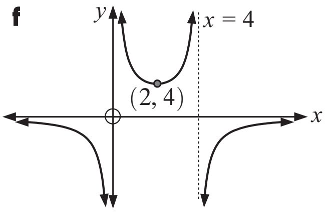
i all real x
ii never

e



i $1 \leqslant x \leqslant 5$
ii $x \leqslant 1$, $x \geqslant 5$

f

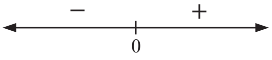


i $2 \leqslant x < 4$, $x > 4$
ii $x < 0$, $0 < x \leqslant 2$

EXERCISE 22D.2

1 **a** $f(x) = x^2$, $f'(x) = 2x$

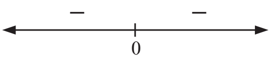
Sign $f'(x)$:



increasing when $x \geqslant 0$,
decreasing when $x \leqslant 0$

b $f(x) = -x^3$, $f'(x) = -3x^2$

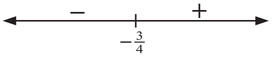
Sign $f'(x)$:



decreasing for all x

c $f(x) = 2x^2 + 3x - 4$, $f'(x) = 4x + 3$

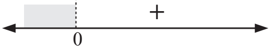
Sign $f'(x)$:



increasing when $x \geqslant -\frac{3}{4}$,
decreasing when $x \leqslant -\frac{3}{4}$

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Sign $f'(x)$:

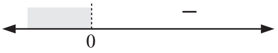


only defined when $x \geqslant 0$,
increasing when $x \geqslant 0$, never decreasing

e $f(x) = \dfrac{2}{\sqrt{x}}$, $f'(x) = -x^{-\frac{3}{2}}$

$= 2x^{-\frac{1}{2}}$ $\quad = \dfrac{-1}{x\sqrt{x}}$

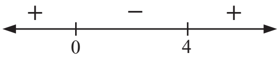
Sign $f'(x)$:



$f(x)$ is only defined for $x > 0$
decreasing when $x > 0$, never increasing

f $f(x) = x^3 - 6x^2$, $f'(x) = 3x^2 - 12x$
$= 3x(x - 4)$

Sign $f'(x)$:

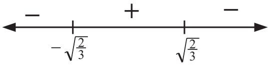


increasing when $x \leqslant 0$ or $x \geqslant 4$,
decreasing when $0 \leqslant x \leqslant 4$

g $f(x) = -2x^3 + 4x$

$$\begin{aligned} f'(x) &= -6x^2 + 4 \\ &= -2(3x^2 - 2) \end{aligned}$$

Sign $f'(x)$:



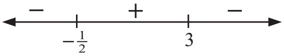
increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,

decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$ or $x \geqslant \sqrt{\frac{2}{3}}$

h $f(x) = -4x^3 + 15x^2 + 18x + 3$

$$\begin{aligned} f'(x) &= -12x^2 + 30x + 18 \\ &= -6(2x^2 - 5x - 3) \\ &= -6(2x + 1)(x - 3) \end{aligned}$$

Sign $f'(x)$:



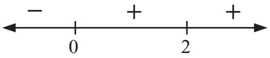
increasing when $-\frac{1}{2} \leqslant x \leqslant 3$,

decreasing when $x \leqslant -\frac{1}{2}$ and $x \geqslant 3$

i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$,

$$\begin{aligned} f'(x) &= 12x^3 + 48x^2 + 48x \\ &= 12x(x^2 - 4x + 4) \\ &= 12x(x - 2)^2 \end{aligned}$$

Sign $f'(x)$:



increasing when $x \geqslant 0$,

decreasing when $x \leqslant 0$

j $f(x) = 2x^3 + 9x^2 + 6x - 7$,

$$\begin{aligned} f'(x) &= 6x^2 + 18x + 6 \\ &= 6(x^2 + 3x + 1) \end{aligned}$$

Sign $f'(x)$:

$x = \frac{-3\pm\sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$

increasing for $x \leqslant \frac{-3-\sqrt{5}}{2}$ or $x \geqslant \frac{-3+\sqrt{5}}{2}$,

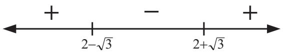
decreasing for $\frac{-3-\sqrt{5}}{2} \leqslant x \leqslant \frac{-3+\sqrt{5}}{2}$

k $f(x) = x^3 - 6x^2 + 3x - 1$,

$$\begin{aligned} f(x) &= 3x^2 - 12x + 3 \\ &= 3(x^2 - 4x + 1) \end{aligned}$$

$x = \frac{4\pm\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

Sign $f'(x)$:



increasing when $x \leqslant 2 - \sqrt{3}$

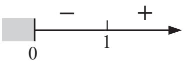
or $x \geqslant 2 + \sqrt{3}$,

decreasing when $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

l $f(x) = x - 2\sqrt{x}, \quad = x - 2x^{\frac{1}{2}}$

$$\begin{aligned} f'(x) &= 1 - x^{-\frac{1}{2}} \\ &= 1 - \frac{1}{\sqrt{x}} \\ &= \frac{\sqrt{x} - 1}{\sqrt{x}} \end{aligned}$$

Sign $f'(x)$:



increasing when $x \geqslant 1$,

decreasing when $0 \leqslant x \leqslant 1$

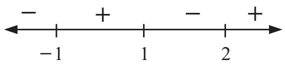
m $y = 3x^4 - 8x^3 - 6x^2 + 24x + 11$

$$\begin{aligned} \frac{dy}{dx} &= 12x^3 - 24x^2 - 12x + 24 \\ &= 12(x^3 - 2x^2 - x + 2) \end{aligned}$$

Using technology, a root is -1,

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 12(x + 1)(x^2 - 3x + 2) \\ &= 12(x + 1)(x - 1)(x - 2) \end{aligned}$$

Sign diagram of $\frac{dy}{dx}$:



So $f(x)$ is increasing for $-1 \leqslant x \leqslant 1$ and $x \geqslant 2$, and decreasing for $x \leqslant -1$ and $1 \leqslant x \leqslant 2$.

n $y = x^4 - 4x^3 + 2x^2 + 4x + 1$,

$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 12x^2 + 4x + 4 \\ &= 4(x^3 - 3x^2 + x + 1) \end{aligned}$$

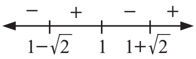
Using technology, a root is 1,

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= 4(x - 1)(x^2 - 2x - 1) \\ &= 0 \end{aligned}$$

when $x = 1$ or $x = \frac{2 \pm \sqrt{8}}{2}$

i.e., $x = 1$ or $x = 1 \pm \sqrt{2}$

Sign diagram of $\frac{dy}{dx}$:



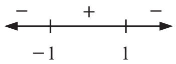
$\therefore$ increasing for $1 - \sqrt{2} \leqslant x \leqslant 1$ and $x \geqslant 1 + \sqrt{2}$, decreasing for $x \leqslant 1 - \sqrt{2}$ and $1 \leqslant x \leqslant 1 + \sqrt{2}$

2 **a** **i** $f(x) = \dfrac{4x}{x^2+1}$,

let $u = 4x, \quad u' = 4,$
$v = x^2+1, \quad v' = 2x$

$$\begin{aligned}\therefore \quad f'(x) &= \frac{4(x^2+1) - 4x \times 2x}{(x^2+1)^2} \\ &= \frac{4x^2+4-8x^2}{(x^2+1)^2} \\ &= \frac{4-4x^2}{(x^2+1)^2} \\ &= \frac{-4(x^2-1)}{(x^2+1)^2} \\ &= \frac{-4(x+1)(x-1)}{(x^2+1)^2}\end{aligned}$$

Sign diagram of $\dfrac{dy}{dx}$:



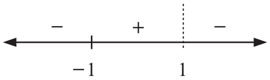
ii $\therefore$ increasing for $-1 \leqslant x \leqslant 1$,
decreasing for $x \leqslant -1$ and $x \geqslant 1$

b **i** $f(x) = \dfrac{4x}{(x-1)^2}$,

let $u = 4x, \quad u' = 4,$
$v = (x-1)^2, \quad v' = 2(x-1)^1$

$$\begin{aligned}\therefore \quad f'(x) &= \frac{4(x-1)^2 - 8x(x-1)}{(x-1)^4} \\ &= \frac{4(x-1)\left((x-1)-2x\right)}{(x-1)^4} \\ &= \frac{4(-1-x)}{(x-1)^3} \\ &= \frac{-4(x+1)}{(x-1)^3}\end{aligned}$$

Sign diagram of $\dfrac{dy}{dx}$:

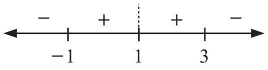


ii $\therefore$ increasing for $-1 \leqslant x < 1$,
decreasing for $x \leqslant -1$ and $x > 1$

c **i** $f(x) = \dfrac{-x^2+4x-7}{x-1}$, let $u = -x^2+4x-7,\ u' = -2x+4,\ v = x-1,\ v' = 1$

$$\begin{aligned}\therefore \quad f'(x) &= \frac{(-2x+4)(x-1) - (-x^2+4x-7)(1)}{(x-1)^2} \\ &= \frac{-2x^2+6x-4+x^2-4x+7}{(x-1)^2} \\ &= \frac{-x^2+2x+3}{(x-1)^2} \\ &= \frac{-(x^2-2x-3)}{(x-1)^2} \\ &= \frac{-(x-3)(x+1)}{(x-1)^2}\end{aligned}$$

Sign diagram of $\dfrac{dy}{dx}$:

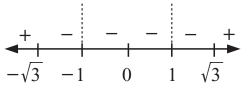


ii $\therefore$ $f(x)$ is increasing for $-1 \leqslant x < 1$ and $1 < x \leqslant 3$, and
decreasing for $x \leqslant -1$ and $x \geqslant 3$

3 **a** $f(x) = \dfrac{x^3}{x^2-1}$, let $u = x^3,\ u' = 3x^2,\ v = x^2-1,\ v' = 2x$

$$\begin{aligned}\therefore \quad f'(x) &= \frac{3x^2(x^2-1) - x^3 \times 2x}{(x^2-1)^2} \\ &= \frac{3x^4-3x^2-2x^4}{(x^2-1)^2} \\ &= \frac{x^2(x^2-3)}{(x^2-1)^2} \\ &= \frac{x^2(x+\sqrt{3})(x-\sqrt{3})}{(x^2-1)^2}\end{aligned}$$

Sign diagram of $\dfrac{dy}{dx}$:



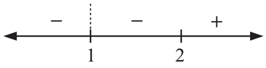
$\therefore$ $f(x)$ is increasing for $x \geqslant \sqrt{3}$ and $x \leqslant -\sqrt{3}$, and
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x \leqslant 0$, $0 \leqslant x < 1$ and $1 < x \leqslant \sqrt{3}$.

b $f(x) = x^2 + \dfrac{4}{x-1}$, $\quad f'(x) = 2x - 4(x-1)^{-2} \times 1$

$= x^2 + 4(x-1)^{-1}$

$$\begin{aligned} f'(x) &= 2x - 4(x-1)^{-2} \times 1 \\ &= 2x - \frac{4}{(x-1)^2} \\ &= \frac{2x(x-1)^2 - 4}{(x-1)^2} \\ &= \frac{2x(x^2-2x+1) - 4}{(x-1)^2} \\ &= \frac{2x^3 - 4x^2 + 2x - 4}{(x-1)^2} \\ &= \frac{(x-2)(2x^2+2)}{(x-1)^2} \end{aligned}$$

Sign of $\dfrac{dy}{dx}$:

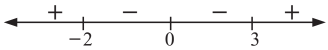


$\therefore$ $f(x)$ is increasing for $x \geqslant 2$, and decreasing for $1 < x \leqslant 2$ and $x < 1$.

EXERCISE 22D.3

1 **a** A is a *local minimum*, B is a *local maximum*, C is a *horizontal inflection.*

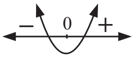
b $f'(x)$ has sign diagram:



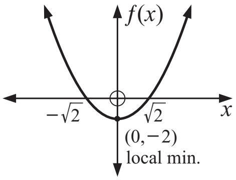
c **i** $f(x)$ is increasing for $x \leqslant -2$ and $x \geqslant 3$ **ii** $f(x)$ is decreasing for $-2 \leqslant x \leqslant 3$.

2 **a** $f(x) = x^2 - 2$ $\therefore$ $f'(x) = 2x$

with sign diagram:

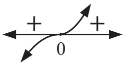


local minimum at (0, −2)

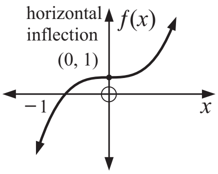


b $f(x) = x^3 + 1$ $\therefore$ $f'(x) = 3x^2$

with sign diagram:



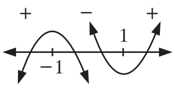
horizontal inflection at (0, 1)



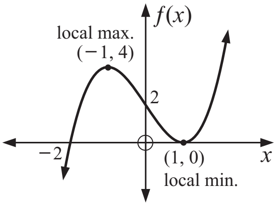
c $f(x) = x^3 - 3x + 2$

$$\begin{aligned} \therefore\ f'(x) &= 3x^2 - 3 \\ &= 3(x^2-1) \\ &= 3(x+1)(x-1) \end{aligned}$$

with sign diagram:



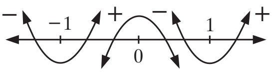
local maximum at (−1, 4),
local minimum at (1, 0)



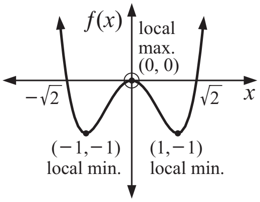
d $f(x) = x^4 - 2x^2$

$$\begin{aligned} \therefore\ f'(x) &= 4x^3 - 4x \\ &= 4x(x^2-1) \\ &= 4x(x+1)(x-1) \end{aligned}$$

with sign diagram:



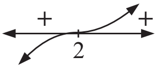
local minima at (−1, −1) and (1, −1),
local maximum at (0, 0)



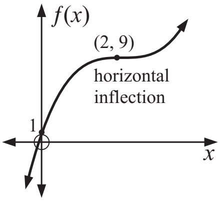
e $f(x) = x^3 - 6x^2 + 12x + 1$

$$\therefore \quad f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x-2)^2$$

with sign diagram:



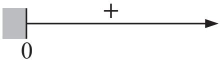
$\therefore$ horizontal inflection at (2, 9)



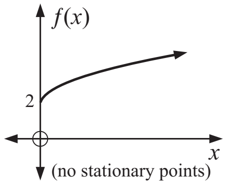
f $f(x) = \sqrt{x} + 2$

$$\therefore \quad f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \neq 0$$

with sign diagram:



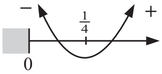
$\therefore$ no stationary points.



g $f(x) = x - \sqrt{x}$

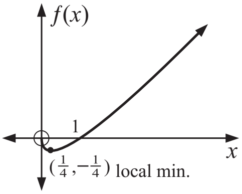
$$\therefore \quad f'(x) = 1 - \tfrac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x} - 1}{2\sqrt{x}}$$

with sign diagram:



$f(x)$ is defined for all $x \geqslant 0$

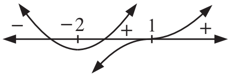
local minimum at $(\frac{1}{4}, -\frac{1}{4})$



h $f(x) = x^4 - 6x^2 + 8x - 3$

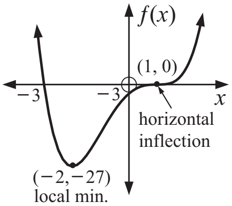
$$\therefore \quad f'(x) = 4x^3 - 12x + 8 = 4(x^3 - 3x + 2) = 4(x-1)(x^2 + x - 2) = 4(x-1)(x+2)(x-1)$$

with sign diagram:



local minimum at (−2, −27),

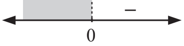
horizontal inflection at (1, 0)



i $f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}}$

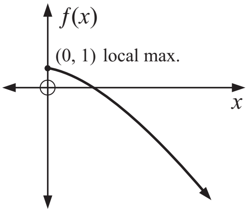
$$\therefore \quad f'(x) = -\tfrac{3}{2}x^{\frac{1}{2}} = \frac{-3\sqrt{x}}{2}$$

with sign diagram:



$f(x)$ is only defined when $x \geqslant 0$

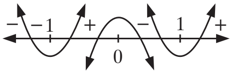
$\therefore$ local maximum at (0, 1)



j $f(x) = x^4 - 2x^2 - 8$

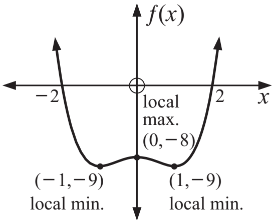
$$\therefore \quad f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$$

with sign diagram:



local minima at (−1, −9) and (1, −9),

local maximum at (0, −8)



3 $f(x) = ax^2 + bx + c, \quad a \neq 0$

$f'(x) = 2ax + b$ and $f(x)$ has a stationary point when $f'(x) = 0$ i.e., $x = -\dfrac{b}{2a}$

There is a local maximum when $a < 0$  and there is a local minimum when $a > 0$ .

4 $f(x) = 2x^3 + ax^2 - 24x + 1, \quad \therefore \quad f'(x) = 6x^2 + 2ax - 24$

But $f'(-4) = 0, \quad \therefore \quad 96 - 8a - 24 = 0$

$\therefore \quad 72 = 8a$ and so $a = 9$.

5 **a** $f(x) = x^3 + ax + b, \quad \therefore \quad f'(x) = 3x^2 + a$

Now $3x^2 + a = 0$ when $x = -2$

$\therefore \quad 12 + a = 0$

i.e., $a = -12$

Now $(-2, 3)$ is a point on the curve.

$\therefore \quad f(x) = x^3 - 12x + b$

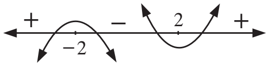
$\therefore \quad 3 = -8 + 24 + b$

$\therefore \quad b = -13$

b $\therefore \quad f(x) = x^3 - 12x - 13$ and $f'(x) = 3x^2 - 12$

$f'(x) = 3(x+2)(x-2)$

where $f'(x)$ has sign diagram:



$\therefore$ there is a local maximum at $(-2, 3)$ and a local minimum at $(2, -29)$

6 Let the cubic polynomial be $P(x) = ax^3 + bx^2 + cx + d$ ...... (1)

$\therefore \quad P'(x) = 3ax^2 + 2bx + c$ ...... (2)

(0, 2) lies on $P(x)$ $\quad \therefore \quad P(0) = 2$ and so $a(0) + b(0) + c(0) + d = 2$

i.e., $d = 2$

Now the tangent is $y = 9x + 2$, $\therefore$ slope at (0, 2) is 9 $\quad \therefore \quad P'(0) = 9$

$\therefore \quad 3a(0) + 2b(0) + c = 9$

$\therefore \quad c = 9$

and a stationary point at $(-1, -7)$ means that $P'(-1) = 0$

$\therefore \quad 3a(-1)^2 + 2b(-1) + c = 0$ {using (2)}

$\therefore \quad 3a - 2b + c = 0$

but since $c = 9$

$3a - 2b = -9$ ...... (3)

Now $(-1, -7)$ also lies on $P(x)$ $\quad \therefore \quad a(-1)^3 + b(-1)^2 + c(-1) + d = -7$

$\therefore \quad -a + b - 9 + 2 = -7$

$\therefore \quad a - b = 0$ ...... (4)

$\therefore \quad a = b$

$\therefore \quad 3a - 2a = -9$ {in (3)}

$\therefore \quad a = -9$

$\therefore \quad a = b = -9$

$\therefore \quad P(x) = -9x^3 - 9x^2 + 9x + 2$

7 **a** $f(x) = x^3 - 12x - 2$, for $-3 \leqslant x \leqslant 5$

$\therefore \quad f'(x) = 3x^2 - 12$

$= 3(x+2)(x-2)$

which is 0 when $x = -2$ or 2

x	-3	-2	2	5
$f(x)$	7	14	-18	63

$\therefore$ maximum value is 63 when $x = 5$, and minimum value is -18, when $x = 2$.

b $f(x) = 4 - 3x^2 + x^3$, for $-2 \leqslant x \leqslant 3$

$$\therefore \quad f'(x) = -6x + 3x^2 = 3x(x-2)$$

which is 0 when $x = 0$ or 2

x	-2	0	2	3
$f(x)$	-16	4	0	4

$\therefore$ maximum value is 4 when $x = 0$ or $x = 3$, minimum value is -16, when $x = -2$.

8 $C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160$

$C'(x) = 0.0021x^2 - 0.3592x + 14.663$

$C'(x) = 0$ when $0.0021x^2 - 0.3592x + 14.663 = 0$

Using technology, $x \doteqdot 103.74$ or $x \doteqdot 67.30$

x	50	67.30	103.74	150
$C(x)$	531.65	546.73	529.80	680.95

$\therefore$ the maximum hourly cost is \$680.95 when 150 hinges are made. The minimum hourly cost is \$529.80 when 104 hinges are made.

EXERCISE 22E.1

1 **a** $y = \dfrac{2x}{x^2 - 4} = \dfrac{2x}{(x+2)(x-2)}$

VAs are $x + 2 = 0$ and $x - 2 = 0$

i.e., $x = -2$ and $x = 2$

HA is $y = 0$ $\{$as $|x| \to \infty$, $y \to 0\}$

b $y = \dfrac{1-x}{(x+2)^2}$

VA is $x + 2 = 0$ i.e., $x = -2$

HA is $y = 0$ $\{$as $|x| \to \infty$, $y \to 0\}$

c $y = \dfrac{3x+2}{x^2+1}$ has no VAs $\{$as $x^2 + 1 = 0$ has no real solutions$\}$

and a HA of $y = 0$ $\{$as $|x| \to \infty$, $y \to 0\}$

2 **a** **i** $f(x) = \dfrac{4x}{x^2+1}$

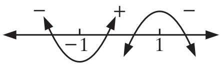
has no VAs $\{$as $x^2 + 1 = 0$ has no real solutions$\}$

and a HA of $y = 0$

$\{$as $|x| \to \infty$, $y \to 0\}$

ii $$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2} = \frac{4 - 4x^2}{(x^2+1)^2} = \frac{4(1+x)(1-x)}{(x^2+1)^2}$$

and has a sign diagram of



$\therefore$ local min. is $\left(-1, \dfrac{4(-1)}{1+1}\right)$

i.e., $(-1, -2)$

and local max is $\left(1, \dfrac{4(1)}{1+1}\right)$

i.e., $(1, 2)$

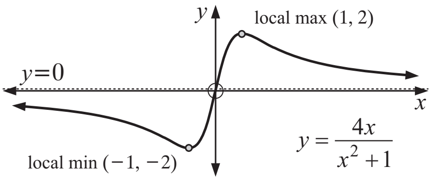
iii when $x = 0$, $f(0) = 0$

$\therefore$ y-intercept is 0

when $y = 0$, $\dfrac{4x}{x^2+1} = 0$

$\therefore$ $x = 0$ and so the x-intercept is 0

iv



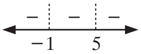
b $f(x) = \dfrac{4x}{x^2-4x-5} = \dfrac{4x}{(x-5)(x+1)}$

i Vertical asymptotes occur when $(x-5)(x+1)=0$, i.e., at $x=-1$ and $x=5$.

Horizontal asymptote is $y=0$ $\{$as $|x| \to \infty$, $y \to 0\}|$

ii $$f'(x) = \frac{4(x^2-4x-5)-4x(2x-4)}{(x^2-4x-5)^2} = \frac{4x^2-16x-20-8x^2+16x}{(x^2-4x-5)^2} = \frac{-4x^2-20}{(x-5)^2(x+1)^2}$$

Since $-4x^2-20$ is always negative we have no turning points, and sign diagram is:



iii When $x=0$, $y=\frac{0}{-5}=0$

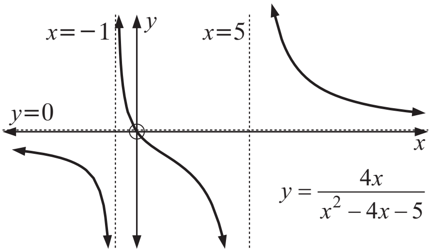
$\therefore$ y-intercept is 0

When $y=0$, $\dfrac{4x}{x^2-4x-5}=0$

$\therefore$ $x=0$

$\therefore$ x-intercept is 0

iv

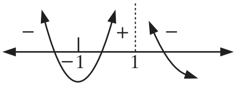


c **i** $f(x) = \dfrac{4x}{(x-1)^2}$

has VA $x-1=0$ i.e., $x=1$ and a HA of $y=0$ $\{$as $|x| \to \infty$, $y \to 0\}$

ii $$f'(x) = \frac{4(x-1)^2-4x \times 2(x-1)1}{(x-1)^4} = \frac{4(x-1)[x-1-2x]}{(x-1)^4} = \frac{4(-x-1)}{(x-1)^3} = \frac{-4(x+1)}{(x-1)^3}$$

which has sign diagram



local min. at $\left(-1, \frac{4(-1)}{(-2)^2}\right)$ i.e., at $(-1, -1)$

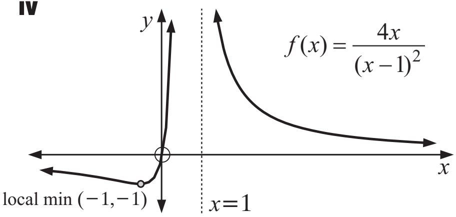
iii when $x=0$, $y=\frac{0}{1}=0$

$\therefore$ y-intercept is 0

when $y=0$, $\dfrac{4x}{(x-1)^2}=0$

$\therefore$ x-intercept is 0

iv



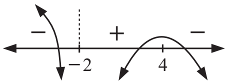
d $f(x) = \dfrac{3x-3}{(x+2)^2}$

i Vertical asymptotes occur when $(x+2)^2 = 0$, i.e., at $x = -2$.

HA is $y = 0$ $\{$as $|x| \to \infty$, $y \to 0\}$

ii
$$\begin{aligned} f'(x) &= \frac{3(x+2)^2 - 2(x+2)(3x-3)}{(x+2)^4} \\ &= \frac{3(x+2) - 2(3x-3)}{(x+2)^3} \\ &= \frac{3x+6-6x+6}{(x+2)^3} \\ &= \frac{12-3x}{(x+2)^3} \\ &= \frac{-3(x-4)}{(x+2)^3} \end{aligned}$$

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(4, \frac{1}{4})$.

iii when $x = 0$, $y = \frac{-3}{2^2} = -\frac{3}{4}$

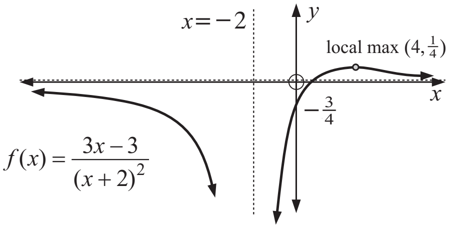
$\therefore$ y-intercept is $-\frac{3}{4}$

when $y = 0$, $3x - 3 = 0$

$\therefore$ $x = 1$

$\therefore$ x-intercept is 1

iv



EXERCISE 22E.2

1 **a** $y = \dfrac{2x^2 - x + 2}{x^2 - 1} = \dfrac{2x^2 - x + 2}{(x+1)(x-1)}$

has VA when $x + 1 = 0$, $x - 1 = 0$

i.e., $x = -1$, $x = 1$

and as $y = \dfrac{2 - \frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}}$

HA is $y = 2$

$\{$as $|x| \to \infty$, $y \to \frac{2}{1}\}$

b $y = \dfrac{-x^2 + 2x - 1}{x^2 + x + 1}$

has a VA when $x^2 + x + 1 = 0$

But $\Delta = 1^2 - 4(1)(1) < 0$ $\therefore$ no real solutions $\therefore$ no VA's exist

and as $y = \dfrac{-1 + \frac{2}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}}$

HA is $y = -1$ $\{$as $|x| \to \infty$, $y = \dfrac{-1}{1}\}$

c $y = \dfrac{3x^2 - x + 2}{(x+2)^2} = \dfrac{3x^2 - x + 2}{x^2 + 4x + 4} = \dfrac{3 - \frac{1}{x} + \frac{2}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}}$

has VA $x + 2 = 0$ i.e., $x = -2$ had HA $y = 3$ $\{$as $|x| \to \infty$, $y \to \frac{3}{1}\}$

2 **a** **i** $y = \dfrac{x^2 - x}{x^2 - x - 6} = \dfrac{x(x-1)}{(x-3)(x+2)}$

has VA when $x - 3 = 0$, $x + 2 = 0$

i.e., $x = 3$, $x = -2$

and since $y = \dfrac{1 - \dfrac{1}{x}}{1 - \dfrac{1}{x} - \dfrac{6}{x^2}}$

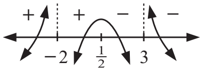
the HA is $y = 1$

{as $|x| \to \infty$, $y \to \frac{1}{1}$}

ii $$\begin{aligned} f'(x) &= \frac{(2x-1)(x^2-x-6) - (x^2-x)(2x-1)}{(x^2-x-6)^2} \\ &= \frac{(2x-1)(x^2-x-6-x^2+x)}{(x^2-x-6)^2} \\ &= \frac{-6(2x-1)}{(x^2-x-6)^2} \end{aligned}$$

Turning points are when $f'(x) = 0$, i.e., when $x = \frac{1}{2}$.

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(\frac{1}{2}, \frac{1}{25})$.

iii When $x = 0$, $y = \frac{0}{-6} = 0$

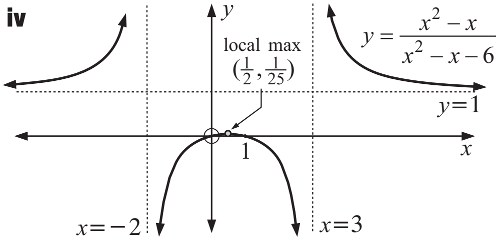
$\therefore$ y-intercept is 0

When $y = 0$, $x(x-1) = 0$

$\therefore$ $x = 0$ or 1

$\therefore$ x-intercepts are 0 and 1

iv



b **i** $y = \dfrac{x^2 - 1}{x^2 + 1} = \dfrac{(x+1)(x-1)}{x^2+1}$

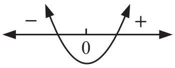
has no VA as $x^2 + 1 = 0$ has no real solutions and since

$y = \dfrac{1 - \dfrac{1}{x^2}}{1 + \dfrac{1}{x^2}}$ the HA is $y = 1$

{as $|x| \to \infty$, $y \to \frac{1}{1}$}

ii $$\begin{aligned} f'(x) &= \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} \\ &= \frac{2x[x^2+1-x^2+1]}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

and has sign diagram



$\therefore$ a local minimum of (0, −1).

iii When $x = 0$, $y = \frac{-1}{1} = -1$

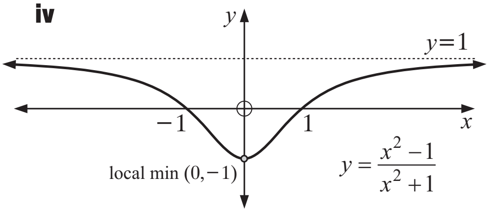
$\therefore$ y-intercept is −1

When $y = 0$, $(x+1)(x-1) = 0$

$\therefore$ $x = \pm 1$

$\therefore$ x-intercepts are ± 1

iv



c **i** $y = \dfrac{x^2 - 5x + 4}{x^2 + 5x + 4} = \dfrac{(x-1)(x-4)}{(x+1)(x+4)}$

has VAs of $x + 1 = 0$ and $x + 4 = 0$

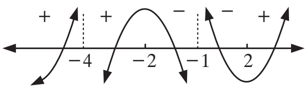
i.e., $x = -1$, $x = -4$

and as $y = \dfrac{1 - \dfrac{5}{x} + \dfrac{4}{x^2}}{1 + \dfrac{5}{x} + \dfrac{4}{x^2}}$ the HA is $y = 1$ {as $|x| \to \infty$, $y \to \frac{1}{1}$}

ii $$\begin{aligned} \frac{dy}{dx} &= \frac{(2x-5)(x^2+5x+4) - (x^2-5x+4)(2x+5)}{(x+1)^2(x+4)^2} \\ &= \frac{[2x^3+10x^2+8x-5x^2-25x-20] - [2x^3-10x^2+8x+5x^2-25x+20]}{(x+1)^2(x+4)^2} \end{aligned}$$

$$= \frac{10x^2-40}{(x+1)^2(x+4)^2}$$

$$= \frac{10(x+2)(x-2)}{(x+1)^2(x+4)^2} \quad \text{which has sign diagram}$$



$\therefore$ local max at $\left(-2, \dfrac{4+10+4}{4-10+4}\right)$ i.e., at $(-2, -9)$

local min at $\left(2, \dfrac{4-10+4}{4+10+4}\right)$ i.e., at $(2, -\frac{1}{9})$

iii When $x=0$, $y=\frac{4}{4}=1$

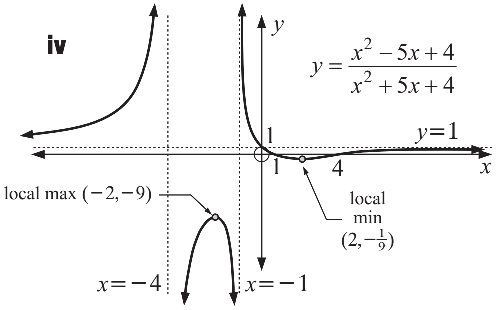
$\therefore$ the y-intercept is 1

When $y=0$, $(x-1)(x-4)=0$

$\therefore$ $x=1$ or 4

$\therefore$ x-intercepts are 1, 4

iv



d **i** $y=\dfrac{x^2-6x+5}{(x+1)^2}=\dfrac{(x-1)(x-5)}{(x+1)^2}$

has VA $x+1=0$ i.e., $x=-1$ and HA $y=1$

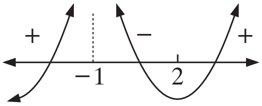
$$\left\{\text{as } y=\frac{1-\frac{6}{x}+\frac{5}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}} \to \tfrac{1}{1} \text{ as } |x|\to\infty\right\}$$

ii $$\frac{dy}{dx} = \frac{(2x-6)(x+1)^2-(x^2-6x+5)2(x+1)^1 1}{(x+1)^4}$$

$$= \frac{(x+1)[2x^2-4x-6-2x^2+12x-10]}{(x+1)^4}$$

$$= \frac{8x-16}{(x+1)^3}$$

$$= \frac{8(x-2)}{(x+1)^3} \quad \text{and has sign diagram}$$



$\therefore$ local minimum at $\left(2, \frac{4-12+5}{3^2}\right)$ i.e., $(2, -\frac{1}{3})$

iii When $x=0$, $y=\frac{5}{1}=5$

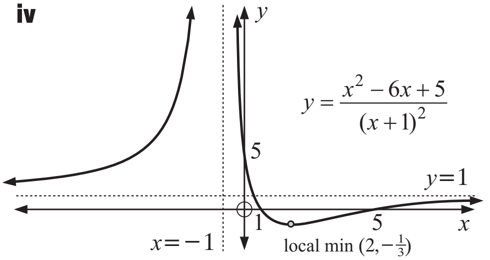
$\therefore$ y-intercept is 5

When $y=0$, $(x-1)(x-5)=0$

$\therefore$ $x=1$ or 5

$\therefore$ x-intercepts are 1, 5

iv



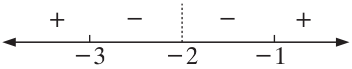
EXERCISE 22E.3

1 **a** **i** $y=\dfrac{x^2+4x+5}{x+2}=\dfrac{(x+2)^2+1}{x+2}=x+2+\dfrac{1}{x+2}$

$\therefore$ there is a vertical asymptote of $x=-2$ $\{$as $x\to-2$, $|y|\to\infty\}$ and an oblique asymptote of $y=x+2$ $\{$as $|x|\to\infty$, $y\to x+2\}$

ii $$\begin{aligned}\frac{dy}{dx} &= 1-\frac{1}{(x+2)^2}\\ &= \frac{(x+2)^2-1}{(x+2)^2}\\ &= \frac{x^2+4x+3}{(x+2)^2}\\ &= \frac{(x+1)(x+3)}{(x+2)^2}\end{aligned}$$

which has sign diagram



$\therefore$ there is a local maximum at $(-3, -2)$
and a local minimum at $(-1, 2)$

iii When $y=0$, $x^2+4x+5=0$

$\Delta = 4^2 - 4\times 5 = -4$

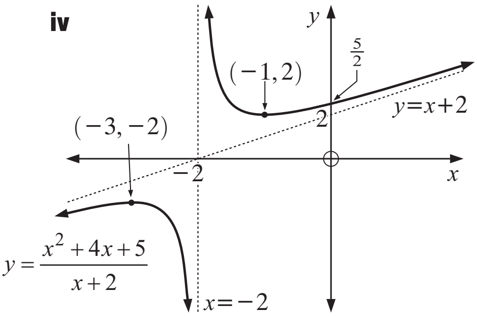
$\therefore$ there are no real roots.

So there are no x-intercepts.

When $x=0$, $y=\frac{5}{2}$

so the y-intercept is $\frac{5}{2}$

iv



b **i** $y=\dfrac{x^2+3x}{x+1}=\dfrac{(x+1)(x+2)-2}{x+1}=x+2-\dfrac{2}{x+1}$

$\therefore$ there is a vertical asymptote of $x=-1$ $\{$as $x\to -1,\ |y|\to\infty\}$
and an oblique asymptote of $y=x+2$ $\{$as $|x|\to\infty,\ y\to x+2\}$

ii $$\begin{aligned}\frac{dy}{dx} &= 1+\frac{2}{(x+1)^2}\\ &= \frac{(x+1)^2+2}{(x+1)^2}\\ &= \frac{x^2+2x+3}{(x+1)^2}\end{aligned}$$

Now x^2+2x+3 has $\Delta = 2^2-4\times 3=-8$

So $\dfrac{dy}{dx}$ is never zero.

$\therefore$ there are no turning points.

iii When $y=0$, $x^2+3x = 0$

$\therefore$ $x(x+3) = 0$

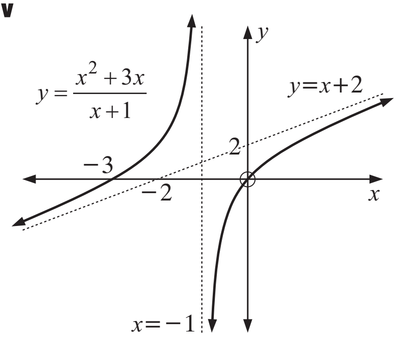
$\therefore$ $x = -3$ or 0

i.e., there are x-intercepts -3 and 0

When $x=0$, $y=0$

so the y-intercept is 0

iv

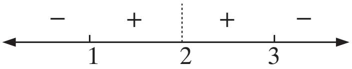


c **i** $y=-2x+1-\dfrac{2}{x-2}$

has a vertical asymptote of $x=2$ $\{$as $x\to 2,\ |y|\to\infty\}$
and an oblique asymptote of $y=-2x+1$ $\{$as $|x|\to\infty,\ y\to -2x+1\}$

ii
$$\begin{aligned}\frac{dy}{dx} &= -2+\frac{2}{(x-2)^2}\\ &= \frac{-2(x-2)^2+2}{(x-2)^2}\\ &= \frac{-2x^2+8x-6}{(x-2)^2}\\ &= \frac{-2(x^2-4x+3)}{(x-2)^2}\\ &= \frac{-2(x-3)(x-1)}{(x-2)^2}\end{aligned}$$

which has sign diagram

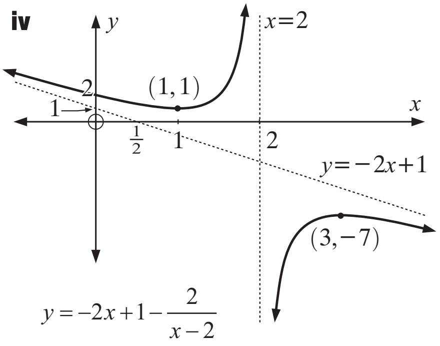


$\therefore$ there is a local minimum at (1, 1)
and a local maximum at (3, −7)

iii When $y=0,\quad -2x+1-\dfrac{2}{x-2}=0$

$$\begin{aligned}\therefore\quad \frac{(-2x+1)(x-2)-2}{x-2} &= 0\\ \therefore\quad \frac{-2x^2+4x+x-2-2}{x-2} &= 0\\ \therefore\quad \frac{-2x^2+5x-4}{x-2} &= 0\\ \therefore\quad -2x^2+5x-4 &= 0\\ \therefore\quad \Delta = 5^2-4(-2)(-4) &< 0\end{aligned}$$

iv



$\therefore$ there are no real roots, so there are no x- intercepts.

When $x=0,\quad y=-2(0)+1-\dfrac{2}{0-2}=2$ so the y-intercept is 2.

EXERCISE 22F

1 **a** $f(x)=x^2+3$

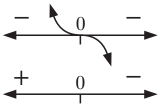
$\therefore\quad f'(x)=2x$ and $f''(x)=2$

Since $f''(x)\neq 0$,
no points of inflection exist.

b $f(x)=2-x^3$

$\therefore\quad f'(x)=-3x^2$

and $f''(x)=-6x$

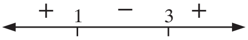


Now $f''(x)=0$ when $x=0$

and $f'(0)=0$

$\therefore$ there is a horizontal inflection at (0, 2)

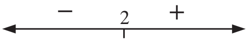
c
$$\begin{aligned}f(x) &= x^3-6x^2+9x+1\\ \therefore\quad f'(x) &= 3x^2-12x+9\\ &= 3(x^2-4x+3)\\ &= 3(x-3)(x-1)\end{aligned}$$



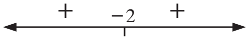
$$\begin{aligned}\text{and}\quad f''(x) &= 6x-12\\ &= 6(x-2)\end{aligned}$$

Now $f''(x)=0$ when $x=2$
and $f'(2)\neq 0$

$\therefore$ there is a non-horizontal inflection at (2, 3)



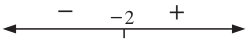
d
$$\begin{aligned}f(x) &= x^3+6x^2+12x+5\\ \therefore\quad f'(x) &= 3x^2+12x+12\\ &= 3(x^2+4x+4)\\ &= 3(x+2)^2\end{aligned}$$



$$\begin{aligned}\text{and}\quad f''(x) &= 6x+12\\ &= 6(x+2)\end{aligned}$$

Now $f''(x)=0$ when $x=-2$
and $f'(2)=0$

$\therefore$ there is a horizontal inflection at (−2, −3)

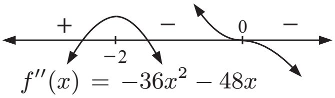


e

$$f(x) = -3x^4 - 8x^3 + 2$$

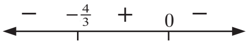
$$\therefore \quad f'(x) = -12x^3 - 24x^2$$

$$= -12x^2(x+2)$$



$$\text{and} \quad f''(x) = -36x^2 - 48x$$

$$= -12x(3x+4)$$



$\therefore$ horizontal inflection at (0, 2),

non-horizontal inflection at $\left(-\frac{4}{3}, \frac{310}{27}\right)$

f

$$f(x) = 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$$

$$\therefore \quad f'(x) = \tfrac{1}{2}x^{-\frac{3}{2}}$$

$$\text{and} \quad f''(x) = -\tfrac{3}{4}x^{-\frac{5}{2}}$$

$$= \frac{-3}{4x^2\sqrt{x}}$$

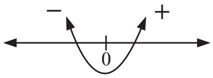
$$f''(x) \neq 0 \quad \text{for all } x$$

$\therefore$ no points of inflection.

2 **a**

$$f(x) = x^2$$

$\therefore \quad f'(x) = 2x$ which has sign diagram:



and $f''(x) = 2$

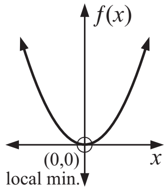
i There is a local minimum at (0, 0).

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing when $x \geqslant 0$, and decreasing when $x \leqslant 0$.

iv $f(x)$ is concave up for all x as $f''(x) > 0$ for all x.

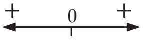
v



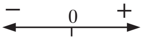
b

$$f(x) = x^3$$

$\therefore \quad f'(x) = 3x^2$ which has sign diagram:



and $f''(x) = 6x$ which has sign diagram:



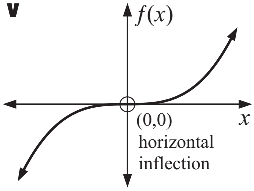
i A horizontal inflection at (0, 0). $\{f'(x) = 0\}$

ii see **i**

iii $f(x)$ is increasing for all x.

iv $f(x)$ is concave up $x \geqslant 0$, and concave down $x \leqslant 0$

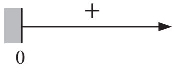
v



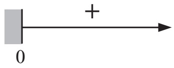
c

$$f(x) = \sqrt{x}$$

$\therefore \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ which has sign diagram:



and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \dfrac{-1}{4x\sqrt{x}}$ which has sign diagram:



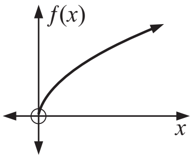
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x \geqslant 0$.

iv $f(x)$ is concave down for all $x \geqslant 0$ as $f''(x) < 0$ for all $x > 0$.

v



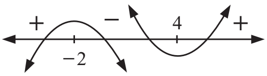
d

$$f(x) = x^3 - 3x^2 - 24x + 1$$

$$\therefore \quad f'(x) = 3x^2 - 6x - 24$$

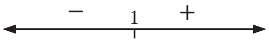
$$= 3(x^2 - 2x - 8)$$

$= 3(x-4)(x+2)$ which has sign diagram:



and $f''(x) = 6x - 6$

$= 6(x-1)$ which has sign diagram:



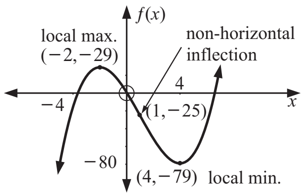
i There is a local maximum at (−2, 29), and a local minimum at (4, −79).

ii There is a non-horizontal inflection at (1, -25).

iii $f(x)$ is increasing for $x \leqslant -2$ or $x \geqslant 4$, and decreasing for $-2 \leqslant x \leqslant 4$.

iv $f(x)$ is concave down when $x \leqslant 1$, and concave up when $x \geqslant 1$.

v

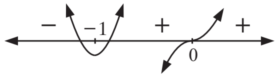


e

$$f(x) = 3x^4 + 4x^3 - 2$$

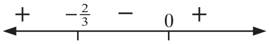
$$\therefore \quad f'(x) = 12x^3 + 12x^2$$

$$= 12x^2(x+1) \quad \text{which has sign diagram:}$$



$$\text{and} \quad f''(x) = 36x^2 + 24x$$

$$= 12x(3x+2) \quad \text{which has sign diagram:}$$



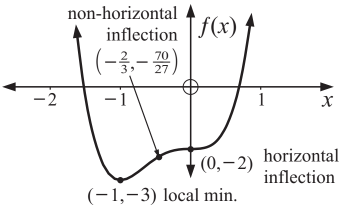
i There is a local minimum at (-1, -3), and a horizontal inflection at (0, -2)

ii There is a non-horizontal inflection at ($-\frac{2}{3}$, $-\frac{70}{27}$) and a horizontal inflection at (0, -2)

iii $f(x)$ is increasing for $x \geqslant -1$, and decreasing for $x \leqslant -1$.

iv $f(x)$ is concave down for $-\frac{2}{3} \leqslant x \leqslant 0$, and concave up for $x \leqslant -\frac{2}{3}$ or $x \geqslant 0$.

v

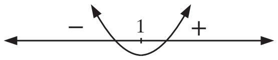


f

$$f(x) = (x-1)^4$$

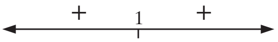
$$\therefore \quad f'(x) = 4(x-1)^3 \times 1$$

$$= 4(x-1)^3 \quad \text{which has sign diagram:}$$



$$\text{and} \quad f''(x) = 12(x-1)^2 \times 1$$

$$= 12(x-1)^2 \quad \text{which has sign diagram:}$$



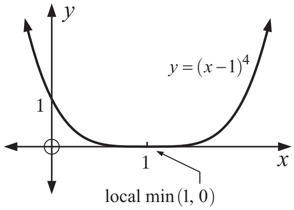
i There is a local minimum at (1, 0).

ii There are no points of inflection.

iii $f(x)$ is increasing for $x \geqslant 1$, and decreasing for $x \leqslant 1$.

iv $f(x)$ is concave up for all x.

v



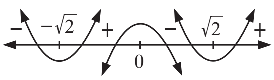
g

$$f(x) = x^4 - 4x^2 + 3$$

$$f'(x) = 4x^3 - 8x$$

$$= 4x(x^2 - 2)$$

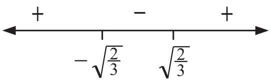
$$= 4x(x+\sqrt{2})(x-\sqrt{2}) \quad \text{which has sign diagram:}$$



$$f''(x) = 12x^2 - 8$$

$$= 4(3x^2 - 2)$$

$$= 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2}) \quad \text{which has sign diagram:}$$

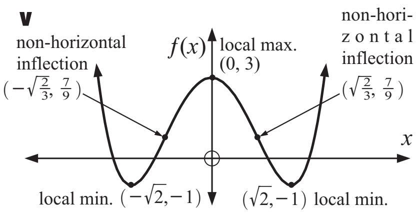


i There is a local maximum at (0, 3), and local min. at ($\sqrt{2}$, -1) and ($-\sqrt{2}$, -1).

ii There are non-horizontal inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$. $\{f''(x) = 0\}$

iii $f(x)$ is increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$, and decreasing for $0 \leqslant x \leqslant \sqrt{2}$ and $x \leqslant -\sqrt{2}$.

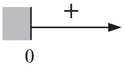
v



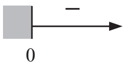
iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$, and concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$.

h $$f(x) = 3 - \frac{4}{\sqrt{x}}, \quad x > 0$$
$$= 3 - 4x^{-\frac{1}{2}}$$

$$\therefore \quad f'(x) = 2x^{-\frac{3}{2}} = \frac{2}{x\sqrt{x}}$$ with sign diag:



and $$f''(x) = -3x^{-\frac{5}{2}} = -\frac{3}{x^2\sqrt{x}}$$ with sign diag:



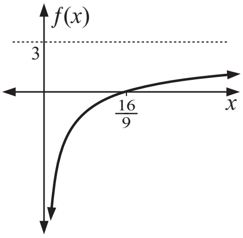
i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x > 0$ as $f'(x) > 0$ for all x.

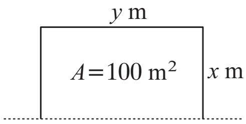
iv $f(x)$ is concave down for all $x > 0$ as $f''(x) < 0$ for all x.

v



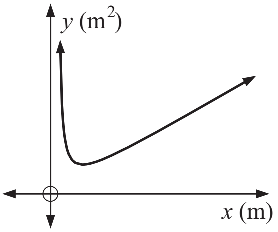
EXERCISE 22G

1 **a**



$$L = 2x + y$$
but $$xy = 100$$
$$\therefore \quad y = \frac{100}{x}$$
$$\therefore \quad L = 2x + \frac{100}{x}$$

b



c $$\frac{dL}{dx} = 2 - 100x^{-2} = 2 - \frac{100}{x^2}$$

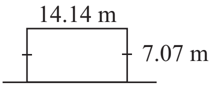
which is 0 when $$\frac{100}{x^2} = 2$$
$$\therefore \quad x^2 = 50$$
$$\therefore \quad x = \sqrt{50} \quad \{x > 0\}$$

$$\frac{d^2L}{dt^2} = 200x^{-3} = \frac{200}{x^3} > 0 \text{ for } x > 0$$

$\therefore$ min $L = 28.28$ m when $x = \sqrt{50}$ m

$$\therefore \quad L_{\min} = 2\sqrt{50} + \frac{100}{\sqrt{50}}$$
$$= 2\sqrt{50} + 2\sqrt{50}$$
$$= 4\sqrt{50}$$
$$= 20\sqrt{2} \text{ m when } x = 5\sqrt{2} \text{ m}$$

d



2 **a** Inner length of box $= 2x$ cm

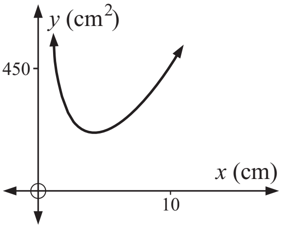
b $$\text{Volume} = 200 \text{ cm}^3$$
$$\therefore \quad x \times 2x \times h = 200$$
$$2x^2h = 200$$
$$\therefore \quad x^2h = 100 \quad \ldots\ldots (1)$$

c From (1) $h = \dfrac{100}{x^2}$.

Now area of inner surface is
$$A(x) = 2(2x \times x) + 2(2x \times h) + 2(x \times h)$$
$$= 4x^2 + 4xh + 2xh$$
$$= 4x^2 + 6xh$$

i.e., $$A(x) = 4x^2 + \frac{600}{x} \text{ cm}^2$$

d



e

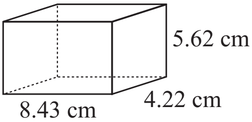
$$\begin{aligned} A(x) &= 4x^2 + 600x^{-1} \\ \therefore\quad A'(x) &= 8x - 600x^{-2} \\ &= 8x - \frac{600}{x^2} \\ \therefore\quad A'(x) &= 0 \quad \text{when} \\ 8x &= \frac{600}{x^2} \\ 8x^3 &= 600 \\ x^3 &= 75 \\ x &= \sqrt[3]{75} \\ x &\doteqdot 4.127 \text{ cm} \end{aligned}$$

$$\begin{aligned} A''(x) &= 8 + 1200x^{-3} \\ &= 8 + \frac{1200}{x^3} \\ \therefore\quad A''(x) &> 0 \qquad \{\text{as } x > 0\} \end{aligned}$$

$\therefore$ minimum when $x \doteqdot 4.22$ cm

$$\begin{aligned} \therefore\quad A_{\min} &= 4(4.217)^2 + \frac{600}{(4.217)} \\ &\doteqdot 213.41 \text{ cm}^2 \end{aligned}$$

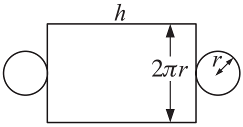
f



3 **a** Volume of can $= \pi r^2 h$

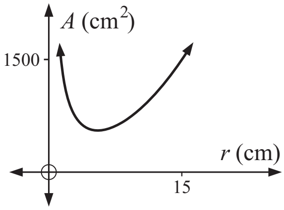
$$\begin{aligned} \therefore\quad 1000 &= \pi r^2 h \quad \text{(in cm)} \\ \therefore\quad h &= \frac{1000}{\pi r^2} \text{ cm} \end{aligned}$$

b Opening the can up we get



$$\begin{aligned} \therefore\quad A(r) &= \pi r^2 + \pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2000}{r} \text{ cm}^2 \end{aligned}$$

c



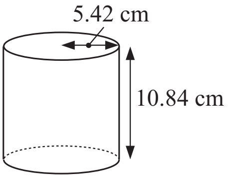
d

$$\begin{aligned} A(r) &= 2\pi r^2 + 2000r^{-1} \\ A'(r) &= 4\pi r - 2000r^{-2} = 4\pi r - \frac{2000}{r^2} \end{aligned}$$

So, $A'(r) = 0$ when

$$\begin{aligned} 4\pi r &= \frac{2000}{r^2} \\ r^3 &= \frac{2000}{4\pi} \\ r &= \sqrt[3]{\frac{500}{\pi}} \\ \therefore\quad r &\doteqdot 5.419 \text{ cm} \end{aligned}$$

e



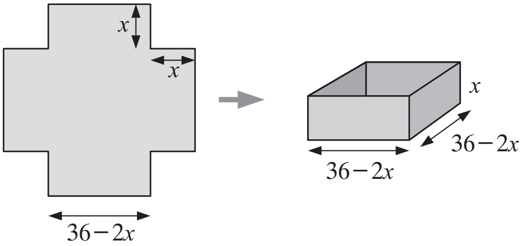
$$A''(r) = 4\pi + 4000r^{-3} = 4\pi + \frac{4000}{r^3}$$

and as $r > 0$, $A''(r) > 0$

$\therefore$ area is a minimum when $r \doteqdot 5.42$ cm

and $h = \dfrac{1000}{\pi r^2} \doteqdot 10.84$ cm

4



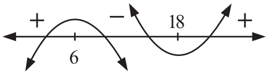
Now volume of container is
$$\begin{aligned} V &= lbd \\ &= x(36-2x)(36-2x) \\ \therefore\quad V &= x(36-2x)^2 \text{ cm}^3 \end{aligned}$$

let $u = x,\quad u' = 1,\quad v = (36-2x)^2,\quad v' = 2(36-2x)(-2)$

$$\begin{aligned} \therefore\quad V'(x) &= (36-2x)^2 - 4x(36-2x) \\ &= (36-2x)[(36-2x)-4x] \\ &= (36-2x)(36-6x) \end{aligned}$$

$\therefore\quad V'(x) = 0$ when $x = 6$ or $x = 18$

Sign diagram of $V'(x)$ is:



$\therefore$ volume is maximised when $x = 6$ cm

$\therefore$ cut out 6 cm × 6 cm squares.

5 **a**
$$\begin{aligned} P &= 2\pi r + 2l \\ \therefore\quad 400 &= 2\pi(x) + 2l \\ \therefore\quad 200 &= \pi x + l \\ \therefore\quad l &= 200 - \pi x \end{aligned}$$

Now $0 \leqslant l \leqslant 200$

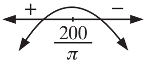
and $l > 0$ means that
$$\begin{aligned} \pi x &< 200 \\ \therefore\quad x &< \frac{200}{\pi} \end{aligned}$$
and $x > 0$
$$\therefore\quad 0 \leqslant x \leqslant \frac{200}{\pi}$$

b Area,
$$\begin{aligned} A &= \pi r^2 + (2x)\times l \\ &= \pi x^2 + 2xl \\ &= \pi x^2 + 2x(200-\pi x) \\ &= \pi x^2 + 400x - 2\pi x^2 \\ \therefore\quad A &= 400x - \pi x^2 \end{aligned}$$

c Now $\dfrac{dA}{dx} = 400 - 2\pi x$

which is 0 when $2\pi x = 400$
$$x = \frac{200}{\pi} \quad\text{and}\quad l = 0$$

Sign diagram of $A'(x)$:



$\therefore$ area will be a maximum when the track is a circle.

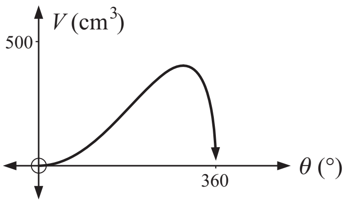
6 **a**
$$\begin{aligned} \text{Arc AC} &= \frac{\theta}{360}\times(2\pi r) \\ &= \frac{\theta}{360}(2\times\pi\times 10) \\ \text{Arc AC} &= \frac{\pi\theta}{18} \end{aligned}$$

b Now arc AC forms the base of the cone.
$$\begin{aligned} \therefore\quad 2\pi r &= \frac{\theta}{360}\times 2\pi\times 10 \\ \therefore\quad r &= \frac{\theta}{36} \end{aligned}$$

c Height of cone $= \sqrt{10^2 - r^2}$ {Pythagoras}
$$\therefore\quad h = \sqrt{100 - \left(\tfrac{\theta}{36}\right)^2}$$

d
$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ V &= \tfrac{1}{3}\pi\left(\tfrac{\theta}{36}\right)^2\sqrt{100-\left(\frac{\theta}{36}\right)^2} \\ &= \frac{\pi\theta^2}{3\times 36^2}\sqrt{\frac{129\,600-\theta^2}{36^2}} \\ &= \frac{\pi\theta^2}{139\,968}\sqrt{129\,600-\theta^2} \end{aligned}$$

e



f Now $V'(\theta) = \dfrac{2\pi\theta}{139\,968}\left(129\,600 - \theta^2\right)^{\frac{1}{2}} + \dfrac{\pi\theta^2}{139\,968}\left(\tfrac{1}{2}\right)(129\,600 - \theta^2)^{-\frac{1}{2}}(-2\theta)$

$$= \frac{\pi\theta}{139\,968}\left(\frac{2\sqrt{129\,600 - \theta^2}}{1}) - \frac{\theta^2}{\sqrt{129\,600 - \theta^2}}\right)$$

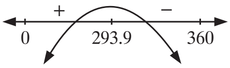
$$= \frac{\pi\theta}{139\,968}\left(\frac{2(129\,600 - \theta^2) - \theta^2}{\sqrt{129\,600 - \theta^2}}\right)$$

and $V'(\theta) = 0$ when $\theta = 0$ or $2(129\,600 - \theta^2) = \theta^2$

$259\,200 - 2\theta^2 = \theta^2$

$\therefore\ 3\theta^2 = 259\,200$

Sign diagram of $V'(\theta)$ is:



$\therefore\ \theta = \sqrt{86\,400}$ {as $\theta > 0$}

$\therefore\ \theta \doteqdot 293.9$

$\therefore$ maximum V occurs when $\theta = 293.9^o$

7 **a** X must lie between A and C (or at A or C).

If $x = 0$, then he rows straight to the shore and runs to C.

If $x = 6$, then he rows straight to C. $\therefore\ 0 \leqslant x \leqslant 6$

b Now $\text{XC} = 6 - x$

$\therefore$ time to row from B to X

$$= \frac{\text{BX}}{8} = \frac{\sqrt{5^2 + x}}{8}$$

and time to run from X to C

$$= \frac{\text{XC}}{17} = \frac{6 - x}{17}$$

$\therefore$ total time $T(x) = \dfrac{\sqrt{25 + x^2}}{8} + \dfrac{6 - x}{17}$ hours

$$= \tfrac{1}{8}(25 + x^2)^{\frac{1}{2}} + \tfrac{6}{17} - \frac{x}{17}$$

c Now $\dfrac{dT}{dx} = \tfrac{1}{16}(25 + x^2)^{-\frac{1}{2}}(2x) - \tfrac{1}{17}$

$\therefore\ \dfrac{dT}{dx} = \dfrac{x}{8\sqrt{25 + x^2}} - \tfrac{1}{17}$

$\therefore\ \dfrac{dT}{dx} = 0$

when $\dfrac{x}{8\sqrt{25 + x^2}} = \tfrac{1}{17}$

$\therefore\ 17x = 8\sqrt{25 + x^2}$

$\therefore\ 289x^2 = 64(25 + x^2)$

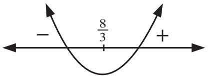
$\therefore\ 289x^2 = 1600 + 64x^2$

$\therefore\ 225x^2 = 1600$

$\therefore\ x^2 = \dfrac{1600}{225}$

$\therefore\ x = \tfrac{40}{15} = \tfrac{8}{3}$ km

Sign diagram of $\dfrac{dT}{dx}$:



Thus, the time taken is a minimum if Peter aims for X such that $x = \frac{8}{3}$ km.

8 Let MX $= x$ km, then XN $= 5 - x$ km

$\therefore$ AX $= \sqrt{4 + x^2}$ km and XB $= \sqrt{1 + (5 - x)^2}$ km {Pythagoras}

Now $P = \text{AX} + \text{XB}$

$$P = (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\therefore\ \frac{dP}{dx} = \tfrac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \tfrac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(2x - 10)$$

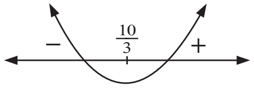
$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

Thus, $\dfrac{dP}{dx} = 0$ when $\dfrac{x}{\sqrt{4 + x^2}} = \dfrac{5 - x}{\sqrt{x^2 - 10x + 26}}$

$\therefore\ \dfrac{x^2}{4 + x^2} = \dfrac{(5 - x)^2}{x^2 - 10x + 26}$ {squaring both sides}

$$\begin{aligned}\therefore\quad x^2(x^2-10x+26) &= (4+x^2)(25-10x+x^2)\\ \therefore\quad x^4-10x^3+26x^2 &= 100-40x+4x^2+25x^2-10x^3+x^4\\ \therefore\quad 3x^2-40x+100 &= 0\\ \therefore\quad (3x-10)(x-10) &= 0\\ \therefore\quad x &= \tfrac{10}{3} \qquad \{\text{as } x \text{ cannot be } 10\}\end{aligned}$$

Sign diagram of $\dfrac{dP}{dx}$ is:



$\therefore$ minimum length pipeline occurs when $x = \frac{10}{3}$ km

9

$$\begin{aligned} V &= \pi r^2 h\\ \therefore\quad 0.1 &= \pi r^2 h \qquad \{\text{as } 100 \text{ L} = 0.1 \text{ m}^3\}\\ \therefore\quad h &= \frac{0.1}{\pi r^2}\end{aligned}$$

$$\text{Now}\quad A = \pi r^2 + (2\pi r)h = \pi r^2 + 2\pi r\left(\frac{0.1}{\pi r^2}\right)$$

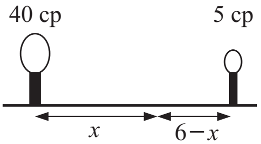
$$\begin{aligned}\text{i.e.,}\quad A(r) &= \pi r^2 + 0.2r^{-1}\\ \therefore\quad A'(r) &= 2\pi r - 0.2r^{-2} = 2\pi r - \frac{0.2}{r^2}\\ \therefore\quad A'(r) &= 0 \quad\text{when}\quad 2\pi r = \frac{0.2}{r^2}\\ r^3 &= \frac{0.2}{2\pi}\\ r &= \sqrt[3]{\tfrac{0.2}{2\pi}} \doteqdot 0.3169 \text{ m}\\ r &\doteqdot 31.7 \text{ cm}\end{aligned}$$

$$\text{Now}\quad A''(r) = 2\pi + 0.4r^{-3} = 2\pi + \frac{0.4}{r^3} \quad \text{which is} > 0 \quad \text{as } r > 0$$

$\therefore$ minimum area occurs when $r \doteqdot 31.7$ cm and $h \doteqdot \dfrac{1}{10\pi(31.69)} \doteqdot 31.7$ cm

$$\therefore\quad r = h \doteqdot 31.7 \text{ cm}$$

10

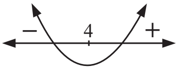


$I \propto \dfrac{s}{d^2}$ where s is the power of the source and d is the distance from it

$$\begin{aligned}\therefore\quad I &= \frac{kS}{d^2} \qquad \{k \text{ is a constant}\}\\ \therefore\quad \text{intensity due to 40 cp} &= \frac{40k}{x^2}\\ \text{and intensity due to 5 cp} &= \frac{5k}{(6-x)^2}\\ \therefore\quad \text{total intensity,}\quad I &= \frac{40k}{x^2} + \frac{5k}{(6-x)^2}\\ &= k[40x^{-2} + 5(6-x)^{-2}]\\ \text{Now}\quad \frac{dI}{dx} &= k[-80x^{-3} - 10(6-x)^{-3}(-1)]\\ &= k\left[\frac{-80}{x^3} + \frac{10}{(6-x)^3}\right]\\ \therefore\quad \frac{dI}{dx} = 0 \quad\text{when}\quad \frac{80}{x^3} &= \frac{10}{(6-x)^3}\\ \text{i.e.,}\quad 8(6-x)^3 &= x^3\end{aligned}$$

$$\begin{aligned}\therefore\quad 2(6-x) &= x \qquad \{\text{finding cube roots}\}\\ \therefore\quad 12-2x &= x\\ \therefore\quad 12 &= 3x\\ \therefore\quad x &= 4\end{aligned}$$

Sign diagram of $\dfrac{dI}{dx}$ is:



$\therefore$ the minimum intensity, i.e., the darkest point occurs when $x = 4$ m i.e., at 4 m from the 40 cp lamp.

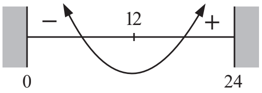
11 **a**

$$\begin{aligned}\text{AB} &= x\text{ m}\\ \therefore\quad \text{BC} &= (24-x)\text{ m}\\ \therefore\quad \text{AC}^2 &= \text{AB}^2+\text{BC}^2 \quad \{\text{Pythagoras}\}\\ &= x^2+(24-x)^2\\ &= x^2+576-48x+x^2\\ &= 2x^2-48x+576\end{aligned}$$

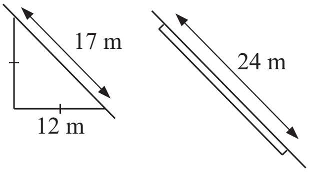
i.e., $[D(x)]^2 = 2x^2-48x+576$ and so $D(x)=\sqrt{2x^2-48x+576}$

b $\therefore\quad \dfrac{d[D(x)]^2}{dx} = 4x-48$

$\therefore\quad \dfrac{d[D(x)]^2}{dx} = 0$ when $x=12$ and the sign diagram is:



c i.e., when $\text{AB} = \text{BC} = 12$ m, $D(x)$ is a minimum, and minimum $D(x) = 12\sqrt{2}$ m $\doteqdot$ 16.97 m.

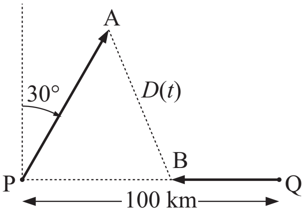


$D(x)$ is a maximum when either $x=0$ or $x=24$, i.e., when the pen ceases to exist and $D(x)=24$ m.

12 **a** Consider each boat's position t hours after 1.00 pm.

$\text{AP} = 12t \qquad \text{BQ} = 8t$

$\therefore\quad \text{PB} = 100-8t$



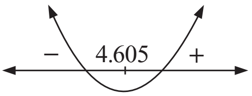
Using the Cosine rule in ΔPAB

$$\begin{aligned}D(t)^2 &= \text{AP}^2+\text{BP}^2-2\text{AP}\times\text{BP}\cos 60^o\\ &= (12t)^2+(100-8t)^2-2(12t)(100-8t)\tfrac{1}{2}\\ &= 144t^2+(100-8t)^2-12t(100-8t)\\ &= 144t^2+10\,000-1600t+64t^2-1200t+96t^2\\ &= 304t^2-2800t+10\,000 \quad\text{and so}\quad D(t)=\sqrt{304t^2-2800t+10\,000}\end{aligned}$$

b Now $\dfrac{d[D(t)]^2}{dt} = 608t-2800$

$\therefore\quad \dfrac{d[D(t)]^2}{dt} = 0$ when $t=\dfrac{2800}{608}\doteqdot 4.605\,26$

and has sign diagram:



$\therefore\quad D(t)$ is a minimum when $t \doteqdot 4.60526$ hours after 1.00 pm

and $[D(t)]^2_{\text{min}} \doteqdot 304\,(4.6053)^2-2800\,(4.6053)+10\,000$

$\therefore\quad [D(t)]^2_{\text{min}} = 3552.63\text{ km}^2$

c The ships are closest when $t = 4.60526$ hours

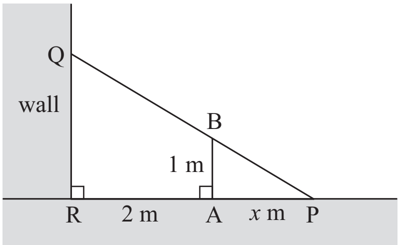
i.e., when the time is 4 hours 36 minutes

$\therefore$ time is approximately 5.36 pm

13 **a** Δs PAB and PRQ are similar.

$$\therefore \quad \frac{PA}{PR} = \frac{PB}{PQ} = \frac{AB}{RQ}$$

$$\therefore \quad \frac{x}{x+2} = \frac{1}{QR} \quad \text{and} \quad \therefore \quad QR = \frac{x+2}{x}$$



b Now $[L(x)]^2 = RP^2 + QR^2$ {Pythagoras}

$$= (x+2)^2 + \left(\frac{x+2}{x}\right)^2$$

$$= (x+2)^2 \times 1 \;+\; (x+2)^2 \times \frac{1}{x^2}$$

$$\therefore \quad [L(x)]^2 = (x+2)^2\left[1 + \frac{1}{x^2}\right] \quad \text{as required}$$

$$\therefore \quad [L(x)]^2 = (x+2)^2\left[1 + x^{-2}\right]$$

c

$$\frac{d[L(x)]^2}{dx} = 2(x+2)\left[1+x^{-2}\right] + (x+2)^2\left[-2x^{-3}\right] \quad \text{\{product rule\}}$$

$$= 2(x+2)\left[1 + x^{-2} - (x+2)x^{-3}\right]$$

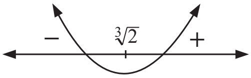
$$= 2(x+2)[1 + x^{-2} - x^{-2} - 2x^{-3}]$$

$$= 2(x+2)(1 - \frac{2}{x^3})$$

$$= 2(x+2)(\frac{x^3-2}{x^3})$$

$$\therefore \quad \frac{d[L(x)]^2}{dx} = 0 \quad \text{when} \quad x = \sqrt[3]{2} \doteqdot 1.2599 \qquad \{\text{as } x > 0 \text{ and } L(x) > 0\}$$

d Sign diagram of $L'(x)$ is:



$\therefore$ the ladder is shortest when $x = \sqrt[3]{2}$ m and minimum $L \doteqdot \sqrt{(x+2)^2(1+\frac{1}{x^2})} \doteqdot 4.16$ m

14 Suppose PN $= x$ m

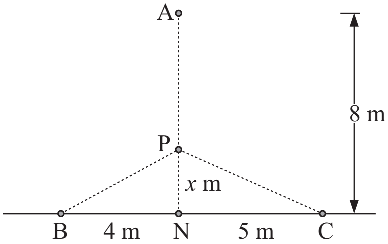
$\therefore$ length of cable $=$ PA $+$ PB $+$ PC

$$\therefore \quad L = 8 - x + \sqrt{x^2+16} + \sqrt{x^2+25}$$

Using technology we graph this and find the minimum,

i.e., the minimum length occurs when $x = 2.577\,98$

i.e., $x \doteqdot 2.578$ m from N

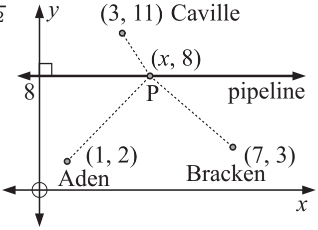


15 Suppose P has coordinates $(x, 8)$

$$\therefore \quad PC = \sqrt{(x-3)^2 + (8-11)^2} \qquad AP = \sqrt{(x-1)^2 + (8-2)^2}$$

$$= \sqrt{x^2 - 6x + 9 + 9} \qquad = \sqrt{x^2 - 2x + 1 + 36}$$

$$= \sqrt{x^2 - 6x + 18} \qquad = \sqrt{x^2 - 2x + 37}$$



$$PB = \sqrt{(x-7)^2 + (8-3)^2}$$

$$= \sqrt{x^2 - 14x + 49 + 25}$$

$$= \sqrt{x^2 - 14x + 74}$$

$\therefore$ length of pipeline $L = \sqrt{x^2 - 6x + 18} + \sqrt{x^2 - 2x + 3} + \sqrt{x^2 - 14x + 74}$

Use technology to graph L and find a minimum. This occurs when $x \doteqdot 3.543\,66$,

i.e., P is at $(3.544, 8)$.

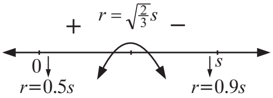
16

$$\begin{aligned} r^2+h^2 &= s^2 \\ \therefore\quad h^2 &= s^2-r^2 \\ h &= \sqrt{s^2-r^2} \\ \text{But}\quad V &= \tfrac{1}{3}\pi r^2 h \\ \therefore\quad V &= \tfrac{1}{3}\pi r^2\sqrt{s^2-r^2} \\ \therefore\quad V^2 &= \frac{\pi^2}{9}r^4(s^2-r^2) \\ &= \frac{\pi^2}{9}\left(r^4s^2-r^6\right) \\ \therefore\quad \frac{dV^2}{dr} &= \frac{\pi^2}{9}\left(4r^3s^2-6r^5\right) \\ &= \frac{\pi^2}{9}\,2r^3(2s^2-3r^2) \end{aligned}$$

$\dfrac{dV^2}{dr}$ is 0 when

$$\begin{aligned} 2s^2-3r^2 &= 0 \\ \therefore\quad 2s^2 &= 3r^2 \qquad \{\text{as } r>0\} \\ \therefore\quad \frac{s^2}{r^2} &= \tfrac{3}{2} \\ \frac{s}{r} &= \sqrt{\tfrac{3}{2}} \\ \text{i.e.,}\quad s:r &= \sqrt{\tfrac{3}{2}}:1 \end{aligned}$$

Sign diagram of $\dfrac{dV^2}{dr}$ is:



$\therefore\quad V$ is a maximum when $s:r=\sqrt{\tfrac{3}{2}}:1=\sqrt{3}:\sqrt{2}$

17 **a**

$$\begin{aligned} \frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1 \\ \therefore\quad x^2b^2+y^2a^2 &= a^2b^2 \quad \{\times \text{ by } a^2b^2\} \\ \therefore\quad a^2y^2 &= a^2b^2-x^2b^2 \\ \therefore\quad y^2 &= \frac{a^2b^2-x^2b^2}{a^2} \\ \therefore\quad y &= \pm\sqrt{b^2-\frac{b^2}{a^2}x^2} \end{aligned}$$

Since A lies in Q₁, $y>0$

$$\begin{aligned} \therefore\quad y &= \sqrt{b^2-\frac{b^2}{a^2}x^2} \\ \therefore\quad y &= \frac{b}{a}\sqrt{a^2-x^2} \end{aligned}$$

b Seating area is

$$\begin{aligned} A &= 2x\times 2y \\ &= 4xy \\ &= 4x\left[\frac{b}{a}\sqrt{a^2-x^2}\right] \\ \therefore\quad A(x) &= \frac{4bx}{a}\sqrt{a^2-x^2} \\ &\qquad\text{as required} \\ \therefore\quad A^2 &= \frac{16b^2x^2}{a^2}(a^2-x^2) \\ &= \frac{16b^2}{a^2}(a^2x^2-x^4) \end{aligned}$$

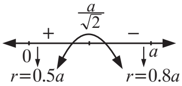
c $\dfrac{dA^2}{dx}=\dfrac{16b^2}{a^2}[2a^2x-4x^3]$

which is 0 when

$$\begin{aligned} 2a^2x-4x^3 &= 0 \\ \therefore\quad 2x(a^2-2x^2) &= 0 \\ \therefore\quad 2x^2 &= a^2 \quad \{\text{as } x>0\} \\ \therefore\quad x &= \pm\frac{a}{\sqrt{2}} \\ \therefore\quad x &= \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in Q}_1\} \end{aligned}$$

[Note that A is a maximum when A^2 is a maximum.]

d Sign diagram of $\dfrac{dA^2}{dx}$ is:



$\therefore$ maximum area occurs when $x=\dfrac{a}{\sqrt{2}}$

$$\begin{aligned} \text{Max. area} &= \frac{4b}{a}\,\frac{a}{\sqrt{2}}\sqrt{a^2-\left(\frac{a}{\sqrt{2}}\right)^2} \\ &= \frac{4b}{\sqrt{2}}\times\sqrt{\frac{a^2}{2}} \\ &= \frac{4b}{\sqrt{2}}\times\frac{a}{\sqrt{2}} \\ &= 2ab \end{aligned}$$

e % occupied $=\dfrac{2ab}{\pi ab}\times 100\% = 63.7\%$

EXERCISE 22H

1 $C(x) = 38\,000 + 250x + x^2$

a

$$\begin{aligned}\text{Cost} &= C(800)\\ &= 38\,000 + 250(800) + (800)^2\\ &= \$878\,000\end{aligned}$$

$$\begin{aligned}\text{Average cost} &= \frac{\$878\,000}{800}\\ &= \$1097.50\end{aligned}$$

$$\begin{aligned}\text{Marginal cost} &= C'(x)\\ &= 250 + 2x\\ \therefore\quad C'(800) &= 250 + 2(800)\\ &= \$1850\end{aligned}$$

b $A(x)$ is minimised when $A(x) = C'(x)$

$$\begin{aligned}\text{i.e.,}\quad \frac{C(x)}{x} &= C'(x)\\ C(x) &= xC'(x)\\ 38\,000 + 250x + x^2 &= x(250 + 2x)\\ 38\,000 + 250x + x^2 &= 250x + 2x^2\\ \therefore\quad 38\,000 &= x^2\\ \therefore\quad x &= \sqrt{38\,000}\quad \{\text{as } x > 0\}\\ \therefore\quad x &\doteq 195\end{aligned}$$

i.e., A is minimised when approximately 195 items are produced.

$$\begin{aligned}\text{Average cost} &= \frac{C(x)}{x}\\ &= \frac{C(195)}{195}\\ &= \frac{\$124\,775}{195}\\ &\doteq \$639.87\end{aligned}$$

2 **a**

$$\begin{aligned}C(x) &= 295 + 24x - 0.08x^2 + 0.0008x^3\\ \text{Average cost,}\quad A(x) &= \frac{C(x)}{x} = \frac{295}{x} + 24 - 0.08x + 0.0008x^2\\ \text{Marginal cost} &= C'(x) = 24 - 0.16x + 0.0024x^2\end{aligned}$$

b Average cost is minimised when $C'(x) = A(x)$

$$\begin{aligned}\therefore\quad 24 - 0.16x + 0.0024x^2 &= \frac{295}{x} + 24 - 0.08x + 0.0008x^2\\ \therefore\quad 24x - 0.16x^2 + 0.0024x^3 &= 295 + 24x - 0.08x^2 + 0.0008x^3\\ \therefore\quad 0.0016x^3 - 0.08x^2 - 295 &= 0\end{aligned}$$

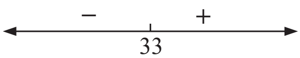
Using technology, $x \doteq 79.311$ and so, $x > 79$ items

$$\begin{aligned}\therefore\quad \text{minimum average cost} &= A(79)\\ &= \frac{295}{79} + 24 - 0.08(79) + 0.0008(79)\\ &= \$26.41\end{aligned}$$

c

$$\begin{aligned}C'(x) &= 24 - 0.16x + 0.0024x^2\\ C''(x) &= -0.16 + 0.0048x\\ \text{which is 0 when } 0.0048x &= 0.16\\ \therefore\quad x &= \frac{0.16}{0.0048} \doteq 33.33\\ \text{i.e.,}\quad x &= 33 \text{ items}\end{aligned}$$

Sign diagram for $C''(x)$ is:



$$\begin{aligned}\therefore\quad C'(x) \text{ is a minimum when } x &= 33 \text{ items}\\ \therefore\quad \text{minimum marginal cost} &= C'(33) = \$21.33\end{aligned}$$

3 Suppose x fittings are produced daily.

$$\therefore \quad C(x) = 1000 + 2x + \frac{5000}{x} = 1000 + 2x + 5000x^{-1} \text{ dollars}$$

$$\therefore \quad C'(x) = 2 - \frac{5000}{x^2}$$

$$C'(x) = 0 \quad \text{when} \quad x^2 = 2500$$

$$\text{i.e.,} \quad x = 50 \qquad \{\text{as } x > 0\}$$

$$C''(x) = 10\,000x^{-3} = \frac{10\,000}{x^3}$$

which is > 0 when $x > 0$

$\therefore$ there is minimum cost when 50 fittings are produced.

4

$$C(x) = 720 + 4x + 0.02x^2 \text{ dollars} \quad \text{and} \quad p(x) = 15 - 0.002x \text{ dollars}$$

$$\therefore \quad R(x) = xp(x) = 15x - 0.002x^2 \quad \text{and} \quad R'(x) = 15 - 0.004x$$

$$\text{If} \quad C'(x) = R'(x) \quad \text{then}$$

$$4 + 0.04x = 15 - 0.004x$$

$$\therefore \quad 0.044x = 11$$

$$\therefore \quad x = 250$$

And as $R''(x) = -0.004$ and $C''(x) = 0.04$ i.e., $R''(250) < C''(250)$

then maximum profit is made when 250 items are produced.

5

$$C(x) = \tfrac{1}{4}x^2 + 8x + 200 \quad \text{and} \quad P(x) = 23 - \tfrac{1}{2}x$$

$$\therefore \quad R(x) = xP(x) = x(23 - \tfrac{1}{2}x) = 23x - \tfrac{1}{2}x^2 \quad \therefore \quad R'(x) = 23 - x$$

$$\text{If} \quad C'(x) = R'(x) \quad \text{then}$$

$$\tfrac{1}{2}x + 8 = 23 - x$$

$$\therefore \quad \tfrac{3}{2}x = 15$$

$$\therefore \quad x = 10$$

And as $R''(x) = -1$ and $C''(x) = \frac{1}{2}$ i.e., $R''(10) < C''(10)$

then maximum profit is made when 10 blankets/day are produced.

6 Cost/hour = running costs + other costs $= \dfrac{v^2}{10} + 62.5$

$$\text{Now} \quad C(v) = \frac{\frac{v^2}{10} + 62.5}{v} \qquad \left\{\text{cost/km} = \frac{\text{cost/hour}}{\text{km/hour}}\right\}$$

$$\therefore \quad C(v) = \frac{v}{10} + \frac{62.5}{v} = 0.1v + 62.5v^{-1}$$

$$\therefore \quad C'(v) = 0.1 - 62.5v^{-2}$$

$$\text{which is } 0 \quad \text{when} \quad 0.1 = \frac{62.5}{v^2}$$

$$\therefore \quad v^2 = 625$$

$$\therefore \quad v = 25 \quad \{\text{as } v > 0\}$$

and $C''(x) = 62.5 \times 2v^{-3} = \dfrac{125}{v^3}$ which is > 0 when $v > 0$

$\therefore$ minimum cost/km occurs when $v = 25$ km/h

EXERCISE 22I

1 **a** $\dfrac{d}{dx}(2y) = 2\,\dfrac{dy}{dx}$

b $\dfrac{d}{dx}(-3y) = -3\,\dfrac{dy}{dx}$

c $\dfrac{d}{dx}(y^3) = 3y^2\,\dfrac{dy}{dx}$

d $$\begin{aligned}\frac{d}{dx}\left(\frac{1}{y}\right) &= \frac{d}{dx}(y^{-1})\\ &= -y^{-2}\,\frac{dy}{dx}\end{aligned}$$

e $\dfrac{d}{dx}(y^4) = 4y^3\,\dfrac{dy}{dx}$

f $$\begin{aligned}\frac{d}{dx}(\sqrt{y}) &= \frac{d}{dx}(y^{\frac{1}{2}})\\ &= \tfrac{1}{2}y^{-\frac{1}{2}}\,\frac{dy}{dx}\end{aligned}$$

g $$\begin{aligned}\frac{d}{dx}\left(\frac{1}{y^2}\right) &= \frac{d}{dx}(y^{-2})\\ &= -2y^{-3}\,\frac{dy}{dx}\end{aligned}$$

h $\dfrac{d}{dx}(xy) = y + x\,\dfrac{dy}{dx}$ {product rule}

i $\dfrac{d}{dx}(x^2y) = 2xy + x^2\,\dfrac{dy}{dx}$ {product rule}

j $\dfrac{d}{dx}(xy^2) = y^2 + x(2y)\,\dfrac{dy}{dx} = y^2 + 2xy\,\dfrac{dy}{dx}$ {product rule}

2 **a** $x^2 + y^2 = 25$ differentiating both sides with respect to x,

$$\begin{aligned}2x + 2y\frac{dy}{dx} &= 0\\ \therefore\quad 2y\frac{dy}{dx} &= -2x\\ \text{i.e.,}\quad \frac{dy}{dx} &= -\frac{2y}{2x} = -\frac{x}{y}\end{aligned}$$

b $x^2 + 3y^2 = 9$ differentiating both sides with respect to x,

$$\begin{aligned}2x + 6y\frac{dy}{dx} &= 0\\ \therefore\quad 6y\frac{dy}{dx} &= -2x\\ \text{i.e.,}\quad \frac{dy}{dx} &= -\frac{2x}{6y} = -\frac{x}{3y}\end{aligned}$$

c $y^2 - x^2 = 8$ differentiating both sides with respect to x,

$$\begin{aligned}2y\frac{dy}{dx} - 2x &= 0\\ \therefore\quad 2y\frac{dy}{dx} &= 2x\\ \text{i.e.,}\quad \frac{dy}{dx} &= \frac{2x}{2y} = \frac{x}{y}\end{aligned}$$

d $x^2 - y^3 = 10$ differentiating both sides with respect to x,

$$\begin{aligned}2x - 3y^2\frac{dy}{dx} &= 0\\ \therefore\quad 3y^2\frac{dy}{dx} &= 2x\\ \text{i.e.,}\quad \frac{dy}{dx} &= \frac{2x}{3y^2}\end{aligned}$$

e $x^2 + xy = 4$ differentiating both sides with respect to x,

$$2x + \left(y + x\frac{dy}{dx}\right) = 0 \quad \text{\{using the product rule\}}$$

$$x\frac{dy}{dx} = -2x - y \quad \text{and so} \quad \frac{dy}{dx} = \frac{-2x - y}{x}$$

f $x^3 - 2xy = 5$ differentiating both sides with respect to x,

$$3x^2 - \left[2y + 2x\frac{dy}{dx}\right] = 0 \quad \text{\{using the product rule\}}$$

$$3x^2 - 2y = 2x\frac{dy}{dx} \quad \text{and so,} \quad \frac{dy}{dx} = \frac{3x^2 - 2y}{2x}$$

3 **a** $x + y^3 = 4y$. Differentiating both sides of the equation with respect to x,

$$\begin{aligned}1 + 3y^2\frac{dy}{dx} &= 4\frac{dy}{dx}\\ \therefore\quad \text{at}\quad y = 1,\quad 1 + 3\frac{dy}{dx} &= 4\frac{dy}{dx}\\ \therefore\quad \frac{dy}{dx} &= 1 \qquad \text{i.e., the slope of the tangent is 1.}\end{aligned}$$

b $x+y=8xy$. Now when $x=\frac{1}{2}$, $\frac{1}{2}+y=4y$ $\therefore$ $y=\frac{1}{6}$

Thus the point of contact is $(\frac{1}{2}, \frac{1}{6})$

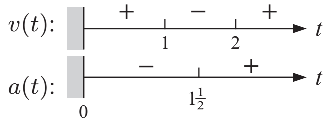
Differentiating both sides of $x+y=8xy$ with respect to x,

$$\begin{aligned} 1+\frac{dy}{dx} &= 8y+8x\frac{dy}{dx} \\ \therefore\quad 1+\frac{dy}{dx} &= 8(\tfrac{1}{6})+8(\tfrac{1}{2})\frac{dy}{dx} \quad \text{at } (\tfrac{1}{2}, \tfrac{1}{6}) \\ \therefore\quad 1+\frac{dy}{dx} &= \tfrac{4}{3}+4\frac{dy}{dx} \\ \therefore\quad -\tfrac{1}{3} &= 3\frac{dy}{dx} \\ \therefore\quad \frac{dy}{dx} &= -\tfrac{1}{9} \qquad \text{i.e., the slope of the tangent is } -\tfrac{1}{9}. \end{aligned}$$

REVIEW SET 22A

1 **a** $s(t) = 2t^3-9t^2+12t-5$ cm, $t \geqslant 0$

$$\begin{aligned} v(t) &= 6t^2-18t+12 \\ &= 6(t^2-3t+2) \\ &= 6(t-2)(t-1) \text{ cm s}^{-1} \end{aligned}$$



and $a(t) = 12t-18$
$= 6(2t-3)$ cm s^{-2}

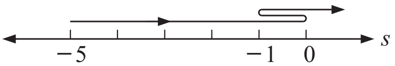
b When $t=0$, $s(0) = -5$ cm
$v(0) = 12$ cm s^{-1}
$a(0) = -18$ cm s^{-2}

c When $t=2$, $s(2) = -1$ cm
$v(2) = 0$ cm s^{-1}
$a(2) = 6$ cm s^{-2}

i.e., when $t=2$, the particle is 1 cm to the left of O, instantaneously at rest and increasing in speed towards O.

d Particle changes direction when $t=1$ and $t=2$ $s(1) = 0$ cm $s(2) = -1$ cm

e



f Speed is increasing when $1 \leqslant t \leqslant \frac{3}{2}$ and $t \geqslant 2$ $\{v(t)$ and $a(t)$ have the same sign$\}$

2 $C(v) = 10v+\dfrac{90}{v}$ dollars/hour

a **i** $t=2$ hours at $v=15$ kmph
$\therefore$ cost $= \$\left(10\times 15+\frac{90}{15}\right)\times 2$
$= \$312.00$

ii $t=5$ hours at $v=24$ kmph
$\therefore$ cost $= \$\left(10\times 24+\frac{90}{24}\right)\times 5$
$= \$1218.75$

b $C'(v) = 10-90v^{-2} = 10-\dfrac{90}{v^2}$

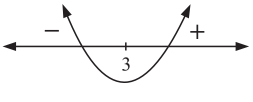
i If $v=10$ kmph
$\therefore$ $C'(10) = 10-\dfrac{90}{10^2}$
$= 10-0.9$
$= \$9.10$ per kmph

ii If $v=6$ kmph
$\therefore$ $C'(6) = 10-\dfrac{90}{6^2}$
$= \$7.50$ per kmph

c Now $C'(v) = 10-\dfrac{90}{v^2} = \dfrac{10v^2-90}{v^2}$

$\therefore$ $C'(v) = 0$ when $v^2 = 9$
i.e., $v = 3$ $\{$as $v>0\}$

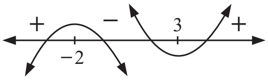
$\therefore$ minimum cost occurs when $v=3$ kmph



3 **a**

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 36x + 7 \\ \therefore\quad f'(x) &= 6x^2 - 6x - 36 \\ &= 6(x^2 - x - 6) \\ &= 6(x-3)(x+2) \end{aligned}$$

and $f'(x)$ has sign diagram:

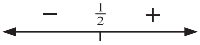


and $f(-2) = 51, \quad f(3) = -74$

So there is a local maximum at $(-2, 51)$, and a local minimum at $(3, -74)$

$$\begin{aligned} f''(x) &= 12x - 6 \\ &= 6(2x-1) \end{aligned}$$

and $f''(x)$ has sign diagram:

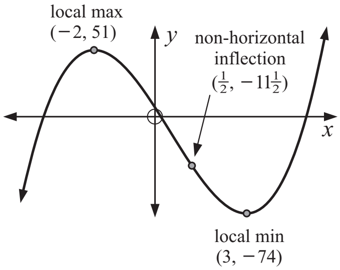


and $f(\frac{1}{2}) = -\frac{23}{2}$ so there is a non-horizontal inflection at $(\frac{1}{2}, -\frac{23}{2})$

b $f(x)$ is increasing when $x \leqslant -2$ or $x \geqslant 3$, and decreasing when $-2 \leqslant x \leqslant 3$.

c $f(x)$ is concave up when $x \geqslant \frac{1}{2}$, and concave down when $x \leqslant \frac{1}{2}$.

d



4 **a** Now if OD $= x$, the coordinates of C are $(x, k - x^2)$

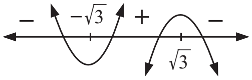
$$\begin{aligned} \therefore\quad \text{area of ABCD} &= 2x \times (k - x^2) \\ \text{i.e.,}\quad A &= 2kx - 2x^3 \end{aligned}$$

b $\dfrac{dA}{dx} = 2k - 6x^2$ and $\dfrac{dA}{dx} = 0$ when AD $= 2\sqrt{3}$ i.e., $x = \sqrt{3}$

$$\begin{aligned} \therefore\quad 2k - 6(\sqrt{3})^2 &= 0 \\ \therefore\quad 2k - 18 &= 0 \\ \therefore\quad 2k &= 18 \\ \therefore\quad k &= 9 \end{aligned}$$

Checking
$$\begin{aligned} \frac{dA}{dx} &= 18 - 6x^2 \\ &= 6(3 - x^2) \\ &= 6(\sqrt{3} + x)(\sqrt{3} - x) \end{aligned}$$

which has sign diagram:



i.e., maximum occurs at $x = \sqrt{3}$ i.e., when AD $= 2\sqrt{3}$

5 **a**

$$\begin{aligned} \text{Volume} &= lbd \\ \therefore\quad x^2 y &= 1 \\ \therefore\quad y &= \frac{1}{x^2}, \quad x > 0 \end{aligned}$$

b

$$\begin{aligned} \text{area} &= x^2 + 4xy \\ \therefore\quad \text{cost} &= (x^2 + 4xy) \times 2 \\ &= 2x^2 + 8xy \end{aligned}$$

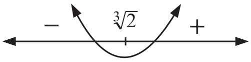
But $y = \dfrac{1}{x^2}$

$\therefore\quad C = 2x^2 + \dfrac{8}{x}$ dollars

c Thus
$$\begin{aligned} \frac{dC}{dx} &= 4x - 8x^{-2} \\ &= 4x - \frac{8}{x^2} \\ &= \frac{4(x^3 - 2)}{x^2} \end{aligned}$$

which is 0 when $x = \sqrt[3]{2}$ m

and $\dfrac{dC}{dx}$ has sign diagram:



$\therefore$ minimum cost when $x = \sqrt[3]{2}$ m i.e., $x = 1.26$

$\therefore\quad y = \dfrac{1}{x^2} \doteqdot 0.630$

and the box is 1.26 m by 1.26 m by 0.630 m

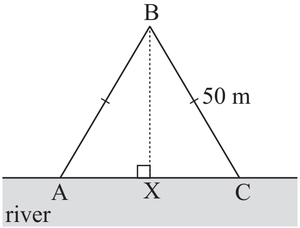
6 **a** $x^2y + 2xy^3 = -18$ Differentiating with respect to x,

$$
\begin{aligned}
\frac{d}{dx}(x^2y) + \frac{d}{dx}(2xy^3) &= \frac{d}{dx}(-18)\\
\therefore\quad \frac{d}{dx}(x^2)y + x^2\frac{dy}{dx} + \frac{d}{dx}(2x)y^3 + 2x\frac{d}{dx}(y^3) &= 0 \quad \{\text{product rule}\}\\
\therefore\quad 2xy + x^2\frac{dy}{dx} + 2y^3 + 6xy^2\frac{dy}{dx} &= 0\\
\therefore\quad \frac{dy}{dx}(x^2 + 6xy^2) &= -2xy - 2y^3\\
\therefore\quad \frac{dy}{dx} &= \frac{-2y(x+y^2)}{x(x+6y^2)}
\end{aligned}
$$

b At (1, −2) $\dfrac{dy}{dx} = \dfrac{-2(-2)(1+(-2))^2}{1(1+6(-2)^2)} = \dfrac{4\times 5}{25} = \frac{4}{5}$

$\therefore$ the tangent has equation $\dfrac{y-(-2)}{x-1} = \frac{4}{5}$ i.e., $y + 2 = \frac{4}{5}x - \frac{4}{5}$ i.e., $y = \frac{4}{5}x - \frac{14}{5}$

REVIEW SET 22B



1 **a** AC $= 2x$ m

Now, ABC is an isosceles triangle.

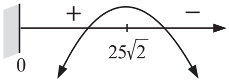
$$
\begin{aligned}
\therefore\quad \text{XC} &= x\\
\text{But,}\quad \text{BC}^2 &= \text{BX}^2 + \text{XC}^2 \quad \{\text{Pythagoras}\}\\
\therefore\quad 2500 &= \text{BX}^2 + x^2\\
\therefore\quad \text{BX} &= \sqrt{2500 - x^2}\\
\therefore\quad A(x) &= \tfrac{1}{2}(2x)\sqrt{2500-x^2} = x\sqrt{2500-x^2}
\end{aligned}
$$

b Now $[A(x)]^2 = x^2(2500 - x^2)$

i.e., $A^2 = 2500x^2 - x^4$

$$
\begin{aligned}
\therefore\quad \frac{dA^2}{dx} &= 5000x - 4x^3\\
&= 4x(1250 - x^2)\\
&= 2x(\sqrt{1250} + x)(\sqrt{1250} - x)
\end{aligned}
$$

Sign diagram for $\dfrac{dA^2}{dx}$ is:



$\therefore$ maximum area occurs when $x = 25\sqrt{2}$ m $\doteqdot$ 35.4 m

The corresponding maximum area $\doteqdot$ 1250 m^2.

2 $s(t) = 15t - \dfrac{60}{(t-1)^2}$ cm, $t \geqslant 0$

a

$$
\begin{aligned}
\therefore\quad s(t) &= 15t - 60(t-1)^{-2} \text{ cm}\\
\therefore\quad v(t) &= 15 + 120(t-1)^{-3} \text{ cm s}^{-1}\\
\therefore\quad a(t) &= -360(t-1)^{-4} \text{ cm s}^{-2}
\end{aligned}
$$

b When $t = 3$,

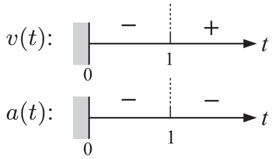
$$
\begin{aligned}
s(t) &= 30 \text{ cm}\\
v(t) &= 30 \text{ cm s}^{-1}\\
a(t) &= -22.5 \text{ cm s}^{-2}
\end{aligned}
$$

The particle is 30 cm right of O, travelling right at 30 cm s^{-1} and is slowing down at 22.5 cm s^{-2}

c $v(t) = 15 + \dfrac{120}{(t-1)^3}$ cm s^{-1}

$$
\begin{aligned}
\therefore\quad v(t) &= 0\\
\text{when}\quad 15 + \frac{120}{(t-1)^3} &= 0\\
\therefore\quad 15(t-1)^3 + 120 &= 0\\
\therefore\quad (t-1)^3 &= -8\\
\therefore\quad t &= -1
\end{aligned}
$$

$a(t) = -360(t-1)^{-4} = \dfrac{-360}{(t-1)^4}$ cm s^{-2} Sign diagrams:

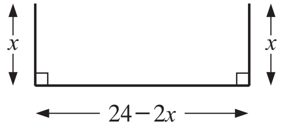


where $(t-1)^4$ is always positive. $\therefore$ $a(t) < 0$ for all $t > 0$

So, speed increases for $0 \leqslant t < 1$ as

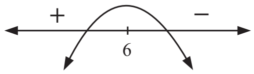
$v(t)$, $a(t)$ have the same sign.

3 Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of cross-section.



$$\begin{aligned} A &= x(24-2x) \\ \therefore\ A &= 24x - 2x^2 \\ \text{and}\quad \frac{dA}{dx} &= 24 - 4x \end{aligned}$$

which is 0 when $x = 6$ Sign diagram of $\dfrac{dA}{dx}$ is:



$\therefore$ maximum water when $x = 6$ cm

i.e., bends must be made 6 cm from each end.

4 **a** (2, −1) lies on the curve, so

$$\begin{aligned} k &= 2^2 - 2(2)(-1)^2 + (-1)^3 \\ &= 4 - 4 - 1 \\ &= -1 \end{aligned}$$

b $x^2 - 2xy^2 + y^3 = -1$

Differentiating with respect to x,

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(2xy^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(-1)$$

$$\therefore\ 2x - \frac{d}{dx}(2x)y^2 - 2x\frac{d}{dx}(y^2) + 3y^2\frac{dy}{dx} = 0$$

$$\therefore\ 2x - 2y^2 - 4xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx}\left(3y^2 - 4xy\right) = 2y^2 - 2x$$

$$\therefore\ \frac{dy}{dx} = \frac{2(y^2 - x)}{y(3y - 4x)}$$

c At (2, −1), $\dfrac{dy}{dx} = \dfrac{2(1-2)}{(-1)(-3-4\times 2)}$

$$= -\tfrac{2}{11}$$

$\therefore$ the slope of the normal is $\frac{11}{2}$

and its equation is $\dfrac{y-(-1)}{x-2} = \frac{11}{2}$

$$\begin{aligned} \therefore\ y + 1 &= \tfrac{11}{2}x - 11 \\ \therefore\ y &= \tfrac{11}{2}x - 12 \\ \text{or}\quad 11x - 2y &= 24 \end{aligned}$$

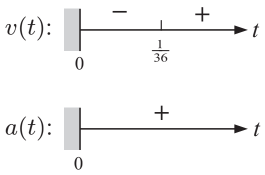
5 $x(t) = 3t - \sqrt{t}$ cm, $t \geqslant 0$

a $x(t) = 3t - t^{\frac{1}{2}}$

$$\therefore\ v(t) = 3 - \tfrac{1}{2}t^{-\frac{1}{2}} = 3 - \frac{1}{2\sqrt{t}} = \frac{6\sqrt{t}-1}{2\sqrt{t}}$$

which is 0 when $\sqrt{t} = \frac{1}{6}$, i.e., $t = \frac{1}{36}$

and $a(t) = \frac{1}{4}t^{-\frac{3}{2}} = \dfrac{1}{4t\sqrt{t}}$ which is always positive



b When $t = 0$, $x(0) = 0$ cm

$v(0)$ is infinitely large and negative, so is not defined

$a(0)$ is infinitely large and positive, so is not defined

i.e., the particle is at O, moving left and slowing down.

c When $t = 9$, $x(9) = 24$ cm $v(9) = \frac{17}{6}$ cm s^{-1} $a(9) = \frac{1}{108}$ cm s^{-2}
the particle is 24 cm right of O, moving right at $\frac{17}{6}$ cm s^{-1} and increasing its speed.

d The particle reverses direction when $t = \frac{1}{36}$ seconds.
$x\left(\frac{1}{36}\right) = \frac{3}{36} - \frac{1}{6} = -\frac{3}{36} = -\frac{1}{12}$, i.e., it is $\frac{1}{12} \doteqdot 0.083$ cm to the left of O.

e The particle's speed decreases when $v(t)$ and $a(t)$ have different signs, i.e., for $\leqslant t \leqslant \frac{1}{36}$.

6 $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

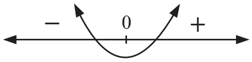
a Cuts x-axis when $y = 0$
i.e., $x^2 - 1 = 0$, $\therefore$ $x = \pm 1$
i.e., at (1, 0) and (−1, 0)

Cuts y-axis when $x = 0$, i.e., (0, −1)

b As $x^2 \geqslant 0$, $x^2 + 1$ can never be 0.
$\therefore$ there are no vertical asymptotes.

c $$f'(x) = \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

Sign diagram of $f'(x)$ is:

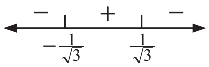


$\therefore$ there is a local minimum at (0, −1)

d $$f''(x) = \frac{4(x^2+1)^2 - 4x \times 2(x^2+1)(2x)}{(x^2+1)^4} = \frac{4(x^2+1) - 16x^2}{(x^2+1)^3} = \frac{4 - 12x^2}{(x^2+1)^3} = \frac{4(1+\sqrt{3}x)(1-\sqrt{3}x)}{(x^2+1)^3}$$

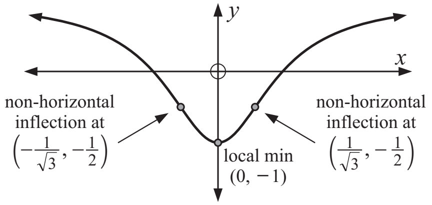
$\therefore$ $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{3}}$

and sign diagram is:



$\therefore$ $f(x)$ has nonstationary inflections at $x = \pm\sqrt{\frac{1}{3}}$.

e



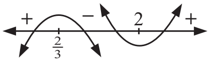
REVIEW SET 22C

1 $$f(x) = x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$$

a Cuts the y-axis when $x = 0$ i.e., at (0, 0)
Cuts the x-axis when $y = 0$ i.e., $x(x-2)^2 = 0$,
i.e., when $x = 0$ or 2
i.e., at (0, 0) and (2, 0)

b $f'(x) = 3x^2 - 8x + 4$
$= (3x - 2)(x - 2)$
which is 0 when $x = \frac{2}{3}$ or 2

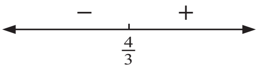
Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$,
and a local minimum at (2, 0)

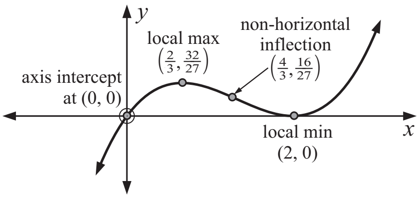
$f''(x) = 6x - 8 = 2(3x - 4)$

Sign diagram of $f''(x)$ is:



$\therefore$ there is a non-horizontal inflection at $(\frac{4}{3}, \frac{16}{27})$

c



2 **a**

$$\begin{aligned} P &= 200 \text{ m} \\ \therefore \quad P &= 2x + 2y + \pi x \\ \therefore \quad 200 &= 2x + 2y + \pi x \\ \therefore \quad 2y &= 200 - 2x - \pi x \\ \therefore \quad y &= 100 - x - \tfrac{\pi}{2}x \end{aligned}$$

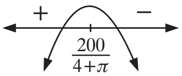
b

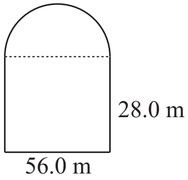
$$\begin{aligned} \text{Area of lawn} &= 2x \times y + \tfrac{1}{2}\pi x^2 \\ &= 2x\left[100 - x - \tfrac{\pi}{2}x\right] + \tfrac{1}{2}\pi x^2 \\ &= 200x - 2x^2 - \pi x^2 + \tfrac{1}{2}\pi x^2 \\ &= 200x - 2x^2 - \tfrac{\pi}{2}x^2 \\ \therefore \quad A &= 200x - \left(2 + \tfrac{\pi}{2}\right)x^2 \text{ m}^2 \end{aligned}$$

c

$$\begin{aligned} \frac{dA}{dx} &= 200 - 2\left(2 + \tfrac{\pi}{2}\right)x \\ &= 200 - (4+\pi)x \quad \text{which is 0 when} \quad (4+\pi)x = 200 \quad \text{i.e.,} \quad x = \frac{200}{4+\pi} \end{aligned}$$

and the sign diagram for $\dfrac{dA}{dx}$ is:





$\therefore$ maximum area occurs when $x = \dfrac{200}{4+\pi} \doteqdot 28.00$ m

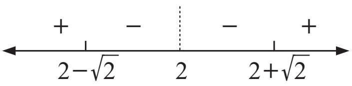
3 **a** $f(x) = \dfrac{x^2+2x}{x-2} = \dfrac{(x-2)(x+4)+8}{x-2} = x + 4 + \dfrac{8}{x-2}$

$\therefore$ there is a vertical asymptote of $x = 2$ $\{$as $x \to 2,\ |f(x)| \to \infty\}$
and an oblique asymptote of $y = x + 4$ $\{$as $|x| \to \infty,\ f(x) \to x + 4\}$

b

$$\begin{aligned} f'(x) &= 1 - \frac{8}{(x-2)^2} \\ &= \frac{(x-2)^2 - 8}{(x-2)^2} \\ &= \frac{x^2 - 4x - 4}{(x-2)^2} \\ \therefore \quad f'(x) &= 0 \quad \text{if} \quad x^2 - 4x - 4 = 0 \\ \therefore \quad x &= \frac{4 \pm \sqrt{(-4)^2 - 4(-4)}}{2} \\ &= 2 \pm 2\sqrt{2} \end{aligned}$$

$\therefore$ $f'(x)$ has sign diagram



$$\begin{aligned} f(2+2\sqrt{2}) &= 2 + 2\sqrt{2} + 4 + \tfrac{8}{2\sqrt{2}} \\ &= 6 + 2\sqrt{2} + 2\sqrt{2} \\ &= 6 + 4\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{and} \quad f(2-2\sqrt{2}) &= 2 - 2\sqrt{2} + 4 + \tfrac{8}{-2\sqrt{2}} \\ &= 6 - 2\sqrt{2} - 2\sqrt{2} \\ &= 6 - 4\sqrt{2} \end{aligned}$$

$\therefore$ $f(x)$ has a local maximum at $(2 - 2\sqrt{2},\ 6 - 4\sqrt{2})$ and a local minimum at $(2 + 2\sqrt{2},\ 6 + 4\sqrt{2})$.

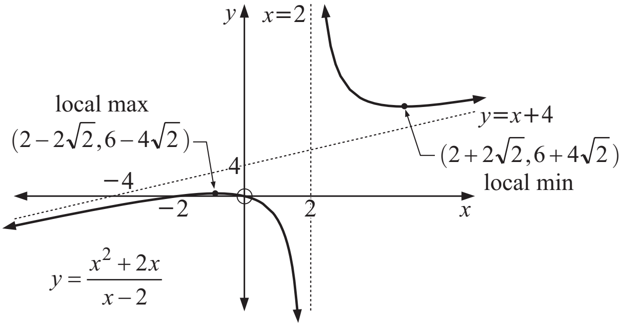
c When $f(x) = 0$,

$$\begin{aligned} x^2 + 2x &= 0 \\ \therefore \quad x(x+2) &= 0 \\ \therefore \quad x &= -2 \text{ or } 0 \end{aligned}$$

$\therefore$ the x-intercepts are -2 and 0

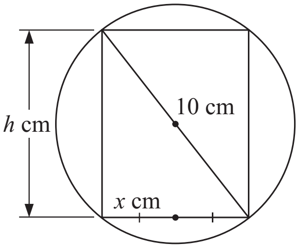
Also, $f(0) = 0$, so the y-intercept is 0

d



e $f(x) = p$ has two real distinct roots if $p < 6 - 4\sqrt{2}$ or $p > 6 + 4\sqrt{2}$

4 **a**



Let the height of the cylinder be h cm.

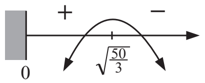
$$\begin{aligned}\therefore \quad (2x)^2 + h^2 &= 10^2 \quad \{\text{Pythagoras}\}\\ \therefore \quad h &= \sqrt{100-4x^2}\\ \therefore \quad V(x) &= \text{area of base} \times \text{height}\\ &= \pi x^2 \times \sqrt{100-4x^2}\\ \text{So,} \quad V(x) &= \pi x^2\sqrt{100-4x^2}\ \text{cm}^2\end{aligned}$$

b Now

$$\begin{aligned}V^2 &= \pi^2x^4(100-4x^2)\\ &= \pi^2(100x^4-4x^6)\\ \therefore \quad \frac{dV^2}{dx} &= \pi^2(400x^3-24x^5)\\ &= 8\pi^2x^3(50-3x^2)\\ &= 8\pi^2x^3(\sqrt{50}+\sqrt{3}x)(\sqrt{50}-\sqrt{3}x)\end{aligned}$$

which is 0 when $x = \sqrt{\frac{50}{3}}$ $\{$as $x > 0\}$

and $\dfrac{dV^2}{dx}$ has sign diagram:



$\therefore$ maximum V occurs when $x = \sqrt{\frac{50}{3}} \doteqdot 4.08$

$\therefore$ radius $= 4.08$ cm, height $= \sqrt{100-4\left(\frac{50}{3}\right)} \doteqdot 5.77$ cm

5 **a** Δs LQX and XPM are similar.

$$\begin{aligned}\therefore \quad \frac{\text{LQ}}{\text{XP}} &= \frac{\text{LX}}{\text{XM}} = \frac{\text{QX}}{\text{PM}}\\ \therefore \quad \frac{\text{LQ}}{1} &= \frac{8}{\text{PM}}\\ \therefore \quad \text{LQ} &= \frac{8}{\text{PM}}\\ \therefore \quad \text{LQ} &= \frac{8}{x}\ \text{km}\end{aligned}$$

b

$$\begin{aligned}L(x) &= \text{LX} + \text{XM}\\ &= \sqrt{\left(\frac{8}{x}\right)^2 + 8^2} + \sqrt{1^2+x^2}\\ &= 8\sqrt{\frac{1}{x^2}+1} + \sqrt{x^2+1}\\ &= \frac{8}{x}\sqrt{1+x^2} + 1\sqrt{1+x^2}\\ &= \sqrt{x^2+1}\left(\frac{8}{x}+1\right)\\ \therefore \quad L(x) &= \sqrt{x^2+1}\left(1+\frac{8}{x}\right)\\ \text{and} \quad L^2 &= \left(x^2+1\right)\left(1+\frac{8}{x}\right)^2\end{aligned}$$

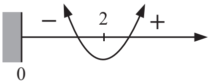
c

$$\begin{aligned}\frac{d[L(x)]^2}{dx} &= 2x\left(1+\frac{8}{x}\right)^2 + (x^2+1)2\left(1+\frac{8}{x}\right)\left(-\frac{8}{x^2}\right) \quad \{\text{product rule}\}\\ &= 2\left(1+\frac{8}{x}\right)\left[x\left(1+\frac{8}{x}\right) - (x^2+1)\left(\frac{8}{x^2}\right)\right]\\ &= 2\left(1+\frac{8}{x}\right)\left[x+8-8-\frac{8}{x^2}\right]\\ &= 2\left(\frac{x+8}{x}\right)\left(\frac{x^3-8}{x^2}\right)\end{aligned}$$

$$\frac{d[L(x)]^2}{dx} = 0 \quad \text{when} \quad x = -8 \quad \text{or} \quad x^3 = 8$$

but $x > 0$, $\therefore$ when $x = 2$

Sign diagram for $\dfrac{d[L(x)]^2}{dx}$ is:



$\therefore$ minimum $L(x)$ occurs when $x = 2$ and the shortest length is $\sqrt{2^2+1}\left(1+\frac{8}{2}\right)$

$$= 5\sqrt{5}$$

$$\doteq 11.2 \text{ km}$$

6 When the box is manufactured its base is $(k-2x)$ by $(k-2x)$ and height is x cm.

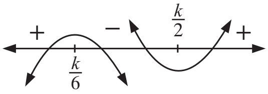
$$\therefore \quad V = x(k-2x)(k-2x)$$

$$\text{i.e.,} \quad V = x(k-2x)^2$$

$$\begin{aligned}\therefore \quad \frac{dV}{dx} &= 1(k-2x)^2 + x\,2(k-2x)^1(-2)\\ &= (k-2x)^2 - 4x(k-2x)\\ &= (k-2x)[k-2x-4x]\\ &= (k-2x)(k-6x)\end{aligned}$$

which is 0 when $x = \frac{k}{6}$ or $\frac{k}{2}$, but $k - 2x > 0$ and so $x < \frac{k}{2}$.

and the sign diagram of $\dfrac{dV}{dx}$ is:



$\therefore$ maximum capacity occurs when $x = \dfrac{k}{6}$.

Chapter 23

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXERCISE 23A

1 **a**
$$\begin{aligned} f(x) &= e^{4x} \\ \therefore\ f'(x) &= 4e^{4x} \end{aligned}$$

b
$$\begin{aligned} f(x) &= e^{x}+3 \\ \therefore\ f'(x) &= e^{x}+0 \\ &= e^{x} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \exp(-2x) \\ \therefore\ f(x) &= e^{-2x} \\ \therefore\ f'(x) &= -2e^{-2x} \end{aligned}$$

d
$$\begin{aligned} f(x) &= e^{\frac{x}{2}} \\ \therefore\ f'(x) &= \tfrac{1}{2}e^{\frac{x}{2}} \end{aligned}$$

e
$$\begin{aligned} f(x) &= 2e^{-\frac{x}{2}} \\ \therefore\ f'(x) &= 2e^{-\frac{x}{2}}\left(-\tfrac{1}{2}\right) \\ &= -e^{-\frac{x}{2}} \end{aligned}$$

f
$$\begin{aligned} f(x) &= 1-2e^{-x} \\ \therefore\ f'(x) &= 0-2e^{-x}(-1) \\ &= 2e^{-x} \end{aligned}$$

g
$$\begin{aligned} f(x) &= 4e^{\frac{x}{2}}-3e^{-x} \\ \therefore\ f'(x) &= 4e^{\frac{x}{2}}\left(\tfrac{1}{2}\right) \\ &\qquad -3e^{-x}(-1) \\ &= 2e^{\frac{x}{2}}+3e^{-x} \end{aligned}$$

h
$$\begin{aligned} f(x) &= \frac{e^{x}+e^{-x}}{2} \\ \therefore\ f(x) &= \tfrac{1}{2}(e^{x}+e^{-x}) \\ \therefore\ f'(x) &= \tfrac{1}{2}(e^{x}+e^{-x}(-1)) \\ &= \tfrac{1}{2}(e^{x}-e^{-x}) \end{aligned}$$

i
$$\begin{aligned} f(x) &= e^{-x^2} \\ \therefore\ f'(x) &= e^{-x^2}(-2x) \\ &= -2xe^{-x^2} \end{aligned}$$

j
$$\begin{aligned} f(x) &= e^{\frac{1}{x}} \\ \therefore\ f'(x) &= e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) \\ &= -\frac{e^{\frac{1}{x}}}{x^2} \end{aligned}$$

k
$$\begin{aligned} f(x) &= 10\left(1+e^{2x}\right) \\ &= 10+10e^{2x} \\ \therefore\ f'(x) &= 0+10e^{2x}(2) \\ &= 20e^{2x} \end{aligned}$$

l
$$\begin{aligned} f(x) &= 20\left(1-e^{-2x}\right) \\ &= 20-20e^{-2x} \\ \therefore\ f'(x) &= 0-20e^{-2x}(-2) \\ &= 40e^{-2x} \end{aligned}$$

m
$$\begin{aligned} f(x) &= e^{2x+1} \\ \therefore\ f'(x) &= e^{2x+1}\times 2 \\ &= 2e^{2x+1} \end{aligned}$$

n
$$\begin{aligned} y &= e^{\frac{x}{4}} \\ \therefore\ \frac{dy}{dx} &= e^{\frac{x}{4}}\times\tfrac{1}{4} \\ &= \tfrac{1}{4}e^{\frac{x}{4}} \end{aligned}$$

o
$$\begin{aligned} y &= e^{1-2x^2} \\ \therefore\ \frac{dy}{dx} &= e^{1-2x^2}(-4x) \\ &= -4xe^{1-2x^2} \end{aligned}$$

p $y = e^{-0.02x}$ $\quad\therefore\ \dfrac{dy}{dx} = e^{-0.02x}\times(-0.02) = -0.02e^{-0.02x}$

2 **a**
$$f(x) = xe^{x}$$
$u=x,\ v=e^{x},\quad u'=1,\ v'=e^{x}$
$$\begin{aligned} \therefore\ f'(x) &= 1e^{x}+e^{x}x \\ &= e^{x}+xe^{x} \end{aligned}$$

b
$$f(x) = x^3e^{-x}$$
$u=x^3,\ v=e^{-x},\quad u'=3x^2,\ v'=-e^{-x}$
$$\begin{aligned} \therefore\ f'(x) &= 3x^2e^{-x}+x^3(-e^{-x}) \\ &= 3x^2e^{-x}-x^3e^{-x} \end{aligned}$$

c
$$f(x) = \frac{e^{x}}{x}$$
$u=e^{x},\ v=x,\quad u'=e^{x},\ v'=1$
$$\begin{aligned} \therefore\ f'(x) &= \frac{e^{x}x-e^{x}(1)}{x^2} \\ &= \frac{xe^{x}-e^{x}}{x^2} \end{aligned}$$

d
$$f(x) = \frac{x}{e^{x}}$$
$u=x,\ v=e^{x},\quad u'=1,\ v'=e^{x}$
$$\begin{aligned} \therefore\ f'(x) &= \frac{1e^{x}-xe^{x}}{(e^{x})^2} \\ &= \frac{e^{x}(1-x)}{(e^{x})^2} \\ &= \frac{1-x}{e^{x}} \end{aligned}$$

e
$$\begin{aligned} f(x) &= x^2e^{3x} \\ &u = x^2, \quad v = e^{3x}, \\ &u' = 2x, \quad v' = 3e^{3x} \\ \therefore \quad f'(x) &= 2xe^{3x} + 3x^2e^{3x} \end{aligned}$$

f
$$f(x) = \frac{e^x}{\sqrt{x}}$$
$$u = e^x, \ v = x^{\frac{1}{2}}, \quad u' = e^x, \ v' = \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$\begin{aligned} \therefore \quad f'(x) &= \frac{e^x\sqrt{x} - \dfrac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \\ &= \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}} \end{aligned}$$

g
$$\begin{aligned} f(x) &= \sqrt{x}e^{-x} \\ &u = x^{\frac{1}{2}}, \ v = e^{-x}, \\ &u' = \tfrac{1}{2}x^{-\frac{1}{2}}, \ v' = -e^{-x} \\ \therefore \quad f'(x) &= \frac{1}{2\sqrt{x}}e^{-x} - \sqrt{x}e^{-x} \end{aligned}$$

h
$$f(x) = \frac{e^x + 2}{e^{-x} + 1}$$
$$u = e^x + 2, \ v' = e^{-x} + 1, \quad u' = e^x, \ v' = -e^{-x}$$
$$\begin{aligned} \therefore \quad f'(x) &= \frac{e^x(e^{-x} + 1) - (e^x + 2)\left(-e^{-x}\right)}{(e^{-x} + 1)^2} \\ &= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2} \\ &= \frac{2 + e^x + 2e^{-x}}{(e^{-x} + 1)^2} \end{aligned}$$

3 **a**
$$\begin{aligned} f(x) &= (e^x + 2)^4 \\ &= u^4 \quad \text{where } u = e^x + 2 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 4u^3(e^x) \\ \therefore \quad f'(x) &= 4(e^x + 2)^3(e^x) \\ &= 4e^x(e^x + 2)^3 \end{aligned}$$

b
$$\begin{aligned} f(x) &= \frac{1}{1 - e^{-x}} \\ &= u^{-1} \quad \text{where } u = 1 - e^{-x} \\ \therefore \quad f'(x) &= \frac{dy}{du}\frac{du}{dx} \\ &= -u^{-2}(e^{-x}) \\ \therefore \quad f'(x) &= -\frac{e^{-x}}{(1 - e^{-x})^2} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \sqrt{e^{2x} + 10} \\ &= u^{\frac{1}{2}} \quad \text{where } u = e^{2x} + 10 \\ \therefore \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \tfrac{1}{2}u^{-\frac{1}{2}}(2e^{2x}) \\ \therefore \quad f'(x) &= \frac{e^{2x}}{\sqrt{e^{2x} + 10}} \end{aligned}$$

d
$$\begin{aligned} f(x) &= \frac{1}{(1 - e^{3x})^2} \\ &= u^{-2} \quad \text{where } u = 1 - e^{3x} \\ \therefore \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= -2u^{-3}(-3e^{3x}) = \frac{6e^{3x}}{u^3} \\ \therefore \quad f'(x) &= \frac{6e^{3x}}{(1 - e^{3x})^3} \end{aligned}$$

e
$$\begin{aligned} f(x) &= \frac{1}{\sqrt{1 - e^{-x}}} \\ &= = u^{-\frac{1}{2}} \quad \text{where } u = 1 - e^{-x} \\ \therefore \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= -\tfrac{1}{2}u^{-\frac{3}{2}}(e^{-x}) = \frac{-e^{-x}}{2u^{\frac{3}{2}}} \\ \therefore \quad f'(x) &= \frac{-e^{-x}}{2(1 - e^{-x})^{\frac{3}{2}}} \end{aligned}$$

f
$$\begin{aligned} f(x) &= x\sqrt{1 - 2e^{-x}} \\ &= x\,u^{\frac{1}{2}} \quad \text{where} \quad u = 1 - 2e^{-x} \\ \therefore \quad f'(x) &= 1\,u^{\frac{1}{2}} + x\,\tfrac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} \\ &= 1\,\sqrt{u} + x\,\tfrac{1}{2}u^{-\frac{1}{2}}\,2e^{-x} \\ &= \frac{\sqrt{1 - 2e^{-x}}}{1} + \frac{xe^{-x}}{\sqrt{1 - 2e^{-x}}} \\ \therefore \quad f'(x) &= \frac{1 - 2e^{-x} + xe^{-x}}{\sqrt{1 - 2e^{-x}}} \end{aligned}$$

4 **a** $y = Ae^{kx}$

i $\frac{dy}{dx} = Ae^{kx}(k) = k(Ae^{kx}) = ky$

ii $\frac{d^2y}{dx^2} = k\frac{dy}{dx} = k(ky) = k^2y$

b Prediction: $\frac{d^ny}{dx^n} = k^ny$

5 $y = 2e^{3x} + 5e^{4x}$ $\therefore$ $\frac{dy}{dx} = 6e^{3x} + 20e^{4x}$ and $\frac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$

Now
$$\begin{aligned}\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y &= \left(18e^{3x} + 80e^{4x}\right) - 7\left(6e^{3x} + 20e^{4x}\right) + 12\left(2e^{3x} + 5e^{4x}\right)\\ &= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x}\\ &= e^{3x}\,[18 - 42 + 24] + e^{4x}\,[80 - 140 + 60]\\ &= e^{3x}(0) + e^{4x}(0)\\ &= 0\end{aligned}$$

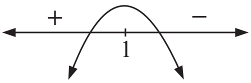
i.e., $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

6 **a** $y = xe^{-x}$ $u = x,\ v = e^{-x},\ u' = 1,\ v' = -e^{-x}$

$\therefore$ $\frac{dy}{dx} = 1e^{-x} - xe^{-x}$ {product rule}

$= e^{-x}(1 - x)$

$= \frac{1-x}{e^x}$ which has sign diagram:



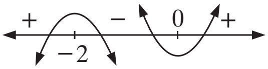
$\therefore$ at $x = 1,\ y = 1e^{-1} = \frac{1}{e}$ we have a maximum turning point.

$\therefore$ has a local maximum at $(1, \frac{1}{e})$

b $y = x^2e^x$ $u = x^2,\ v = e^x,\ u' = 2x,\ v' = e^x$

$\therefore$ $\frac{dy}{dx} = 2xe^x + x^2e^x$ {product rule}

$= xe^x(2 + x)$ which has sign diagram:



$\therefore$ at $x = -2,\ y = 4e^{-2}$, we have a maximum turning point

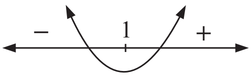
and at $x = 0,\ y = 0$, we have a minimum turning point.

$\therefore$ we have a local maximum at $(-2, \frac{4}{e^2})$, and a local minimum at (0, 0).

c $y = \frac{e^x}{x}$ $u = e^x,\ v = x,\ u' = e^x,\ v' = 1$

$\therefore$ $\frac{dy}{dx} = \frac{e^xx - e^x(1)}{x^2}$ {quotient rule}

$= \frac{e^x(x-1)}{x^2}$ which has sign diagram:



$\therefore$ at $x = 1,\ y = \frac{e}{1} = e$ we have a minimum turning point.

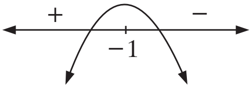
$\therefore$ we have a local minimum at $(1, e)$

d $y = e^{-x}(x + 2)$ $u = e^{-x},\ v = x + 2,\ u' = -e^{-x},\ v' = 1$

$\therefore$ $\frac{dy}{dx} = -e^{-x}(x + 2) + e^{-x}$ {product rule}

$= e^{-x}(-x - 2 + 1)$

$= e^{-x}(-x - 1)$ which has sign diagram:



$\therefore$ at $x = -1,\ y = e(-1 + 2) = e$ we have a maximum turning point.

$\therefore$ we have a local maximum at $(-1, e)$

EXERCISE 23B

1 **a**
$$\begin{aligned} N &= 50e^{2t} \\ \therefore\ \ln N &= \ln(50e^{2t}) && \{\ln ab = \ln a + \ln b\} \\ &= \ln 50 + \ln e^{2t} && \{\text{as } \ln e^n = n\} \\ \therefore\ \ln N &= \ln 50 + 2t \end{aligned}$$

b
$$\begin{aligned} P &= 8.69e^{-0.0541t} \\ \therefore\ \ln P &= \ln(8.69e^{-0.0541t}) \\ &= \ln 8.69 + \ln e^{-0.0541t} \\ \therefore\ \ln P &= \ln 8.69 - 0.0541t \end{aligned}$$

c
$$\begin{aligned} S &= a^2e^{-kt} \\ \therefore\ \ln S &= \ln\left(a^2e^{-kt}\right) \\ &= \ln a^2 + \ln e^{-kt} \\ \therefore\ \ln S &= 2\ln a - kt && \{\ln a^n = n\ln a\} \end{aligned}$$

2 **a**
$$\begin{aligned} \ln e^2 &= 2\ln e \\ &= 2(1) \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} \ln\sqrt{e} &= \ln e^{\frac{1}{2}} \\ &= \tfrac{1}{2}\ln e \\ &= \tfrac{1}{2} \end{aligned}$$

c
$$\begin{aligned} \ln\left(\frac{1}{e}\right) &= \ln e^{-1} \\ &= -1\ln e \\ &= -1 \end{aligned}$$

d
$$\begin{aligned} \ln\left(\frac{1}{\sqrt{e}}\right) &= \ln e^{-\frac{1}{2}} \\ &= -\tfrac{1}{2}\ln e \\ &= -\tfrac{1}{2}(1) \\ &= -\tfrac{1}{2} \end{aligned}$$

e $e^{\ln 3} = 3$

f
$$\begin{aligned} e^{2\ln 3} &= e^{\ln 3^2} \\ &= e^{\ln 9} \\ &= 9 \end{aligned}$$

g
$$\begin{aligned} e^{-\ln 5} &= e^{\ln 5^{-1}} \\ &= e^{\ln \frac{1}{5}} \\ &= \tfrac{1}{5} \end{aligned}$$

h
$$\begin{aligned} e^{-2\ln 2} &= e^{\ln 2^{-2}} \\ &= e^{\ln \frac{1}{4}} \\ &= \tfrac{1}{4} \end{aligned}$$

3 **a**
$$\begin{aligned} \text{Let } 2 &= e^x \\ \therefore\ \ln 2 &= x \\ \therefore\ 2 &= e^{\ln 2} \end{aligned}$$

b
$$\begin{aligned} \text{Let } 10 &= e^x \\ \therefore\ \ln 10 &= x \\ \therefore\ 10 &= e^{\ln 10} \end{aligned}$$

c
$$\begin{aligned} \text{Let } a &= e^x \\ \therefore\ \ln a &= x \\ \therefore\ a &= e^{\ln a} \end{aligned}$$

d
$$\begin{aligned} \text{Let } a^x &= e^x \\ \therefore\ \ln a^x &= x \\ \therefore\ x\ln a &= x \\ \therefore\ a^x &= e^{x\ln a} \end{aligned}$$

4 **a**
$$\begin{aligned} e^x &= 2 \\ \therefore\ \ln e^x &= \ln 2 \\ x &= \ln 2 \end{aligned}$$

b $e^x = -2$ has no solutions
as $e^x > 0$ for all x

c $e^x = 0$ has no solutions
as $e^x > 0$ for all x

d
$$\begin{aligned} e^{2x} &= 2e^x \\ \therefore\ e^x(e^x - 2) &= 0 \\ \therefore\ e^x &= 2 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 2 \end{aligned}$$

e
$$\begin{aligned} e^x &= e^{-x} \\ \therefore\ x &= -x \\ \therefore\ 2x &= 0 \\ \therefore\ x &= 0 \end{aligned}$$

f
$$\begin{aligned} e^{2x} - 5e^x + 6 &= 0 \\ \therefore\ (e^x - 3)(e^x - 2) &= 0 \\ \therefore\ e^x &= 3 \text{ or } 2 \\ \therefore\ x &= \ln 3 \text{ or } \ln 2 \end{aligned}$$

g
$$\begin{aligned} e^x + 2 &= 3e^{-x} \\ \therefore\ e^{2x} + 2e^x &= 3 \quad \{\times e^x\} \\ \therefore\ e^{2x} + 2e^x - 3 &= 0 \\ \therefore\ (e^x + 3)(e^x - 1) &= 0 \\ \therefore\ e^x &= -3 \text{ or } 1 \\ \therefore\ e^x &= 1 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 1 \\ \therefore\ x &= 0 \end{aligned}$$

h
$$\begin{aligned} 1 + 12e^{-x} &= e^x \\ \therefore\ e^x + 12 &= e^{2x} \quad \{\times e^x\} \\ \therefore\ e^{2x} - e^x - 12 &= 0 \\ \therefore\ (e^x - 4)(e^x + 3) &= 0 \\ \therefore\ e^x &= 4 \text{ or } -3 \\ \therefore\ e^x &= 4 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 4 \end{aligned}$$

i
$$\begin{aligned} e^x + e^{-x} &= 3 \\ \therefore\ e^{2x} + 1 &= 3e^x \quad \{\times e^x\} \\ \therefore\ e^{2x} - 3e^x + 1 &= 0 \\ \therefore\ e^x &= \frac{3 \pm \sqrt{9-4}}{2} \end{aligned}$$

$$\begin{aligned} \therefore\ e^x &= \frac{3 \pm \sqrt{5}}{2} \\ \therefore\ x &= \ln\left(\frac{3+\sqrt{5}}{2}\right) \text{ or } \ln\left(\frac{3-\sqrt{5}}{2}\right) \\ &\doteqdot 0.962 \quad \text{or} \quad -0.962 \end{aligned}$$

5 **a**
$$\begin{aligned} y &= 2^x \\ \therefore\ \frac{dy}{dx} &= 2^x \ln 2 \end{aligned}$$

b
$$\begin{aligned} y &= 5^x \\ \therefore\ \frac{dy}{dx} &= 5^x \ln 5 \end{aligned}$$

c
$$\begin{aligned} y &= x2^x \\ \therefore\ \frac{dy}{dx} &= \frac{d}{dx}(x)2^x + x\frac{d}{dx}(2^x) \\ &\qquad \{\text{product rule}\} \\ &= 2^x + x2^x \ln 2 \end{aligned}$$

d
$$\begin{aligned} y &= x^3 6^{-x} = \frac{x^3}{6^x} \\ \therefore\ \frac{dy}{dx} &= \frac{\frac{d}{dx}(x^3)6^x - x^3\frac{d}{dx}(6^x)}{6^{2x}} \\ &= \frac{3x^2 6^x - x^3 \times 6^x \ln 6}{6^{2x}} \\ &= \frac{x^2(3 - x\ln 6)}{6^x} \end{aligned}$$

e
$$\begin{aligned} y &= \frac{2^x}{x} \\ \therefore\ \frac{dy}{dx} &= \frac{\frac{d}{dx}(2^x)x - 2^x\frac{d}{dx}(x)}{x^2} \\ &= \frac{2^x \ln 2 \times x - 2^x}{x^2} \\ &= \frac{2^x(x\ln 2 - 1)}{x^2} \end{aligned}$$

f
$$\begin{aligned} y &= \frac{x}{3^x} \\ \therefore\ \frac{dy}{dx} &= \frac{\frac{d}{dx}(x)3^x - x\frac{d}{dx}(3^x)}{3^{2x}} \\ &= \frac{3^x - x \times 3^x \ln 3}{3^{2x}} \\ &= \frac{1 - x\ln 3}{3^x} \end{aligned}$$

6 **a** $y_1 = e^x$, $y_2 = e^{2x} - 6$ meet when
$$\begin{aligned} y_1 &= y_2 \\ \therefore\ e^x &= e^{2x} - 6 \\ \therefore\ e^{2x} - e^x - 6 &= 0 \\ \therefore\ (e^x - 3)(e^x + 2) &= 0 \\ \therefore\ e^x &= 3 \text{ or } -2 \\ \therefore\ e^x &= 3 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 3 \quad \text{and } y = e^x = 3 \quad \therefore \text{ meet at } (\ln 3,\ 3) \end{aligned}$$

b $y_1 = 2e^x + 1$, $y_1 = 7 - e^x$ meet when
$$\begin{aligned} y_1 &= y_2 \\ \therefore\ 2e^x + 1 &= 7 - e^x \\ \therefore\ 3e^x &= 6 \\ \therefore\ e^x &= 2 \\ \therefore\ x &= \ln 2 \quad \text{and} \quad y = 7 - e^x = 7 - 2 = 5 \end{aligned}$$

$\therefore$ meet at $(\ln 2,\ 5)$

c $y_1 = 3 - e^x$, $y_2 = 5e^{-x} - 3$ meet when
$$\begin{aligned} y_1 &= y_2 \\ \therefore\ 3 - e^x &= 5e^{-x} - 3 \\ \therefore\ 3e^x - e^{2x} &= 5 - 3e^x \quad \{\times \text{ by } e^x\} \\ \therefore\ e^{2x} - 6e^x + 5 &= 0 \\ \therefore\ (e^x - 5)(e^x - 1) &= 0 \\ \therefore\ e^x &= 1 \text{ or } 5 \\ \therefore\ x &= 0 \text{ or } \ln 5 \\ \text{and}\ \ y &= 3 - e^0 \qquad \text{or} \qquad 3 - e^{\ln 5} \\ &= 3 - 1 \qquad\qquad\quad = 3 - 5 \\ &= 2 \qquad\qquad\qquad\ \ = -2 \end{aligned}$$

i.e., meet at $(0,\ 2)$ and $(\ln 5,\ -2)$

7 **a** $y = e^x - 3e^{-x}$

cuts the x axis at P when $y = 0$

$$\begin{aligned} \therefore\ e^x - 3e^{-x} &= 0 \\ \therefore\ e^{2x} - 3 &= 0 \qquad \{\times \text{ each term by } e^x\} \\ \therefore\ e^{2x} &= 3 \\ \therefore\ 2x &= \ln 3 \\ \therefore\ x &= \tfrac{1}{2}\ln 3 \qquad \therefore\ \text{P is } (\tfrac{1}{2}\ln 3,\ 0) \end{aligned}$$

cuts the y-axis at Q when $x = 0$

$$\begin{aligned} \therefore\ y &= e^0 - 3e^0 \\ &= 1 - 3 \\ &= -2 \qquad \therefore\ \text{Q is } (0,\ -2) \end{aligned}$$

b

$$\begin{aligned} \frac{dy}{dx} &= e^x + 3e^{-x} \\ &= e^x + \frac{3}{e^x} \end{aligned}$$

Since $e^x > 0$ for all x

$\dfrac{dy}{dx} > 0$ for all x

$\therefore$ the function is increasing for all x

c

$$\begin{aligned} \frac{dy}{dx} &= e^x + 3x^{-x} \\ \therefore\ \frac{d^2y}{dx^2} &= e^x - 3e^{-x} \\ &= y \end{aligned}$$

Above x-axis $y > 0$ $\therefore$ $\dfrac{d^2y}{dx^2} > 0$

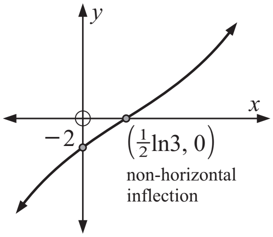
$\therefore$ the function is concave up

Below x-axis $y < 0$ $\therefore$ $\dfrac{d^2y}{dx^2} < 0$

$\therefore$ the function is concave down

i.e., a non-horizontal inflection occurs when $y = 0$

d



8 **a** For $f(x) = e^x - 3$ and $g(x) = 3 - \dfrac{5}{e^x}$

the x-intercept occurs when $y = 0$

$$\begin{aligned} \therefore\ e^x - 3 &= 0 \\ \therefore\ e^x &= 3 \\ \therefore\ x &= \ln 3 \end{aligned}$$

i.e., at $(\ln 3,\ 0)$

and

$$\begin{aligned} 3 - \frac{5}{e^x} &= 0 \\ \therefore\ \frac{3e^x - 5}{e^x} &= 0 \\ \therefore\ 3e^x - 5 &= 0 \\ \therefore\ e^x &= \tfrac{5}{3} \\ \therefore\ x &= \ln \tfrac{5}{3} \end{aligned}$$

i.e., at $(\ln \frac{5}{3},\ 0)$

the y-intercept occurs when $x = 0$

$$\begin{aligned} \therefore\ y &= e^0 - 3 \\ y &= -2 \end{aligned}$$

i.e., at $(0,\ -2)$

and

$$\begin{aligned} y &= 3 - \frac{5}{e^0} \\ &= 3 - 5 \\ &= -2 \end{aligned}$$

i.e., $(0,\ -2)$

b as $x \to +\infty$ $f(x) \to +\infty$ as $x \to +\infty$ $g(x) \to 3$ (below)

$x \to -\infty$ $f(x) \to -3$ (above) $x \to -\infty$ $g(x) \to -\infty$

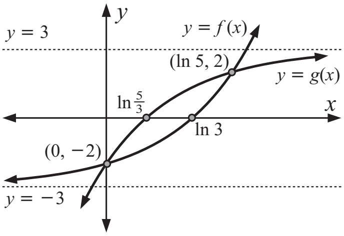
c $f(x)$ and $g(x)$ meet when

$$\begin{aligned} e^x-3 &= 3-5e^{-x} \\ \therefore\ e^{2x}-3e^x &= 3e^x-5 \quad \{\times\ e^x\} \\ \therefore\ e^{2x}-6e^x+5 &= 0 \\ \therefore\ (e^x-5)(e^x-1) &= 0 \\ \text{i.e., when}\quad e^x &= 5 \text{ or } 1 \\ \therefore\ x &= \ln 5 \text{ or } 0 \end{aligned}$$

$$\begin{aligned} \text{when}\quad x=\ln 5,\quad f(x) &= e^{\ln 5}-3=5-3=2 \\ \text{when}\quad x=0,\quad f(x) &= e^0-3=1-3=-2 \end{aligned}$$

$\therefore$ $f(x)$ and $g(x)$ meet at $(0,\ -2)$ and $(\ln 5,\ 2)$

d



EXERCISE 23C

1 **a**

$$\begin{aligned} y &= \ln(7x) \\ \therefore\ y &= \ln 7+\ln x \\ \therefore\ \frac{dy}{dx} &= 0+\frac{1}{x}=\frac{1}{x} \end{aligned}$$

or

$$\begin{aligned} y &= \ln(7x) \\ \therefore\ \frac{dy}{dx} &= \frac{7 \leftarrow f'(x)}{7x \leftarrow f(x)} \\ &= \frac{1}{x} \end{aligned}$$

b

$$\begin{aligned} y &= \ln(2x+1) \\ \therefore\ \frac{dy}{dx} &= \frac{2 \quad\leftarrow f'(x)}{2x+1 \leftarrow f(x)} \end{aligned}$$

c

$$\begin{aligned} y &= \ln(x-x^2) \\ \therefore\ \frac{dy}{dx} &= \frac{1-2x \leftarrow f'(x)}{x-x^2 \leftarrow f(x)} \end{aligned}$$

d

$$\begin{aligned} y &= 3-2\ln x \\ \therefore\ \frac{dy}{dx} &= 0-2\left(\frac{1}{x}\right) \\ &= -\frac{2}{x} \end{aligned}$$

e

$$\begin{aligned} y &= x^2\ln x \\ \therefore\ \frac{dy}{dx} &= 2x\ln x+x^2\left(\frac{1}{x}\right) \\ &= 2x\ln x+x \end{aligned}$$

f

$$\begin{aligned} y &= \frac{\ln x}{2x} \\ \therefore\ \frac{dy}{dx} &= \frac{\left(\frac{1}{x}\right)2x-\ln x(2)}{(2x)^2} \\ &= \frac{2-2\ln x}{4x^2} \\ &= \frac{1-\ln x}{2x^2} \end{aligned}$$

g

$$\begin{aligned} y &= e^x\ln x \\ \therefore\ \frac{dy}{dx} &= e^x\ln x+\frac{e^x}{x} \end{aligned}$$

h

$$\begin{aligned} y &= (\ln x)^2 \\ \therefore\ \frac{dy}{dx} &= 2\,(\ln x)^1\left(\frac{1}{x}\right) \\ &= \frac{2\ln x}{x} \end{aligned}$$

i

$$\begin{aligned} y &= \sqrt{\ln x}=(\ln x)^{\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\right) \\ &= \frac{1}{2x\sqrt{\ln x}} \end{aligned}$$

j

$$\begin{aligned} y &= e^{-x}\ln x \\ \therefore\ \frac{dy}{dx} &= -e^{-x}\ln x+e^{-x}\left(\frac{1}{x}\right) \\ &= \frac{e^{-x}}{x}-e^{-x}\ln x \end{aligned}$$

k

$$\begin{aligned} y &= \sqrt{x}\ln 2x \\ \therefore\ \frac{dy}{dx} &= \frac{1}{2\sqrt{x}}\ln 2x+\sqrt{x}\left(\frac{1}{x}\right) \\ &= \frac{\ln 2x}{2\sqrt{x}}+\frac{1}{\sqrt{x}} \end{aligned}$$

l $y = \dfrac{2\sqrt{x}}{\ln x} \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{1}{\sqrt{x}}\ln x - 2\sqrt{x}\left(\frac{1}{x}\right)}{(\ln x)^2} = \dfrac{\frac{1}{\sqrt{x}}\ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2} = \dfrac{\ln x - 2}{\sqrt{x}(\ln x)^2}$

2 **a**

$$\begin{aligned} y &= x\ln 5 \\ \therefore \quad \frac{dy}{dx} &= \ln 5 \end{aligned}$$

b

$$\begin{aligned} y &= \ln(x^3) = 3\ln x \\ \therefore \quad \frac{dy}{dx} &= 3\left(\frac{1}{x}\right) = \frac{3}{x} \end{aligned}$$

c

$$\begin{aligned} y &= \ln(x^4 + x) \\ \therefore \quad \frac{dy}{dx} &= \frac{4x^3 + 1}{x^4 + x} \end{aligned}$$

d

$$\begin{aligned} y &= \ln(10 - 5x) \\ \therefore \quad \frac{dy}{dx} &= \frac{-5}{10 - 5x} = \frac{1}{x - 2} \end{aligned}$$

e

$$\begin{aligned} y &= [\ln(2x+1)]^3 \\ \therefore \quad \frac{dy}{dx} &= 3\,[\ln(2x+1)]^2 \times \frac{2}{2x+1} \\ &= \frac{6\,[\ln(2x+1)]^2}{2x+1} \end{aligned}$$

f

$$\begin{aligned} y &= \frac{\ln(4x)}{x} \\ \therefore \quad \frac{dy}{dx} &= \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2} \\ &= \frac{1 - \ln(4x)}{x^2} \end{aligned}$$

g

$$\begin{aligned} y &= \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) \\ \therefore \quad y &= -\ln x \quad \therefore \quad \frac{dy}{dx} = -\frac{1}{x} \end{aligned}$$

h

$$\begin{aligned} y &= \ln(\ln x) \\ \therefore \quad \frac{dy}{dx} &= \frac{\frac{1}{x}}{\ln(\ln x)} = \frac{1}{x\ln(\ln x)} \end{aligned}$$

i $y = \dfrac{1}{\ln x} = [\ln x]^{-1} \quad \therefore \quad \dfrac{dy}{dx} = -1\,[\ln x]^{-2} \times \dfrac{1}{x} = \dfrac{-1}{x\,[\ln x]^2}$

3 **a**

$$\begin{aligned} y &= \ln\sqrt{1 - 2x} \\ &= \ln(1 - 2x)^{\frac{1}{2}} \\ &= \tfrac{1}{2}\ln(1 - 2x) \\ \therefore \quad \frac{dy}{dx} &= \tfrac{1}{2} \times \frac{-2}{1 - 2x} \\ &= \frac{1}{2x - 1} \end{aligned}$$

b

$$\begin{aligned} y &= \ln\frac{1}{(2x+3)} \\ &= -\ln(2x+3) \\ \therefore \quad \frac{dy}{dx} &= -\frac{2}{2x+3} \end{aligned}$$

c

$$\begin{aligned} y &= \ln(e^x\sqrt{x}) \\ &= \ln e^x + \ln x^{\frac{1}{2}} \\ &= \ln e^x + \tfrac{1}{2}\ln x \\ &= x + \tfrac{1}{2}\ln x \\ \therefore \quad \frac{dy}{dx} &= 1 + \tfrac{1}{2}\left(\frac{1}{x}\right) \\ \therefore \quad \frac{dy}{dx} &= 1 + \frac{1}{2x} \end{aligned}$$

d

$$\begin{aligned} y &= \ln(x\sqrt{2 - x}) \\ &= \ln x + \ln(2 - x)^{\frac{1}{2}} \\ &= \ln x + \tfrac{1}{2}\ln(2 - x) \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{x} + \tfrac{1}{2}\left(\frac{-1}{2 - x}\right) \\ &= \frac{1}{x} - \frac{1}{2(2 - x)} \end{aligned}$$

e

$$\begin{aligned} y &= \ln\left(\frac{x+3}{x-1}\right) \\ &= \ln(x+3) - \ln(x-1) \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{x+3} - \frac{1}{x-1} \end{aligned}$$

f

$$\begin{aligned} y &= \ln\left(\frac{x^2}{3 - x}\right) \\ &= \ln x^2 - \ln(3 - x) \\ &= 2\ln x - \ln(3 - x) \\ \therefore \quad \frac{dy}{dx} &= \frac{2}{x} - \frac{-1}{3 - x} \\ &= \frac{2}{x} + \frac{1}{3 - x} \end{aligned}$$

g

$$\begin{aligned} f(x) &= \ln\left((3x - 4)^3\right) \\ &= 3\ln(3x - 4) \\ \therefore \quad f'(x) &= 3 \times \frac{3}{3x - 4} \\ &= \frac{9}{3x - 4} \end{aligned}$$

h

$$\begin{aligned} f(x) &= \ln\left(x(x^2 + 1)\right) \\ &= \ln x + \ln(x^2 + 1) \\ \therefore \quad f'(x) &= \frac{1}{x} + \frac{2x}{x^2 + 1} \end{aligned}$$

i

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2 + 2x}{x - 5}\right) \\ &= \ln(x^2 + 2x) \\ &\quad - \ln(x - 5) \\ f'(x) &= \frac{2x + 2}{x^2 + 2x} - \frac{1}{x - 5} \end{aligned}$$

4 **a** For this question, we remember that $\log_a x = \dfrac{\log_e x}{\log_e a} = \dfrac{\ln x}{\ln a}$

i $y = \log_2 x = \dfrac{\ln x}{\ln 2}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{x \ln 2}$

ii $y = \log_{10} x = \dfrac{\ln x}{\ln 10}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{x \ln 10}$

iii $y = x \log_3 x = \dfrac{x \ln x}{\ln 3}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{\frac{d}{dx}(x)\ln x + x\frac{d}{dx}(\ln x)}{\ln 3}\\ &= \frac{\ln x + x\left(\frac{1}{x}\right)}{\ln 3}\\ &= \frac{1+\ln x}{\ln 3}\end{aligned}$$

b $$\begin{aligned} y &= 2^x\\ &= (e^{\ln 2})^x\\ &= e^{x\ln 2}\end{aligned} \qquad \begin{aligned}\therefore \frac{dy}{dx} &= e^{x\ln 2} \times \ln 2\\ &= 2^x \ln 2\end{aligned}$$

c $$\begin{aligned} y &= a^x\\ &= (e^{\ln a})^x\\ &= e^{x\ln a}\end{aligned} \qquad \begin{aligned}\therefore \frac{dy}{dx} &= e^{x\ln a} \times \ln a\\ &= a^x \ln a\end{aligned}$$

5 $f(x) = \ln(2x-1) - 3$

a when $y = 0$,
$$\begin{aligned} \ln(2x-1) &= 3\\ \therefore\ 2x - 1 &= e^3\\ \therefore\ 2x &= e^3 + 1\\ \therefore\ x &= \frac{e^3+1}{2} \doteqdot 10.54\end{aligned}$$
i.e., the x intercept is $\dfrac{e^3+1}{2}$.

b $f(0) = \ln(-1)$ cannot be found as $\ln(-1)$ is not defined. $\therefore$ there is no y-intercept.

c $f'(x) = \dfrac{2}{2x-1}$, $\therefore$ $f'(1) = \dfrac{2}{2-1} = 2$ $\therefore$ slope of tangent $= 2$

d $\ln(2x-1)$ has meaning provided $2x - 1 > 0$ i.e., $2x > 1$ and so $x > \frac{1}{2}$

$\therefore$ $f(x)$ has meaning provided $x > \frac{1}{2}$

e $f'(x) = 2(2x-1)^{-1}$

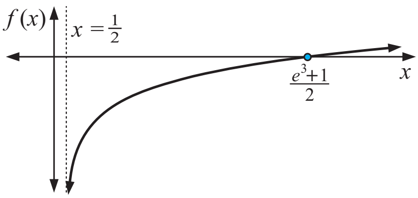
$\therefore$ $f''(x) = -2(2x-1)^{-2}2 = \dfrac{-4}{(2x-1)^2}$

provided $f(x)$ has meaning

i.e., provided $x > \frac{1}{2}$, $f''(x) < 0$

$\therefore$ $f(x)$ is always concave down for $x > \frac{1}{2}$

f

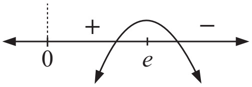


6 Consider $f(x) = \dfrac{\ln x}{x}$ Let $u = \ln x$, $v = x$, $u' = \dfrac{1}{x}$, $v' = 1$

$$\therefore\ f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1-\ln x}{x^2}$$

Now $f'(x) = 0$ when
$$\begin{aligned} 1 - \ln x &= 0\\ \therefore\ \ln x &= 1\\ \therefore\ x &= e^1, \text{ and } f(e) = \frac{\ln e}{e} = \frac{1}{e}\end{aligned}$$

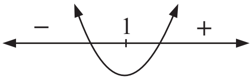
Sign diagram of $f'(x)$ is:



$\therefore$ there is a max. turning point at $\left(e, \dfrac{1}{e}\right)$, i.e., $f(x) \leqslant \dfrac{1}{e}$ for all x, i.e., $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all x

7 $f(x) = x - \ln x$

$\therefore \quad f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$ and the sign diagram of $f'(x)$ is:



$\therefore \quad f(x)$ has a minimum turning point at $(1,\ 1-\ln 1)$, i.e., $(1,\ 1)$

$\therefore \quad f(x) \geqslant 1$ for all $x > 0$

$x - \ln x \geqslant 1$

i.e., $\ln x \leqslant x - 1$ for all $x > 0$

EXERCISE 23D

1 $f(x) = e^{-x} \quad \therefore \quad f(1) = e^{-1}$ i.e., the point of contact is $(1,\ \frac{1}{e})$.

Now $f'(x) = -e^{-x}$

$f'(1) = -e^{-1} = \dfrac{-1}{e}$ and is the slope of the tangent at $(1,\ \frac{1}{e})$.

$\therefore$ tangent has equation $\dfrac{y - \frac{1}{e}}{x-1} = -\dfrac{1}{e} \quad \therefore \quad e(y - \frac{1}{e}) = -(x-1)$

$\therefore \quad ey - 1 = -x + 1$

i.e., $x + ey = 2$ or $y = -\dfrac{1}{e}x + \dfrac{2}{e}$

2 $y = \ln(2-x)$, so when $x = -1$, $y = \ln 3$ i.e., the point of contact is $(-1,\ \ln 3)$.

Now $\dfrac{dy}{dx} = \dfrac{-1}{2-x}$

$\therefore$ when $x = -1$, $\dfrac{dy}{dx} = -\dfrac{1}{2+1} = -\frac{1}{3}$ and is the slope of the tangent at $(-1,\ \ln 3)$.

$\therefore$ tangent has equation $\dfrac{y - \ln 3}{x+1} = -\frac{1}{3}$ i.e., $3(y - \ln 3) = -(x+1)$

$\therefore \quad 3y - 3\ln 3 = -x - 1$

i.e., $x + 3y = 3\ln 3 - 1$

3 $y = x^2 e^x$, so when $x = 1$, $y = e$, i.e., the point of contact is $(1,\ e)$.

Now $\dfrac{dy}{dx} = x^2 e^x + 2xe^x$

$\therefore$ when $x = 1$, $\dfrac{dy}{dx} = e + 2e = 3e$ and is the slope of the tangent at $(1,\ e)$.

$\therefore$ the tangent has equation $\dfrac{y-e}{x-1} = 3e$ i.e., $y - e = 3e(x-1)$

$\therefore \quad y - e = 3ex - 3e$

$\therefore \quad y - 3ex = -2e$

i.e., $3ex - y = 2e$

Cuts the x-axis when $y = 0 \quad \therefore \quad 3ex = 2e$ i.e., $x = \frac{2}{3}$ i.e., at A$(\frac{2}{3},\ 0)$

Cuts the y-axis when $x = 0$ i.e., $-y = 2e$ i.e., at B$(0,\ -2e)$

4 $y = \ln\sqrt{x}$ $\quad \therefore$ when $y = -1$, $-1 = \frac{1}{2}\ln x$

$= \ln x^{\frac{1}{2}}$ $\quad \therefore \quad \ln x = -2$

$= \frac{1}{2}\ln x$ $\quad \therefore \quad x = e^{-2}$

$\therefore \quad x = \dfrac{1}{e^2}$ i.e., the point of contact is $\left(\dfrac{1}{e^2},\ -1\right)$

Now $\dfrac{dy}{dx} = \frac{1}{2}\,\dfrac{1}{x} = \dfrac{1}{2x} \quad \therefore$ at $\left(\dfrac{1}{e^2},\ -1\right)$, $\dfrac{dy}{dx} = \dfrac{1}{2e^{-2}} = \dfrac{e^2}{2}$

$$\therefore \text{ the tangent has slope } = \frac{e^2}{2} \quad \text{and} \quad \text{the normal has slope } = -\frac{2}{e^2}$$

$$\therefore \text{ the normal has equation } \frac{y+1}{x-\frac{1}{e^2}} = -\frac{2}{e^2} \quad \text{i.e., } e^2(y+1) = -2\left(x-\frac{1}{e^2}\right)$$

$$\therefore \; e^2y + e^2 = -2x + \frac{2}{e^2}$$

$$\therefore \; 2x + e^2y = -e^2 + \frac{2}{e^2} \quad \text{or} \quad y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$$

5 $y = e^x$

When $x = a$, $y = e^a$, i.e., the point of contact is (a, e^a).

Now $\frac{dy}{dx} = e^x$ and so at (a, e^a), $\frac{dy}{dx} = e^a$ and is the slope of the tangent at (a, e^a).

$\therefore$ the tangent has equation $\frac{y-e^a}{x-a} = e^a$ i.e., $y - e^a = e^a(x-a)$ ......$*$

if the tangent is through the origin, (0, 0) must satisfy $*$

$$\therefore \; 0 - e^a = e^a(0-a)$$
$$\therefore \; -e^a = -ae^a$$
$$\therefore \; e^a(a-1)0 = 0$$
$$\therefore \; a = 1 \quad \{\text{as } e^a > 0\}$$

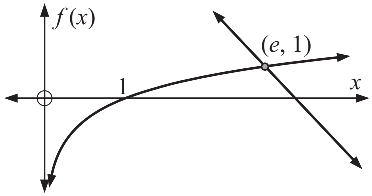
So the equation of the tangent is $y - e = ex - e$ i.e., $y = ex$

6 **a** $f(x) = \ln x$ is defined for all $x > 0$.

b $f'(x) = \frac{1}{x}$ which is > 0 for all $x > 0$, $\therefore$ $f(x)$ is increasing on $x > 0$.

$f''(x) = -x^{-2} = \frac{-1}{x^2}$ which is < 0 for all $x > 0$, $\therefore$ $f(x)$ is concave down on $x > 0$.

c



At $y = 1$, $1 = \ln x$

$\therefore$ $x = e^1 = e$,

i.e., the point of contact is $(e, 1)$

and $\frac{dy}{dx} = \frac{1}{x}$

$\therefore$ at $(e, 1)$, $\frac{dy}{dx} = \frac{1}{e}$

$\therefore$ the slope of the tangent is $\frac{1}{e}$, and the slope of the normal is $-e$

$\therefore$ the equation of the normal is $\frac{y-1}{x-e} = -e$

$$\therefore \; y - 1 = -e(x-e)$$
$$\therefore \; y - 1 = -ex + e^2$$
$$\therefore \; ex + y = 1 + e^2$$

7 $y = 3e^{-x}$ and $y = 2 + e^x$ meet when

$$3e^{-x} = 2 + e^x$$
$$\therefore \; 3 = 2e^x + e^{2x} \quad \{\times \text{ by } e^x\}$$
$$\therefore \; e^{2x} + 2e^x - 3 = 0$$
$$\therefore \; (e^x+3)(e^x-1) = 0$$
$$\therefore \; e^x = -3 \text{ or } 1 \quad \{\text{as } e^x > 0\}$$
$$\therefore \; x = 0 \quad \{\text{as } e^x > 0\}$$

and when $x = 0$, $y = 3e^0 = 3$

When $y = 2 + e^x$, $\frac{dy}{dx} = e^x$ $\therefore$ at (0, 3), $\frac{dy}{dx} = e^0 = 1$

i.e., the slope of the tangent is 1

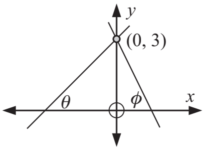
and $1 = \tan\theta$ where θ is the angle between the tangent and the x-axis, $\therefore \quad \theta = 45^o$

When $y = 3e^{-x}$, $\dfrac{dy}{dx} = -3e^{-x}$ $\therefore$ at $(0, 3)$, $\dfrac{dy}{dx} = -3$

and $\tan\phi = 3$ where ϕ is the angle between the tangent and the x-axis

$\therefore \quad \phi = \tan^{-1}(3) \doteqdot 71.6^o$

$\therefore$ angle between the tangents $\doteqdot (180 - (71.57 + 45))^o \doteqdot 63.4^o$



8 **a** $W = 20e^{-kt}$ and when $t = 50$ hours, $W = 10$ g

$$\therefore \quad 20e^{-50k} = 10$$
$$\therefore \quad e^{-50k} = \tfrac{1}{2}$$
$$\therefore \quad -50k = \ln\tfrac{1}{2} = -\ln 2 \quad \text{and so} \quad k = \tfrac{1}{50}\ln 2 \doteqdot 0.0139$$

b **i** When $t = 0$,

$$W = 20e^{-kt}$$
$$= 20e^{0}$$
$$\therefore \quad W = 20 \text{ g}$$

ii When $t = 24$,

$$W = 20e^{-24k}$$
$$= 20e^{-24\frac{\ln 2}{50}}$$
$$\doteqdot 14.3 \text{ g}$$

iii When $t = 1$ week

$$= 7 \times 24 \text{ hours}$$
$$= 168 \text{ hours}$$
$$W = 20e^{-168\frac{\ln 2}{50}} \doteqdot 1.95 \text{ g}$$

c When $W = 1$ g,

$$20e^{-\frac{\ln 2}{50}\times t} = 1$$
$$\therefore \quad e^{-\frac{\ln 2}{50}\times t} = 0.05$$
$$\therefore \quad -\frac{\ln 2}{50} \times t = \ln 0.05$$
$$\therefore \quad t = \frac{-50\ln 0.05}{\ln 2} \doteqdot 216.1 \text{ hours}$$

d $$\frac{dW}{dt}$$
$$= 20e^{-kt}(-k)$$
$$= \left(-20\frac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50}t}$$

i When $t = 100$ hours,

$$\frac{dW}{dt} = \left(\frac{-20\ln 2}{50}\right)e^{-2\ln 2}$$
$$\doteqdot -0.0693 \text{ g/h}$$

ii When $t = 1000$ hours,

$$\frac{dW}{dt} = \left(\frac{-20\ln 2}{50}\right)e^{-2\ln 2}$$
$$\doteqdot -2.64 \times 10^{-7} \text{ g/h}$$

e $\dfrac{dW}{dt} = -k(20e^{-kt}) = -kW$ $\quad \therefore \quad \dfrac{dW}{dt} \;\alpha\; W$

9 $T = 5 + 95e^{-kt}$ $\quad {}^oC$

a $T = 20^oC$ when $t = 15$

$$\therefore \quad 20 = 5 + 95e^{-15k}$$
$$\therefore \quad 15 = 95e^{-15k}$$
$$\therefore \quad \ln\left(\tfrac{15}{95}\right) = -15k$$
$$\therefore \quad k = \frac{\ln\left(\frac{15}{95}\right)}{-15} \doteqdot 0.1231$$

b When $t = 0$,

$$T = 5 + 95e^{0}$$
$$\therefore \quad T = 5 + 95$$
$$= 100^o \text{ C}$$

c $$\frac{dT}{dt} = 0 + 95e^{-kt}(-k)$$
$$= -(95e^{kt})k$$
$$= -k(T-5)$$
$$\therefore \quad \frac{dT}{dt} \propto T - 5$$

d $$\frac{dT}{dt} = -95e^{-kt} \times k$$
$$\doteqdot -11.6902e^{-0.1231t}$$

i When $t = 0$,

$$\frac{dT}{dt} \doteqdot -11.69$$

$\therefore$ temperature is decreasing at 11.7^oC/min

ii When $t = 10$,

$$\frac{dT}{dt} \doteqdot -11.6902e^{-1.231}$$
$$\doteqdot -3.415$$

$\therefore$ temperature is decreasing at 3.42^oC/min

iii When $t = 20$,

$$\frac{dT}{dt} \doteqdot -11.6902e^{-2.462}$$
$$\doteqdot -0.998$$

$\therefore$ temperature is decreasing at 0.998^oC/min

10 $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$

a Planted when $t = 0$

$\therefore\ H(0) = 20\ln(2) + 30 \doteqdot 43.9$ cm

b When $H = 1$ m $= 100$ cm

$$\begin{aligned}\therefore\ 20\ln(3t+2)+30 &= 100\\ \therefore\ 20\ln(3t+2) &= 70\\ \therefore\ \ln(3t+2) &= 3.5\\ \therefore\ 3t+2 &= e^{3.5}\\ \therefore\ 3t &= e^{3.5}-2\\ \therefore\ t &= \frac{e^{3.5}-2}{3}\text{ years}\\ \therefore\ t &\doteqdot 10.4\text{ years}\end{aligned}$$

c $$\begin{aligned}\frac{dH}{dt} &= \frac{20}{(3t+2)} \times 3\\ &= \frac{60}{3t+2}\quad\text{cm/year}\end{aligned}$$

i When $t = 3$,

$$\frac{dH}{dt} = \frac{60}{11} \doteqdot 5.454$$

$\therefore$ it is growing at 5.45 cm/year

ii When $t = 10$,

$$\frac{dH}{dt} = \frac{60}{32} = 1.875$$

$\therefore$ it is growing at 1.88 cm/year

11 $A = s(1-e^{-kt})$, $t \geqslant 0$

a When $t = 0$,

$$\begin{aligned}A &= s(1-e^0)\\ &= s(1-1)\\ &= 0\end{aligned}$$

b When $t = 3$, $A = 5$ and $s = 10$

$$\begin{aligned}\therefore\ 5 &= 10(1-e^{-3k})\\ \therefore\ 0.5 &= 1-e^{-3k}\\ \therefore\ e^{-3k} &= 0.5\\ \therefore\ e^{3k} &= 2\\ \therefore\ 3k &= \ln 2\\ \therefore\ k &= \tfrac{1}{3}\ln 2\\ \therefore\ k &\doteqdot 0.231\end{aligned}$$

c When $t = 5$ and $s = 10$

$$\begin{aligned}\frac{dA}{dt} &= ske^{-kt}\\ &= 10\left(\tfrac{1}{3}\ln 2\right)\left(e^{-\frac{5}{3}\ln 2}\right)\\ &\doteqdot 0.728\quad\text{units/hour}\end{aligned}$$

d Now $$\begin{aligned}\frac{dA}{dt} &= sk\left(e^{-kt}\right)\\ &= k\left(se^{-kt}\right)\\ &= -k\left(-se^{-kt}\right)\\ &= -k(A-s)\end{aligned}$$

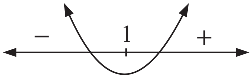
$$\therefore\ \frac{dA}{dt} \propto A - s$$

12 $f(x) = \dfrac{e^x}{x}$

a No x-intercept since $e^x \neq 0$, i.e., $f(x) \neq 0$, and no y-intercept since $\dfrac{e^0}{0}$ is undefined.

b as $x \to +\infty$ $f(x) \to \infty$, and as $x \to -\infty$ $f(x) \to 0$ (below)

c By the quotient rule, $f'(x) = \dfrac{e^x x - e^x(1)}{x^2} = \dfrac{e^x(x-1)}{x^2}$ with sign diagram:



and when $x = 1$, $f(x) = \dfrac{e^1}{1} = e$

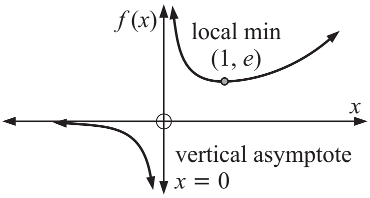
$\therefore$ local minimum when $x = 1$

i.e., local minimum at (1, e).

Note: Vertical asymptote exists at $x = 0$,

i.e., as $x \to 0$ (from above), $y \to +\infty$, as $x \to 0$ (from below), $y \to -\infty$

d



e Now $f'(x) = \dfrac{e^x(x-1)}{x^2}$

$\therefore\ f'(-1) = \dfrac{e^{-1}(-1-1)}{(-1)^2} = -\dfrac{2}{e}$

$\therefore$ slope of the tangent is $= -\dfrac{2}{e}$

and when $x = -1,\ y = \dfrac{e^{-1}}{-1} = -\dfrac{1}{e}$

$\therefore$ equation of tangent is $\dfrac{y - \left(-\frac{1}{e}\right)}{x-(-1)} = -\dfrac{2}{e}$ i.e., $\dfrac{y + \frac{1}{e}}{x+1} = -\dfrac{2}{e}$

$$\therefore\ e\left(y + \frac{1}{e}\right) = -2(x+1)$$
$$ey + 1 = -2x - 2$$
$$\therefore\ 2x + ey = -3$$

13 **a** $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, $t \geqslant 0$

$v(t) = 100 - 40e^{-\frac{t}{5}}$ cm/sec

$a(t) = 8e^{-\frac{t}{5}}$ cm/sec^2

b When $t = 0$,

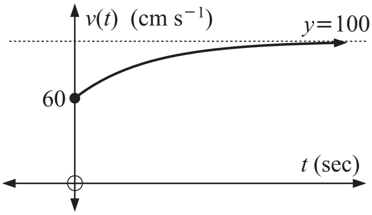
$s(0) = 200$ cm $\qquad v(0) = 60$ cm/sec

$a(0) = 8$ cm/sec^2

c as $t \to +\infty$, $e^{-\frac{t}{5}} \to 0$,

$\therefore\ v(t) \to 100$ cm/sec (below)

d



e When $v(t) = 80$ cm/sec,

$$100 - 40e^{-\frac{t}{5}} = 80$$
$$\therefore\ -40e^{-\frac{t}{5}} = -20$$
$$\therefore\ e^{-\frac{t}{5}} = 0.5$$
$$\therefore\ -\frac{t}{5} = \ln 0.5$$
$$\therefore\ t = -5\ln 0.5 \doteqdot 3.47 \text{ sec}$$

14 **a** $A(t) = t\ln t + 1,\quad 0 < t \leqslant 5$

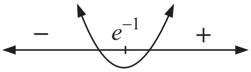
$\therefore\ A'(t) = \ln t + t \times \dfrac{1}{t} + 0$ {product rule}

$= \ln t + 1$

which is 0 when $\ln t = -1$

i.e., $t = e^{-1}$

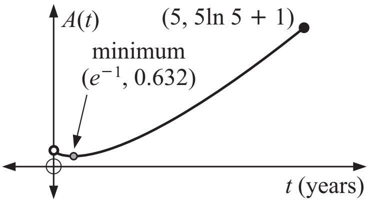
and the sign diagram of $A'(t)$ is:



$\therefore\ A(t)$ is a minimum when $t = \dfrac{1}{e} \doteqdot 0.3679$ years

i.e., at 4.41 months old.

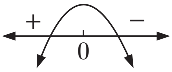
b



15 **a** $f(x) = \dfrac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$, $\quad\therefore\ f'(x) = \dfrac{1}{\pi\sqrt{2}}\ e^{-\frac{1}{2}x^2}(-x) = \dfrac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$

which is 0 only when $x = 0$, and when $x = 0$, $f(0) = \dfrac{1}{\pi\sqrt{2}}$

and $f'(x)$ has sign diagram:



$\therefore$ there is a local maximum at $\left(0, \dfrac{1}{\pi\sqrt{2}}\right)$

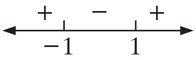
and the function is increasing for $x \leqslant 0$, and decreasing for $x \geqslant 0$

b $f'(x)=\dfrac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}=\dfrac{1}{\pi\sqrt{2}}\left(-xe^{-\frac{1}{2}x^2}\right)$

$$\therefore\quad f''(x)=\frac{1}{\pi\sqrt{2}}\left((-1)\,e^{-\frac{1}{2}x^2}+(-x)\,e^{-\frac{1}{2}x^2}\,(-x)\right)\quad\{\text{product rule}\}$$

$$=\frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}\left(x^2-1\right)$$

$$=\frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}\,(x+1)\,(x-1)\quad\text{which has sign diagram:}$$



When $x=1$, $f(x)=\dfrac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}}=\dfrac{1}{\pi\sqrt{2e}}$ and also when $x=-1$, $f(x)=\dfrac{1}{\pi\sqrt{2e}}$

$\therefore$ there are points of inflection at $\left(1,\ \dfrac{1}{\pi\sqrt{2e}}\right)$ and $\left(-1,\ \dfrac{1}{\pi\sqrt{2e}}\right)$

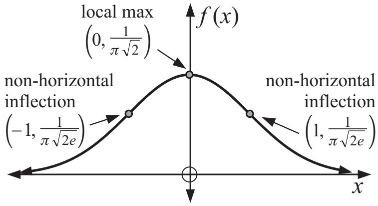
c as $x\to\infty$, $e^{-\frac{1}{2}x^2}\to 0$ (above),

$\therefore$ $f(x)\to 0$ (above),

as $x\to-\infty$, $e^{-\frac{1}{2}x^2}\to 0$ (above),

$\therefore$ $f(x)\to 0$ (above)

d



16 $C(x)=4\ln x+\left(\dfrac{30-x}{10}\right)^2,\quad x\geqslant 10\quad\therefore\quad C'(x)=\dfrac{4}{x}+2\left(\dfrac{30-x}{10}\right)\left(-\frac{1}{10}\right)$

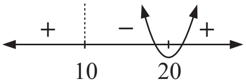
$$=\frac{4}{x}-\frac{30-x}{50}$$

$$=\frac{200-x(30-x)}{50x}$$

$$=\frac{200-30x+x^2}{50x}$$

$$=\frac{(x-10)(x-20)}{50x}$$

which has sign diagram:



$\therefore$ the minimum cost occurs when $x=20$, i.e., 20 kettles/day are produced

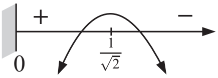
17 Let coordinates of D be $(x,\,0)$, where $x>0$. $\therefore$ the coordinates of C are $(x,\,e^{-x^2})$

$\therefore$ area ABCD $=2xe^{-x^2}$ $\quad\therefore\quad \dfrac{dA}{dx}=2e^{-x^2}+2xe^{-x^2}(-2x)$ {product rule}

$$=2e^{-x^2}\left(1-2x^2\right)$$

$$=2e^{-x^2}(1+\sqrt{2}x)(1-\sqrt{2}x)$$

and $\dfrac{dA}{dx}$ has sign diagram:



$\therefore$ area is a maximum when $x=\frac{1}{\sqrt{2}}$, and so C is $\left(\frac{1}{\sqrt{2}},\,e^{-\frac{1}{2}}\right)$.

18 $P(x)=R(x)-C(x)$

$$\therefore\quad P(x)=\left[1000\ln\left(1+\frac{x}{400}\right)+600\right]-[x(1.5)+300]$$

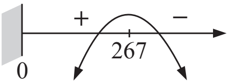
$$=1000\ln\left(1+0.0025x\right)-1.5x+300$$

$$\therefore\quad P'(x)=1000\left(\frac{0.0025}{1+0.0025x}\right)-1.5=\frac{2.5}{1+0.0025x}-1.5$$

which is 0 when $\dfrac{2.5}{1+0.0025x} = \dfrac{3}{2}$

$$\therefore \quad 3+0.0075x = 5$$
$$\therefore \quad 0.0075x = 2$$
$$\therefore \quad x = \frac{2}{0.0075} \doteqdot 266.7 \quad \text{and} \quad P(266) \doteqdot 410.83$$
$$P(267) \doteqdot 410.83$$

Sign diagram of $P'(x)$:



$\therefore$ to maximise profit, 266 or 267 torches/day should be produced.

19 **a** $y = ax^2$ $(a > 0)$ touches $y = \ln x$ when $ax^2 = \ln x$

If they touch at $x = b$

then $ab^2 = \ln b$ ...... (1)

Now if $y = ax^2$ then $\dfrac{dy}{dx} = 2ax$ $\quad$ and if $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ at $x = b$, $\dfrac{dy}{dx} = 2ab$ $\quad$ $\therefore$ at $x = b$, $\dfrac{dy}{dx} = \dfrac{1}{b}$

and the slope of the tangent at $x = b$ to the curve $y = ax^2$ is also the slope of the tangent to $y = \ln x$ at $x = b$, $\therefore$ $\dfrac{1}{b} = 2ab$ ...... (2)

b Now $ab^2 = \frac{1}{2}$ {from (2)}

$ab^2 = \ln b$ {from (1)}

$\therefore$ $\ln b = \frac{1}{2}$

$\therefore$ $b = e^{\frac{1}{2}}$ i.e., $b = \sqrt{e}$ and when $x = b = \sqrt{e}$, $y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$

$\therefore$ point of contact is $(\sqrt{e}, \frac{1}{2})$

c $a = \dfrac{1}{2b^2}$ {from (2)} $\quad \therefore \quad a = \dfrac{1}{2(\sqrt{e})^2} = \dfrac{1}{2e}$

d The tangent has slope $2ab = 2\left(\dfrac{1}{2e}\right)\sqrt{e} = \dfrac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$\therefore$ tangent is $\dfrac{y-\frac{1}{2}}{x-\sqrt{e}} = \dfrac{1}{\sqrt{e}}$ $\quad \therefore \quad y - \frac{1}{2} = \dfrac{1}{\sqrt{e}}\left(x - \sqrt{e}\right)$

$\therefore \quad y - \frac{1}{2} = \dfrac{1}{\sqrt{e}}x - 1$ $\quad$ i.e., $y = xe^{-\frac{1}{2}} - \frac{1}{2}$

20
$$P(t) = \frac{50\,000}{1+1000e^{-0.5t}}, \quad 0 \leqslant t \leqslant 25$$
$$= 50\,000(1+1000e^{-0.5t})^{-1}$$

$$\therefore \quad P'(t) = -50\,000(1+1000e^{-0.5t})^{-2}\left(-500e^{-0.5t}\right)$$
$$= 2.5\times10^7 e^{-0.5t}(1+1000e^{-0.5t})^{-2}$$

Now the wasp population is growing the fastest when $\dfrac{dP}{dt}$ is a maximum.

Using technology, the graph of $P'(t)$ can be drawn and the maximum obtained.

Maximum occurs when $x = 13.8155$, i.e., $x \doteqdot 13.8$ weeks

21 $f(t) = ate^{bt^2} = (at)\,e^{bt^2}$

$\therefore\quad f'(t) = ae^{bt^2} + ate^{bt^2}(2bt) = ae^{bt^2}(1+2bt^2)$ {product rule}

$\therefore\quad f'(t)$ is 0 when $1+2bt^2=0$ but $t=2$ at this point,

$$\therefore\quad 1+8b = 0$$
$$b = -\tfrac{1}{8}$$

$\therefore\quad f(t) = ate^{-\frac{t^2}{8}}$ and $f(2)=1$, $\quad\therefore\quad 2ae^{-\frac{4}{8}} = 1$

$\therefore\quad ae^{-\frac{1}{2}} = \frac{1}{2}$ and so $a = \dfrac{\sqrt{e}}{2}$

REVIEW SET 23A

1 **a**

$$\begin{aligned} y &= e^{x^3+2} \\ &= e^u \quad \text{where } u = x^3+2 \\ \therefore\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= e^u \times 3x^2 \\ &= 3x^2e^u \\ &= 3x^2e^{x^3+2} \end{aligned}$$

b

$$\begin{aligned} y &= \frac{e^x}{x^2} = \frac{u}{v} \\ \text{Now}\quad \frac{dy}{dx} &= \frac{u'v-uv'}{v^2} \\ \therefore\quad \frac{dy}{dx} &= \frac{e^x\,x^2 - e^x\,2x}{x^4} \\ &= \frac{xe^x(x-2)}{x^4} \\ &= \frac{e^x(x-2)}{x^3} \end{aligned}$$

c

$$x^3 + xy^4 = xe^y$$

Differentiating with respect to x,

$$\begin{aligned} \frac{d}{dx}(x^3) + \frac{d}{dx}(xy^4) &= \frac{d}{dx}(xe^y) \\ \therefore\quad 3x^2 + \frac{d}{dx}(x)y^4 + x\frac{d}{dx}(y^4) &= \frac{d}{dx}(x)e^y + x\frac{d}{dx}(e^y) \\ \therefore\quad 3x^2 + y^4 + 4xy^3\frac{dy}{dx} &= e^y + xe^y\frac{dy}{dx} \\ \therefore\quad \frac{dy}{dx}(4xy^3 - xe^y) &= e^y - 3x^2 - y^4 \\ \therefore\quad \frac{dy}{dx} &= \frac{e^y - 3x^2 - y^4}{4xy^3 - xe^y} \end{aligned}$$

2 $y = e^{-x^2}$ and when $x=1$, $y = e^{-x^2} = e^{-1} = \dfrac{1}{e}$, $\therefore$ point of contact is $\left(1, \dfrac{1}{e}\right)$

$$\text{Now}\quad \frac{dy}{dx} = -2xe^{-x^2}$$

$$\therefore\quad \text{at}\quad x=1,\quad \frac{dy}{dx} = -2e^{-1}$$

$\therefore$ the slope of the tangent $= -\dfrac{2}{e}$ and the slope of the normal $= \dfrac{e}{2}$

$\therefore$ equation of the normal is $\dfrac{y-\frac{1}{e}}{x-1} = \dfrac{e}{2}$ i.e., $2\left(y - \dfrac{1}{e}\right) = e(x-1)$

$$\begin{aligned} \therefore\quad 2y - \frac{2}{e} &= ex - e \\ \therefore\quad 2ey - 2 &= e^2x - e^2 \\ \therefore\quad e^2x - 2ey &= e^2 - 2 \quad \text{or} \quad y = \frac{e}{2}x - \frac{e}{2} + \frac{1}{e} \end{aligned}$$

3 Graphs meet when

$$\begin{aligned} e^x + 3 &= 9 - e^{-x} \\ \therefore\quad e^{2x} + 3e^x &= 9e^x - 1 \quad \{\times e^x\} \\ \therefore\quad e^{2x} - 6e^x + 1 &= 0 \\ \therefore\quad e^x &= \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2} \\ \therefore\quad x &= \ln(3+2\sqrt{2}) \text{ or } \ln(3-2\sqrt{2}) \end{aligned}$$

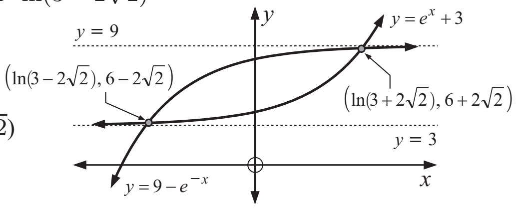
and $y = e^{\ln(3+2\sqrt{2})} + 3 = 6 + 2\sqrt{2}$ or

$y = e^{\ln(3-2\sqrt{2})} + 3 = 6 - 2\sqrt{2}$

i.e., graphs meet at $(\ln(3+2\sqrt{2}),\ 6+2\sqrt{2})$

and $(\ln(3-2\sqrt{2}),\ 6-2\sqrt{2})$

i.e., (1.76, 8.83) and (−1.76, 3.17)



4 **a** $f(x) = \dfrac{e^x}{x-1}$ $\therefore$ when $x = 0$, $f(x) = \dfrac{e^0}{-1} = -1$ and so the y-intercept is at (0, −1)

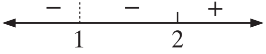
no x-intercept exist as $\dfrac{e^x}{x-1} \neq 0$

b $f(x)$ is defined for all $x \neq 1$

c Now

$$\begin{aligned} f'(x) &= \frac{e^x(x-1) - e^x(1)}{(x-1)^2} \quad \{\text{quotient rule}\} \\ &= \frac{e^x(x-2)}{(x-1)^2} \end{aligned}$$

and has sign diagram:



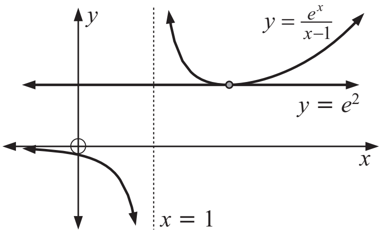
$\therefore$ $f(x)$ is decreasing for $x < 1$ and $1 < x \leqslant 2$, and increasing for $x \geqslant 2$.

$$\begin{aligned} f''(x) &= \frac{[e^x(x-2) + e^x\,(1)](x-1)^2 - e^x(x-2)[2(x-1)^1]}{(x-1)^4} \quad \{\text{product and quotient rules}\} \\ &= \frac{[e^x(x-2+1)](x-1)^2] \ - \ 2e^x(x-2)(x-1)}{(x-1)^4} \\ &= \frac{e^x(x-1)(x-1)^2 \ - \ 2e^x(x-2)(x-1)}{(x-1)^4} \\ &= \frac{e^x(x-1)\left[(x-1)^2 \ - \ 2(x-2)\right]}{(x-1)^4} \\ &= \frac{e^x(x-1)\left[x^2 - 2x + 1 - 2x + 4\right]}{(x-1)^4} \\ &= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3} \quad \text{where the quadratic has } \Delta < 0 \end{aligned}$$

The sign diagram of $f''(x)$ is:

$\therefore$ concave down for all $x \leqslant 1$
and concave up for all $x \geqslant 1$.

d



e Using **c** we have a local minimum at

$(2,\ e^2)$ $\quad\{\text{as } f(2) = \dfrac{e^2}{2-1} = e^2\}$

$\therefore$ the tangent at $x = 2$ is horizontal and is $y = e^2$.

5 $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$

a when planted $t = 0$ $\quad \therefore \quad H(0) = 60 + 40\ln(1) = 60 + 40(0) = 60$ cm

b **i** When $H(t) = 150$ cm,

$$\begin{aligned} \therefore \quad 60 + 40\ln(2t+1) &= 150 \\ \therefore \quad 40\ln(2t+1) &= 90 \\ \therefore \quad \ln(2t+1) &= \tfrac{90}{40} = 2.25 \\ \therefore \quad 2t+1 &= e^{2.25} \\ \therefore \quad 2t &= e^{2.25} - 1 \\ \therefore \quad t &= \tfrac{1}{2}(e^{2.25} - 1) \\ \therefore \quad t &\doteqdot 4.244 \text{ years} \end{aligned}$$

ii When $H(t) = 300$ cm,

$$\begin{aligned} \therefore \quad 60 + 40\ln(2t+1) &= 300 \\ \therefore \quad 40\ln(2t+1) &= 240 \\ \therefore \quad \ln(2t+1) &= 6 \\ \therefore \quad 2t+1 &= e^{6} \\ \therefore \quad 2t &= e^{6} - 1 \\ \therefore \quad t &= \tfrac{1}{2}(e^{6} - 1) \\ \therefore \quad t &\doteqdot 201.2 \text{ years} \end{aligned}$$

c $H'(t) = 40\left(\dfrac{2}{2t+1}\right) = \dfrac{80}{2t+1}$ cm/year

i When $t = 2$, $H'(2) = \frac{80}{5} = 16$ cm/year

ii When $t = 20$, $H'(20) = \frac{80}{41} \doteqdot 1.95$ cm/year

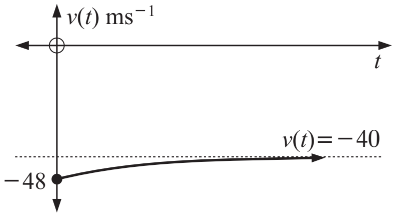
6 $s(t) = 80e^{-\frac{t}{10}} - 40t$ metres, $t \geqslant 0$

a $\therefore \quad v(t) = -8e^{-\frac{t}{10}} - 40$ m/sec and $a(t) = 0.8e^{-\frac{t}{10}}$ m/sec^2

b When $t = 0$, $s(0) = 80$ m $\quad v(0) = -48$ m/sec $\quad a(0) = 0.8$ m/sec^2

c as $t \to \infty$, $e^{-\frac{t}{10}} \to 0$, $\therefore \quad v(t) \to -40$ m/sec (below)

d



e when $v(t) = -44$ m/sec

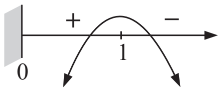
$$\begin{aligned} -8e^{-\frac{t}{10}} - 40 &= -44 \\ \therefore \quad -8e^{-\frac{t}{10}} &= -4 \\ \therefore \quad e^{-\frac{t}{10}} &= 0.5 \\ \therefore \quad -\tfrac{t}{10} &= \ln 0.5 \\ \therefore \quad t &= -10\ln 0.5 \\ \therefore \quad t &\doteqdot 6.93 \text{ seconds} \end{aligned}$$

7 Let the coordinates of B be $(x, 0)$

$\therefore$ the coordinates of A are (x, e^{-x})

$$\begin{aligned} \therefore \quad \text{area OABC} &= xe^{-x} \\ \therefore \quad \frac{dA}{dx} &= (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\} \\ &= e^{-x}(1-x) \\ &= \frac{1-x}{e^{x}} \end{aligned}$$

and has sign diagram:



$\therefore$ has a local maximum when $x = 1$

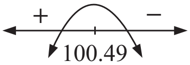
and when $x = 1$, $y = 1e^{-1} = \dfrac{1}{e}$

$\therefore$ the coordinates of A are $\left(1, \dfrac{1}{e}\right)$

8
$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= \left[200\ln\left(1+\tfrac{x}{100}\right)+1000\right] - \left[(x-100)^2+200\right]\\ &= 200\ln(1+0.01x)-(x-100)^2+800\\ \frac{dP}{dx} &= 200\left(\frac{0.01}{1+0.01x}\right)-2(x-100)^1\\ &= \frac{2}{1+0.01x}-\frac{2(x-100)}{1}\\ &= \frac{2-2(x-100)(1+0.01x)}{1+0.01x}\\ &= \frac{2-2(x+0.01x^2-100-x)}{1+0.01x}\\ &= \frac{2-0.02x^2+200}{1+0.01x}\\ &= \frac{202-0.02x^2}{1+0.01x}\end{aligned}$$

which is 0 when
$$\begin{aligned}0.02x^2 &= 202\\ \therefore\ x^2 &= 10\,100\\ \therefore\ x &= \sqrt{10\,100}\quad \{\text{as } x>0\}\\ \therefore\ x &\doteqdot 100.49\end{aligned}$$

and sign diagram of $\dfrac{dP}{dx}$ is:



$\therefore$ maximum profit when $x \doteqdot 100.49$

Now $P(100) = \$938.63$ and $P(101) = \$938.63$

$\therefore$ maximum profit of $938.63 when 100 or 101 shirts are made.

REVIEW SET 23B

1 **a**
$$\begin{aligned}y &= \ln(x^3-3x)\\ \therefore\ y &= \ln u \quad\text{where}\quad u = x^3-3x\end{aligned}$$

Now
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \frac{1}{u}(3x^2-3)\\ \therefore\ \frac{dy}{dx} &= \frac{3x^2-3}{x^3-3x}\end{aligned}$$

b
$$\begin{aligned}y &= \ln\left(\frac{x+3}{x^2}\right)\\ &= \ln(x+3)-\ln x^2\\ &= \ln(x+3)-2\ln x\\ \therefore\ \frac{dy}{dx} &= \frac{1}{x+3}-\frac{2}{x}\end{aligned}$$

c
$$\begin{aligned}e^{x+y} &= xy^2\\ \therefore\ e^xe^y &= xy^2\end{aligned}$$

Differentiating with respect to x,

$$\begin{aligned}\frac{d}{dx}(e^xe^y) &= \frac{d}{dx}(xy^2)\\ \therefore\ \frac{d}{dx}(e^x)e^y+e^x\frac{d}{dx}(e^y) &= \frac{d}{dx}(x)y^2+x\frac{d}{dx}(y^2)\quad\{\text{product rule}\}\\ \therefore\ e^xe^y+e^xe^y\frac{dy}{dx} &= y^2+2xy\frac{dy}{dx}\\ \therefore\ e^{x+y}\left(1+\frac{dy}{dx}\right) &= y^2+2xy\frac{dy}{dx}\end{aligned}$$

2
$$y = \ln(x^2+3)$$
$$\therefore\ y = \ln u \quad \text{where} \quad u = x^2+3$$
Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{u} \times 2x$

$\therefore\ \dfrac{dy}{dx} = \dfrac{2x}{x^2+3}$ and when $x = 0$, $\dfrac{dy}{dx} = 0$, i.e., the slope of the tangent is 0.

But at $x = 0$, $y = \ln(0+3) = \ln 3$

$\therefore$ tangent is $y = \ln 3$ which does not cut the x-axis.

3 **a**
$$\begin{aligned} e^{2x} &= 3e^x \\ \therefore\ e^{2x} - 3e^x &= 0 \\ \therefore\ e^x(e^x - 3) &= 0 \\ \therefore\ e^x &= 0 \text{ or } 3 \\ \therefore\ e^x &= 3 \quad \{\text{as } e^x > 0\} \\ \therefore\ x &= \ln 3 \end{aligned}$$

b
$$\begin{aligned} e^{2x} - 7e^x + 12 &= 0 \\ \therefore\ (e^x - 3)(e^x - 4) &= 0 \\ \therefore\ e^x &= 3 \text{ or } 4 \\ \therefore\ x &= \ln 3 \text{ or } \ln 4 \end{aligned}$$

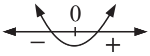
4 $f(x) = e^x - x$

a $f'(x) = e^x - 1$

which is 0 when $e^x = 1$

i.e., $x = 0$

Sign diagram of $f'(x)$ is:



and at $x = 0$, $f(0) = e^0 - 0 = 1$

$\therefore$ there is a local minimum at (0, 1)

b as $x \to +\infty$, $e^x \to \infty$ faster than x $\therefore$ $f(x) \to +\infty$,

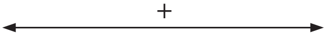
as $x \to -\infty$, $e^x \to 0$

$\therefore$ $f(x) \to -x$ (from above)

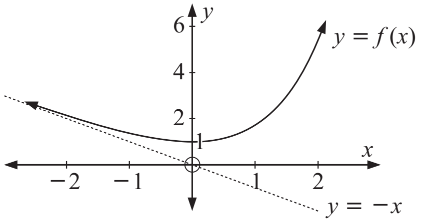
c $f''(x) = e^x$

$\therefore$ $f''(x) > 0$ for all x

$\therefore$ $f(x)$ is concave up for all x



d



e Since a local minimum exists at (0, 1), $f(x) \geqslant 1$ for all x.
$$\therefore\ e^x - x \geqslant 1$$
$$\therefore\ e^x \geqslant x + 1 \quad \text{for all } x$$

5 **a** $f(x) = \ln(e^x + 3)$
$$\begin{aligned} \therefore\ y &= \ln u \quad \text{where} \quad u = e^x + 3 \\ \therefore\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u} \times e^x \\ &= \frac{e^x}{u} \\ \therefore\ f'(x) &= \frac{e^x}{e^x+3} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \ln\left(\frac{(x+2)^3}{x}\right) \\ &= \ln(x+2)^3 - \ln x \\ &= 3\ln(x+2) - \ln x \\ \therefore\ f'(x) &= \frac{3}{x+2} - \frac{1}{x} \\ &= \frac{3x - (x+2)}{x(x+2)} \\ \therefore\ f'(x) &= \frac{2x-2}{x(x+2)} \end{aligned}$$

6 **a**

$$\begin{aligned} 3e^x - 5 &= -2e^{-x} \\ \therefore \quad 3e^{2x} - 5e^x &= -2 \qquad \{\times \text{ by } e^x\} \\ \therefore \quad 3e^{2x} - 5e^x + 2 &= 0 \\ \therefore \quad (3e^x - 2)(e^x - 1) &= 0 \\ \therefore \quad e^x &= \tfrac{2}{3} \text{ or } 1 \\ \therefore \quad x &= \ln \tfrac{2}{3} \text{ or } 0 \end{aligned}$$

b

$$\begin{aligned} 2\ln x - 3\ln\left(\frac{1}{x}\right) &= 10 \\ \therefore \quad 2\ln x - 3\ln(x^{-1}) &= 10 \\ \therefore \quad 2\ln x + 3\ln x &= 10 \\ \therefore \quad 5\ln x &= 10 \\ \therefore \quad \ln x &= 2 \\ \therefore \quad x &= e^2 \end{aligned}$$

7 $s(t) = 25t - 10\ln t$ cm $\quad t \geqslant 1$

a

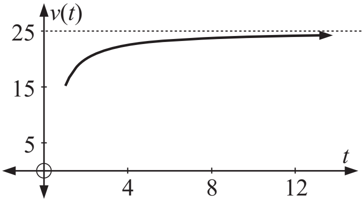
$$\begin{aligned} v(t) &= 25 - \tfrac{10}{t} \text{ cm/min} \\ \therefore \quad a(t) &= 10t^{-2} \\ &= \frac{10}{t^2} \text{ cm/min}^2 \end{aligned}$$

b when $t = e$,

$$\begin{aligned} s(e) &= 25e - 10\ln e = 25e - 10 \text{ cm} \doteqdot 57.96 \text{ cm} \\ v(e) &= 25 - \frac{10}{e} \text{ cm/min} \doteqdot 21.32 \text{ cm/min} \\ a(e) &= \frac{10}{e^2} \text{ cm/min}^2 \doteqdot 1.35 \text{ cm/min}^2 \end{aligned}$$

c as $t \to +\infty$, $\dfrac{10}{t} \to 0$, $\therefore$ $v(t) \to 25$ cm/min (below)

d



e when $v(t) = 12$ cm/min,

$$\begin{aligned} 25 - \frac{10}{t} &= 12 \\ \therefore \quad \frac{10}{t} &= 13 \\ \therefore \quad t &= \tfrac{10}{13} \text{ minutes} \end{aligned}$$

8

$$\begin{aligned} C(x) &= 10\ln x + \left(20 - \frac{x}{10}\right)^2 = 10\ln x + 400 - 4x + \frac{x^2}{100} \\ \therefore \quad C'(x) &= \frac{10}{x} - 4 + \frac{x}{50} \\ &= \frac{500 - 200x + x^2}{50x} \end{aligned}$$

which is 0 when $x^2 - 200x + 500 = 0$

i.e., when $x = \dfrac{200 \pm \sqrt{38\,000}}{2} \doteqdot 2.53$ or 197.47

But $x \geqslant 50$, $x \doteqdot 197.47$ and $C(197) \doteqdot 52.92$

$C(198) \doteqdot 52.92$

Now $C''(x) = -10x^{-2} + \frac{1}{50}$, $\therefore$ $C''(197.47) \doteqdot -0.1(197.47)^{-2} + 0.02 \doteqdot 0.02$ which is > 0

$\therefore$ minimum cost when $x \doteqdot 197.47$

i.e., need to produce 197 or 198 clocks/day

Chapter 24

DERIVATIVES OF CIRCULAR FUNCTIONS AND RELATED RATES

EXERCISE 24A

1 **a**

$$\begin{aligned} y &= \sin(2x) \\ \therefore \quad \frac{dy}{dx} &= \cos(2x)\frac{d}{dx}(2x) \\ &= 2\cos(2x) \end{aligned}$$

b

$$\begin{aligned} y &= \sin x + \cos x \\ \therefore \quad \frac{dy}{dx} &= \cos x - \sin x \end{aligned}$$

c

$$\begin{aligned} y &= \cos(3x) - \sin x \\ \therefore \quad \frac{dy}{dx} &= -\sin(3x) \times 3 - \cos x \\ &= -3\sin(3x) - \cos x \end{aligned}$$

d

$$\begin{aligned} y &= \sin(x+1) \\ \therefore \quad \frac{dy}{dx} &= \cos(x+1)\frac{d}{dx}(x+1) \\ &= 1\cos(x+1) \\ &= \cos(x+1) \end{aligned}$$

e

$$\begin{aligned} y &= \cos(3-2x) \\ \therefore \quad \frac{dy}{dx} &= -\sin(3-2x) \times -2 \\ &= 2\sin(3-2x) \end{aligned}$$

f

$$\begin{aligned} y &= \tan(5x) \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{\cos^2(5x)} \times 5 \\ &= \frac{5}{\cos^2(5x)} \end{aligned}$$

g

$$\begin{aligned} y &= \sin\left(\tfrac{x}{2}\right) - 3\cos x \\ \therefore \quad \frac{dy}{dx} &= \tfrac{1}{2}\cos\left(\tfrac{x}{2}\right) + 3\sin x \end{aligned}$$

h

$$\begin{aligned} y &= 3\tan(\pi x) \\ \therefore \quad \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(\pi x)} \times \pi \\ &= \frac{3\pi}{\cos^2(\pi x)} \end{aligned}$$

i

$$\begin{aligned} y &= 4\sin x - \cos(2x) \\ \therefore \quad \frac{dy}{dx} &= 4\cos x + \sin(2x) \times 2 \\ &= 4\cos x + 2\sin(2x) \end{aligned}$$

2 **a**

$$\begin{aligned} y &= x^2 + \cos x \\ \therefore \quad \frac{dy}{dx} &= 2x - \sin x \end{aligned}$$

b

$$\begin{aligned} y &= \tan x - 3\sin x \\ \therefore \quad \frac{dy}{dx} &= \frac{1}{\cos^2 x} - 3\cos x \end{aligned}$$

c

$$\begin{aligned} y &= e^x \cos x \\ \therefore \quad \frac{dy}{dx} &= e^x(-\sin x) + e^x \cos x \\ &= e^x \cos x - e^x \sin x \end{aligned}$$

d

$$\begin{aligned} y &= e^{-x} \sin x \\ \therefore \quad \frac{dy}{dx} &= -e^{-x} \sin x + e^{-x} \cos x \end{aligned}$$

e

$$\begin{aligned} y &= \ln(\sin x) \\ \therefore \quad \frac{dy}{dx} &= \frac{\cos x}{\sin x} \\ &= \cot x \end{aligned}$$

f

$$\begin{aligned} y &= e^{2x} \tan x \\ \therefore \quad \frac{dy}{dx} &= 2e^{2x} \tan x + e^{2x} \times \frac{1}{\cos^2 x} \\ \therefore \quad \frac{dy}{dx} &= 2e^{2x} \tan x + \frac{e^{2x}}{\cos^2 x} \end{aligned}$$

g

$$\begin{aligned} y &= \sin(3x) \\ \therefore \quad \frac{dy}{dx} &= 3\cos(3x) \end{aligned}$$

h

$$\begin{aligned} y &= \cos(\tfrac{x}{2}) \\ \therefore \quad \frac{dy}{dx} &= -\tfrac{1}{2}\sin(\tfrac{x}{2}) \end{aligned}$$

i
$$\begin{aligned} y &= 3\tan(2x) \\ \therefore\ \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(2x)} \times 2 \\ &= \frac{6}{\cos^2(2x)} \end{aligned}$$

j
$$\begin{aligned} y &= x\cos x \\ \therefore\ \frac{dy}{dx} &= 1 \times \cos x + x(-\sin x) \\ &= \cos x - x\sin x \end{aligned}$$

k
$$\begin{aligned} y &= \frac{\sin x}{x} \\ \therefore\ \frac{dy}{dx} &= \frac{x(\cos x) - \sin x \times 1}{x^2} \\ &= \frac{x\cos x - \sin x}{x^2} \end{aligned}$$

l
$$\begin{aligned} y &= x\tan x \\ \therefore\ \frac{dy}{dx} &= 1 \times \tan x + x \times \frac{1}{\cos^2 x} \\ \therefore\ \frac{dy}{dx} &= \tan x + \frac{x}{\cos^2 x} \end{aligned}$$

3 **a**
$$\begin{aligned} y &= \sin(x^2) \\ \therefore\ \frac{dy}{dx} &= 2x\cos(x^2) \end{aligned}$$

b
$$\begin{aligned} y &= \cos\left(\sqrt{x}\right) \\ &= \cos(x^{\frac{1}{2}}) \\ \therefore\ \frac{dy}{dx} &= -\sin(x^{\frac{1}{2}}) \times \tfrac{1}{2}x^{-\frac{1}{2}} \\ &= -\frac{1}{2\sqrt{x}}\sin(\sqrt{x}) \end{aligned}$$

c
$$\begin{aligned} y &= \sqrt{\cos x} \\ &= (\cos x)^{\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x) \\ &= -\frac{\sin x}{2\sqrt{\cos x}} \end{aligned}$$

d
$$\begin{aligned} y &= \sin^2 x \\ &= (\sin x)^2 \\ \therefore\ \frac{dy}{dx} &= 2\sin x\cos x \end{aligned}$$

e
$$\begin{aligned} y &= \cos^3 x \\ &= (\cos x)^3 \\ \therefore\ \frac{dy}{dx} &= 3\cos^2 x \times (-\sin x) \\ &= -3\sin x\cos^2 x \end{aligned}$$

f
$$\begin{aligned} y &= \cos x\sin(2x) \\ \therefore\ \frac{dy}{dx} &= (-\sin x)\sin(2x) + \cos x(2\cos(2x)) \\ &= -\sin x\sin(2x) + 2\cos x\cos(2x) \end{aligned}$$

g
$$\begin{aligned} y &= \cos(\cos x) \\ \therefore\ \frac{dy}{dx} &= -\sin(\cos x) \times (-\sin x) \\ &= \sin x\sin(\cos x) \end{aligned}$$

h
$$\begin{aligned} y &= \cos^3 4x \\ &= (\cos(4x))^3 \\ \therefore\ \frac{dy}{dx} &= 3(\cos(4x))^2 \times (-4\sin(4x)) \\ &= -12\sin(4x)\cos^2(4x) \end{aligned}$$

i
$$\begin{aligned} y &= \frac{1}{\sin x} \\ &= (\sin x)^{-1} \\ \therefore\ \frac{dy}{dx} &= -1(\sin x)^{-2} \times \cos x \\ &= -\frac{\cos x}{\sin^2 x} \end{aligned}$$

j
$$\begin{aligned} y &= \frac{1}{\cos(2x)} \\ &= (\cos(2x))^{-1} \\ \therefore\ \frac{dy}{dx} &= -1(\cos(2x))^{-2} \times (-2\sin(2x)) \\ &= \frac{2\sin(2x)}{\cos^2(2x)} \end{aligned}$$

k

$$\begin{aligned} y &= \frac{2}{\sin^2(2x)} \\ &= 2(\sin(2x))^{-2} \\ \therefore\ \frac{dy}{dx} &= -4(\sin(2x))^{-3} \times 2\cos(2x) \\ &= -\frac{8\cos(2x)}{\sin^3(2x)} \end{aligned}$$

l

$$\begin{aligned} y &= \frac{8}{\tan^3\left(\frac{x}{2}\right)} = 8\left[\tan\left(\tfrac{x}{2}\right)\right]^{-3} \\ \therefore\ \frac{dy}{dx} &= -24\left[\tan\left(\tfrac{x}{2}\right)\right]^{-4} \times \tfrac{1}{2} \times \frac{1}{\cos^2\left(\frac{x}{2}\right)} \\ &= \frac{-12}{\cos^2\left(\frac{x}{2}\right)\tan^4\left(\frac{x}{2}\right)} \end{aligned}$$

4 **a** If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$, $\dfrac{d^2y}{dx^2} = -\sin x$, $\dfrac{d^3y}{dx^3} = -\cos x$ and $\dfrac{d^4y}{dx^4} = \sin x$

b Successive derivatives will cycle through the pattern,

so $\dfrac{d^ny}{dx^n}$ may only take the values found in **a**.

$$\begin{tikzcd} & \sin x \arrow[dr] & \\ -\cos x \arrow[ur] & & \cos x \arrow[dl] \\ & -\sin x \arrow[ul] & \end{tikzcd}$$

5 **a** If $y = \sin(2x+3)$, then $\dfrac{dy}{dx} = 2\cos(2x+3)$ and $\dfrac{d^2y}{dx^2} = -4\sin(2x+3)$

$$\therefore\ \frac{d^2y}{dx^2} + 4y = -4\sin(2x+3) + 4\sin(2x+3) = 0$$

b

$$\begin{aligned} \text{If}\quad y &= 2\sin x + 3\cos x \\ \text{then}\quad y' &= 2\cos x - 3\sin x \\ \text{and}\quad y'' &= -2\sin x - 3\cos x \\ \therefore\ y'' + y &= -2\sin x - 3\cos x + 2\sin x + 3\cos x = 0 \end{aligned}$$

c

$$\begin{aligned} y &= \frac{\cos x}{1+\sin x} \\ \therefore\ \frac{dy}{dx} &= \frac{(-\sin x)(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\ &= \frac{-1-\sin x}{(1+\sin x)^2} \quad \{\text{as}\ \ \sin^2 x + \cos^2 x = 1\} \\ &= -\frac{(1+\sin x)}{(1+\sin x)^2} \\ &= \frac{-1}{1+\sin x} \end{aligned}$$

Since $\dfrac{-1}{1+\sin x}$ never equals 0, there are no horizontal tangents.

6 **a**

$$\begin{aligned} y &= \sin x \\ \therefore\ \frac{dy}{dx} &= \cos x \end{aligned}$$

$\therefore$ At $x = 0$, the tangent has slope $\cos 0 = 1$

$\therefore$ the equation is $\dfrac{y-0}{x-0} = 1$

i.e., $y = x$

b

$$\begin{aligned} y &= \tan x \\ \therefore\ \frac{dy}{dx} &= \frac{1}{\cos^2 x} \end{aligned}$$

$\therefore$ at $x = 0$, the tangent has slope $\dfrac{1}{\cos^2 0} = 1$

so the equation is $\dfrac{y-0}{x-0} = 1$

i.e., $y = x$

c $y = \cos x$ and at $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$

$\therefore \quad \dfrac{dy}{dx} = -\sin x$

At $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$ and the tangent has slope $-\sin\frac{\pi}{6} = -\frac{1}{2}$

$\therefore$ the normal has slope 2,

so its equation is $\dfrac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$

i.e. $y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3}$

i.e. $2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

d $y = \dfrac{1}{\sin(2x)} = (\sin(2x))^{-1}$

$$\therefore \quad \frac{dy}{dx} = -1(\sin(2x))^{-2} \times 2\cos(2x) = -\frac{2\cos(2x)}{(\sin(2x))^2}$$

At $x = \frac{\pi}{4}$, $y = 1$ and the tangent has slope $-\dfrac{2\cos\frac{\pi}{2}}{\left(\sin\frac{\pi}{2}\right)^2} = 0$

$\therefore$ the slope of the normal is undefined, so the normal is $x = \frac{\pi}{4}$.

7 $d = 9.3 + 6.8\cos(0.507t)$ m $\quad \therefore \quad \dfrac{dd}{dt} = -6.8\sin(0.507t) \times 0.507 = -3.4476\sin(0.507t)$

a When $t = 8$, $\dfrac{dd}{dt} \doteq 2.731 > 0$

$\therefore$ the tide is rising.

b When $t = 8$, the tide is rising at the rate of 2.731 m per hour.

8 **a** $V(t) = 340\sin(100\pi t)$

$$\therefore \quad \frac{dV}{dt} = 340\cos(100\pi t) \times 100\pi = 34\,000\pi\cos(100\pi t)$$

When $t = 0.01$,

$$\frac{dV}{dt} = 34\,000\pi \times \cos\pi = -34\,000\pi \text{ units/second}$$

b $V(t)$ is a maximum when $\sin(100\pi t) = 1$.

This occurs when $100\pi t = \frac{\pi}{2}$, and at this time,

$$\frac{dV}{dt} = 34\,000\pi\cos(100\pi t) = 34\,000\pi\cos(\tfrac{\pi}{2}) = 0 \text{ units/second}$$

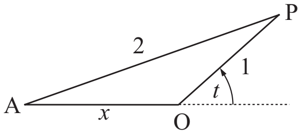
9 **a** The distance from A$(-x, 0)$ to P$(\cos t, \sin t)$ is fixed at 2 m.

$\therefore \quad (\cos t + x)^2 + \sin^2 t = 2^2$

$\therefore \quad (\cos t + x)^2 = 4 - \sin^2 t$

$\therefore \quad x + \cos t = \pm\sqrt{4 - \sin^2 t}$

$\therefore$ since $x > 0$, $x = \sqrt{4 - \sin^2 t} - \cos t$



b Now
$$\frac{dx}{dt} = \tfrac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2\sin t\cos t) + \sin t = \frac{-\sin t\cos t}{\sqrt{4 - \sin^2 t}} + \sin t$$

i When $t = 0$,

$\sin t = 0$ and $\cos t = 1$

$\therefore \quad \dfrac{dx}{dt} = 0 + 0 = 0$

ii When $t = \frac{\pi}{2}$,

$\sin t = 1$ and $\cos t = 0$

$\therefore \quad \dfrac{dx}{dt} = 0 + \sin\frac{\pi}{2} = 1$

iii When $t = \frac{2\pi}{3}$,

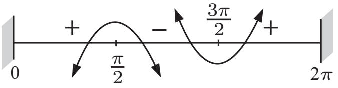
$\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = -\frac{1}{2}$

$$\therefore \quad \frac{dx}{dt} = \frac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2} \doteq 1.106$$

10 **a** If $y=\sin x$ then $\dfrac{dy}{dx}=\cos x$

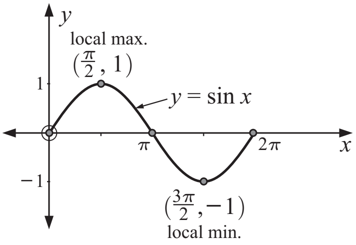
Stationary points occur when $\dfrac{dy}{dx}=0$, i.e., when $x=\frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $\dfrac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{2}, 1)$.

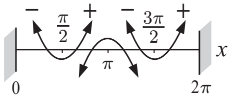
Local minimum at $(\frac{3\pi}{2}, -1)$.



b If $y=\cos(2x)$ then $\dfrac{dy}{dx}=-2\sin(2x)$

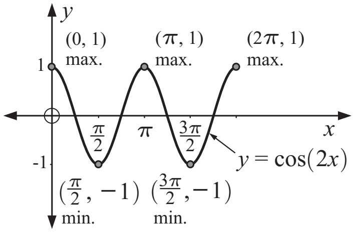
$$\begin{aligned}\frac{dy}{dx}=0 \quad \text{when} \quad -2\sin(2x) &= 0\\ \therefore \quad \sin(2x) &= 0\\ \therefore \quad 2x &= k\pi \quad \text{for any integer } k\\ \therefore \quad x &= \frac{k\pi}{2}\\ \therefore \quad x &= 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2} \text{ or } 2\pi \quad \text{for} \quad 0 \leqslant x \leqslant 2\pi\end{aligned}$$

Sign diagram for $\dfrac{dy}{dx}$ is:



Local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$.

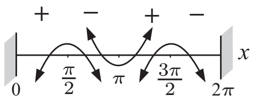
Local minima at $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{2}, -1)$.



c If $y=\sin^2 x$ then $\dfrac{dy}{dx}=2\sin x\cos x=\sin(2x)$

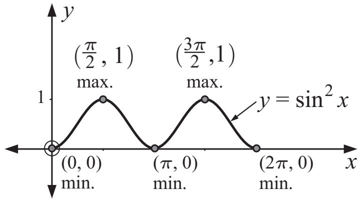
$\therefore \dfrac{dy}{dx}=0$ when $\sin(2x)=0$
i.e., when $x=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ or 2π for $0 \leqslant x \leqslant 2\pi$
{using **b**}

Sign diagram for $\dfrac{dy}{dx}$ is:



Local mimima at $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$.

Local maxima at $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.



11 **a** $f(x) = \dfrac{1}{\cos x}$ for $0 \leqslant x \leqslant 2\pi$

$\therefore$ $f(x)$ is undefined whenever

$\cos x = 0$

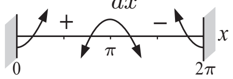
i.e., $x = \frac{\pi}{2}, \frac{3\pi}{2}$

b

$$\begin{aligned} f(x) &= (\cos x)^{-1} \\ \therefore\quad f'(x) &= -1(\cos x)^{-2}(-\sin x) \\ &= \frac{\sin x}{\cos^2 x} \\ \therefore\quad f'(x) &= 0 \quad \text{when } \sin x = 0 \end{aligned}$$

i.e., $x = 0, \pi, 2\pi$

Sign diagram for $\dfrac{dy}{dx}$ is:



Local minima at $(0, 1)$, $(2\pi, 1)$

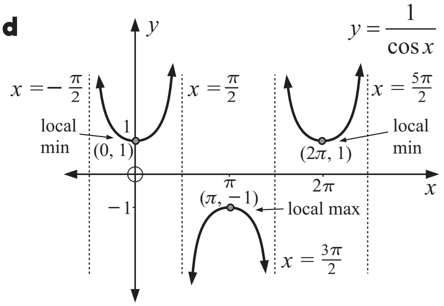
Local maximum at $(\pi, -1)$

c

$$\begin{aligned} f(x) &= \frac{1}{\cos x} \\ \therefore\quad f(x+2\pi) &= \frac{1}{\cos(x+2\pi)} \\ &= \frac{1}{\cos x} \\ &= f(x) \end{aligned}$$

i.e., $f(x)$ has a period of 2π.

d



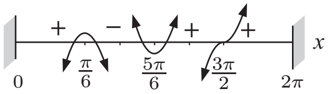
12 If $y = \sin(2x) + 2\cos x$

$$\begin{aligned} \text{then}\quad \frac{dy}{dx} &= 2\cos(2x) + (-2\sin x) \\ &= 2[1 - 2\sin^2 x] - 2\sin x \\ &= 2 - 4\sin^2 x - 2\sin x \end{aligned}$$

At the stationary points, $\dfrac{dy}{dx} = 0$.

$$\begin{aligned} \therefore\quad -4\sin^2 x - 2\sin x + 2 &= 0 \\ \therefore\quad 2\sin^2 x + \sin x - 1 &= 0 \\ \therefore\quad (2\sin x - 1)(\sin x + 1) &= 0 \\ \therefore\quad \sin x = \tfrac{1}{2} \text{ or } \sin x &= -1 \\ \therefore\quad x = \tfrac{\pi}{6}, \tfrac{5\pi}{6} \text{ or } x &= \tfrac{3\pi}{2} \end{aligned}$$

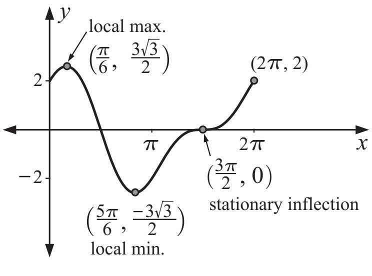
Sign diagram for $\dfrac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$.

Local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$.

Stationary inflection at $(\frac{3\pi}{2}, 0)$.



13 $x(t) = 1 - 2\cos t$ cm

$\therefore \quad v(t) = x'(t) = 2\sin t$ and $a(t) = v'(t) = 2\cos t$

a When $t = 0$

$$\begin{aligned} x(0) &= 1 - 2\cos(0) \\ &= -1 \text{ cm} \\ v(0) &= 2\sin(0) \\ &= 0 \text{ cm/s} \\ a(0) &= 2\cos(0) \\ &= 2 \text{ cm/s}^2 \end{aligned}$$

b When $t = \frac{\pi}{4}$

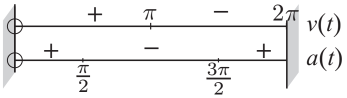
$$\begin{aligned} x(\tfrac{\pi}{4}) &= 1 - \tfrac{2}{\sqrt{2}} = 1 - \sqrt{2} \text{ cm} \\ v(\tfrac{\pi}{4}) &= \tfrac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s} \\ a(\tfrac{\pi}{4}) &= \tfrac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s}^2 \end{aligned}$$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm/s with increasing speed.

c The particle stops momentarily before it reverses direction. Therefore we need to look for the points where the velocity equals zero, i.e., when

$$\begin{aligned} v(t) = 2\sin t &= 0 \\ \therefore \quad \sin t &= 0 \\ \therefore \quad t &= 0,\ \pi,\ 2\pi \quad (0 \leqslant t \leqslant 2\pi) \end{aligned}$$

Sign diagrams for $v(t)$ and $a(t)$ are:



d The particle's speed is increasing when $v(t) = 2\sin t$ and $a(t) = 2\cos t$ have the same sign.

$$\begin{aligned} \text{Considering} \quad a(t) &= 0 \\ \text{i.e.,} \quad 2\cos t &= 0 \\ \cos t &= 0 \\ t &= \tfrac{\pi}{2}, \tfrac{3\pi}{2} \end{aligned}$$

i.e., $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

The particle reverses direction at $t = 0,\ \pi,\ 2\pi \quad (0 \leqslant t \leqslant 2\pi)$

At $t = 0$, $x(0) = -1$ cm At $t = \pi$, $x(\pi) = 3$ cm At $t = 2\pi$, $x(2\pi) = -1$ cm

EXERCISE 24B

1 **a** $y = (\cos x)^{-1} = \dfrac{1}{\cos x} = \sec x$

Now $\dfrac{dy}{dx} = -\dfrac{(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos x \times \cos x} = \dfrac{\tan x}{\cos x} = \sec x \tan x$

$\therefore \quad \dfrac{d}{dx}(\sec x) = \sec x \tan x$ as required.

b $y = \cot x = \dfrac{\cos x}{\sin x} \quad \therefore$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x)\sin x - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x} \qquad \{\text{quotient rate}\} \\ &= \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= -\csc^2 x \quad \text{as required} \end{aligned}$$

2 **a** $y = x \sec x \quad \therefore$

$$\begin{aligned} \frac{dy}{dx} &= \sec x + x \sec x \tan x \quad \{\text{product rule}\} \\ &= \sec x (x \tan x + 1) \end{aligned}$$

b $y = e^x \cot x \quad \therefore$

$$\begin{aligned} \frac{dy}{dx} &= e^x \cot x + e^x(-\csc^2 x) \quad \{\text{product rule}\} \\ &= e^x(\cot x - \csc^2 x) \end{aligned}$$

c $y = 4\sec(2x) \quad \therefore$

$$\begin{aligned} \frac{dy}{dx} &= 4\sec(2x)\tan(2x)(2) \\ &= 8\sec(2x)\tan(2x) \end{aligned}$$

d $y = e^{-x}\cot\left(\frac{x}{2}\right) \quad \therefore \quad \begin{aligned}[t] \frac{dy}{dx} &= -e^{-x}\cot\left(\tfrac{x}{2}\right) + e^{-x}\left(-\csc^2\left(\tfrac{x}{2}\right)\right)\left(\tfrac{1}{2}\right) \quad \{\text{product rule}\} \\ &= -e^{-x}\left(\cot\left(\tfrac{x}{2}\right) + \tfrac{1}{2}\csc^2\left(\tfrac{x}{2}\right)\right) \end{aligned}$

e $y = x^2\csc x \quad \therefore \quad \begin{aligned}[t] \frac{dy}{dx} &= 2x\csc x + x^2(-\csc x\cot x) \quad \{\text{product rule}\} \\ &= x\csc x\,(2 - x\cot x) \end{aligned}$

f $y = x\sqrt{\csc x} = x(\csc x)^{\frac{1}{2}} \quad \therefore \quad \begin{aligned}[t] \frac{dy}{dx} &= (\csc x)^{\frac{1}{2}} + \tfrac{1}{2}x(\csc x)^{-\frac{1}{2}}(-\csc x\cot x) \\ &= (\csc x)^{\frac{1}{2}} - \tfrac{1}{2}x(\csc x)^{\frac{1}{2}}\cot x \\ &= \sqrt{\csc x}\left(1 - \tfrac{1}{2}x\cot x\right) \end{aligned}$

g $y = \ln(\sec x) \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{\frac{d}{dx}(\sec x)}{\sec x} = \dfrac{\cancel{\sec x}\tan x}{\cancel{\sec x}} = \tan x$

h $y = x\csc(x^2) \quad \therefore \quad \begin{aligned}[t] \frac{dy}{dx} &= \csc(x^2) + x\left[-\csc(x^2)\cot(x^2)\right](2x) \\ &= \csc(x^2)(1 - 2x^2\cot(x^2)) \end{aligned}$

i $y = \dfrac{\cot x}{\sqrt{x}} = x^{-\frac{1}{2}}\cot x \quad \therefore \quad \begin{aligned}[t] \frac{dy}{dx} &= -\tfrac{1}{2}x^{-\frac{3}{2}}\cot x + x^{-\frac{1}{2}}(-\csc^2 x) \quad \{\text{product rule}\} \\ &= -\frac{\cot x + 2x\csc^2 x}{2x\sqrt{x}} \\ &= -\frac{\cos x\sin x + 2x}{2x\sqrt{x}\sin^2 x} \end{aligned}$

3 **a** $y = \sec x \quad \therefore \quad \dfrac{dy}{dx} = \sec x\tan x$

When $x = \frac{\pi}{4}$, $y = \sec\frac{\pi}{4} = \dfrac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$

and $\dfrac{dy}{dx} = \sec\frac{\pi}{4}\tan\frac{\pi}{4} = \sqrt{2}\times 1 = \sqrt{2}$

$\therefore$ the tangent has equation $\dfrac{y-\sqrt{2}}{x-\frac{\pi}{4}} = \sqrt{2} \quad \therefore \quad y - \sqrt{2} = x\sqrt{2} - \frac{\pi\sqrt{2}}{4}$

$\therefore \quad y = x\sqrt{2} - \frac{\pi\sqrt{2}}{4} + \sqrt{2}$

b $y = \cot\left(\frac{x}{2}\right) \quad \therefore \quad \dfrac{dy}{dx} = -\csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)$

When $x = \frac{\pi}{3}$, $\frac{x}{2} = \frac{\pi}{6} \quad \therefore \quad y = \cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$

and $\dfrac{dy}{dx} = -\dfrac{1}{2\sin^2\left(\frac{\pi}{6}\right)} = -\dfrac{1}{2\left(\frac{1}{2}\right)^2} = -2$

$\therefore$ the tangent has equation $\dfrac{y-\sqrt{3}}{x-\frac{\pi}{3}} = -2 \quad \therefore \quad y - \sqrt{3} = -2x + \frac{2\pi}{3}$

$\therefore \quad y = -2x + \frac{2\pi}{3} + \sqrt{3}$

4 **a** $y = \csc x \quad \therefore \quad \dfrac{dy}{dx} = -\csc x\cot x$

When $x = \frac{\pi}{6}$, $y = \dfrac{1}{\sin\frac{\pi}{6}} = 2$ and $\dfrac{dy}{dx} = -\csc\frac{\pi}{6}\cot\frac{\pi}{6} = -2\sqrt{3}$

$\therefore$ the normal has slope $\frac{1}{2\sqrt{3}}$

and its equation is $\dfrac{y-2}{x-\frac{\pi}{6}} = \dfrac{1}{2\sqrt{3}} \quad \therefore \quad 2\sqrt{3}y - 4\sqrt{3} = x - \frac{\pi}{6}$

$\therefore \quad x - 2\sqrt{3}y = \frac{\pi}{6} - 4\sqrt{3}$

b $y = \sqrt{\sec\left(\frac{x}{3}\right)} = \left(\sec\left(\frac{x}{3}\right)\right)^{\frac{1}{2}}$ $\quad\therefore\quad \dfrac{dy}{dx} = \frac{1}{2}\left(\sec\left(\frac{x}{3}\right)^{-\frac{1}{2}}\right)\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\left(\frac{1}{3}\right)$

$$= \frac{\sqrt{\sec\left(\frac{x}{3}\right)}\tan\left(\frac{x}{3}\right)}{6}$$

When $x = \pi$, $\dfrac{x}{3} = \dfrac{\pi}{3}$

$$\therefore \quad y = \frac{1}{\sqrt{\cos\left(\frac{\pi}{3}\right)}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$$

and $$\frac{dy}{dx} = \frac{\sqrt{\sec\left(\frac{\pi}{3}\right)}\tan\left(\frac{\pi}{3}\right)}{6} = \frac{\sqrt{2}\sqrt{3}}{6} = \frac{1}{\sqrt{6}}$$

$\therefore$ the normal has slope $-\sqrt{6}$

and its equation is $\dfrac{y-\sqrt{2}}{x-\pi} = -\sqrt{6}$ $\quad\therefore\quad y - \sqrt{2} = -x\sqrt{6} + \pi\sqrt{6}$

$$\therefore \quad \sqrt{6}x + y = \pi\sqrt{6} + \sqrt{2}$$

EXERCISE 24C.1

1 Refer to page **641** of the text.

2

a $\arccos(1) = 0$ **b** $\arcsin(-1) = -\frac{\pi}{2}$ **c** $\arctan(1) = \frac{\pi}{4}$

d $\arctan(-1) = -\frac{\pi}{4}$ **e** $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ **f** $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

g $\arctan(\sqrt{3}) = \frac{\pi}{3}$ **h** $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$ **i** $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

j $\sin^{-1}(-0.767) \doteqdot -0.874$ **k** $\cos^{-1}(0.327) \doteqdot 1.24$ **l** $\tan^{-1}(-50) \doteqdot -1.55$

3 **a** If $\arcsin x = \frac{\pi}{3}$ then $x = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

b If $\arctan(3x) = -\frac{\pi}{4}$ then $3x = \tan\left(-\frac{\pi}{4}\right) = -1$ $\quad\therefore\quad x = -\frac{1}{3}$

4 Now $$\tan\left(\arctan(5) - \arctan\left(\tfrac{2}{3}\right)\right) = \frac{\tan(\arctan(5)) - \tan\left(\arctan\left(\frac{2}{3}\right)\right)}{1 + \tan(\arctan(5))\tan\left(\arctan\left(\frac{2}{3}\right)\right)} \quad \{\text{using formula}\}$$

$$= \frac{5 - \frac{2}{3}}{1 + 5 \times \frac{2}{3}} = \frac{\frac{13}{3}}{\frac{13}{3}} = 1$$

But $\tan\left(\frac{\pi}{4}\right) = 1$, so $\arctan(5) - \arctan\left(\frac{2}{3}\right) = \frac{\pi}{4}$

$*$ Note there is an error in the question in early print runs of the text.

EXERCISE 24C.2

1 If $y = \arccos x$ then $x = \cos y$

$$\therefore \quad \frac{dy}{dx} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}, \quad x \in [-1, 1]$$

2 **a** $y = \arctan(2x)$

$$\therefore \quad \frac{dy}{dx} = 2 \times \frac{1}{1+(2x)^2} = \frac{2}{1+4x^2}$$

b $y = \arccos(3x)$

$$\therefore \quad \frac{dy}{dx} = 3 \times \left(\frac{-1}{\sqrt{1-(3x)^2}}\right) = -\frac{3}{\sqrt{1-9x^2}}$$

c

$$\begin{aligned} y &= \arcsin\left(\tfrac{x}{4}\right) \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{4} \times \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}} \\ &= \frac{1}{4\sqrt{1-\frac{x^2}{16}}} \\ &= \frac{1}{\sqrt{16-x^2}} \end{aligned}$$

d

$$\begin{aligned} y &= \arccos\left(\tfrac{x}{5}\right) \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{5} \times \left(\frac{-1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\right) \\ &= -\frac{1}{5\sqrt{1-\frac{x^2}{25}}} \\ &= -\frac{1}{\sqrt{25-x^2}} \end{aligned}$$

e

$$\begin{aligned} y &= \arctan(x^2) \\ \therefore\ \frac{dy}{dx} &= 2x \times \frac{1}{1+(x^2)^2} \\ &= \frac{2x}{1+x^4} \end{aligned}$$

f

$$\begin{aligned} y &= \arccos(\sin x) \\ \therefore\ \frac{dy}{dx} &= \cos x \times \left(-\frac{1}{\sqrt{1-\sin^2 x}}\right) \\ &= -\frac{\cos x}{\cos x} \\ &= -1 \end{aligned}$$

3 **a** $y = x\arcsin x$

$$\begin{aligned} \therefore\ \frac{dy}{dx} &= \arcsin x + x\left(\frac{1}{\sqrt{1-x^2}}\right) \quad \{\text{product rule}\} \\ &= \arcsin x + \frac{x}{\sqrt{1-x^2}} \end{aligned}$$

b $y = e^x \arccos x$

$$\begin{aligned} \therefore\ \frac{dy}{dx} &= e^x \arccos x + e^x\left(-\frac{1}{\sqrt{1-x^2}}\right) \quad \{\text{product rule}\} \\ &= e^x \arccos x - \frac{e^x}{\sqrt{1-x^2}} \end{aligned}$$

c $y = e^{-x}\arctan x$

$$\begin{aligned} \therefore\ \frac{dy}{dx} &= -e^{-x}\arctan x + e^{-x}\left(\frac{1}{1+x^2}\right) \quad \{\text{product rule}\} \\ &= -e^{-x}\arctan x + \frac{e^{-x}}{1+x^2} \end{aligned}$$

4 **a** If $y = \arcsin\left(\frac{x}{a}\right)$,

$$\begin{aligned} \text{then } \frac{dy}{dx} &= \left(\tfrac{1}{a}\right)\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \\ &= \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}} \\ &= \frac{1}{\sqrt{a^2-x^2}} \quad \text{as required,} \end{aligned}$$

and this is defined for $x \in [-a, a]$.

b If $y = \arctan\left(\frac{x}{a}\right)$,

$$\begin{aligned} \text{then } \frac{dy}{dx} &= \tfrac{1}{a}\left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right) \\ &= \tfrac{a}{a^2}\left(\frac{1}{1+\frac{x^2}{a^2}}\right) \\ &= \frac{a}{a^2+x^2} \quad \text{as required,} \end{aligned}$$

and this is defined for $x \in \mathcal{R}$.

c If $y = \arccos\left(\dfrac{x}{a}\right)$,

$$\begin{aligned} \text{then } \frac{dy}{dx} &= \frac{1}{a}\left(-\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right) = -\frac{1}{a\sqrt{1-\frac{x^2}{a^2}}} \\ &= -\frac{1}{\sqrt{a^2-x^2}} \end{aligned}$$

and this is defined for $x \in [-a, a]$.

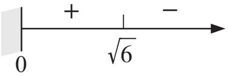
5 **a** $\tan\alpha = \dfrac{2}{x}$ and $\tan(\alpha+\theta) = \dfrac{3}{x}$

b Now $\theta = (\alpha+\theta) - \alpha = \arctan\left(\frac{3}{x}\right) - \arctan\left(\frac{2}{x}\right)$

c
$$\begin{aligned}\frac{d\theta}{dx} &= \left(-\frac{3}{x^2}\right)\left(\frac{1}{1+\left(\frac{3}{x}\right)^2}\right) - \left(-\frac{2}{x^2}\right)\left(\frac{1}{1+\left(\frac{2}{x}\right)^2}\right)\\ &= -\frac{3}{x^2+9} + \frac{2}{x^2+4}\\ &= \frac{2}{x^2+4} - \frac{3}{x^2+9} \quad \text{as required.}\end{aligned}$$

$$\begin{aligned}\frac{d\theta}{dx} = 0 \quad \text{when} \quad 2(x^2+9) - 3(x^2+4) &= 0\\ \therefore \quad 2x^2 + 18 - 3x^2 - 12 &= 0\\ \therefore \quad x^2 &= 6\\ \therefore \quad x &= \sqrt{6} \qquad \{x > 0\}\end{aligned}$$

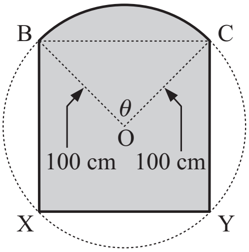
and $\dfrac{d\theta}{dx}$ has sign diagram:



d The maximum viewing angle occurs when $x = \sqrt{6}$, i.e., when Sonia is $\sqrt{6}$ m from the wall.

EXERCISE 24D

1



Using the cosine rule in ΔBCO,

$$\begin{aligned}\text{BC}^2 &= 10^2 + 10^2 - 2 \times 10 \times 10\cos\theta\\ \therefore \quad \text{BC} &= \sqrt{200 - 200\cos\theta}\\ \text{and XY} &= \text{BC}\\ \text{Now} \quad \text{BY}^2 &= \text{BX}^2 + \text{XY}^2 \qquad \{\text{Pythagoras}\}\\ \therefore \quad 400 &= \text{BX}^2 + (200 - 200\cos\theta)\\ \therefore \quad \text{BX}^2 &= 200 + 200\cos\theta\\ \therefore \quad \text{BX} &= \sqrt{200 + 200\cos\theta}\end{aligned}$$

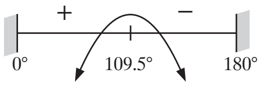
The shaded area is equal to the area of the sector plus $\frac{3}{4}$ of the area of BCYX

$$\begin{aligned}\therefore \quad A &= \tfrac{1}{2}(10)^2\,\theta + \tfrac{3}{4}[\text{BX} \times \text{BC}]\\ &= 50\theta + \tfrac{3}{4}\sqrt{200+200\cos\theta}\sqrt{200-200\cos\theta}\\ &= 50\theta + \tfrac{3}{4} \times 200\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}\\ &= 50\theta + 150\sqrt{1-\cos^2\theta}\\ &= 50\theta + 150\sin\theta\\ &= 50(\theta + 3\sin\theta) \quad \text{as required}\end{aligned}$$

$$\therefore \quad \frac{dA}{d\theta} = 50 + 150\cos\theta = 50(1 + 3\cos\theta),$$

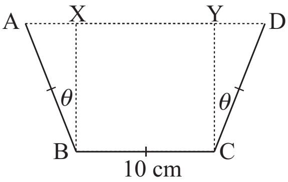
which is zero when $\cos\theta = -\frac{1}{3}$

The sign diagram of $\dfrac{dA}{d\theta}$ is:



$\therefore$ since $0 < \theta < 180$, max A is when $\theta \doteqdot 109.5^o$.

2 **a**



The triangles have height $10\cos\theta$ and width $10\sin\theta$.

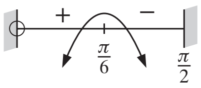
$\therefore$ area A
$= $ area of Δs + area of rectangle
$= 2\times\frac{1}{2}\times 10\cos\theta\times 10\sin\theta \ + \ 10\times 10\cos\theta$
$= 100\sin\theta\cos\theta + 100\cos\theta$
$= 100\cos\theta(1+\sin\theta)$

b
$$\begin{aligned}\frac{dA}{d\theta} &= -100(2\sin\theta-1)(\sin\theta+1)\\ &= 100(-\sin\theta(1+\sin\theta)+\cos\theta\times\cos\theta)\\ &= 100(-\sin\theta-\sin^2\theta+\cos^2\theta)\\ &= 100(-\sin\theta-\sin^2\theta+1-\sin^2\theta)\\ &= -100(2\sin^2\theta+\sin\theta-1)\\ &= -100(2\sin\theta-1)(\sin\theta+1)\end{aligned}$$

$\therefore \ \frac{dA}{d\theta}=0$ when $2\sin\theta-1 = 0$ *or* $\sin\theta+1=0$
i.e., $\sin\theta=\frac{1}{2}$ *or* $\sin\theta=-1$

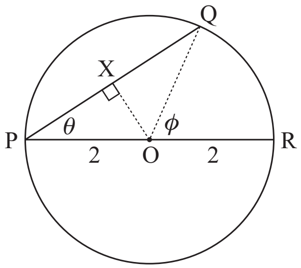
c Using **b**, $\frac{dA}{d\theta}=0$ when $\theta=\frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$.

Sign diagram for $\frac{dA}{d\theta}$ is:



so the maximum area is when $\theta=\frac{\pi}{6}=30^o$

3



$\frac{\text{PX}}{2}=\cos\theta, \ \therefore \ \text{PQ}=2\text{PX}=4\cos\theta$

$\therefore$ the time taken to row from P to Q is $\frac{4\cos\theta}{3}$ hours

Now $\phi=2\theta$ {angle at the centre}
But, arc length $\text{QR}_{\text{arc}}=2\phi$,
$\therefore \ \text{QR}_{\text{arc}}=4\theta$,

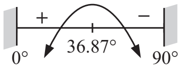
and the time taken to walk from Q to R is $\frac{4\theta}{5}$

$\therefore$ total time from P to R, $T=\frac{4}{3}\cos\theta+\frac{4\theta}{5}$

$\therefore \ \frac{dT}{d\theta}=-\frac{4}{3}\sin\theta+\frac{4}{5}$

$\therefore \ \frac{dT}{d\theta}=0$ when $-\frac{4}{3}\sin\theta=-\frac{4}{5}$
$\therefore \ \sin\theta=\frac{3}{5}$
$\therefore \ \theta \doteqdot 0.6435$ radians
i.e., $\theta \doteqdot 36.87^o$

sign diagram of $\frac{dT}{d\theta}$ is:

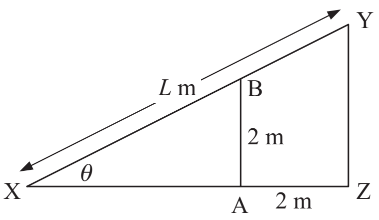


i.e., the maximum time is when $\theta \doteqdot 36.87^o$

and the maximum time is $\frac{4}{3}\cos 0.6435+\frac{4}{5}\times 0.6435$
$\doteqdot 1.581$ hours
$\doteqdot 1$ hour 34 min 53 sec

4 **a** $\tan\theta = \dfrac{2}{\text{AX}}$ and $\sin\theta = \dfrac{2}{\text{BX}}$

$\therefore\ \text{AX} = \dfrac{2}{\tan\theta} = \dfrac{2\cos\theta}{\sin\theta}$ and $\text{BX} = \dfrac{2}{\sin\theta}$



From the similar Δ's, $\dfrac{L}{\text{BX}} = \dfrac{\text{AX}+2}{\text{AX}} = 1 + \dfrac{2}{\text{AX}}$

$$\therefore\ L = \text{BX} + \frac{2\text{BX}}{\text{AX}}$$

$$\therefore\ L = \frac{2}{\sin\theta} + \frac{2\left(\dfrac{2}{\sin\theta}\right)}{\left(\dfrac{2\cos\theta}{\sin\theta}\right)}$$

$$\therefore\ L = \frac{2}{\sin\theta} + 2\left(\frac{2}{\sin\theta}\right)\left(\frac{\sin\theta}{2\cos\theta}\right)$$

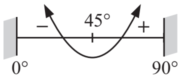
$$\therefore\ L = \frac{2}{\sin\theta} + \frac{2}{\cos\theta}, \text{ as required.}$$

b Now $L = 2(\cos\theta)^{-1} + 2(\sin\theta)^{-1}$

$$\begin{aligned}\therefore\ \frac{dL}{d\theta} &= -2(\cos\theta)^{-2}(-\sin\theta) + -2(\sin\theta)^{-2}(\cos\theta)\\ &= \frac{2\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}\\ &= \frac{2\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta}\end{aligned}$$

c Now $\dfrac{dL}{d\theta} = 0$ when $2\sin^3\theta - 2\cos^3\theta = 0$

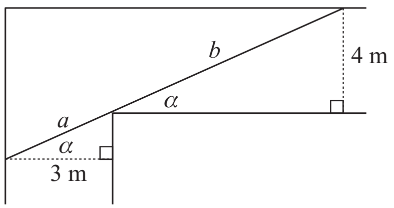
$$\begin{aligned}\therefore\ 2\sin^3\theta &= 2\cos^3\theta\\ \therefore\ \tan^3\theta &= 1\\ \therefore\ \tan\theta &= 1\\ \therefore\ \theta &= 45^o \quad \text{since} \quad 0 < \theta < 90^o\end{aligned}$$

Sign diagram of $\dfrac{dL}{d\theta}$ is: 

$\therefore$ the shortest ladder is required when $\theta = 45^o$

and $L_{\min} = \dfrac{2}{\frac{1}{\sqrt{2}}} + \dfrac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ m

5



$\cos\alpha = \dfrac{3}{a}$ and $\sin\alpha = \dfrac{4}{b}$

$\therefore\ a = \dfrac{3}{\cos\alpha}$ and $b = \dfrac{4}{\sin\alpha}$

Now $L = a + b$

$$\therefore\ L = \frac{3}{\cos\alpha} + \frac{4}{\sin\alpha}$$

$$\therefore\ L = 3(\cos\alpha)^{-1} + 4(\sin\alpha)^{-1}$$

$$\begin{aligned}\therefore\ \frac{dL}{d\alpha} &= -3(\cos\alpha)^{-2}(-\sin\alpha) + 4(-(\sin\alpha)^{-2})\cos\alpha\\ &= \frac{3\sin\alpha}{\cos^2\alpha} - \frac{4\cos\alpha}{\sin^2\alpha}\\ &= \frac{3\sin^3\alpha - 4\cos^3\alpha}{\cos^2\alpha\sin^2\alpha}\end{aligned}$$

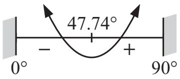
$$\therefore \quad \frac{dL}{d\alpha} = 0 \quad \text{when} \quad 3\sin^3\alpha - 4\cos^3\alpha = 0$$

$$\text{i.e.,} \quad 3\sin^3\alpha = 4\cos^3\alpha$$

$$\tan^3\alpha = \tfrac{4}{3}$$

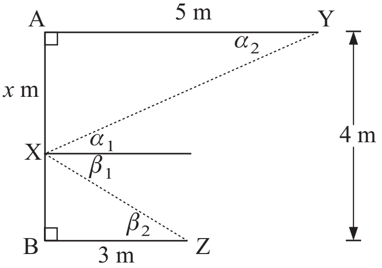
$$\therefore \quad \tan\alpha = \sqrt[3]{\tfrac{4}{3}} \quad \text{and so} \quad \alpha \doteqdot 47.74^o$$

Sign diagram of $\dfrac{dL}{d\alpha}$ is:



$\therefore$ AB is minimised when $\alpha = 47.74^o$ and $L = \dfrac{3}{\cos\alpha} + \dfrac{4}{\sin\alpha} \doteqdot 9.866$ m

6



Now $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {alternate angles}

$$\therefore \quad \theta = \alpha + \beta$$

Let $\text{AX} = x$ m

$$\therefore \quad \text{XB} = (4 - x) \text{ m}$$

$$\therefore \quad \tan\alpha = \frac{x}{5} \quad \therefore \quad x = 5\tan\alpha$$

$$\text{and} \quad \tan\beta = \frac{4-x}{3}$$

$$\text{Now} \quad \tan\theta = \tan(\alpha+\beta)$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\text{i.e.,} \quad \tan\theta = \frac{\frac{x}{5} + \frac{4-x}{3}}{1 - \frac{x}{5}\left(\frac{4-x}{3}\right)}$$

$$\text{i.e.,} \quad \tan\theta = \frac{3x + 20 - 5x}{15 - x(4-x)} \qquad \{\text{multiplying top and bottom by 15}\}$$

$$\therefore \quad \tan\theta = \frac{20 - 2x}{x^2 - 4x + 15}$$

Differentiating both sides with respect to x

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dx} = \frac{-2(x^2 - 4x + 15) - (20 - 2x)(2x - 4)}{(x^2 - 4x + 15)^2} \qquad \{\text{Chain and Quotient rules}\}$$

$$\therefore \quad \frac{1}{\cos^2\theta}\frac{d\theta}{dx} = \frac{[-2x^2 + 8x - 30 - 40x + 80 + 4x^2 - 8x]}{(x^2 - 4x + 15)^2}$$

$$\therefore \quad \frac{1}{\cos^2\theta}\frac{d\theta}{dx} = \frac{[2x^2 - 40x + 50]}{(x^2 - 4x + 15)^2}$$

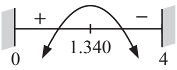
$$\therefore \quad \frac{d\theta}{dx} = 2\cos^2\theta \frac{[x^2 - 20x + 25]}{(x^2 - 4x + 15)^2}$$

Now by inspection, $\theta < 90^o$, so $\dfrac{d\theta}{dx} = 0$ when $x^2 - 20x + 25 = 0$

$\therefore$ $x \doteqdot 1.3397$ or $x \doteqdot 18.660$ (where 18.660 is not physically possible)

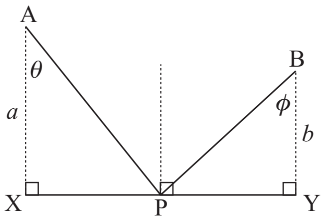
$\therefore$ $x \doteqdot 1.340$ m from A

Sign diagram for $\dfrac{d\theta}{dx}$ is:



$\therefore$ θ is a maximum when $x \doteqdot 1.340$ m from A

7 **a**



$$\frac{a}{\text{AP}} = \cos\theta \quad \text{and} \quad \frac{b}{\text{BP}} = \cos\phi$$

$$\text{AP} = \frac{a}{\cos\theta} \quad \text{and} \quad \text{BP} = \frac{b}{\cos\phi}$$

Now $L = \text{AP} + \text{BP}$

$$\therefore \quad L = \frac{a}{\cos\theta} + \frac{b}{\cos\phi}$$

b

$$L = a(\cos\theta)^{-1} + b(\cos\phi)^{-1}$$

$$\frac{dL}{d\theta} = -a(\cos\theta)^{-2}(-\sin\theta) - b(\cos\phi)^{-2}(-\sin\phi)\frac{d\phi}{d\theta}$$

$$= \frac{a\sin\theta}{\cos^2\theta} + \frac{b\sin\phi}{\cos^2\phi}\frac{d\phi}{d\theta}, \quad \text{as required.}$$

c Now $\dfrac{\text{XP}}{a} = \tan\theta$ and $\dfrac{\text{YP}}{b} = \tan\phi$

$\therefore$ $\text{XP} = a\tan\theta$ and $\text{YP} = b\tan\phi$

$\therefore$ $\text{XY} = a\tan\theta + b\tan\phi$

But XY is a fixed distance, so $a\tan\theta + b\tan\phi$ is a constant,

i.e., $a\tan\theta + b\tan\phi = c$

$$\therefore \quad \frac{a}{\cos^2\theta} + \frac{b}{\cos^2\phi}\frac{d\phi}{d\theta} = 0 \qquad \{\text{differentiating with respect to } \theta\}$$

$$\therefore \quad \frac{b}{\cos^2\phi}\frac{d\phi}{d\theta} = -\frac{a}{\cos^2\theta}$$

$$\therefore \quad \frac{d\phi}{d\theta} = -\frac{a\cos^2\phi}{b\cos^2\theta}, \quad \text{as required.}$$

d

$$\frac{dL}{d\theta} = \frac{a\sin\theta}{\cos^2\theta} + \frac{b\sin\phi}{\cos^2\phi}\left(\frac{-a\cos^2\phi}{b\cos^2\theta}\right) \qquad \{\text{using } \textbf{b} \text{ and } \textbf{c}\}$$

$$= \frac{a\sin\theta}{\cos^2\theta} - \frac{a\sin\phi}{\cos^2\theta}$$

$$= \frac{a(\sin\theta - \sin\phi)}{\cos^2\theta}$$

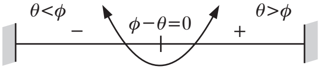
$$\therefore \quad \frac{dL}{d\theta} = 0 \quad \text{when} \quad \sin\theta - \sin\phi = 0$$

i.e., when $\sin\phi = \sin\theta$

e Since $\dfrac{dL}{d\theta} = 0$ when $\sin\phi = \sin\theta$,

$L = \text{AP} + \text{PB}$ is either a maximum or minimum when $\phi = \theta$

Sign diagram of $\dfrac{dL}{d\theta}$ is:



$\therefore$ $\text{AP} + \text{PB}$ is a minimum when $\theta - \phi = 0$

i.e., when $\theta = \phi$

$\therefore$ it will be cheapest for the pump house to be located at the point such that $\theta = \phi$.

EXERCISE 24E

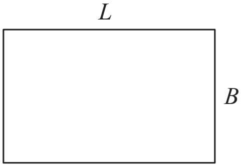
1 $ab^3 = 40, \quad \therefore \quad \frac{da}{dt}b^3 + a(3b^2)\frac{db}{dt} = 0$

Particular case: when $a = 5, \quad b = 2$ and $\frac{db}{dt} = +1, \quad 8\frac{da}{dt} + 5(12)(1) = 0$

$$\therefore \quad \frac{da}{dt} = -\tfrac{60}{8} = -7.5$$

$\therefore$ a decreases at 7.5 units per second

2



$$A = LB = 100 \text{ cm}^2$$

$$\therefore \quad \frac{dL}{dt}B + L\frac{dB}{dt} = 0$$

Particular case: When a square, $L = B = 10$ cm and $\frac{dL}{dt} = -1$

$$\therefore \quad 10\frac{dB}{dt} + 10(-1) = 0$$

$$\therefore \quad \frac{dB}{dt} = 1 \text{ cm/min}$$

$\therefore$ the breadth is increasing at 1 cm/min

3 $A = \pi r^2 \quad \therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

and $r = 1 \times t$

Particular cases:

a When $t = 2, \quad r = 2$ and $\frac{dr}{dt} = 1$ then $\frac{dA}{dt} = 2\pi(2)(1) = 4\pi$ m^2/sec

b When $t = 4, \quad r = 4$ and $\frac{dr}{dt} = 1$ then $\frac{dA}{dt} = 2\pi(4)(1) = 8\pi$ m^2/sec

4 $V = \frac{4}{3}\pi r^3 \quad \therefore \quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = 6\pi$ m^3/min

$$\therefore \quad \frac{dr}{dt} = \frac{6\pi}{4\pi r^2} = \frac{3}{2r^2} \text{ m/min}$$

Now $A = 4\pi r^2$

$$\therefore \quad \frac{dA}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r \times \frac{3}{2r^2}$$

Particular case: At $r = 2$, $\frac{dA}{dt} = \frac{8\pi \times 2 \times 3}{2 \times 4}$ m^2/min $= 6\pi$ m^2/min

i.e., the surface area is increasing at 6π m^2/min.

5 $pv^{\frac{3}{2}} = 400 \quad \therefore \quad \frac{dp}{dt}v^{\frac{3}{2}} + \frac{3}{2}pv^{\frac{1}{2}}\frac{dv}{dt} = 0$

Particular case: When $p = 50$ Nm^{-2}, $v^{\frac{3}{2}} = 8$ and so $v = 4$

$$\therefore \quad 3(8) + \tfrac{3}{2}(50)^2\frac{dv}{dt} = 0 \quad \text{as} \quad \frac{dp}{dt} = +3 \text{ Nm}^{-2}$$

$$\therefore \quad \frac{dv}{dt} = -\tfrac{24}{150} \text{ m}^3\text{/min}$$

i.e., the volume is decreasing at 0.16 m^3/min.

6 $V = \frac{1}{3}\pi r^2 h$ and $r = 3h$

$$\therefore\ V = \tfrac{1}{3}\pi(3h)^2 h = 3\pi h^3 \quad ..... *$$

Particular case: After 1 min, the volume $V = 3\pi(20)^3 \text{ cm}^3 = 24\,000\pi \text{ cm}^3$

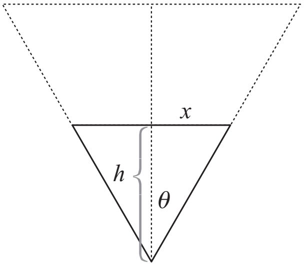
$$\text{i.e., } \frac{dV}{dt} = 24\,000\pi \text{ cm}^3\text{/min}$$

$$\text{But } \frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt} \quad \{\text{from } *\}$$

$$\therefore \text{ when } h = 20,\quad 24\,000\pi = 9\pi \times (20^2) \times \frac{dh}{dt}$$

$$\therefore\ \frac{dh}{dt} = \frac{24\,000\pi}{400 \times 9\pi} = \tfrac{20}{3} \text{ cm/min}$$

7



$$\theta = 30^o$$

$$\therefore\ \frac{x}{h} = \tan 30^o$$

$$x = h\tan 30^o = \frac{h}{\sqrt{3}}$$

$$\therefore\ V = \frac{h}{\sqrt{3}} \times h \times 600 = 200\sqrt{3}h^2 \text{ cm}^3$$

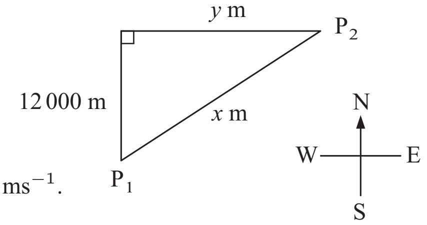
$$\therefore\ \frac{dV}{dt} = 400\sqrt{3}h\frac{dh}{dt}$$

Particular case: When $h = 20$, $-100\,000 = 400\sqrt{3}(20)\dfrac{dh}{dt}$ as $\dfrac{dV}{dt} = -0.1 \text{ m}^3 = -100\,000 \text{ cm}^3$

$$\therefore\ \frac{dh}{dt} = \frac{-100\,000}{400\sqrt{3} \times 20} = -\tfrac{25}{6}\sqrt{3} \text{ cm/min}$$

$\therefore$ the water level is falling at $\frac{25\sqrt{3}}{6}$ cm/min

8 Let P_1 in the diagram be the faster jet and P_2 be the slower jet. Let y m be the distance that P_2 is ahead of P_1, and x m be the distance between them.



Now $x^2 = y^2 + (12\,000)^2$ {Pythagoras}

$$\therefore\ 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

Particular case:

As P_1 is behind P_2, then it is catching up at a rate of 50 ms^{-1}.

$\therefore$ distance y is decreasing at a rate of 50 ms^{-1}

$$\therefore\ \frac{dy}{dt} = -50 \text{ ms}^{-1}$$

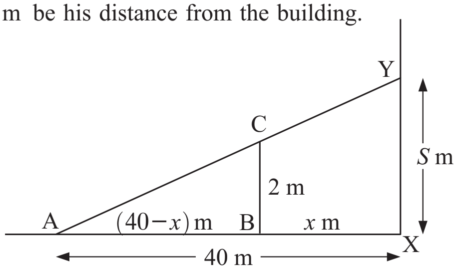
When $y = 5000$, $x = 13\,000$

$$\therefore\ 26\,000 \times \frac{dx}{dt} = 10\,000 \times (-50)$$

$$\frac{dx}{dt} = \tfrac{10}{26} \times (-50) = -\tfrac{250}{13} \text{ ms}^{-1}$$

$\therefore$ their separation is decreasing at $\frac{250}{13}$ ms^{-1}.

9 Let S m be the height of the person's shadow and x m be his distance from the building.



ΔABC and ΔAXY are similar

$$\therefore \quad \frac{\text{AB}}{\text{AX}} = \frac{\text{BC}}{\text{XY}}$$

$$\therefore \quad \frac{40-x}{40} = \frac{2}{S}$$

$$\therefore \quad S = \frac{80}{40-x}$$

$$= 80(40-x)^{-1}$$

$$\therefore \quad \frac{dS}{dt} = -80(40-x)^{-2}(-1)\frac{dx}{dt} = \frac{80}{(40-x)^2}\frac{dx}{dt}$$

Particular cases: **a** When $x = 20$ m, $\frac{dS}{dt} = \frac{80}{(40-20)^2}(-1)$ as $\frac{dx}{dt} = -1 \text{ ms}^{-1}$

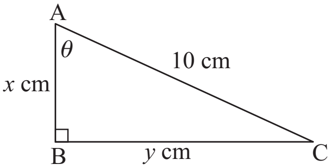
$$\therefore \quad \frac{dS}{dt} = -\tfrac{80}{400} = -0.2$$

i.e., the person's shadow is shortening at 0.2 ms^{-2}

b When $x = 10$ m, $\frac{dS}{dt} = \frac{80}{(40-10)^2}(-1) = -\frac{80}{900} = -\frac{8}{90}$

i.e., the person's shadow is shortening at $\frac{8}{90} \text{ ms}^{-1}$

10



$$\cos\theta = \frac{x}{10}$$

$$\therefore \quad -\sin\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt} \qquad \{\text{differentiating with respect to } t\}$$

If side AB increases at 0.1 cm/sec, i.e., $\frac{dx}{dt} = 0.1$ cm/sec,

Particular case: When ABC is isosceles, $\theta = 45^o$

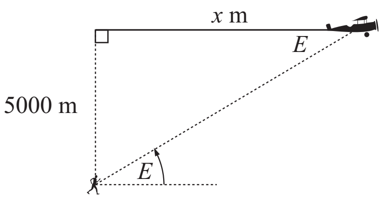
$$\therefore \quad \sin\theta = \tfrac{1}{\sqrt{2}}$$

$$\therefore \quad -\tfrac{1}{\sqrt{2}}\frac{d\theta}{dt} = \tfrac{1}{10} \times 0.1$$

$$\therefore \quad \frac{d\theta}{dt} = -\tfrac{\sqrt{2}}{100} \text{ radians/sec}$$

i.e., $\measuredangle$CAB is decreasing at $\frac{\sqrt{2}}{100}$ radians per second

11



$$\text{Now} \quad \tan E = \frac{5000}{x}$$

$$\therefore \quad \tan E = 5000x^{-1}$$

Differentiating with respect to t,

$$\sec^2 E\frac{dE}{dt} = -5000x^{-2}\frac{dx}{dt}$$

a *Particular case:* When $E = 60^o$,

$$\cos E = \tfrac{1}{2} \text{ and } \tan E = \sqrt{3} = \frac{5000}{x} \text{ and } \therefore \quad x = \tfrac{5000}{\sqrt{3}}$$

$$\therefore \quad 2^2\frac{dE}{dt} = -5000 \times \left(\tfrac{\sqrt{3}}{5000}\right)^2 \times 200$$

$$\therefore \quad \frac{dE}{dt} = -0.03$$

i.e., the angle of elevation is decreasing at 0.03 radians per second

b *Particular case:* When $E = 30^o$

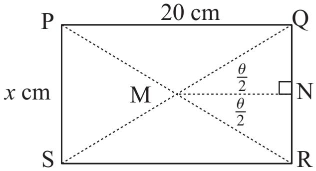
$$\cos E = \tfrac{\sqrt{3}}{2} \quad \text{and} \quad \tan E = \tfrac{1}{\sqrt{3}} = \frac{5000}{x} \quad \text{and} \quad \therefore \quad x = 5000\sqrt{3} \text{ m}$$

$$\therefore \quad \left(\tfrac{2}{\sqrt{3}}\right)^2 \frac{dE}{dt} = -5000 \times \left(5000\sqrt{3}\right)^{-2} \times 200$$

$$\therefore \quad \frac{dE}{dt} = -5000 \times \left(5000\sqrt{3}\right)^{-2} \times 200 \times \tfrac{3}{4} = -0.01$$

i.e., the angle of elevation is decreasing at 0.01 radians per second

12



Let N be the midpoint of QR in isosceles triangle QMR

$\therefore$ MN = 10 cm

Let QR = x cm,

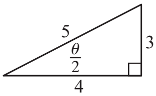
and let $\measuredangle$QMR $= \theta$.

$$\text{Now in triangle MNQ,} \quad \tan\left(\frac{\theta}{2}\right) = \frac{\text{QN}}{\text{MN}} = \frac{\frac{x}{2}}{10} = \frac{x}{20}$$

$$\therefore \quad \tfrac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\therefore \quad \frac{d\theta}{dt} = \tfrac{1}{10}\cos^2\left(\frac{\theta}{2}\right)\frac{dx}{dt} \quad \text{where} \quad \frac{dx}{dt} = 2 \text{ cms}^{-1}$$

Particular case: When $x = 15$ cm, $\tan\left(\frac{\theta}{2}\right) = \frac{15}{20} = \frac{3}{4}$

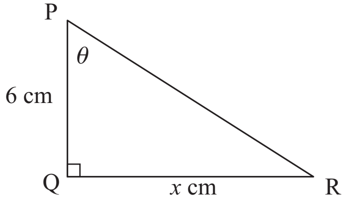


$$\therefore \quad \cos\left(\frac{\theta}{2}\right) = \tfrac{4}{5}$$

$$\therefore \quad \frac{d\theta}{dt} = \tfrac{1}{10}\left(\tfrac{4}{5}\right)^2 2 = 0.128$$

i.e., θ is increasing at 0.128 radians per second

13



Let QR $= x$ cm and the angle at P be Q.

$$\text{Then} \quad \tan\theta = \frac{x}{6}$$

$$\therefore \quad \sec^2\theta\,\frac{d\theta}{dt} = \tfrac{1}{6}\frac{dx}{dt}$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{\cos^2\theta}{6}\frac{dx}{dt} \quad \text{and} \quad \frac{dx}{dt} = 2 \text{ cm/min}$$

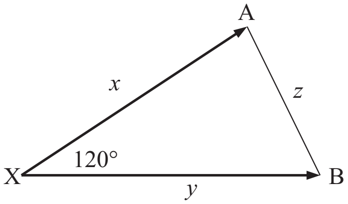
Particular case: When $x = 8$ cm, PR = 10 cm

Now $\cos\theta = \frac{6}{10}$

$$\therefore \quad \frac{d\theta}{dt} = \left(\tfrac{6}{10}\right)^2 \times \tfrac{1}{6} \times 2 = 0.12$$

$\therefore$ θ is increasing at a rate of 0.12 radians per minute.

14 Let x and y be the distances the trains have travelled (respectively) at time t, and let z be the distance between them.



$$\text{Then} \quad z^2 = x^2 + y^2 - 2xy\cos 120^o$$

$$z^2 = x^2 + y^2 + xy$$

$$\therefore \quad 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt}$$

Particular case: After 2 minutes, i.e., $t = 120$ secs,

$$x = 1440 \text{ m}, \quad y = 1920 \text{ m}, \quad \frac{dx}{dt} = 12 \text{ ms}^{-1}, \quad \frac{dy}{dt} = 16 \text{ ms}^{-1}$$

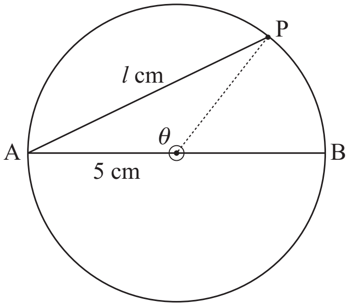
$$\therefore \quad z^2 = 1440^2 + 1920^2 + 1440 \times 1920 = 8\,524\,800$$

$$\therefore \quad 2\sqrt{8\,524\,800}\,\frac{dz}{dt} = 2880(12) + 3840(16) + (12)1920 + 1440(16) = 1.4208 \times 10^5$$

$$\therefore \quad 2\sqrt{8\,524\,800}\,\frac{dz}{dt} = 142080 \quad \therefore \quad \frac{dz}{dt} \doteqdot 24.33$$

$\therefore$ z is increasing at 24.33 ms^{-1}

15



Let AP $= l$ cm and let $\measuredangle$AOP $= \theta$

$$\therefore \quad l^2 = 5^2 + 5^2 - 2 \times 5 \times 5\cos\theta \quad \{\text{cosine rule}\}$$

$$\therefore \quad l^2 = 50 - 50\cos\theta$$

$$\therefore \quad 2l\,\frac{dl}{dt} = 50\sin\theta\,\frac{d\theta}{dt}$$

$$\therefore \quad \frac{dl}{dt} = \frac{25\sin\theta}{l}\,\frac{d\theta}{dt}$$

Now the point moves at one revolution every 10 seconds.

$$\therefore \quad \frac{d\theta}{dt} = \tfrac{2\pi}{10} = \tfrac{\pi}{5} \text{ radians/sec}$$

Particular cases:

a If AP $= l = 5$ cm, $\dfrac{dl}{dt} > 0$,

then $\theta = \frac{\pi}{3}$

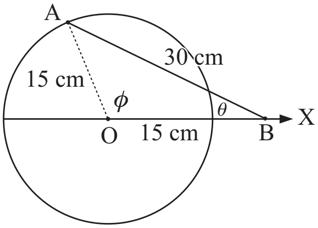
$$\therefore \quad \frac{dl}{dt} = \frac{25\sin(\frac{\pi}{3})}{5} \times \tfrac{\pi}{5} = \tfrac{\sqrt{3}}{2}\pi \text{ cms}^{-1}$$

b If P is at B, then $l = 10$ cm

and $\theta = \pi$

$$\therefore \quad \frac{dl}{dt} = \frac{25\sin\pi}{10} \times \tfrac{\pi}{5} = 0 \text{ cms}^{-1}$$

16



Let $\measuredangle$AOB $= \phi$ and $\measuredangle$ABO $= \theta$

$$\text{Then} \quad \frac{d\phi}{dt} = -100 \text{ revolutions per second} = -200\pi \text{ radians per second}$$

{clockwise rotation}

$$\text{Also,} \quad \frac{30}{\sin\phi} = \frac{15}{\sin\theta} \quad \{\text{sine rule}\}$$

$$\therefore \quad \sin\phi = 2\sin\theta$$

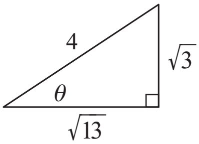
$$\therefore \quad \cos\phi\,\frac{d\phi}{dt} = 2\cos\theta\,\frac{d\theta}{dt} \qquad \text{and so} \quad \frac{d\theta}{dt} = \tfrac{1}{2}\,\frac{\cos\phi}{\cos\theta}\,\frac{d\phi}{dt}$$

Particular cases:

a When $\measuredangle$AOX $= 120^o$, i.e., $\phi = \frac{2\pi}{3}$

then $\cos\phi = -\frac{1}{2}$ and $\sin\phi = \frac{\sqrt{3}}{2}$

$$\therefore \quad \sin\theta = \tfrac{1}{2}\sin\phi = \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{\sqrt{3}}{4}$$



and so $\cos\theta = \frac{\sqrt{13}}{4}$

$$\begin{aligned}\therefore\quad \frac{d\theta}{dt} &= \tfrac{1}{2}\frac{\cos\phi}{\cos\theta}\frac{d\phi}{dt}\\ &= \tfrac{1}{2}\times\frac{-\frac{1}{2}}{\frac{\sqrt{13}}{4}}\times(-200\pi) \qquad \left\{\text{as } \frac{d\phi}{dt} = -200\pi\ ^{c}/\text{sec}\right\}\\ &= \tfrac{200\pi}{\sqrt{13}}, \qquad \text{i.e., } \theta \text{ is increasing at } \tfrac{200\pi}{\sqrt{13}} \text{ radians per second}\end{aligned}$$

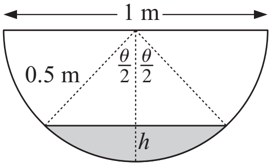
b When $\measuredangle AOX = 180^o$, i.e., $\phi = \pi$,

then $\cos\phi = -1$ and $\sin\phi = 0$

$\therefore\ \sin\theta = 0$ and $\cos\theta = 1$

$$\begin{aligned}\therefore\quad \frac{d\theta}{dt} &= \tfrac{1}{2}\frac{\cos\phi}{\cos\theta}\frac{d\phi}{dt}\\ &= \tfrac{1}{2}\times\tfrac{-1}{1}\times(-200\pi)\\ &= 100\pi, \qquad \text{i.e., } \theta \text{ is increasing at } 100\pi \text{ radians per second}\end{aligned}$$

17



Denote the radius of the semicircle $r = \frac{1}{2}$ m.

Let h be the depth and V be the volume of water in the trough at time t.

a The cross-sectional area of water in the trough

$$\begin{aligned}&= \text{area of sector} - \text{area of triangle}\\ &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\\ &= \tfrac{1}{2}r^2(\theta - \sin\theta)\\ &= \tfrac{1}{8}(\theta - \sin\theta)\end{aligned}$$

$$\begin{aligned}\therefore\ \text{the volume of water,}\quad V &= \text{area of water} \times \text{length of trough}\\ &= \tfrac{1}{8}(\theta - \sin\theta)\times 8\\ &= \theta - \sin\theta \quad \text{as required}\end{aligned}$$

b Now $\dfrac{dV}{dt} = \dfrac{d\theta}{dt} - \cos\theta\dfrac{d\theta}{dt}$ $\quad\therefore\ \dfrac{dV}{dt} = \dfrac{d\theta}{dt}(1-\cos\theta)$ $\quad$ where $\dfrac{dV}{dt} = 0.1\ \text{m}^3/\text{min}$

Particular case: When $h = 0.25$ m,

$$\cos\left(\frac{\theta}{2}\right) = \frac{r-h}{r} = \frac{\frac{1}{2}-\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \tfrac{1}{2} \qquad \textbf{Note: } \frac{\theta}{2} = 60^o$$

$$\text{and}\ \therefore\ \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\left(\tfrac{1}{2}\right)^2 - 1 = -\tfrac{1}{2}$$

$$\begin{aligned}\therefore\quad 0.1 &= \frac{d\theta}{dt}\left(1-(-\tfrac{1}{2})\right) = \tfrac{1}{15} \text{ radians per minute}\\ \therefore\quad \tfrac{1}{10} &= \frac{d\theta}{dt}\left(\tfrac{3}{2}\right)\\ \frac{d\theta}{dt} &= \tfrac{1}{15} \qquad \text{i.e., } \theta \text{ is increasing at } \tfrac{1}{15} \text{ radians per minute}\end{aligned}$$

$$\text{Now}\quad \cos\left(\frac{\theta}{2}\right) = \frac{\frac{1}{2}-h}{\frac{1}{2}} = 1-2h,$$

$$\text{so}\quad -\sin\left(\frac{\theta}{2}\right)\tfrac{1}{2}\frac{d\theta}{dt} = -2\frac{dh}{dt} \quad \text{and so} \quad -\left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{30}\right) = -2\frac{dh}{dt}$$

$$\therefore\quad \frac{dh}{dt} = \tfrac{\sqrt{3}}{120} \qquad \text{i.e., } h \text{ is increasing at } \tfrac{\sqrt{3}}{120} \text{ metres per minute.}$$

REVIEW SET 24A

1 **a** $$\frac{d}{dx}(\sin(5x)\ln x) = \frac{d}{dx}(\sin(5x))\ln x + \sin(5x)\frac{d}{dx}(\ln x) \quad \{\text{product rule}\}$$
$$= 5\cos(5x)\ln x + \frac{\sin(5x)}{x}$$

b $$\frac{d}{dx}(\sin x\cos(2x)) = \frac{d}{dx}(\sin x)\cos(2x) + \sin x\frac{d}{dx}(\cos(2x)) \quad \{\text{product rule}\}$$
$$= \cos x\cos(2x) + \sin x(-2\sin(2x))$$
$$= \cos x\cos(2x) - 2\sin x\sin(2x)$$

c $$\frac{d}{dx}\left(e^{-2x}\tan x\right) = \frac{d}{dx}\left(e^{-2x}\right)\tan x + e^{-2x}\frac{d}{dx}(\tan x) \quad \{\text{product rule}\}$$
$$= -2e^{-2x}\tan x + e^{-2x}\sec^2 x$$

2 $y = x\tan x \quad \therefore \quad \dfrac{dy}{dx} = 1\times\tan x + x\times\dfrac{1}{\cos^2 x}$

Now $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} =$ and $\tan\frac{\pi}{4} = 1$

$\therefore$ at $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$ and $\dfrac{dy}{dx} = 1 + \frac{\pi}{2}$

$\therefore$ the equation of the tangent is $\dfrac{y-\frac{\pi}{4}}{x-\frac{\pi}{4}} = 1+\frac{\pi}{2} \quad \therefore \quad y - \frac{\pi}{4} = (1+\frac{\pi}{2})(x-\frac{\pi}{4})$

$$= x - \tfrac{\pi}{4} + \tfrac{\pi}{2}x - \tfrac{\pi^2}{8}$$
$$\therefore \quad y = (1+\tfrac{\pi}{2})x - \tfrac{\pi^2}{8}$$
$$\therefore \quad 2y = (2+\pi)x - \tfrac{\pi^2}{4}$$
$$\therefore \quad (2+\pi)x - 2y = \tfrac{\pi^2}{4} \quad \text{as required}$$

3 **a** $$f(x) = 3\sin x - 4\cos(2x)$$
$$\therefore \quad f'(x) = 3\cos x + 8\sin(2x) \quad \text{and} \quad f''(x) = -3\sin x + 16\cos(2x)$$

b $$f(x) = x^{\frac{1}{2}}\cos(4x)$$
$$\therefore \quad f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) + x^{\frac{1}{2}}(-4\sin(4x))$$
$$= \tfrac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x)$$
$$\text{and} \quad f''(x) = -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) + \tfrac{1}{2}x^{-\frac{1}{2}}(-4\sin 4x)) - \left[2x^{-\frac{1}{2}}\sin(4x) + 4x^{\frac{1}{2}}\times 4\cos(4x)\right]$$
$$= -\tfrac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)$$

4 $x(t) = 3 + \sin(2t)$ cm, $t \geqslant 0$ sec

a $x'(t) = 0 + 2\cos(2t) \qquad \therefore \quad x(0) = 3$ cm

$x''(t) = -4\sin(2t) \qquad x'(0) = 2$ cm/sec

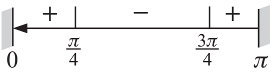
$x''(0) = 0$ cm/sec^2

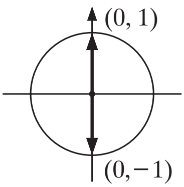
$\therefore$ initially the particle is 3 cm right of O, moving right at a speed of 2 cm/sec.

b $x'(t) = 0$ when $2\cos(2t) = 0$

$$\therefore \quad \cos(2t) = 0$$
$$\therefore \quad 2t = \tfrac{\pi}{2} + k\pi$$
$$\therefore \quad t = \tfrac{\pi}{4} \text{ or } \tfrac{3\pi}{4} \qquad \{\text{as } 0 \leqslant t \leqslant \pi\}$$

$\therefore$ reverses direction at $t = \frac{\pi}{4}, \frac{3\pi}{4}$

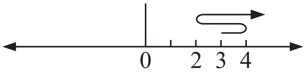




c $x(0) = 3, \quad x\left(\frac{\pi}{4}\right) = 3+\sin\left(\frac{\pi}{2}\right) = 4$

$x\left(\frac{3\pi}{4}\right) = 3+\sin\left(\frac{3\pi}{2}\right) = 3-1 = 2 \quad x(\pi) = 3+\sin(2\pi) = 3$

$\therefore$ total distance travelled $= 1+2+1 = 4$ cm.



5 **a** $f(x) = \sqrt{\cos x}, \quad 0 \leqslant x \leqslant 2\pi$

$\therefore$ $f(x)$ is meaningful when $\cos x \geqslant 0$,

i.e., when $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

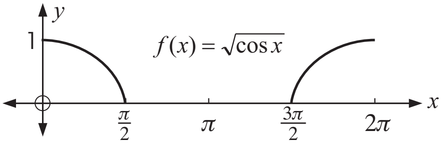
b $f(x) = (\cos x)^{\frac{1}{2}}$

$\therefore \quad f'(x) = \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$

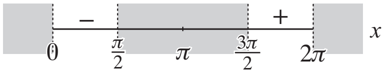
$= \dfrac{-\sin x}{2\sqrt{\cos x}}$

$\therefore \quad f'(x) = 0$ when $-\sin x = 0$

i.e., $x = 0, \pi, 2\pi$ etc.



Sign diagram for $f'(x)$ is:



$f(x)$ is increasing for $\frac{3\pi}{2} < x < 2\pi \qquad f(x)$ is decreasing for $0 < x < \frac{\pi}{2}$

6 **a** $s(t) = 30+\cos(\pi t)$ cm, $t \geqslant 0$

$\therefore \quad v(t) = -\pi\sin(\pi t)$

So, $v(0) = 0$ cms^{-1}

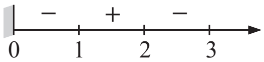
$v(\frac{1}{2}) = -\pi$ cms^{-1}

$v(1) = 0$ cms^{-1}

$v(1\frac{1}{2}) = \pi$ cms^{-1}

$v(2) = 0$ cms^{-1}

sign diagram of $v(t)$:

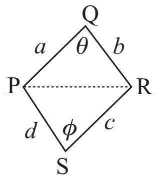


b The cork is falling when $v(t) \leqslant 0$

i.e., $0 \leqslant t \leqslant 1, \quad 2 \leqslant t \leqslant 3$, etc.

i.e., $2n \leqslant t \leqslant 2n+1 \quad n = 0, 1, 2, 3, ...$

7 **a**



Using the cosine rule,

in Δ PQR, $PR^2 = a^2+b^2-2ab\cos\theta$

in Δ PSR, $PR^2 = c^2+d^2-2cd\cos\phi$

$\therefore \quad a^2+b^2-2ab\cos\theta = c^2+d^2-2cd\cos\phi$

Now a, b, c and d are constants, so differentiating with respect to ϕ,

$\therefore \quad 2ab\sin\theta\,\dfrac{d\theta}{d\phi} = 2cd\sin\phi$

$\therefore \quad \dfrac{d\theta}{d\phi} = \dfrac{2cd\sin\phi}{2ab\sin\theta} = \dfrac{cd\sin\phi}{ab\sin\theta}$ as required

b Area of quadrilateral, $A =$ area of Δ PQR $+$ area of Δ PSR

$= \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\phi$

$\therefore \quad \dfrac{dA}{d\phi} = \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{d\phi} + \frac{1}{2}cd\cos\phi$

$= \frac{1}{2}ab\cos\theta\left(\dfrac{cd\sin\phi}{ab\sin\theta}\right) + \frac{1}{2}cd\cos\phi$ {using **a**}

$= \frac{1}{2}cd\left[\dfrac{\cos\theta\sin\phi}{\sin\theta} + \cos\phi\right]$

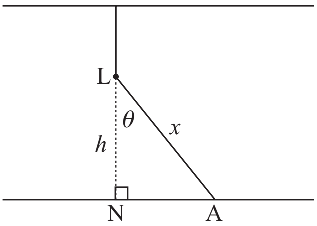
$$= \frac{cd}{2\sin\theta}(\sin\phi\cos\theta + \cos\phi\sin\theta)$$

$$= \frac{cd}{2\sin\theta}\sin(\phi+\theta)$$

$$\therefore \quad \frac{dA}{d\phi} = 0 \quad \text{when} \quad \sin(\phi+\theta) = 0 \quad \text{i.e., when} \quad \phi+\theta = 2\pi$$

$\therefore$ the area of PQRS is a maximum when the opposite angles are supplementary, i.e., when PQRS is a cyclic quadrilateral.

8



a

$$\sin\theta = \frac{\text{NA}}{x} = \frac{1}{x}$$

$$\therefore \quad \frac{1}{x^2} = \sin^2\theta$$

$$\therefore \quad \text{at A,} \quad I = \frac{\sqrt{8}\cos\theta}{x^2}$$

$$= \sqrt{8}\cos\theta\sin^2\theta$$

b

$$\frac{dI}{d\theta} = \sqrt{8}(-\sin\theta)\sin^2\theta + \sqrt{8}\cos\theta(2\sin\theta\cos\theta)$$

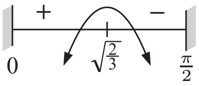
$$= \sqrt{8}\sin\theta[2\cos^2\theta - \sin^2\theta]$$

$$= \sqrt{8}\sin\theta[2(1-\sin^2\theta) - \sin^2\theta]$$

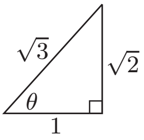
$$= \sqrt{8}\sin\theta[2 - 3\sin^2\theta]$$

$$\frac{dI}{d\theta} = 0 \quad \text{when} \quad \sin\theta = \sqrt{\tfrac{2}{3}}, \quad 0 < \theta < \tfrac{\pi}{2}$$

and the sign diagram of $\frac{dI}{d\theta}$ is:



$\therefore$ the maximum illumination at A is obtained when $\sin\theta = \sqrt{\frac{2}{3}}$.



$$\therefore \quad x = \frac{1}{\sin\theta} = \sqrt{\tfrac{3}{2}}$$

$$\therefore \quad h = \sqrt{x^2 - \text{NA}^2} = \sqrt{\tfrac{3}{2} - 1} = \tfrac{1}{\sqrt{2}}$$

i.e., the bulb is $\frac{1}{\sqrt{2}}$ m above the floor.

c

$$\cos\theta = \frac{h}{x}$$

$$\therefore \quad I = \frac{\sqrt{8}\cos\theta}{x^2} = \frac{\sqrt{8}\times\frac{h}{x}}{x^2} = \frac{\sqrt{8}h}{x^3}$$

$$\text{Now} \quad x^2 = h^2 + 1$$

$$x^3 = \left(h^2+1\right)^{\frac{3}{2}}$$

$$\therefore \quad I = \sqrt{8}h\left(h^2+1\right)^{-\frac{3}{2}}$$

$$\therefore \quad \frac{dI}{dt} = \sqrt{8}\frac{dh}{dt}\left(h^2+1\right)^{-\frac{3}{2}} - \tfrac{3}{2}\sqrt{8}h\left(h^2+1\right)^{-\frac{5}{2}}\left(2h\frac{dh}{dt}\right)$$

$$= \sqrt{8}\frac{dh}{dt}\left[\left(h^2+1\right)^{-\frac{3}{2}} - 3h^2\left(h^2+1\right)^{-\frac{5}{2}}\right]$$

Particular case: $\quad \frac{dh}{dt} = -0.1 \text{ ms}^{-1}$, and when $h = 1$ m

$$\frac{dI}{dt} = \sqrt{8}(-0.1)\left(2^{-\frac{3}{2}} - 3\times 2^{-\frac{5}{2}}\right) = 0.05$$

$\therefore$ the illumination is increasing at 0.05 units/second.

REVIEW SET 24B

1 **a** $y=\dfrac{x}{\sqrt{\sec x}}=x\,(\cos x)^{\frac{1}{2}}\quad\therefore\quad \dfrac{dy}{dx}=(\cos x)^{\frac{1}{2}}+x\times\frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$ {product rule}

$$=\sqrt{\cos x}-\frac{x\sin x}{2\sqrt{\cos x}}$$

b $y=e^x\cot(2x)\quad\therefore\quad\dfrac{dy}{dx}=e^x\cot(2x)+e^x\times 2\left(-\csc^2(2x)\right)$

$$=e^x\left(\cot(2x)-2\csc^2(2x)\right)$$

c $y=\arccos\left(\frac{x}{2}\right)\quad\therefore\quad\dfrac{dy}{dx}=\frac{1}{2}\left(\dfrac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)=-\dfrac{1}{2\sqrt{1-\frac{x^2}{4}}}=-\dfrac{1}{\sqrt{4-x^2}}$

2 **a** If $y=\sec x$ then $\dfrac{dy}{dx}=\sec x\tan x$

When $x=\frac{\pi}{3}$, $y=\sec\frac{\pi}{3}=2$ and $\dfrac{dy}{dx}=\sec\frac{\pi}{3}\tan\frac{\pi}{3}=2\sqrt{3}$

$\therefore$ the tangent has equation $\dfrac{y-2}{x-\frac{\pi}{3}}=2\sqrt{3}\quad\therefore\quad y-2=2\sqrt{3}x-\frac{2\pi}{\sqrt{3}}$

$$\therefore\quad y=2\sqrt{3}x-\tfrac{2\pi}{\sqrt{3}}+2$$

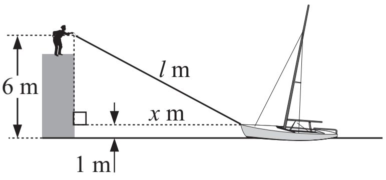
b If $y=\arctan x$ then $\dfrac{dy}{dx}=\dfrac{1}{1+x^2}$. When $x=\sqrt{3}$, $y=\arctan\sqrt{3}=\frac{\pi}{3}$

and $\dfrac{dy}{dx}=\dfrac{1}{1+\left(\sqrt{3}\right)^2}=\dfrac{1}{1+3}=\frac{1}{4}$

$\therefore$ the normal has slope -4, so its equation is $\dfrac{y-\frac{\pi}{3}}{x-\sqrt{3}}=-4\quad\therefore\quad y-\frac{\pi}{3}=-4x+4\sqrt{3}$

$$\therefore\quad y=-4x+4\sqrt{3}+\tfrac{\pi}{3}$$

3 Let l m be the length of rope and x m be the distance of the boat from the jetty.



Then $x^2+5^2=l^2$

$$\therefore\quad 2x\frac{dx}{dt}=2l\frac{dl}{dt}$$

Particular case: When $x=15$ m, $l=\sqrt{15^2+5^2}=\sqrt{250}$ and $\dfrac{dl}{dt}=-40$ m/min

$$\therefore\quad 2(15)\frac{dx}{dt}=2\sqrt{250}(-40)\quad\text{m/minute}$$

$$\therefore\quad \frac{dx}{dt}=-\frac{80\sqrt{250}}{30}=-42.16$$

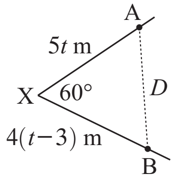
$\therefore$ the boat is approaching the jetty at 42.16 m/minute

4 **Note:** the question should have stated that B passes through X 3 *seconds* after A passes through X.

Let t be the number of seconds after A passes through X. In this time, A travels $5t$ m.

Now B passes through X when $t=3$.

$\therefore$ for $t>3$, B is $4(t-3)$ m from X.



Using the cosine rule,

$$D^2=25t^2+16(t-3)^2-2\times 5t\times 4(t-3)\times\cos 60^o$$
$$=25t^2+16(t-3)^2-20t(t-3)$$

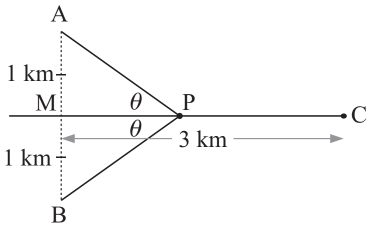
$$\therefore\quad 2D\frac{dD}{dt}=50t+32(t-3)-20(t-3)-20t$$

when $5t = 20, \quad t = 4$ and $D^2 = 25 \times 16 + 16 - 20 \times 4 = 336$

$$\therefore \quad 2\sqrt{336}\,\frac{dD}{dt} = 200 + 32 - 20 - 20 \times 4 = 132$$

$$\therefore \quad \frac{dD}{dt} = \tfrac{66}{\sqrt{336}} \doteqdot 3.60 \text{ m/s}$$

5



a The length of cable required $= (\text{PA} + \text{PB} + \text{PC})$ km

i If P is at M, then PA = PB = 1 km and PC = 3 km

$\therefore$ 5 km of cable is required

ii If P is at C, then PA $= \text{PB} = \sqrt{1^2 + 3^2} = \sqrt{10}$ km and PC = 0 km

$\therefore$ $2\sqrt{10}$ km of cable is required

b

$$\text{Now} \quad \sin\theta = \frac{1}{\text{AP}} = \frac{1}{\text{BP}} \quad \text{and} \quad \tan\theta = \frac{1}{\text{MP}}$$

$$\therefore \quad \text{AP} = \text{BP} = \frac{1}{\sin\theta} \quad \text{and} \quad \text{MP} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\therefore \quad \text{AP} + \text{BP} + \text{CP} = \frac{2}{\sin\theta} + (\text{CM} - \text{MP}) \quad \therefore \quad L = \frac{2}{\sin\theta} + 3 - \frac{\cos\theta}{\sin\theta} \quad \text{as required}$$

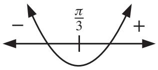
c Since $L = 2(\sin\theta)^{-1} + 3 - (\tan\theta)^{-1}$,

$$\frac{dL}{d\theta} = -2\cos\theta(\sin\theta)^{-2} + (\tan\theta)^{-2} \times \frac{1}{\cos^2\theta}$$

$$= \frac{-2\cos\theta}{\sin^2\theta} + \frac{1}{\cos^2\theta}\,\frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{1 - 2\cos\theta}{\sin^2\theta} \quad \text{as required}$$

$\therefore \quad \dfrac{dL}{d\theta} = 0$ if $\cos\theta = \frac{1}{2}$, i.e., $\theta = \frac{\pi}{3}$ and sign diagram of $\dfrac{dL}{d\theta}$ is:

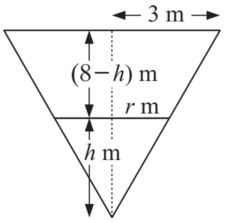


$\therefore$ the minimum length of cable is required when $\theta = \frac{\pi}{3}$

Then $\sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$,

so $L_{\min} = \frac{4}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}} = (3 + \sqrt{3})$ km as required

6



a

$$V(r) = \tfrac{1}{3}\pi r^2 h$$

$$\text{Using similar triangles} \quad \frac{h}{r} = \tfrac{8}{3}$$

$$\therefore \quad h = \frac{8r}{3}$$

$$\therefore \quad V(r) = \tfrac{1}{3}\pi r^2\left(\frac{8r}{3}\right) = \tfrac{8\pi}{9}r^3 \text{ m}^3$$

b *Particular case:* When $h = 5, \quad r = \dfrac{3h}{8} = \frac{15}{8}$ and $\dfrac{dV}{dt} = -0.2 = -\frac{1}{5}$ m^3/min

$$\text{Now} \quad \frac{dV}{dt} = \tfrac{8\pi}{3}r^2\,\frac{dr}{dt} \qquad \therefore \quad -\tfrac{1}{5} = \tfrac{8\pi}{3}\left(\tfrac{15}{8}\right)^2\frac{dr}{dt}$$

$$\therefore \quad -\tfrac{1}{5} = \tfrac{225}{24}\pi\,\frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = -\tfrac{8}{375\pi}$$

$$\therefore \quad \frac{dr}{dt} = -0.006\,79 \quad \text{i.e., radius is decreasing at } 0.006\,79 \text{ m/min}$$

Chapter 25

INTEGRATION

EXERCISE 25A

1 **a** **i** $\frac{d}{dx}(x^2) = 2x$

$\therefore \frac{d}{dx}\left(\tfrac{1}{2}x^2\right) = x$

$\therefore$ the antiderivative of x is $\tfrac{1}{2}x^2$

ii $\frac{d}{dx}(x^3) = 3x^2$

$\therefore \frac{d}{dx}\left(\tfrac{1}{3}x^3\right) = x^2$

$\therefore$ the antiderivative of x^2 is $\tfrac{1}{3}x^3$

iii $\frac{d}{dx}(x^6) = 6x^5$

$\therefore \frac{d}{dx}\left(\tfrac{1}{6}x^6\right) = x^5$

$\therefore$ the antiderivative of x^5 is $\tfrac{1}{6}x^6$

iv $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$

$\therefore$ the antiderivative of x^{-2} is $-x^{-1}$ or $-\frac{1}{x}$

v $\frac{d}{dx}(x^{-3}) = -3x^{-4}$

$\therefore \frac{d}{dx}\left(-\tfrac{1}{3}x^{-3}\right) = x^{-4}$

$\therefore$ the antiderivative of x^{-4} is $-\tfrac{1}{3}x^{-3}$

vi $\frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \tfrac{4}{3}x^{\frac{1}{3}}$

$\therefore \frac{d}{dx}\left(\tfrac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$

$\therefore$ the antiderivative of $x^{\frac{1}{3}}$ is $\tfrac{3}{4}x^{\frac{4}{3}}$

viii $\frac{d}{dx}(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{-\frac{1}{2}}$ $\quad\therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of of $x^{-\frac{1}{2}}$ is $2x^{\frac{1}{2}} = 2\sqrt{x}$

b the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$

2 **a** **i** $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

$\therefore \frac{d}{dx}\left(\tfrac{1}{2}e^{2x}\right) = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\tfrac{1}{2}e^{2x}$

ii $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

$\therefore \frac{d}{dx}\left(\tfrac{1}{5}e^{5x}\right) = e^{5x}$

$\therefore$ the antiderivative of e^{5x} is $\tfrac{1}{5}e^{5x}$

iii $\frac{d}{dx}(e^{\frac{1}{2}x}) = \tfrac{1}{2}e^{\frac{1}{2}x}$

$\therefore \frac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$

$\therefore$ the antiderivative of $e^{\frac{1}{2}x}$ is $2e^{\frac{1}{2}x}$

iv $\frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$

$\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$

$\therefore$ the antiderivative of $e^{0.01x}$ is $100e^{0.01x}$

v $\frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$

$\therefore \frac{d}{dx}\left(\tfrac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$

$\therefore$ the antiderivative of $e^{\pi x}$ is $\tfrac{1}{\pi}e^{\pi x}$

vi $\frac{d}{dx}\left(e^{\frac{x}{3}}\right) = \tfrac{1}{3}e^{\frac{x}{3}}$

$\therefore \frac{d}{dx}\left(3e^{\frac{x}{3}}\right) = e^{\frac{x}{3}}$

$\therefore$ the antiderivative of $e^{\frac{x}{3}}$ is $3e^{\frac{x}{3}}$

b the antiderivative of e^{kx} is $\frac{1}{k}e^{kx}$

3 **a**

$$\frac{d}{dx}(x^3+x^2) = 3x^2+2x$$
$$\frac{d}{dx}\left(2x^3+2x^2\right) = 6x^2+4x$$
$\therefore$ antiderivative of $6x^2+4x$ is $2x^3+2x^2$

b

$$\frac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$$
$$\frac{d}{dx}\left(\tfrac{1}{3}e^{3x+1}\right) = e^{3x+1}$$
$\therefore$ antiderivative of e^{3x+1} is $\frac{1}{3}e^{3x+1}$

c

$$\begin{aligned}\frac{d}{dx}(x\sqrt{x}) = \frac{d}{dx}(x^{\frac{3}{2}}) &= \tfrac{3}{2}x^{\frac{1}{2}}\\ &= \tfrac{3}{2}\sqrt{x}\\ \therefore\quad \frac{d}{dx}\left(\tfrac{2}{3}x\sqrt{x}\right) &= \sqrt{x}\end{aligned}$$
$\therefore$ antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$

d

$$\begin{aligned}\frac{d}{dx}((2x+1)^4) &= 4(2x+1)^3\times 2\\ &= 8(2x+1)^3\\ \therefore\quad \frac{d}{dx}\left(\tfrac{1}{8}(2x+1)^4\right) &= (2x+1)^3\end{aligned}$$
$\therefore$ antiderivative of $(2x+1)^3$ is $\frac{1}{8}(2x+1)^4$

EXERCISE 25B.1

1

$$\begin{aligned} y &= x^7\\ \therefore\quad \frac{dy}{dx} &= 7x^6\end{aligned}$$
$$\begin{aligned}\therefore\quad \int 7x^6\,dx &= x^7+c_1\\ \therefore\quad 7\int x^6\,dx &= x^7+c_1\\ \therefore\quad \int x^6\,dx &= \tfrac{1}{7}x^7+c\end{aligned}$$

2

$$\begin{aligned} y &= x^3+x^2\\ \therefore\quad \frac{dy}{dx} &= 3x^2+2x\end{aligned}$$
$$\therefore\quad \int 3x^2+2x\,dx = x^3+x^2+c$$

3

$$\begin{aligned} y &= e^{2x+1}\\ \therefore\quad \frac{dy}{dx} &= 2e^{2x+1}\end{aligned}$$
$$\begin{aligned}\therefore\quad \int 2e^{2x+1}\,dx &= e^{2x+1}+c_1\\ \therefore\quad 2\int e^{2x+1}\,dx &= e^{2x+1}+c_1\\ \therefore\quad \int e^{2x+1}\,dx &= \tfrac{1}{2}e^{2x+1}+c\end{aligned}$$

4

$$\begin{aligned} y &= (2x+1)^4\\ \therefore\quad \frac{dy}{dx} &= 4(2x+1)^3\times 2\\ &= 8(2x+1)^3\end{aligned}$$
$$\begin{aligned}\therefore\quad \int 8(2x+1)^3\,dx &= (2x+1)^4+c_1\\ \therefore\quad 8\int (2x+1)^3\,dx &= (2x+1)^4+c_1\\ \therefore\quad \int (2x+1)^3\,dx &= \tfrac{1}{8}(2x+1)^4+c\end{aligned}$$

5

$$\begin{aligned} y &= x\sqrt{x} = x^{\frac{3}{2}}\\ \therefore\quad \frac{dy}{dx} &= \tfrac{3}{2}x^{\frac{1}{2}} = \tfrac{3}{2}\sqrt{x}\end{aligned}$$
$$\begin{aligned}\therefore\quad \int \tfrac{3}{2}\sqrt{x}\,dx &= x\sqrt{x}+c_1\\ \therefore\quad \tfrac{3}{2}\int \sqrt{x}\,dx &= x\sqrt{x}+c_1\\ \therefore\quad \int \sqrt{x}\,dx &= \tfrac{2}{3}x\sqrt{x}+c\end{aligned}$$

6

$$\begin{aligned} y &= \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}\\ \therefore\quad \frac{dy}{dx} &= -\tfrac{1}{2}x^{-\frac{3}{2}} = -\tfrac{1}{2}\left(\frac{1}{x\sqrt{x}}\right)\end{aligned}$$
$$\begin{aligned}\therefore\quad \int -\tfrac{1}{2}\left(\frac{1}{x\sqrt{x}}\right)dx &= \frac{1}{\sqrt{x}}+c_1\\ \therefore\quad -\tfrac{1}{2}\int \frac{1}{x\sqrt{x}}\,dx &= \frac{1}{\sqrt{x}}+c_1\\ \therefore\quad \int \frac{1}{x\sqrt{x}}\,dx &= -\frac{2}{\sqrt{x}}+c\end{aligned}$$

7 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.

$$\begin{aligned}\therefore\quad \frac{d}{dx}(F(x)+G(x)) &= f(x)+g(x)\\ \therefore\quad \int (f(x)+g(x))\,dx &= F(x)+G(x)+c\\ &= (F(x)+c_1)+(G(x)+c_2)\\ &= \int f(x)\,dx + \int g(x)\,dx\end{aligned}$$

8

$$\begin{aligned} y &= \sqrt{1-4x} \\ &= (1-4x)^{\frac{1}{2}} \\ \therefore\ \frac{dy}{dx} &= \tfrac{1}{2}(1-4x)^{-\frac{1}{2}}(-4) \\ &= -2(1-4x)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \therefore\ \int \frac{-2}{\sqrt{1-4x}}\,dx &= \sqrt{1-4x}+c_1 \\ \therefore\ -2\int \frac{1}{\sqrt{1-4x}}\,dx &= \sqrt{1-4x}+c_1 \\ \therefore\ \int \frac{1}{\sqrt{1-4x}}\,dx &= -\tfrac{1}{2}\sqrt{1-4x}+c \end{aligned}$$

9

$$\begin{aligned} \frac{d}{dx}(\ln(5-3x+x^2)) &= \frac{2x-3}{5-3x+x^2} \\ \therefore\ \int \frac{2x-3}{5-3x+x^2}\,dx &= \ln\,|\,5-3x+x^2\,|+c_1 \\ \therefore\ \int \frac{4x-6}{5-3x+x^2}\,dx &= 2\ln(5-3x+x^2)+c \qquad \{5-3x+x^2>0 \text{ for all } x \text{ as } a>0 \text{ and } \Delta=-11<0\} \end{aligned}$$

10

$$\begin{aligned} \frac{d}{dx}(2^x) &= \frac{d}{dx}(e^{\ln 2})^x \qquad \{\text{since} \quad 2^x=(e^{\ln 2})^x\} \\ &= e^{(\ln 2)x}\times \ln 2 \\ &= 2^x \ln 2 \end{aligned}$$

$$\begin{aligned} \therefore\ \int 2^x \ln 2\,dx &= 2^x+c_1 \\ \therefore\ \ln 2\int 2^x\,dx &= 2^x+c_1 \\ \therefore\ \int 2^x\,dx &= \frac{2^x}{\ln 2}+c \end{aligned}$$

11

$$\begin{aligned} \frac{d}{dx}(x\ln x) &= 1\times \ln x + x\times \frac{1}{x} \\ &= \ln x + 1 \end{aligned}$$

$$\begin{aligned} \therefore\ \int(\ln x+1)\,dx &= x\ln x+c_1 \\ \therefore\ \int \ln x\,dx + x &= x\ln x + c_1 \\ \therefore\ \int \ln x\,dx &= x\ln x - x + c \end{aligned}$$

EXERCISE 25B.2

1 **a**

$$\begin{aligned} &\int(x^4-x^2-x+2)\,dx \\ &= \tfrac{1}{5}x^5-\tfrac{1}{3}x^3-\tfrac{1}{2}x^2+2x+c \end{aligned}$$

b

$$\begin{aligned} &\int(\sqrt{x}+e^x)\,dx \\ &= \int(x^{\frac{1}{2}}+e^x)\,dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}+e^x+c \\ &= \tfrac{2}{3}x^{\frac{3}{2}}+e^x+c \end{aligned}$$

c

$$\begin{aligned} &\int(3e^x-\frac{1}{x})\,dx \\ &= 3e^x-\ln|x|+c \end{aligned}$$

d

$$\begin{aligned} &\int(x\sqrt{x}-\frac{2}{x})\,dx \\ &= \int(x^{\frac{3}{2}}-\frac{2}{x})\,dx \\ &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}}-2\ln|x|+c \\ &= \tfrac{2}{5}x^{\frac{5}{2}}-2\ln|x|+c \end{aligned}$$

e

$$\begin{aligned} &\int\left(\frac{1}{x\sqrt{x}}+\frac{4}{x}\right)dx \\ &= \int(x^{-\frac{3}{2}}+\frac{4}{x})\,dx \\ &= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}+4\ln|x|+c \\ &= -\frac{2}{\sqrt{x}}+4\ln|x|+c \end{aligned}$$

f

$$\begin{aligned} &\int(\tfrac{1}{2}x^3-x^4+x^{\frac{1}{3}})\,dx \\ &= \tfrac{1}{2}\frac{x^4}{4}-\frac{x^5}{5}+\frac{x^{\frac{4}{3}}}{\frac{4}{3}}+c \\ &= \tfrac{1}{8}x^4-\tfrac{1}{5}x^5+\tfrac{3}{4}x^{\frac{4}{3}}+c \end{aligned}$$

g

$$\begin{aligned} &\int\left(x^2+\frac{3}{x}\right)dx \\ &= \frac{x^3}{3}+3\ln|x|+c \end{aligned}$$

h

$$\begin{aligned} &\int\left(\frac{1}{2x}+x^2-e^x\right)dx \\ &= \tfrac{1}{2}\ln|x|+\frac{x^3}{3}-e^x+c \end{aligned}$$

i

$$\begin{aligned} &\int\left(5e^x+\tfrac{1}{3}x^3-\frac{4}{x}\right)dx \\ &= 5e^x+\tfrac{1}{3}\frac{x^4}{4}-4\ln|x|+c \\ &= 5e^x+\tfrac{1}{12}x^4-4\ln|x|+c \end{aligned}$$

2 **a** $\int (x^2+3x-2)\,dx$

$= \dfrac{x^3}{3}+\dfrac{3x^2}{2}-2x+c$

b $\displaystyle\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\,dx$

$= \int x^{\frac{1}{2}} - x^{-\frac{1}{2}}\ dx$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

c $\displaystyle\int \left(2e^x - \frac{1}{x^2}\right)\,dx$

$= \int \left(2e^x - x^{-2}\right)\ dx$

$= 2\int e^x\ dx - \dfrac{x^{-1}}{-1} + c$

$= 2e^x + \dfrac{1}{x} + c$

d $\displaystyle\int \left(\frac{1-4x}{x\sqrt{x}}\right)\,dx$

$= \displaystyle\int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}}\right)\,dx$

$= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}})\,dx$

$= \dfrac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \dfrac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -\dfrac{2}{\sqrt{x}} - 8\sqrt{x} + c$

e $\int (2x+1)^2\ dx$

$= \int (4x^2+4x+1)\ dx$

$= \dfrac{4x^3}{3} + \dfrac{4x^2}{2} + x + c$

$= \frac{4}{3}x^3 + 2x^2 + x + c$

f $\displaystyle\int \frac{x^2+x-3}{x}\ dx$

$= \displaystyle\int \left(x+1-\frac{3}{x}\right)\ dx$

$= \dfrac{x^2}{2} + x - 3\ln|x| + c$

g $\displaystyle\int \frac{2x-1}{\sqrt{x}}\ dx$

$= \int \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\ dx$

$= \dfrac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

h $\displaystyle\int \frac{x^2-4x+10}{x^2\sqrt{x}}\,dx$

$= \displaystyle\int \left(\frac{x^2}{x^2\sqrt{x}} - \frac{4x}{x^2\sqrt{x}} + \frac{10}{x^2\sqrt{x}}\right)\,dx$

$= \int \left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}} + 10x^{-\frac{5}{2}}\right)\ dx$

$= \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} - \dfrac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + \dfrac{10x^{-\frac{3}{2}}}{-\frac{3}{2}} + c$

$= 2\sqrt{x} + \dfrac{8}{\sqrt{x}} - \dfrac{20}{3x\sqrt{x}} + c$

i $\int (x+1)^3\,dx$

$= \int (x^3+3x^2+3x+1)\ dx$

$= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

3 **a** $\dfrac{dy}{dx} = 6$

$\therefore\ y = \int 6\ dx$

$\therefore\ y = 6x + c$

b $\dfrac{dy}{dx} = 4x^2$

$\therefore\ y = \int 4x^2\ dx$

$\therefore\ y = \frac{4}{3}x^3 + c$

c $\dfrac{dy}{dx} = 5x - x^2$

$\therefore\ y = \int (5x - x^2)\ dx$

$\therefore\ y = \frac{5}{2}x^2 - \frac{1}{3}x^3 + c$

d $\dfrac{dy}{dx} = \dfrac{1}{x^2} = x^{-2}$

$\therefore\ y = \int x^{-2}\ dx$

$= \dfrac{x^{-1}}{-1} + c$

$\therefore\ y = -\dfrac{1}{x} + c$

e $\dfrac{dy}{dx} = 2e^x - 5$

$\therefore\ y = \int (2e^x - 5)\ dx$

$\therefore\ y = 2e^x - 5x + c$

f $\dfrac{dy}{dx} = 4x^3 + 3x^2$

$\therefore\ y = \int (4x^3 + 3x^2)\ dx$

$= \dfrac{4x^4}{4} + \dfrac{3x^3}{3} + c$

$\therefore\ y = x^4 + x^3 + c$

4 **a**
$$\begin{aligned}\frac{dy}{dx} &= (1-2x)^2\\ \therefore\quad y &= \int(1-2x)^2\,dx\\ &= \int\left(1-4x+4x^2\right)dx\\ &= x-\frac{4x^2}{2}+\frac{4x^3}{3}+c\\ &= x-2x^2+\tfrac{4}{3}x^3+c\end{aligned}$$

b
$$\begin{aligned}\frac{dy}{dx} &= \sqrt{x}-\frac{2}{\sqrt{x}}\\ &= x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\\ \therefore\quad y &= \int\left(x^{\frac{1}{2}}-2x^{-\frac{1}{2}}\right)dx\\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}+c\\ &= \tfrac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+c\end{aligned}$$

c
$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2+2x-5}{x^2}\\ &= 1+2x^{-1}-5x^{-2}\\ \therefore\quad y &= \int\left(1+2x^{-1}-5x^{-2}\right)dx = x+2\ln|x|-\frac{5x^{-1}}{-1}+c = x+2\ln|x|+\frac{5}{x}+c\end{aligned}$$

5 **a**
$$\begin{aligned}f'(x) &= x^3-5x+3\\ \therefore\quad f(x) &= \int(x^3-5x+3)\,dx\\ &= \frac{x^4}{4}-\frac{5x^2}{2}+3x+c\end{aligned}$$

b
$$\begin{aligned}f'(x) &= 2\sqrt{x}(1-3x)\\ &= 2x^{\frac{1}{2}}-6x^{\frac{3}{2}}\\ \therefore\quad f(x) &= \int\left(2x^{\frac{1}{2}}-6x^{\frac{3}{2}}\right)dx\\ &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{6x^{\frac{5}{2}}}{\frac{5}{2}}+c\\ &= \tfrac{4}{3}x^{\frac{3}{2}}-\tfrac{12}{5}x^{\frac{5}{2}}+c\end{aligned}$$

c
$$\begin{aligned}f'(x) &= 3e^x-\frac{4}{x}\\ \therefore\quad f(x) &= \int\left(3e^x-\frac{4}{x}\right)dx\\ &= 3e^x-4\ln|x|+c\end{aligned}$$

6 **a** $f'(x)=2x-1$ and $f(0)=3$
$$\begin{aligned}f(x) &= \int(2x-1)\,dx\\ &= \frac{2x^2}{2}-x+c\\ &= x^2-x+c\end{aligned}$$
But $f(0)=3$

$\therefore\quad 0-0+c=3$

i.e., $c=3$

$\therefore\quad f(x)=x^2-x+3$

b $f'(x)=3x^2+2x$ and $f(2)=5$
$$\begin{aligned}f(x) &= \int(3x^2+2x)\,dx\\ &= \frac{3x^3}{3}+\frac{2x^2}{2}+c\\ &= x^3+x^2+c\end{aligned}$$
But $f(2)=5$

$\therefore\quad 8+4+c=5$

i.e., $c=-7$

$\therefore\quad f(x)=x^3+x^2-7$

c $f'(x)=e^x+\dfrac{1}{\sqrt{x}}$ and $f(1)=1$
$$\begin{aligned}f'(x) &= e^x+x^{-\frac{1}{2}}\\ f(x) &= \int\left(e^x+x^{-\frac{1}{2}}\right)dx\\ &= e^x+2x^{\frac{1}{2}}+c\end{aligned}$$
But $f(1)=1$

$\therefore\quad e^1+2+c=1$

i.e., $c=-1-e$

$\therefore\quad f(x)=e^x+2\sqrt{x}-1-e$

d $f'(x)=x-\dfrac{2}{\sqrt{x}}$ and $f(1)=2$
$$\begin{aligned}f'(x) &= x-2x^{-\frac{1}{2}}\\ f(x) &= \int\left(x-2x^{-\frac{1}{2}}\right)dx\\ &= \frac{x^2}{2}-\frac{2x^{\frac{1}{2}}}{\frac{1}{2}}+c\\ &= \tfrac{1}{2}x^2-4\sqrt{x}+c\end{aligned}$$
But $f(1)=2$

$\therefore\quad \tfrac{1}{2}-4+c=2$ i.e., $c=\tfrac{11}{2}$

$\therefore\quad f(x)=\tfrac{1}{2}x^2-4\sqrt{x}+\tfrac{11}{2}$

7 **a** $f''(x) = 2x+1 \qquad f'(1)=3 \qquad f(2)=7$

Now $f'(x) = \dfrac{2x^2}{2} + x + c = x^2 + x + c$

But $f'(1) = 3$

$\therefore \; 1+1+c = 3$

$\therefore \; c = 1$

$\therefore \; f'(x) = x^2+x+1$

$\therefore \; f(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2} + x + k$

and $f(2) = 7$

$\therefore \; \frac{8}{3} + 2 + 2 + k = 7$

$k = 7 - 4 - \frac{8}{3}$

$\therefore \; k = \frac{1}{3}$

$\therefore \; f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}} \qquad f'(1) = 12 \qquad f(0) = 5$

Now $f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$\therefore \; f'(x) = \dfrac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$

But $f'(1) = 12 \;\; \therefore \;\; 10+6+c = 12$ and so $c = -4$

$\therefore \; f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$

$\therefore \; f(x) = \dfrac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \dfrac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$

But $f(0) = 5 \;\; \therefore \;\; k = 5 \qquad \therefore \;\; f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c $f''(x) = 2x \qquad (1,\,0)$ and $(0,\,5)$ lie on the curve

Now $f'(x) = \dfrac{2x^2}{2} + c = x^2 + c$ and $f(x) = \dfrac{x^3}{3} + cx + k$

But $f(0) = 5 \;\; \therefore \;\; 0+0+k = 5$ and so $k = 5$

and $f(1) = 0 \;\; \therefore \;\; \frac{1}{3} + c + 5 = 0$ and so $c = -5\frac{1}{3}$

$\therefore \; f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 25C

1 **a** $\int (2x+5)^3 \, dx$

$= \frac{1}{2} \times \dfrac{(2x+5)^4}{4} + c$

$= \frac{1}{8}(2x+5)^4 + c$

b $\displaystyle\int \frac{1}{(3-2x)^2} \, dx$

$= \int (3-2x)^{-2} \, dx$

$= \frac{1}{-2} \times \dfrac{(3-2x)^{-1}}{-1} + c$

$= \dfrac{1}{2(3-2x)} + c$

c $\displaystyle\int \frac{4}{(2x-1)^4} \, dx$

$= \int 4\,(2x-1)^{-4} \, dx$

$= 4(\frac{1}{2}) \times \dfrac{(2x-1)^{-3}}{-3} + c$

$= -\frac{2}{3}(2x-1)^{-3} + c$

d $\int (4x-3)^7 \, dx$

$= \frac{1}{4} \times \dfrac{(4x-3)^8}{8} + c$

$= \frac{1}{32}(4x-3)^8 + c$

e $\int \sqrt{3x-4} \, dx$

$= \int (3x-4)^{\frac{1}{2}} \, dx$

$= \frac{1}{3} \times \dfrac{(3x-4)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + c$

f $\displaystyle\int \frac{10}{\sqrt{1-5x}} \, dx$

$= \int 10(1-5x)^{-\frac{1}{2}} \, dx$

$= 10(\frac{1}{-5}) \times \dfrac{(1-5x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -4(1-5x)^{\frac{1}{2}} + c$

g $\int 3(1-x)^4\ dx$

$= 3\int (1-x)^4\ dx$

$= 3(\frac{1}{-1}) \times \dfrac{(1-x)^5}{5} + c$

$= -\frac{3}{5}(1-x)^5 + c$

h $\displaystyle\int \frac{4}{\sqrt{3-4x}}\ dx = \int 4(3-4x)^{-\frac{1}{2}}\ dx$

$= 4(\frac{1}{-4}) \times \dfrac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -2\sqrt{3-4x} + c$

2 **a** $\dfrac{dy}{dx} = \sqrt{2x-7}$

$= (2x-7)^{\frac{1}{2}}$

$\therefore\ \ y = \frac{1}{2} \times \dfrac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{1}{3}(2x-7)^{\frac{3}{2}} + c$

But $y = 11$ when $x = 8$

$\therefore\ \ \frac{1}{3}(9)^{\frac{3}{2}} + c = 11$

$\frac{1}{3}(27) + c = 11$

$9 + c = 11$ and so $c = 2$

$\therefore\ \ y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$

b $f(x)$ has tangent-slope function $\dfrac{4}{\sqrt{1-x}}$

i.e., $f'(x) = \dfrac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}}$

$f(x) = 4(\frac{1}{-1}) \times \dfrac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -8\sqrt{1-x} + c$

But $y = 11$ when $x = -3$

$\therefore\ \ -8\sqrt{1-(-3)} + c = 11$

$\therefore\ \ -8\sqrt{4} + c = 11$

$\therefore\ \ -16 + c = 11$ and $c = 5$

$\therefore\ \ f(x) = 5 - 8\sqrt{1-x}$

When $x = -8$, $y = 5 - 8\sqrt{1-(-8)} = 5 - 8(3) = -19$, $\therefore$ point is $(-8, -19)$.

3 **a** $\int 3(2x-1)^2\ dx$

$= 3\int (2x-1)^2\ dx$

$= 3\ (\dfrac{1}{2})\ \dfrac{(2x-1)^3}{3} + c$

$= \frac{1}{2}(2x-1)^3 + c$

b $\int (x^2-x)^2\ dx$

$= \int \left(x^4 - 2x^3 + x^2\right)\ dx$

$= \dfrac{x^5}{5} - \dfrac{2x^4}{4} + \dfrac{x^3}{3} + c$

$= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

c $\int (1-3x)^3\ dx$

$= (\dfrac{1}{-3})\ \dfrac{(1-3x)^4}{4} + c$

$= -\frac{1}{12}(1-3x)^4 + c$

d $\int (1-x^2)^2\ dx$

$= \int (1 - 2x^2 + x^4)\ dx$

$= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

e $\int 4\sqrt{5-x}\ dx$

$= 4\int (5-x)^{\frac{1}{2}}\ dx$

$= 4\ (\dfrac{1}{-1})\ \dfrac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c$

f $\int (x^2+1)^3\ dx$

$= \int (x^6 + 3x^4 + 3x^2 + 1)\ dx$

$= \dfrac{x^7}{7} + \dfrac{3x^5}{5} + \dfrac{3x^3}{3} + x + c$

$= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

4 **a** $\int \left(2e^x + 5e^{2x}\right)\ dx$

$= 2e^x + 5(\frac{1}{2})e^{2x} + c$

$= 2e^x + \frac{5}{2}e^{2x} + c$

b $\int \left(x^2 - 2e^{-3x}\right)\ dx$

$= \dfrac{x^3}{3} - 2(\frac{1}{-3})e^{-3x} + c$

$= \frac{1}{3}x^3 + \frac{2}{3}e^{-3x} + c$

c $\int \left(\sqrt{x} + 4e^{2x} - e^{-x}\right)\ dx$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + 4(\frac{1}{2})e^{2x} - (-1)e^{-x} + c$

$= \frac{2}{3}x^{\frac{3}{2}} + 2e^{2x} + e^{-x} + c$

d $\displaystyle\int \frac{1}{2x-1}\ dx$

$= \frac{1}{2}\ln|2x-1| + c$

e $\displaystyle\int \frac{5}{1-3x}\ dx$

$= 5\displaystyle\int \frac{1}{1-3x}\ dx$

$= 5(\frac{1}{-3})\ln|1-3x| + c$

$= -\frac{5}{3}\ln|1-3x| + c$

f $\displaystyle\int \left(e^{-x} - \frac{4}{2x+1}\right)\ dx$

$= \frac{1}{-1}e^{-x} - 4(\frac{1}{2})\ln|2x+1| + c$

$= -e^{-x} - 2\ln|2x+1| + c$

g
$$\begin{aligned}&\int (e^x+e^{-x})^2\;dx\\ &=\int\left(e^{2x}+2+e^{-2x}\right)\,dx\\ &=\tfrac{1}{2}e^{2x}+2x+(\tfrac{1}{-2})e^{-2x}+c\\ &=\tfrac{1}{2}e^{2x}+2x-\tfrac{1}{2}e^{-2x}+c\end{aligned}$$

h
$$\begin{aligned}&\int (e^{-x}+2)^2\;dx\\ &=\int\left(e^{-2x}+4e^{-x}+4\right)\,dx\\ &=\tfrac{1}{-2}e^{-2x}+4(\tfrac{1}{-1})e^{-x}+4x+c\\ &=-\tfrac{1}{2}e^{-2x}-4e^{-x}+4x+c\end{aligned}$$

i
$$\begin{aligned}\int\left(x-\frac{5}{1-x}\right)dx &= \frac{x^2}{2}-5(\tfrac{1}{-1})\ln|1-x|+c\\ &=\tfrac{1}{2}x^2+5\ln|1-x|+c\end{aligned}$$

5 **a**
$$\begin{aligned}\frac{dy}{dx}&=(1-e^x)^2\\ &=1-2e^x+e^{2x}\\ \therefore\quad y&=x-2e^x+\tfrac{1}{2}e^{2x}+c\end{aligned}$$

b
$$\begin{aligned}\frac{dy}{dx}&=1-2x+\frac{3}{x+2}\\ \therefore\quad y&=x-\frac{2x^2}{2}+3\ln|x+2|+c\\ &=x-x^2+3\ln|x+2|+c\end{aligned}$$

c
$$\frac{dy}{dx}=e^{-2x}+\frac{4}{2x-1}\qquad \therefore\quad \begin{aligned}y&=\tfrac{1}{-2}e^{-2x}+4(\tfrac{1}{2})\ln|2x-1|+c\\ &=-\tfrac{1}{2}e^{-2x}+2\ln|2x-1|+c\end{aligned}$$

6 Differentiating Tracy's answer gives

$$\begin{aligned}&\frac{d}{dx}\left(\tfrac{1}{4}\ln|4x|+c\right)\\ &=\tfrac{1}{4}\left(\frac{1}{4x}\right)\times 4+0\\ &=\frac{1}{4x}\end{aligned}$$

The derivative of Nadine's answer is

$$\begin{aligned}&\frac{d}{dx}(\tfrac{1}{4}\ln|x|+c)\\ &=\tfrac{1}{4}\left(\frac{1}{x}\right)+0\\ &=\frac{1}{4x}\end{aligned}$$

$\therefore$ both answers give the correct derivative and both are correct.

Note that this result occurs because $\log 4x=\log 4+\log x$ $\therefore$ their answers differ by a constant ($\log 4$) which is just part of the constant *c*.

7 **a**
$$\begin{aligned}f'(x)&=2e^{-2x}\qquad f(0)=3\\ \therefore\quad f(x)&=2(\tfrac{1}{-2})e^{-2x}+c\\ &=-e^{-2x}+c\end{aligned}$$

But $f(0)=3$
$$\begin{aligned}\therefore\quad -e^0+c&=3\\ \therefore\quad c&=4\\ \therefore\quad f(x)&=-e^{-2x}+4\end{aligned}$$

b
$$\begin{aligned}f'(x)&=2x-\frac{2}{1-x}\qquad f(-1)=3\\ \therefore\quad f(x)&=\frac{2x^2}{2}-\tfrac{2}{-1}\ln|1-x|+c\\ &=x^2+2\ln|1-x|+c\end{aligned}$$

But $f(-1)=3$
$$\begin{aligned}\therefore\quad 1+2\ln|2|+c&=3\\ \therefore\quad c&=2-2\ln 2\\ \therefore\quad f(x)&=x^2+2\ln|1-x|+2-2\ln 2\end{aligned}$$

c
$$\begin{aligned}f'(x)&=\sqrt{x}+\tfrac{1}{2}e^{-4x}\\ \therefore\quad f'(x)&=x^{\frac{1}{2}}+\tfrac{1}{2}e^{-4x}\\ \text{and}\quad f(x)&=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\tfrac{1}{2}(\tfrac{1}{-4})e^{-4x}+c\\ &=\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{8}e^{-4x}+c\end{aligned}$$

But $f(1)=0$
$$\begin{aligned}\therefore\quad \tfrac{2}{3}-\tfrac{1}{8}e^{-4}+c&=0\\ \therefore\quad c&=\tfrac{1}{8}e^{-4}-\tfrac{2}{3}\\ \therefore\quad f(x)&=\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{8}e^{-4x}+\tfrac{1}{8}e^{-4}-\tfrac{2}{3}\end{aligned}$$

8 $\dfrac{3}{x+2} - \dfrac{1}{x-2}$

$= \dfrac{3(x-2) - 1(x+2)}{(x+2)(x-2)}$

$= \dfrac{3x-6-x-2}{x^2-4}$

$= \dfrac{2x-8}{x^2-4}$

$\therefore \displaystyle\int \frac{2x-8}{x^2-4}\,dx$

$= \displaystyle\int \left(\frac{3}{x+2} - \frac{1}{x-2}\right) dx$

$= 3\ln|x+2| - \ln|x-2| + c$

9 $\dfrac{1}{2x-1} - \dfrac{1}{2x+1}$

$= \dfrac{1(2x+1) - 1(2x-1)}{(2x-1)(2x+1)}$

$= \dfrac{2x+1-2x+1}{(2x-1)(2x+1)}$

$= \dfrac{2}{4x^2-1}$

$\therefore \displaystyle\int \frac{2}{4x^2-1}\,dx$

$= \displaystyle\int \left(\frac{1}{2x-1} - \frac{1}{2x+1}\right) dx$

$= \frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|2x+1| + c$

EXERCISE 25D

1 **a** Let $u = x^3+1$, $\therefore \dfrac{du}{dx} = 3x^2$

$\therefore \int 3x^2(x^3+1)^4\,dx$

$= \int u^4 \dfrac{du}{dx}\,dx$

$= \int u^4\,du$

$= \frac{1}{5}u^5 + c$

$= \frac{1}{5}(x^3+1)^5 + c$

b Let $u = x^2+3$ $\therefore \dfrac{du}{dx} = 2x$

$\therefore \displaystyle\int \frac{2x}{\sqrt{x^2+3}}\,dx$

$= \int \left(\left(x^2+3\right)^{-\frac{1}{2}} \times 2x\right) dx$

$= \int u^{-\frac{1}{2}} \dfrac{du}{dx}\,dx$

$= \int u^{-\frac{1}{2}}\,du$

$= \dfrac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{u} + c$

$= 2\sqrt{x^2+3} + c$

c Let $u = x^3+x$ $\therefore \dfrac{du}{dx} = 3x^2+1$

$\therefore \int \sqrt{x^3+x}\,(3x^2+1)\,dx$

$= \int \sqrt{u}\,\dfrac{du}{dx}\,dx$

$= \int u^{\frac{1}{2}}\,du$

$= \dfrac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{3}(x^3+x)^{\frac{3}{2}} + c$

d Let $u = 2+x^4$ $\therefore \dfrac{du}{dx} = 4x^3$

$\therefore \int 4x^3(2+x^4)^3\,dx$

$= \int u^3 \dfrac{du}{dx}\,dx$

$= \int u^3\,du$

$= \dfrac{u^4}{4} + c$

$= \frac{1}{4}(2+x^4)^4 + c$

e Let $u = x^3 + 2x + 1 \quad \therefore \quad \dfrac{du}{dx} = 3x^2 + 2$

$$\begin{aligned}
\therefore \quad & \int (x^3 + 2x + 1)^4 (3x^2 + 2)\ dx \\
&= \int u^4\ \frac{du}{dx}\ dx \\
&= \int u^4\ du \\
&= \frac{u^5}{5} + c \\
&= \tfrac{1}{5}(x^3 + 2x + 1)^5 + c
\end{aligned}$$

f Let $u = 3x^3 - 1 \quad \therefore \quad \dfrac{du}{dx} = 9x^2$

$$\begin{aligned}
\therefore \quad & \int \frac{x^2}{(3x^3 - 1)^4}\ dx \\
&= \int (3x^3 - 1)^{-4} \times x^2)\ dx \\
&= \int u^{-4} \left(\tfrac{1}{9}\,\frac{du}{dx}\right) dx \\
&= \tfrac{1}{9} \int u^{-4}\ du \\
&= \tfrac{1}{9}\,\frac{u^{-3}}{-3} + c \quad \text{i.e.,} \quad -\frac{1}{27(3x^3 - 1)^3} + c
\end{aligned}$$

g Let $u = 1 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}
\therefore \quad & \int \frac{x}{(1 - x^2)^5}\ dx \\
&= \int \left((1 - x^2)^{-5} \times x\right) dx \\
&= \int u^{-5} \left(-\tfrac{1}{2}\,\frac{du}{dx}\right) dx \\
&= -\tfrac{1}{2} \int u^{-5}\ du \\
&= -\tfrac{1}{2}\,\frac{u^{-4}}{-4} + c \\
&= \frac{1}{8(1 - x^2)^4} + c
\end{aligned}$$

h Let $u = x^2 + 4x - 3 \quad \therefore \quad \dfrac{du}{dx} = 2x + 4$

$$\begin{aligned}
\therefore \quad & \int \frac{x + 2}{(x^2 + 4x - 3)^2}\ dx \\
&= \int (x^2 + 4x - 3)^{-2} (x + 2))\ dx \\
&= \int u^{-2} \left(\tfrac{1}{2}\,\frac{du}{dx}\right) dx \\
&= \tfrac{1}{2} \int u^{-2}\ du \\
&= \tfrac{1}{2}\,\frac{u^{-1}}{-1} + c \\
&= \frac{-1}{2(x^2 + 4x - 3)} + c
\end{aligned}$$

i Let $u = x^2 + x \quad \therefore \quad \dfrac{du}{dx} = 2x + 1 \quad \therefore$

$$\begin{aligned}
& \int x^4 (x + 1)^4 (2x + 1)\ dx \\
&= \int (x^2 + x)^4 (2x + 1)\ dx \quad \{\text{as } a^4 b^4 = (ab)^4\} \\
&= \int u^4\ \frac{du}{dx}\ dx = \int u^4\ du \\
&= \tfrac{1}{5} u^5 + c \\
&= \tfrac{1}{5}(x^2 + x)^5 + c
\end{aligned}$$

2 **a** Let $u = 1 - 2x \quad \therefore \quad \dfrac{du}{dx} = -2$

$$\begin{aligned}
\therefore \quad \int -2e^{1-2x}\ dx &= \int e^u\ \frac{du}{dx}\ dx \\
&= \int e^u\ du \\
&= e^u + c \\
&= e^{1-2x} + c
\end{aligned}$$

b Let $u = x^2 \quad \therefore \quad \dfrac{du}{dx} = 2x$

$$\begin{aligned}
\therefore \quad \int 2xe^{x^2}\ dx &= \int e^u\ \frac{du}{dx}\ dx \\
&= \int e^u\ du \\
&= e^u + c \\
&= e^{x^2} + c
\end{aligned}$$

c Let $u = x^3 + 1 \quad \therefore \quad \dfrac{du}{dx} = 3x^2$

$$\begin{aligned}
\therefore \quad & \int x^2 e^{x^3+1}\ dx \\
&= \int e^u \left(\tfrac{1}{3}\,\frac{du}{dx}\right) dx \\
&= \tfrac{1}{3} \int e^u\ du \\
&= \tfrac{1}{3} e^u + c \\
&= \tfrac{1}{3} e^{x^3+1} + c
\end{aligned}$$

d Let $u = \sqrt{x} \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

$$\begin{aligned}
\therefore \quad & \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx \\
&= \int e^u \left(2\,\frac{du}{dx}\right) dx \\
&= 2 \int e^u\ du \\
&= 2e^u + c \\
&= 2e^{\sqrt{x}} + c
\end{aligned}$$

e Let $u = x - x^2 \quad \therefore \quad \dfrac{du}{dx} = 1 - 2x$

$$\begin{aligned}\therefore \ &\textstyle\int (2x-1)e^{x-x^2}\,dx\\ &= \textstyle\int e^u \left(-\dfrac{du}{dx}\right) dx\\ &= \textstyle\int -e^u\,du\\ &= -e^u + c\\ &= -e^{x-x^2} + c\end{aligned}$$

f Let $u = \dfrac{x-1}{x} = 1 - x^{-1} \quad \therefore \quad \dfrac{du}{dx} = 0 - (-1)x^{-2} = \dfrac{1}{x^2}$

$$\begin{aligned}\therefore \int \frac{e^{\frac{x-1}{x}}}{x^2}\,dx &= \textstyle\int e^u \dfrac{du}{dx}\,dx\\ &= \textstyle\int e^u\,du\\ &= e^u + c\\ &= e^{\frac{x-1}{x}} + c\end{aligned}$$

3 **a** Let $u = x^2 + 1 \quad \therefore \quad \dfrac{du}{dx} = 2x$

$$\begin{aligned}\therefore \ &\int \frac{2x}{x^2+1}\,dx\\ &= \int \frac{1}{x^2+1}\,(2x)\,dx\\ &= \int \frac{1}{u}\,\frac{du}{dx}\,dx\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + c\\ &= \ln(x^2+1) + c \quad \text{as} \quad x^2 + 1 > 0\end{aligned}$$

b Let $u = 2 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore \ &\int \frac{x}{2-x^2}\,dx\\ &= \int \left(\frac{1}{2-x^2} \times x\right) dx\\ &= \int \frac{1}{u}\left(-\tfrac{1}{2}\,\frac{du}{dx}\right) dx\\ &= -\tfrac{1}{2}\int \frac{1}{u}\,du\\ &= -\tfrac{1}{2}\ln|u| + c\\ &= -\tfrac{1}{2}\ln\left|2 - x^2\right| + c\end{aligned}$$

c Let $u = x^2 - 3x \quad \therefore \quad \dfrac{du}{dx} = 2x - 3$

$$\begin{aligned}\therefore \ &\int \frac{2x-3}{x^2-3x}\,dx\\ &= \int \frac{1}{x^2-3x}\,(2x-3)\,dx\\ &= \int \frac{1}{u}\,\frac{du}{dx}\,dx\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + c\\ &= \ln\left|x^2 - 3x\right| + c\end{aligned}$$

d Let $u = x^3 - x \quad \therefore \quad \dfrac{du}{dx} = 3x^2 - 1$

$$\begin{aligned}\therefore \ &\int \frac{6x^2-2}{x^3-x}\,dx\\ &= \int \frac{1}{x^3-3x}\,(6x^2-2)\,dx\\ &= \int \frac{1}{u}\left(2\,\frac{du}{dx}\right) dx\\ &= 2\int \frac{1}{u}\,du\\ &= 2\ln|u| + c\\ &= 2\ln\left|x^3 - x\right| + c\end{aligned}$$

e Let $u = 5x - x^2 \quad \therefore \quad \dfrac{du}{dx} = 5 - 2x$

$$\begin{aligned}\therefore \ &\int \frac{4x-10}{5x-x^2}\,dx\\ &= \int \frac{1}{5x-x^2}\,(4x-10)\,dx\\ &= \int \frac{1}{u}\left(-2\,\frac{du}{dx}\right) dx\\ &= -2\int \frac{1}{u}\,dx\\ &= -2\ln|u| + c\\ &= -2\ln\left|5x - x^2\right| + c\end{aligned}$$

f Let $u = x^3 - 3x \quad \therefore \quad \dfrac{du}{dx} = 3x^2 - 3$

$$\begin{aligned}\therefore \ &\int \frac{1-x^2}{x^3-3x}\,dx\\ &= \int \frac{1}{x^3-3x}\,(1-x^2)\,dx\\ &= \int \frac{1}{u}\left(-\tfrac{1}{3}\,\frac{du}{dx}\right) dx\\ &= -\tfrac{1}{3}\int \frac{1}{u}\,du\\ &= -\tfrac{1}{3}\ln|u| + c\\ &= -\tfrac{1}{3}\ln\left|x^3 - 3x\right| + c\end{aligned}$$

4 **a** Let $u = 3 - x^3 \quad \therefore \quad \dfrac{du}{dx} = -3x^2$

$$\begin{aligned}\therefore \quad f(x) &= \textstyle\int x^2(3-x^3)^2 \; dx \\ &= \int u^2 \left(-\tfrac{1}{3}\,\frac{du}{dx}\right) dx \qquad \{\} \\ &= -\tfrac{1}{3}\textstyle\int u^2 \; du \\ &= -\tfrac{1}{3} \times \frac{u^3}{3} + c \\ &= -\tfrac{1}{9}(3-x^3)^3 + c\end{aligned}$$

b Let $u = \ln x \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{4}{x \ln x} \; dx \\ &= \int u^{-1} \left(4\,\frac{du}{dx}\right) dx \\ &= 4 \int \frac{1}{u} \; du \\ &= 4 \ln |u| + c \\ &= 4 \ln |\ln x| + c\end{aligned}$$

c Let $u = 1 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore f(x) &= \textstyle\int x\sqrt{1-x^2} \; dx \\ &= \int \sqrt{u} \left(-\tfrac{1}{2}\,\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\textstyle\int u^{\frac{1}{2}} \; du \\ &= -\tfrac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= -\tfrac{1}{3}u^{\frac{3}{2}} + c \\ &= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + c\end{aligned}$$

d Let $u = 1 - x^2 \quad \therefore \quad \dfrac{du}{dx} = -2x$

$$\begin{aligned}\therefore \quad f(x) &= \textstyle\int xe^{1-x^2} \; dx \\ &= \int e^u \left(-\tfrac{1}{2}\,\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\textstyle\int e^u \; du \\ &= -\tfrac{1}{2}e^u + c \\ &= -\tfrac{1}{2}e^{1-x^2} + c\end{aligned}$$

e Let $u = x^3 - x \quad \therefore \quad \dfrac{du}{dx} = 3x^2 - 1$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{1-3x^2}{x^3-x} \; dx \\ &= \int \frac{1}{u} \left(-\frac{du}{dx}\right) dx \\ &= -\int \frac{1}{u}\, du \\ &= -\ln |u| + c \\ &= -\ln \left|x^3 - x\right| + c\end{aligned}$$

f Let $u = \ln x \quad \therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

$$\begin{aligned}\therefore \quad f(x) &= \int \frac{(\ln x)^3}{x} \; dx \\ &= \textstyle\int u^3 \, \frac{du}{dx}\, dx \\ &= \textstyle\int u^3 \; du \\ &= \frac{u^4}{4} + c \\ &= \tfrac{1}{4}(\ln x)^4 + c\end{aligned}$$

EXERCISE 25E.1

1 **a**

$$\begin{aligned} &\textstyle\int_0^1 x^3 \; dx \\ &= \left[\frac{x^4}{4}\right]_0^1 \\ &= \tfrac{1}{4} - 0 \\ &= \tfrac{1}{4}\end{aligned}$$

b

$$\begin{aligned} &\textstyle\int_0^2 (x^2 - x)\, dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^2 \\ &= (\tfrac{8}{3} - 2) - (0 - 0) \\ &= \tfrac{2}{3}\end{aligned}$$

c

$$\begin{aligned} &\textstyle\int_0^1 e^x \; dx \\ &= [e^x]_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \\ &\doteqdot 1.718\end{aligned}$$

d $\displaystyle\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx$

$= \int_1^4 \left(x - 3x^{-\frac{1}{2}}\right)\ dx$

$= \left[\dfrac{x^2}{2} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$

$= \left[\dfrac{x^2}{2} - 6\sqrt{x}\right]_1^4$

$= \left[\frac{16}{2} - 12\right] - (\frac{1}{2} - 6)$

$= 1\frac{1}{2}$

e $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\ dx$

$= \int_4^9 \left(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\right)\ dx$

$= \left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}}\right]_4^9$

$= \left[\frac{2}{3}(27) - 6(3)\right] - \left[\frac{2}{3}(8) - 6(2)\right]$

$= (18 - 18) - (\frac{16}{3} - 12)$

$= 6\frac{2}{3}$

f $\displaystyle\int_1^3 \frac{1}{x}\ dx$

$= [\ln|x|]_1^3$

$= \ln 3 - \ln 1$

$= \ln 3 - 0$

$= \ln 3$

$\doteq 1.099$

g $\int_1^2 (e^{-x} + 1)^2\ dx$

$= \int_1^2 (e^{-2x} + 2e^{-x} + 1)\ dx$

$= \left[(\frac{1}{-2})e^{-2x} + 2(\frac{1}{-1})e^{-x} + x\right]_1^2$

$= \left[-\dfrac{e^{-2x}}{2} - 2e^{-x} + x\right]_1^2$

$= \left(-\dfrac{e^{-4}}{2} - 2e^{-2} + 2\right) - \left(-\dfrac{e^{-2}}{2} - 2e^{-1} + 1\right)$

$\doteq 1.524$

h $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\ dx$

$= \int_2^6 (2x-3)^{-\frac{1}{2}}\ dx$

$= \left[\frac{1}{2}\dfrac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^6$

$= \left[\sqrt{2x-3}\right]_2^6$

$= \sqrt{9} - \sqrt{1}$

$= 2$

i $\int_0^1 e^{1-x}\ dx \quad = \left[(\frac{1}{-1})e^{1-x}\right]_0^1$

$= \left(\dfrac{e^0}{-1}\right) - \left(\dfrac{e^1}{-1}\right)$

$= -1 + e$

$\doteq 1.718$

2 **a** In $\displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\ dx$

we let $u = x^2 + 2$

$\therefore \quad \dfrac{du}{dx} = 2x$

and when $x = 1, \quad u = 3$

when $x = 2, \quad u = 6$

$\therefore \displaystyle\int_1^2 \frac{x}{(x^2+2)^2}\ dx = \int_1^2 u^{-2}\left(\frac{1}{2}\frac{du}{dx}\right) dx$

$= \frac{1}{2}\int_3^6 u^{-2}\ du$

$= \frac{1}{2}\left[\dfrac{u^{-1}}{-1}\right]_3^6$

$= \frac{1}{2}\left[-\frac{1}{6} - (-\frac{1}{3})\right]$

$= \frac{1}{12}$

b In $\int_0^1 x^2 e^{x^3+1}\ dx$

we let $u = x^3 + 1$

$\therefore \quad \dfrac{du}{dx} = 3x^2$

and when $x = 0, \quad u = 1$

when $x = 1, \quad u = 2$

$\therefore \int_0^1 x^2 e^{x^3+1}\ dx = \int_0^1 e^u\left(\frac{1}{3}\frac{du}{dx}\right) dx$

$= \frac{1}{3}\int_1^2 e^u\ du$

$= \frac{1}{3}\left[e^u\right]_1^2$

$= \frac{1}{3}(e^2 - e) \qquad (\doteq 1.557)$

c In $\int_0^3 x\sqrt{x^2+16}\;dx$

we let $u = x^2+16$

$\therefore \quad \dfrac{du}{dx} = 2x$

and when $x=0, \quad u=16$

when $x=3, \quad u=25$

$$
\begin{aligned}
\therefore \quad \int_0^3 x\sqrt{x^2+16}\;dx &= \int_0^3 u^{\frac{1}{2}}\left(\tfrac{1}{2}\,\frac{du}{dx}\right)\,dx \\
&= \tfrac{1}{2}\int_{16}^{25} u^{\frac{1}{2}}\,du \\
&= \tfrac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{16}^{25} \\
&= \tfrac{1}{2}\times\tfrac{2}{3}\left[u^{\frac{3}{2}}\right]_{16}^{25} \\
&= \tfrac{1}{3}(125-64) = 20\tfrac{1}{3}
\end{aligned}
$$

d In $\int_1^2 xe^{-2x^2}\;dx$

we let $u = -2x^2$

$\therefore \quad \dfrac{du}{dx} = -4x$

and when $x=1, \quad u=-2$

when $x=2, \quad u=-8$

$$
\begin{aligned}
\therefore \quad \int_1^2 xe^{-2x^2}\;dx &= \int_1^2 e^u\;\left(-\tfrac{1}{4}\,\frac{du}{dx}\right)\,dx \\
&= -\tfrac{1}{4}\int_{-2}^{-8} e^u\;du \\
&= -\tfrac{1}{4}\left[e^u\right]_{-2}^{-8} \\
&= -\tfrac{1}{4}(e^{-8}-e^{-2}) \\
&\doteqdot 0.0337
\end{aligned}
$$

e In $\displaystyle\int_2^3 \frac{x}{2-x^2}\;dx$

we let $u = 2-x^2$

$\therefore \quad \dfrac{du}{dx} = -2x$

and when $x=2, \quad u=-2$

when $x=3, \quad u=-7$

$$
\begin{aligned}
\therefore \quad \int_2^3 \frac{x}{2-x^2} &= \int_2^3 \frac{1}{u}\left(-\tfrac{1}{2}\,\frac{du}{dx}\right)\,dx \\
&= -\tfrac{1}{2}\int_{-2}^{-7} \frac{1}{u}\,du \\
&= -\tfrac{1}{2}\left[\ln|u|\right]_{-2}^{-7} \\
&= -\tfrac{1}{2}(\ln 7 - \ln 2) \\
&= -\tfrac{1}{2}\ln(\tfrac{7}{2}) \qquad (\doteqdot -0.6264)
\end{aligned}
$$

f In $\displaystyle\int_1^2 \frac{\ln x}{x}\;dx$

we let $u = \ln x$

$\therefore \quad \dfrac{du}{dx} = \dfrac{1}{x}$

and when $x=1, \quad u=0$

when $x=2, \quad u=\ln 2$

$$
\begin{aligned}
\therefore \quad \int_1^2 \frac{\ln x}{x}\;dx &= \int_1^2 u\;\frac{du}{dx}\;dx \\
&= \int_0^{\ln 2} u\;du \\
&= \left[\frac{u^2}{2}\right]_0^{\ln 2} \\
&= \frac{(\ln 2)^2}{2} - 0 \\
&\doteqdot 0.2402
\end{aligned}
$$

g In $\displaystyle\int_0^1 \frac{1-3x^2}{1-x^3+x}\;dx$ we let $u = 1-x^3+x$

$\therefore \quad \dfrac{du}{dx} = -3x^2+1$ and when $x=0, \quad u=1$

when $x=1, \quad u=1$

$$
\begin{aligned}
\therefore \quad \int_0^1 \frac{1-3x^2}{1-x^3+x}\;dx &= \int_0^1 \frac{1}{u}\,\frac{du}{dx}\;dx \\
&= \int_1^1 \frac{1}{u}\;du \\
&= 0 \qquad \{\text{since limits of integration are the same}\}
\end{aligned}
$$

h In $\displaystyle\int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx$ we let $u = x^3-x^2+2x$

$\therefore\ \dfrac{du}{dx} = 3x^2-2x+2$ and when $x=2,\ u=8$

when $x=4,\ u=56$

$$\begin{aligned}\therefore \int_2^4 \frac{6x^2-4x+4}{x^3-x^2+2x}\,dx &= \int_2^4 \frac{1}{u}\left(2\frac{du}{dx}\right)dx\\ &= 2\textstyle\int_8^{56} \frac{1}{u}\,du\\ &= 2\,[\ln|u|]_8^{56}\\ &= 2(\ln 56-\ln 8)\\ &= 2\ln 7 \qquad (\doteqdot 3.892)\end{aligned}$$

i In $\int_0^1 (x^2+2x)^n(x+1)\,dx$ we let $u = x^2+2x$

$\therefore\ \dfrac{du}{dx} = 2x+2$ and when $x=0,\ u=0$

when $x=1,\ u=3$

$$\therefore\ \textstyle\int_0^1 (x^2+2x)^n(x+1)\,dx = \int_0^1 u^n\left(\frac{1}{2}\frac{du}{dx}\right)dx = \frac{1}{2}\int_0^3 u^n\,du$$

Now if $n \neq -1$, the integral $= \frac{1}{2}\left[\dfrac{u^{n+1}}{n+1}\right]_0^3 = \frac{1}{2}\left(\dfrac{3^{n+1}}{n+1}\right)$

but if $n=-1$, the integral $= \frac{1}{2}\displaystyle\int_0^3 \frac{1}{u}\,du = \frac{1}{2}\,[\ln|u|]_0^3$ which is undefined as $\ln 0$ is not defined

EXERCISE 25E.2

1 **a** $\int_1^4 \sqrt{x}\,dx = 4.667$

$\int_1^4 (-\sqrt{x})\,dx = -4.667$

b $\int_0^1 x^7\,dx = 0.125 = \frac{1}{8}$

$\int_0^1 (-x^7)\,dx = -0.125 = -\frac{1}{8}$

2 **a** $\int_0^1 x^2dx = \frac{1}{3}$ **b** $\int_1^2 x^2dx = \frac{7}{3}$ **c** $\int_0^2 x^2dx = \frac{8}{3}$ **d** $\int_0^1 3x^2dx = 1$

3 **a** $\int_0^2 (x^3-4x)\,dx = -4$ **b** $\int_2^3 (x^3-4x)\,dx = 6\frac{1}{4}$ **c** $\int_0^3 (x^3-4x)\,dx = 2\frac{1}{4}$

4 **a** $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ **b** $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

and $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

5 **a** $\int_0^1 x^2dx = \frac{1}{3}$ **b** $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$

c $\int_0^1 (x^2+\sqrt{x})\,dx = 1$

this indicates that $\int_0^1 (x^2+\sqrt{x})\,dx = \int_0^1 x^2\,dx + \int_0^1 \sqrt{x}\,dx$

and in general $\displaystyle\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b (f(x)+g(x))\,dx$

EXERCISE 25E.3

1 **a** $\int_0^3 f(x)\,dx = 2+3+1.5 = 6.5$ **b** $\int_3^7 f(x)\,dx = -\left(\frac{3}{2}+3+\frac{5}{2}+2\right) = -9$

c $\int_2^4 f(x)dx = 1.5-1.5 = 0$ **d** $\int_0^7 f(x)\,dx = 6.5-9 = -2.5$

2 **a** $\int_0^4 f(x)\,dx = \frac{1}{2}\pi(2)^2 = 2\pi$

b $\int_4^6 f(x)\,dx = -(2\times 2) = -4$

c $\int_6^8 f(x)\,dx = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

d $\int_0^8 f(x)\,dx = 2\pi + (-4) + \frac{\pi}{2} = \frac{5\pi}{2} - 4$

3 **a**
$$\begin{aligned}&\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx\\ &= \int_2^7 f(x)\,dx\end{aligned}$$

b
$$\begin{aligned}&\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx\\ &= \int_1^9 g(x)\,dx\end{aligned}$$

4 **a** $\int_1^3 f(x)\,dx + \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx$

$$\therefore \int_3^6 f(x)\,dx = \int_1^6 f(x)\,dx - \int_1^3 f(x)\,dx = (-3) - (2) = -5$$

b $\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx$

$$\begin{aligned}\therefore \int_2^4 f(x)\,dx &= \int_0^6 f(x)\,dx - \int_4^6 f(x)\,dx - \int_0^2 f(x)\,dx\\ &= (7) - (-2) - (5)\\ &= 4\end{aligned}$$

REVIEW SET 25A

1 **a**
$$\begin{aligned}\int \frac{4}{\sqrt{x}}\,dx &= 4\int x^{-\frac{1}{2}}\,dx\\ &= 4\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= 8\sqrt{x} + c\end{aligned}$$

b
$$\begin{aligned}\int \frac{3}{1-2x}\,dx &= 3\int \frac{1}{1-2x}\,dx\\ &= 3(\tfrac{1}{-2})\ln|1-2x| + c\\ &= -\tfrac{3}{2}\ln|1-2x| + c\end{aligned}$$

c
$$\begin{aligned}\int xe^{1-x^2}\,dx &= \int e^u\left(-\tfrac{1}{2}\frac{du}{dx}\right)du \qquad \{\text{letting } u = 1-x^2 \quad \therefore \frac{du}{dx} = -2x\}\\ &= -\tfrac{1}{2}\int e^u\,du\\ &= -\tfrac{1}{2}e^u + c\\ &= -\tfrac{1}{2}e^{1-x^2} + c\end{aligned}$$

2 **a**
$$\begin{aligned}&\int_{-5}^{-1}\sqrt{1-3x}\,dx\\ &= \int_{-5}^{-1}(1-3x)^{\frac{1}{2}}\,dx\\ &= \left[\tfrac{1}{-3}\times\frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{-1}\\ &= -\tfrac{2}{9}\left[(1-3x)^{\frac{3}{2}}\right]_{-5}^{-1}\\ &= -\tfrac{2}{9}\left[4^{\frac{3}{2}} - 16^{\frac{3}{2}}\right]\\ &= -\tfrac{2}{9}[8-64]\\ &= 12\tfrac{4}{9}\end{aligned}$$

b In $\displaystyle\int_0^1 \frac{4x^2}{(x^3+2)^3}\,dx$

we let $u = x^3 + 2$

$\therefore \dfrac{du}{dx} = 3x^2$ and when $x = 0,\ u = 2$

when $x = 1,\ u = 3$

$$\begin{aligned}\therefore \int_0^1 \frac{4x^2}{(x^3+2)^3}\,dx \quad \int_0^1 \frac{1}{u^3}\left(\tfrac{4}{3}\frac{du}{dx}\right)dx\\ &= \tfrac{4}{3}\int_2^3 u^{-3}\,du\\ &= \tfrac{4}{3}\left[\frac{u^{-2}}{-2}\right]_2^3\\ &= \tfrac{4}{3}\left[-\frac{1}{2u^2}\right]_2^3\\ &= \tfrac{4}{3}\left[(-\tfrac{1}{18}) - (-\tfrac{1}{8})\right]\\ &= \tfrac{5}{54}\end{aligned}$$

3 $y = \sqrt{x^2-4} = (x^2-4)^{\frac{1}{2}}$

$$\begin{aligned}\therefore\quad \frac{dy}{dx} &= \tfrac{1}{2}\left(x^2-4\right)^{-\frac{1}{2}} \times 2x\\ &= \frac{x}{(x^2-4)^{\frac{1}{2}}}\\ &= \frac{x}{\sqrt{x^2-4}}\end{aligned}$$

$$\therefore\quad \int \frac{x}{\sqrt{x^2-4}}\,dx = \sqrt{x^2-4}+c$$

4 $y = f(x) \qquad f(0) = -1 \qquad f(1) = 13$

$$\begin{aligned} f''(x) &= 18x+10\\ \therefore\quad f'(x) &= 9x^2+10x+c\\ \text{and}\quad f(x) &= 3x^3+5x^2+cx+d\\ \text{But}\quad f(0) &= -1 \quad \therefore\quad d=-1\\ \therefore\quad f(x) &= 3x^3+5x^2+ax-1\end{aligned}$$

$$\begin{aligned}\text{But}\quad f(1) &= 13\\ \therefore\quad 3+5+a-1 &= 13\\ \therefore\quad a+7 &= 13\\ \therefore\quad a &= 6\\ \therefore\quad f(x) &= 3x^3+5x^2+6x-1\end{aligned}$$

5 **a** $\dfrac{4x-3}{2x+1} = \dfrac{2(2x+1)-5}{2x+1} = 2+\dfrac{-5}{2x+1} \qquad \therefore\quad A = 2 \quad B=-5$

b
$$\begin{aligned}\int_0^2 \frac{4x-3}{2x+1}\,dx &= \int_0^2 \left(2-5\left(\frac{1}{2x+1}\right)\right)\,dx\\ &= \left[2x-5\left(\tfrac{1}{2}\right)\ln|2x+1|\right]_0^2\\ &= [4-\tfrac{5}{2}\ln 5]-[0-\tfrac{5}{2}\ln 1]\\ &= 4-\tfrac{5}{2}\ln 5\\ &\doteqdot -0.0236\end{aligned}$$

6 **a**
$$\begin{aligned}&\int_3^4 \frac{1}{\sqrt{2x+1}}\,dx\\ &= \int_3^4 (2x+1)^{-\frac{1}{2}}\,dx\\ &= \left[\tfrac{1}{2}\frac{(2x+1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_3^4\\ &= \left[\sqrt{2x+1}\right]_3^4\\ &= \sqrt{9}-\sqrt{7}\\ &= 3-\sqrt{7}\end{aligned}$$

b
$$\begin{aligned}&\int_0^1 x^2e^{x^3+1}\,dx\\ &= \int_1^2 e^u\left(\tfrac{1}{3}\frac{du}{dx}\right)dx\\ &= \tfrac{1}{3}\int_1^2 e^u\,du\\ &= \tfrac{1}{3}\times[e^u]_1^2\\ &= \tfrac{1}{3}(e^2-e)\end{aligned}$$

$u = x^3+1 \qquad \dfrac{du}{dx} = 3x^2$

$u(0) = 1, \quad u(1) = 2$

7 $\displaystyle\int_0^a e^{1-2x}\,dx = \frac{e}{4}$

$$\begin{aligned}\therefore\quad \left[\tfrac{1}{-2}e^{1-2x}\right]_0^a &= \frac{e}{4}\\ \therefore\quad -\tfrac{1}{2}e^{1-2a}--\tfrac{1}{2}e^1 &= \frac{e}{4}\\ \therefore\quad -\tfrac{1}{2}e^{1-2a}+\frac{e}{2} &= \frac{e}{4}\\ \therefore\quad \tfrac{1}{2}e^{1-2a} &= \frac{e}{2}-\frac{e}{4}\\ \therefore\quad \tfrac{1}{2}e^{1-2a} &= \frac{e}{4}\\ \therefore\quad e^{1-2a} &= \frac{e}{2}\end{aligned}$$

$$\begin{aligned}\therefore\quad 1-2a &= \ln\left(\frac{e}{2}\right) = \ln e-\ln 2\\ \therefore\quad 1-2a &= 1-\ln 2\\ \therefore\quad 2a &= \ln 2\\ \therefore\quad a &= \tfrac{1}{2}\ln 2\\ \therefore\quad a &= \ln 2^{\frac{1}{2}}\\ \text{i.e.,}\quad a &= \ln\sqrt{2}\end{aligned}$$

REVIEW SET 25B

1 **a**
$$\int(2e^{-x} - \frac{1}{x} + 3)\ dx$$
$$= (\tfrac{1}{-1})2e^{-x} - \ln|x| + 3x + c$$
$$= -2e^{-x} - \ln|x| + 3x + c$$

b
$$\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2\ dx$$
$$= \int\left(x - 2 + \frac{1}{x}\right)\ dx$$
$$= \frac{x^2}{2} - 2x + \ln|x| + c$$

2 **a**
$$\int_1^2(x^2-1)^2\ dx$$
$$= \int_1^2(x^4 - 2x^2 + 1)\ dx$$
$$= \left[\frac{x^5}{5} - \frac{2x^3}{3} + x\right]_1^2$$
$$= \left[\tfrac{32}{5} - \tfrac{16}{3} + 2\right] - \left[\tfrac{1}{5} - \tfrac{2}{3} + 1\right]$$
$$= \tfrac{31}{5} - \tfrac{14}{3} + 1$$
$$= 2\tfrac{8}{15}$$

b In $\int_1^2 x(x^2-1)^2\ dx$

We let $u = x^2 - 1$ $\therefore$ $\dfrac{du}{dx} = 2x$

and when $x = 1$, $u = 0$

when $x = 2$, $u = 3$

$$\therefore\ \int_1^2 x(x^2-1)^2\ dx = \int_0^3 u^2(\tfrac{1}{2}\frac{du}{dx}\ dx)$$
$$= \tfrac{1}{2}\int_0^3 u^2\ du$$
$$= \tfrac{1}{2}\left[\frac{u^3}{3}\right]_0^3$$
$$= \tfrac{1}{2}\left(\tfrac{27}{3} - 0\right)$$
$$= 4\tfrac{1}{2}$$

3
$$f(x) = (3x^2+x)^3$$
$$\therefore\ f'(x) = 3(3x^2+x)^2(6x+1)$$
$$\therefore\ \int 3(3x^2+x)^2(6x+1)\ dx = (3x^2+x)^3 + c_1$$
$$\therefore\ 3\int(3x^2+x)^2(6x+1)\ dx = (3x^2+x)^3 + c_1$$
$$\therefore\ \int(3x^2+x)^2(6x+1)\ dx = \tfrac{1}{3}(3x^2+x)^3 + c$$

4 $f'(x) = x^2 - 3x + 2$ $\therefore$ $f(x) = \dfrac{x^3}{3} - \dfrac{3x^2}{2} + 2x + c$

But $f(1) = 3$ $\therefore$ $\tfrac{1}{3} - \tfrac{3}{2} + 2 + c = 3$
$$c = 1 - \tfrac{1}{3} + 1\tfrac{1}{2}$$
$$\therefore\ c = 2\tfrac{1}{6}$$

i.e., $f(x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x + 2\tfrac{1}{6}$

5
$$\int_2^3 \frac{1}{\sqrt{3x-4}}\,dx$$
$$= \int_2^3 (3x-4)^{-\frac{1}{2}}\ dx$$
$$= \left[\tfrac{1}{3}\frac{(3x-4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3$$
$$= \left[\tfrac{2}{3}\sqrt{3x-4}\right]_2^3$$
$$= \tfrac{2}{3}\sqrt{5} - \tfrac{2}{3}\sqrt{2}$$
$$= \tfrac{2}{3}(\sqrt{5} - \sqrt{2})$$

6 **a**

$$\begin{aligned} f''(x) &= 3x^2+2x \\ \therefore\quad f'(x) &= \frac{3x^3}{3}+\frac{2x^2}{2}+c \\ &= x^3+x^2+c \\ \text{and}\quad f(x) &= \frac{x^4}{4}+\frac{x^3}{3}+cx+d \\ \text{But}\quad f(0) &= 3 \qquad \therefore\quad d=3 \\ \text{and}\quad f(x) &= \frac{x^4}{4}+\frac{x^3}{3}+cx+3 \end{aligned}$$

$$\begin{aligned} \text{Now}\quad f(2)=3 \qquad \therefore\quad 4+\tfrac{8}{3}+2c+3 &= 3 \\ \therefore\quad \tfrac{20}{3} &= -2c \\ \therefore\quad c &= -\tfrac{10}{3} \qquad \text{Thus,}\quad f(x)=\tfrac{1}{4}x^4+\tfrac{1}{3}x^3-\tfrac{10}{3}x+3 \end{aligned}$$

b Now $f'(2)=2^3+2^2-\frac{10}{3}=12-\frac{10}{3}=\frac{26}{3}$

$\therefore$ normal has slope $-\frac{3}{26}$

$$\begin{aligned} \therefore\quad \text{equation is}\quad \frac{y-3}{x-2} &= -\tfrac{3}{26} \\ \therefore\quad y-3 &= -\tfrac{3}{26}(x-2) \\ \therefore\quad y &= -\tfrac{3}{26}x+\tfrac{6}{26}+3, \quad \text{i.e.,}\quad 3x+26y=84 \end{aligned}$$

7

$$\begin{array}{c|cccc} 2 & 1 & 0 & -3 & 2 \\ & 0 & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \multicolumn{1}{|c}{4} \end{array} \qquad \begin{aligned} \therefore\quad & \frac{x^3-3x+2}{x-2}=x^2+2x+1+\frac{4}{x-2} \\ \therefore\quad & A=1,\quad B=2,\quad C=1,\quad D=4 \end{aligned}$$

$$\begin{aligned} \text{and}\quad \int\frac{x^3-3x+2}{x-2}\,dx &= \int x^2+2x+1+\frac{4}{x-2}\,dx \qquad \textit{or} \qquad = \int (x+1)^2+\frac{4}{x-2}\,dx \\ &= \frac{x^3}{3}+\frac{2x^2}{2}+x+4\ln|x-2|+c \qquad = \frac{(x+1)^3}{3}+4\ln|x-2|+c \\ &= \tfrac{1}{3}x^3+x^2+x+4\ln|x-2|+c \end{aligned}$$

8 **a**

$$\begin{aligned} \int\frac{1}{x+2}dx-\int\frac{2}{x-1}dx &= \ln|x+2|-2\ln|x-1|+c_1 \\ &= \ln\left(\frac{|x+2|}{(x-1)^2}\right)+c_1 \end{aligned}$$

b

$$\frac{1}{x+2}-\frac{2}{x-1}=\frac{(x-1)-2(x+2)}{(x+2)(x-1)}=\frac{-x-5}{(x+2)(x-1)}=-\frac{x+5}{(x+2)(x-1)}$$

$$\begin{aligned} \therefore\quad \int\frac{x+5}{(x+2)(x-1)}\,dx &= -\left[\int\frac{1}{x+2}dx-\int\frac{2}{x-1}\,dx\right] \\ &= -\ln\left(\frac{|x+2|}{(x-1)^2}\right)+c \\ &= \ln\left(\frac{(x-1)^2}{|x+2|}\right)+c \end{aligned}$$

REVIEW SET 25C

1 **a** $$\frac{dy}{dx} = (x^2-1)^2$$
$$\therefore\ y = \int (x^2-1)^2\ dx$$
$$= \int (x^4-2x^2+1)\ dx$$
$$= \tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 + x + c$$

b $$\frac{dy}{dx} = 400 - 20e^{-\frac{x}{2}} \quad \therefore\ y = \int (400 - 20e^{-\frac{x}{2}})\ dx$$
$$= 400x - \frac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c$$
$$= 400x + 40e^{-\frac{x}{2}} + c$$

2 **a** $$\int_{-2}^{0} \frac{4}{2x-1}\ dx$$
$$= 4\int_{-2}^{0} \frac{1}{2x-1}\ dx$$
$$= 4\left[\left(\tfrac{1}{2}\right)\ln|2x-1|\right]_{-2}^{0}$$
$$= 2\left[\ln|2x-1|\right]_{-2}^{0}$$
$$= 2\left[\ln|-1| - \ln|-5|\right]$$
$$= 2[0 - \ln 5]$$
$$= -2\ln 5$$
$$\doteqdot -3.219$$

b In $\displaystyle\int_0^1 \frac{10x}{\sqrt{3x^2+1}}\ dx$

we let $u = 3x^2+1$, $\therefore\ \dfrac{du}{dx} = 6x$

When $x = 0$, $u = 1$. When $x = 1$, $u = 4$.

$$\therefore\ \int_0^1 \frac{10x}{\sqrt{3x^2+1}}\ dx = \int_0^1 u^{-\frac{1}{2}}\left(\tfrac{5}{3}\frac{du}{dx}\right) dx$$
$$= \tfrac{5}{3}\int_1^4 u^{-\frac{1}{2}}\ du$$
$$= \tfrac{5}{3}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$
$$= \tfrac{10}{3}\left(\sqrt{4}-\sqrt{1}\right)$$
$$= \tfrac{10}{3}$$

3 $$\frac{d}{dx}(\ln x)^2 = 2(\ln x)^1\left(\frac{1}{x}\right)$$
$$= \frac{2\ln x}{x}$$
$$\therefore\ \int \frac{2\ln x}{x}\ dx = (\ln x)^2 + c_1$$
$$\therefore\ \int \frac{\ln x}{x}\ dx = \tfrac{1}{2}(\ln x)^2 + c$$

4 $$f''(x) = 4x^2 - 3$$
$$\therefore\ f'(x) = \frac{4x^3}{3} - 3x + c$$

But $f'(0) = 6 \quad \therefore\ c = 6$

i.e., $f'(x) = \dfrac{4x^3}{3} - 3x + 6$

$$\therefore\ f(x) = \tfrac{4}{3}\frac{x^4}{4} - \frac{3x^2}{2} + 6x + d$$
$$\therefore\ f(x) = \tfrac{1}{3}x^4 - \tfrac{3}{2}x^2 + 6x + d$$

But $f(2) = 3 \quad \therefore\ \tfrac{16}{3} - 6 + 12 + d = 3$

$$\therefore\ d = -3 - \tfrac{16}{3} = -\tfrac{25}{3}$$
$$\therefore\ f(x) = \tfrac{1}{3}x^4 - \tfrac{3}{2}x^2 + 6x - \tfrac{25}{3}$$

and $f(3) = 27 - \tfrac{27}{2} + 18 - \tfrac{25}{3}$

$$= 23\tfrac{1}{6}$$

5 $\int(2x+3)^n\,dx = \frac{1}{2}\frac{(2x+3)^{n+1}}{n+1}+c \quad \text{if} \quad n \neq -1$

i.e., $\frac{(2x+3)^{n+1}}{2(n+1)}+c \quad \text{if} \quad n \neq -1$

But if $n=-1$, $\int(2x+3)^{-1}\,dx = \int \frac{1}{2x+3}\,dx = \frac{1}{2}\ln|2x+3|+c$

i.e., $\int(2x+3)^n\,dx = \begin{cases} \frac{(2x+3)^{n+1}}{2(n+1)}+c & \text{if} \quad n\neq -1 \\ \frac{1}{2}\ln|2x+3|+c & \text{if} \quad n=-1 \end{cases}$

6 **a**
$$\begin{aligned} &(e^x+2)^3 \\ &= (e^x)^3+3(e^x)^2(2)+3(e^x)(2)^2+(2)^3 \\ &= e^{3x}+6e^{2x}+12e^x+8 \end{aligned}$$

b
$$\begin{aligned} \int_0^1 (e^x+2)^3\,dx &= \left[\tfrac{1}{3}e^{3x}+3e^{2x}+12e^x+8x\right]_0^1 \\ &= \left(\tfrac{1}{3}e^3+3e^2+12e+8\right)-\left(\tfrac{1}{3}+3+12\right) \\ &= \tfrac{1}{3}e^3+3e^2+12e-7\tfrac{1}{3} \quad (\doteqdot 54.14......) \end{aligned}$$

c By technology $\int_0^1 (e^x+2)^3\,dx = 54.14.....$ {using $\text{f}_n\text{Int}(.....)$}

7
$$f'(x) = 2\sqrt{x}+\frac{a}{\sqrt{x}} = 2x^{\frac{1}{2}}+ax^{-\frac{1}{2}}$$
$$\therefore \quad f(x) = \tfrac{4}{3}x^{\frac{3}{2}}+2ax^{\frac{1}{2}}+c = \frac{4x\sqrt{x}}{3}+2a\sqrt{x}+c$$

Now $f(0)=2 \quad \therefore \quad c=2 \quad \therefore \quad f(x) = \frac{4x\sqrt{x}}{3}+2a\sqrt{x}+2$

Also $f(1)=4 \quad \therefore \quad \frac{4}{3}+2a+2=4 \quad \therefore \quad 2a=\frac{2}{3} \quad \therefore \quad a=\frac{1}{3}$

Now $f'(x) = 2\sqrt{x}+\frac{1}{3\sqrt{x}} = \frac{6x+1}{3\sqrt{x}} > 0$ since $f(x)$ is only defined for $x>0$

and for $x>0$, $6x+1$ and $3\sqrt{x}$ are >0

$\therefore$ the function has no stationary point as $f'(x)\neq 0$ for any value of x.

8
$$\begin{aligned} \int_a^{2a}(x^2+ax+2)\,dx &= \frac{73a}{2} \\ \therefore \quad \left[\frac{x^3}{3}+\frac{ax^2}{2}+2x\right]_a^{2a} &= \frac{73a}{2} \\ \therefore \quad \left(\frac{8a^3}{3}+\frac{a}{2}(4a^2)+4a\right)-\left(\frac{a^3}{3}+\frac{a^3}{2}+2a\right) &= \frac{73a}{2} \\ \frac{8a^3}{3}+2a^3+4a-\frac{a^3}{3}-\frac{a^3}{2}-2a &= \frac{73a}{2} \\ \therefore \quad 16a^3+12a^3+24a-2a^3-3a^3-12a &= 219a \\ \therefore \quad 23a^3-207a &= 0 \\ \therefore \quad 23a(a^2-9) &= 0 \\ \therefore \quad 23a(a+3)(a-3) &= 0 \\ \therefore \quad a=0 \quad \text{or} \quad a&=\pm 3 \end{aligned}$$

Chapter 26

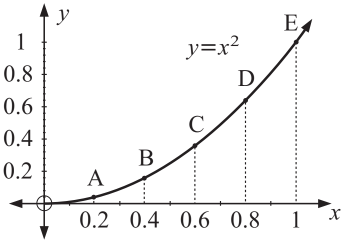
INTEGRATION (AREAS AND OTHER APPLICATIONS)

EXERCISE 26A.1

1 **a** Divide the region into strips of width $\frac{1}{5}$ unit.

$O(0,\ 0)$ $A(\frac{1}{5},\ \frac{1}{25})$ $B(\frac{2}{5},\ \frac{4}{25})$

$C(\frac{3}{5},\ \frac{9}{25})$ $D(\frac{4}{5},\ \frac{16}{25})$ $E(1,\ 1)$



Now $A < A_1 + A_2 + A_3 + A_4 + A_5$

i.e., $A < \dfrac{\left(0+\frac{1}{25}\right)}{2}\frac{1}{5} + \dfrac{\left(\frac{1}{25}+\frac{4}{25}\right)}{2}\frac{1}{5} + \ldots\ldots + \dfrac{\left(\frac{16}{25}+\frac{25}{25}\right)}{2}\frac{1}{5}$

$\therefore\ A < \frac{1}{10}\left(\frac{85}{25}\right)$

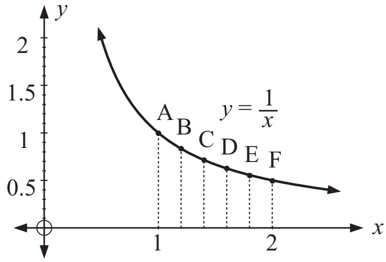
$\therefore\ A < 0.34$

and $A \doteqdot 0.34$ units^2 is an estimate of the area

b Divide the region into strips of width $\frac{1}{5}$ unit.

$A(1,\ 1)$ $B(\frac{6}{5},\ \frac{5}{6})$ $C(\frac{7}{5},\ \frac{5}{7})$

$D(\frac{8}{5},\ \frac{5}{8})$ $E(\frac{9}{5},\ \frac{5}{9})$ $F(2,\ \frac{1}{2})$



Now $A > A_1 + A_2 + A_3 + A_4 + A_5$

i.e., $A > \dfrac{\left(1+\frac{5}{6}\right)}{2}\frac{1}{5} + \dfrac{\left(\frac{5}{6}+\frac{5}{7}\right)}{2}\frac{1}{5} + \ldots\ldots + \dfrac{\left(\frac{5}{9}+\frac{1}{2}\right)}{2}\frac{1}{5}$

$\therefore\ A > \frac{1}{10}\left(\frac{11}{6}+\frac{65}{42}+\frac{75}{56}+\frac{85}{72}+\frac{19}{18}\right)$

$\therefore\ A > 0.6956$

and $A \doteqdot 0.70$ units^2 is an estimate of the area

2 **a** Using 10 vertical strips in question **1 a**

$$A < \frac{\left(\frac{0}{100}+\frac{1}{100}\right)\frac{1}{10}}{2} + \frac{\left(\frac{1}{100}+\frac{4}{100}\right)\frac{1}{10}}{2} + \ldots. + \frac{\left(\frac{81}{100}+\frac{100}{100}\right)\frac{1}{10}}{2}$$

$$< \frac{1}{20}\left(\frac{1}{100}+\frac{5}{100}+\frac{13}{100}+\frac{25}{100}+\frac{41}{100}+\frac{61}{100}+\frac{85}{100}+\frac{113}{100}+\frac{145}{100}+\frac{181}{100}\right)$$

$$< \frac{1}{20}\left(\frac{670}{100}\right)$$

$$< 0.335$$

and $A \doteqdot 0.335$ units^2 is an estimate of the area

b Using 10 vertical strips in question **1 b**

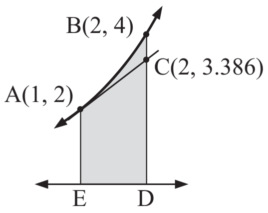
$$A > \frac{\left(\frac{10}{10}+\frac{10}{11}\right)\frac{1}{10}}{2} + \frac{\left(\frac{10}{11}+\frac{10}{12}\right)\frac{1}{10}}{2} + \ldots\ldots + \frac{\left(\frac{10}{19}+\frac{10}{20}\right)\frac{1}{10}}{2}$$

$$\begin{aligned} \therefore\quad A &> \tfrac{10}{20}\left((\tfrac{1}{10}+\tfrac{1}{11})+(\tfrac{1}{11}+\tfrac{1}{12})+ \text{......} +(\tfrac{1}{19}+\tfrac{1}{20})\right) \\ &> \tfrac{1}{2}\left(\tfrac{21}{110}+\tfrac{23}{132}+\tfrac{25}{156}+\tfrac{27}{182}+\tfrac{29}{210}+\tfrac{31}{240}+\tfrac{33}{272}+\tfrac{35}{306}+\tfrac{37}{342}+\tfrac{39}{380}\right) \\ &> 0.693\,771 \end{aligned}$$

and $A \doteqdot 0.6938$ is an estimate of the area

EXERCISE 26A.2

1 **a**



The lower bound is the area ACDE

$= \dfrac{2+3.386}{2}$

$= 2.693$

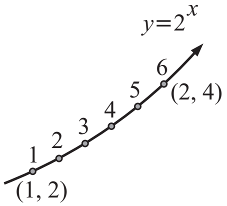
The upper bound is the area ABDE

$= \dfrac{2+4}{2}$

$= 3$

b Without the tangent and using 5 subdivisions on [1, 2]

$n = 5$



Point	*Coordinates*
1	(1, 2)
2	(1.2, $2^{1.2}$)
3	(1.4, $2^{1.4}$)
4	(1.6, $2^{1.6}$)
5	(1.8, $2^{1.8}$)
6	(2, 4)

$$\begin{aligned} A_L &= \tfrac{1}{5}(2^1)+\tfrac{1}{5}(2^{1.2})+\tfrac{1}{5}(2^{1.4})+\tfrac{1}{5}(2^{1.6})+\tfrac{1}{5}(2^{1.8}) \\ &= \tfrac{1}{5}(2+2^{1.2}+2^{1.4}+2^{1.6}+2^{1.8}) \\ &\doteqdot 2.690 \end{aligned}$$

$$\begin{aligned} A_U &= \tfrac{1}{5}(2^{1.2}+2^{1.4}+2^{1.6}+2^{1.8}+2^2) \\ &\doteqdot 3.090 \end{aligned}$$

c Using the provided software:

n	A_L	A_U
10	2.786 55	2.986 55
50	2.865 44	2.905 44
100	2.875 40	2.895 40
500	2.883 39	2.887 39
5000	2.885 19	2.885 59

The upper and lower sums converge to 2.885 (approx.)

2 **a** $y = 1 + e^x$ [0, 1] choosing 5 subdivisions on this [0, 1].

Point	*Coordinates*
1	(0, 2)
2	(0.2, $1+e^{0.2}$)
3	(0.4, $1+e^{0.4}$)
4	(0.6, $1+e^{0.6}$)
5	(0.8, $1+e^{0.8}$)
6	(1, $1+e^1$)

$$\begin{aligned} A_L &= 0.2\left(1+e^0\right)+0.2(1+e^{0.2})+ \text{......} +0.2(1+e^{0.8}) \\ &= 0.2\left(5+(e^0+e^{0.2}+e^{0.4}+e^{0.6}+e^{0.8})\right) \\ &\doteqdot 2.552 \end{aligned}$$

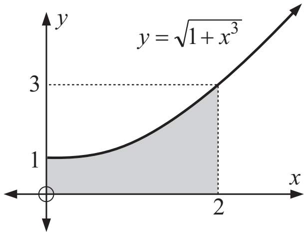
$$\begin{aligned} \therefore\quad A_U &= 0.2(1+e^{0.2})+0.2(1+e^{0.4})+ \text{......} +0.2(1+e^1) \\ &= 0.2\left(5+\left(e^{0.2}+e^{0.4}+e^{0.6}+e^{0.8}+e^1\right)\right) \\ &\doteqdot 2.896 \end{aligned}$$

b

n	A_L	A_U
100	2.709 70	2.726 89
1000	2.717 42	2.719 14
10 000	2.718 20	2.718 37
100 000	2.718 27	2.718 29

c It appears that as $t \to \infty$ both upper and lower sums converge to e, where $e \doteqdot 2.718\,28$

3 **a**



b

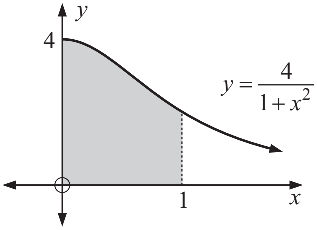
n	A_L	A_U
100	3.221 38	3.261 38
10 000	3.241 11	3.241 51

c A good estimate would be the average

i.e., $\dfrac{3.241\,11 + 3.241\,51}{2}$

$\therefore \quad \int_0^2 \sqrt{1+x^3}dx \doteqdot 3.2413$

4 **a**



b

n	A_L	A_U
100	3.131 58	3.151 58
1000	3.140 59	3.142 59
10 000	3.141 49	3.141 69

c Area $= \int_0^1 \dfrac{4}{1+x^2}\,dx \;= \dfrac{3.141\,69 + 3.141\,49}{2}$

$\doteqdot 3.141\,59$

$\doteqdot 3.1416$

(We suspect this is π)

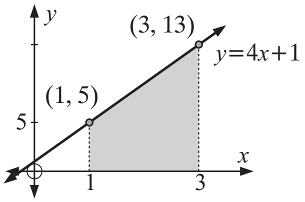
5 For the curve $y = \sqrt[3]{x^2+2}$, $y > 0$ for $0 \leqslant x \leqslant 5$.

Using the software provided

n	A_L	A_U
100	10.159 79	10.246 80
1000	10.198 87	10.207 57
10 000	10.202 78	10.203 65

A good estimate is $\int_0^5 \sqrt[3]{x^2+2}\,dx = \dfrac{10.202\,78 + 10.203\,65}{2} \doteqdot 10.203\,22$

6 **a**



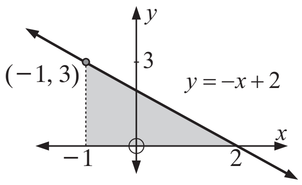
$\int_1^3 (1+4x)\,dx$

$=$ area of the shaded trapezium

$= \left(\dfrac{5+13}{2}\right) \times 2$

$= 18$

b



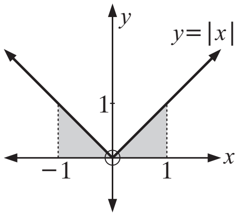
$\int_{-1}^2 (2-x)\,dx$

$=$ area of shaded triangle

$= \frac{1}{2}(3 \times 3)$

$= 4.5$

c



$\int_{-1}^{1} |x|\,dx =$ shaded area shown

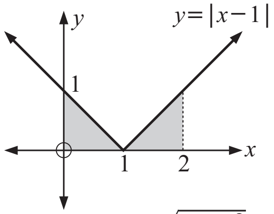
Due to symmetry we need to only consider the RHS.

$\therefore\ \int_{0}^{1} |x|\,dx =$ area of a shaded Δ

$= \frac{1}{2}(1\times 1) = 0.5$

$\therefore\ \int_{-1}^{1} |x|\,dx = 2\times 0.5 = 1$

d



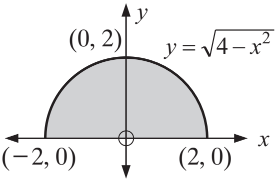
$y = \sqrt{4-x^2}$

$\therefore\ \int_{0}^{2} |x-1|\,dx = 2\int_{1}^{2} |x-1|\,dx$ {by symmetry}

$= 2\times\left[\frac{1}{2}(1\times 1)\right]$

$= 1$

e



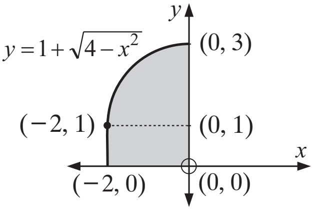
$\int_{-2}^{2} \sqrt{4-x^2}\,dx$

= area of semi-circle radius 2

$= \frac{1}{2}\left(\pi\times 2^2\right)$

$= 2\pi$

f



$\int_{-2}^{0} \left(1+\sqrt{4-x^2}\right) dx$

= area of rectangle

+ area of quarter of circle with radius 2

$= 2\times 1 + \frac{1}{4}(\pi\times 2^2)$

$= 2+\pi$

EXERCISE 26B

1 **a** $\int_{a}^{a} f(x)\,dx = F(a) - F(a) = 0$

graphically: $\int_{a}^{a} f(x)\,dx =$ area of the strip between $x=a$ and $x=a$, i.e., width $= 0$ $\therefore$ area $= 0$

b $\int_{a}^{b} c\,dx = F(b) - F(a)$

$= cb - ca$ {antideriv. of c is cx}

$= c(b-a)$

c $\int_{b}^{a} f(x)\,dx = F(a) - F(b)$

$= -[F(b) - F(a)]$

$= -\int_{a}^{b} f(x)\,dx$

d If $\dfrac{d}{dx}F(x) = f(x)$ then

$\dfrac{d}{dx}cF(x) = cf(x)$

$\therefore\ \int_{a}^{b} cf(x)\,dx = c[F(b) - F(a)]$

$= c\int_{a}^{b} f(x)\,dx$

e $\int_{a}^{b}(f(x)+g(x))\,dx = [F(b)+G(b)] - [F(a)+G(a)]$

$= [F(b) - F(a)] + [G(b) - G(a)]$

$= \int_{a}^{b} f(x)\,dx + \int_{a}^{b} g(x)\,dx$

2 **a** $f(x) = x^3$ has antiderivative

$F(x) = \dfrac{x^4}{4}$

So, area $= \int_{0}^{1} x^3\,dx$

$= F(1) - F(0)$

$= \frac{1}{4} - 0$

$= \frac{1}{4}$ units^2

b $f(x) = x^3$ has antiderivative

$F(x) = \dfrac{x^4}{4}$

So, area $= \int_{1}^{2} x^3\,dx$

$= F(2) - F(1)$

$= \frac{16}{4} - \frac{1}{4}$

$= 3\frac{3}{4}$ units^2

c $f(x) = x^2 + 3x + 2$ has antiderivative

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x$$

$$\begin{aligned} \text{So, area} &= \textstyle\int_1^3 (x^2 + 3x + 2)\ dx \\ &= F(3) - F(1) \\ &= (\tfrac{27}{3} + \tfrac{27}{2} + 6) - (\tfrac{1}{3} + \tfrac{3}{2} + 2) \\ &= 24\tfrac{2}{3}\ \text{units}^2 \end{aligned}$$

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}x\sqrt{x}$$

$$\begin{aligned} \text{So the area} &= \textstyle\int_0^2 \sqrt{x}\ dx \\ &= F(2) - F(0) \\ &= \tfrac{2}{3} \times 2\sqrt{2} - 0 \\ &= \tfrac{4\sqrt{2}}{3}\ \text{units}^2 \end{aligned}$$

e $f(x) = e^x$ has antiderivative

$F(x) = e^x$

$$\begin{aligned} \text{So, area} &= \textstyle\int_0^{1.5} e^x\ dx \\ &= F(1.5) - F(0) \\ &= e^{1.5} - e^0 \\ &= e^{1.5} - 1 \\ &\doteqdot 3.482\ \text{units}^2 \end{aligned}$$

f $f(x) = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$\begin{aligned} \text{So, area} = \int_1^4 \frac{1}{\sqrt{x}}\ dx &= F(4) - F(1) \\ &= 2\sqrt{4} - 2\sqrt{1} \\ &= 2\ \text{units}^2 \end{aligned}$$

g $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative

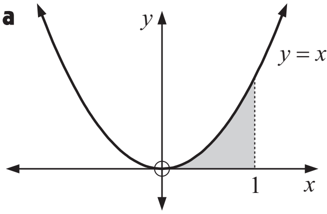
$$F(x) = \frac{x^4}{4} + \frac{2x^3}{3} + \frac{7x^2}{2} + 4x$$

So the area

$$\begin{aligned} &= \int_1^{1.25} x^3 + 2x^2 + 7x + 4\ dx \\ &= F(1.25) - F(1) \\ &= [12.381\,18 - 8.416\,67] \\ &\doteqdot 3.965\ \text{units}^2 \end{aligned}$$

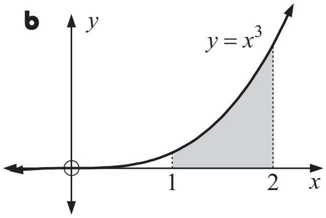
EXERCISE 26C

1 **a**



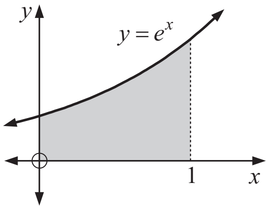
$$\begin{aligned} \text{Area} &= \textstyle\int_0^1 x^2\ dx \\ &= \left[\frac{x^3}{3}\right]_0^1 \\ &= \tfrac{1}{3} - 0 = \tfrac{1}{3}\ \text{units}^2 \end{aligned}$$

b



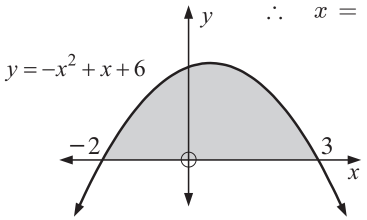
$$\begin{aligned} \text{Area} &= \textstyle\int_1^2 x^3\ dx \\ &= \left[\frac{x^4}{4}\right]_1^2 \\ &= \tfrac{16}{4} - \tfrac{1}{4} = 3\tfrac{3}{4}\ \text{units}^2 \end{aligned}$$

c



$$\begin{aligned} \text{Area} &= \textstyle\int_0^1 e^x\ dx \\ &= [e^x]_0^1 \\ &= e - 1 \\ &\doteqdot 1.718\ \text{units}^2 \end{aligned}$$

d



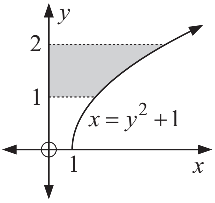
Cuts the x-axis at $y = 0$

$$\begin{aligned} \therefore\quad 6 + x - x^2 &= 0 \\ \therefore\quad (3 - x)(2 + x) &= 0 \\ \therefore\quad x &= 3 \text{ or } -2 \end{aligned}$$

i.e., x-intercepts are 3 and -2

$$\begin{aligned} \text{Area} &= \textstyle\int_{-2}^3 (6 + x - x^2)\ dx \\ &= \left[6x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^3 \\ &= (18 + \tfrac{9}{2} - 9) - (-12 + 2 + \tfrac{8}{3}) \\ &= 20\tfrac{5}{6}\ \text{units}^2 \end{aligned}$$

e



Area

$$= \int_1^2 x\,dy$$

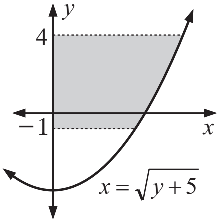
$$= \int_1^2 (y^2+1)\,dy$$

$$= \left[\frac{y^3}{3}+y\right]_1^2$$

$$= \left(\tfrac{8}{3}+2\right)-\left(\tfrac{1}{3}+1\right)$$

$$= 3\tfrac{1}{3} \text{ units}^2$$

f



Area

$$= \int_{-1}^{4} x\,dy$$

$$= \int_{-1}^{4} (y+5)^{\frac{1}{2}}\,dy$$

$$= \left[\frac{1}{1}\,\frac{(y+5)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{4}$$

$$= \left[\tfrac{2}{3}(y+5)^{\frac{3}{2}}\right]_{-1}^{4}$$

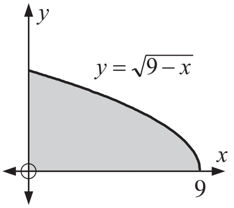
$$= \tfrac{2}{3}\,(9)^{\frac{3}{2}} - \tfrac{2}{3}\,(4)^{\frac{3}{2}}$$

$$= \tfrac{2}{3}(27-8)$$

$$= \tfrac{2}{3}\times 19$$

$$= 12\tfrac{2}{3} \text{ units}^2$$

2 **a**



Area

$$= \int_0^9 \left((9-x)^{\frac{1}{2}}-0\right)\,dx$$

$$= \int_0^9 (9-x)^{\frac{1}{2}}\,dx$$

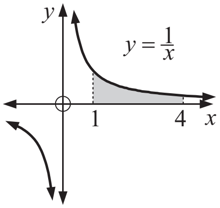
$$= \left[\left(\frac{1}{-1}\right)\frac{(9-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$

$$= -\tfrac{2}{3}\left[(9-x)^{\frac{3}{2}}\right]_0^9$$

$$= -\tfrac{2}{3}[0-27]$$

$$= 18 \text{ units}^2$$

b



Area

$$= \int_1^4 \left(\frac{1}{x}-0\right)\,dx$$

$$= \int_1^4 \frac{1}{x}\,dx$$

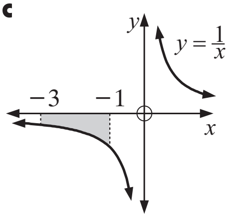
$$= [\ln|x|]_1^4$$

$$= \ln 4 - \ln 1$$

$$= \ln 4 - 0$$

$$\doteqdot 1.386 \text{ units}^2$$

c



Area

$$= \int_{-3}^{-1} \left(0-\frac{1}{x}\right)\,dx$$

$$= -\int_{-3}^{-1} \left(\frac{1}{x}\right)\,dx$$

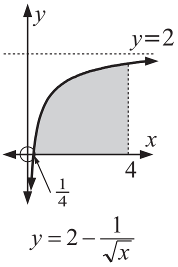
$$= -\,[\ln|x|]_{-3}^{-1}$$

$$= -(\ln 1 - \ln 3)$$

$$= 0 + \ln 3$$

$$\doteqdot 1.099 \text{ units}^2$$

d



Area

$$= \int_{\frac{1}{4}}^{4} \left(2-\frac{1}{\sqrt{x}}-0\right)\,dx$$

$$= \int_{\frac{1}{4}}^{4} (2-x^{-\frac{1}{2}})\,dx$$

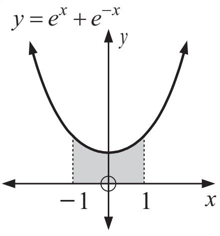
$$= \left[2x-\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\frac{1}{4}}^{4}$$

$$= \left[2x-2\sqrt{x}\right]_{\frac{1}{4}}^{4}$$

$$= (8-4)-(\tfrac{1}{2}-1)$$

$$= 4\tfrac{1}{2} \text{ units}^2$$

e



Area

$$= \int_{-1}^{1} (e^x+e^{-x})\,dx$$

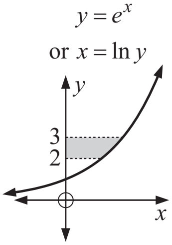
$$= \left[e^x-e^{-x}\right]_{-1}^{1}$$

$$= (e-e^{-1})-(e^{-1}-e)$$

$$= 2e-\frac{2}{e}$$

$$\doteqdot 4.701 \text{ units}^2$$

f



Area

$$= \int_2^3 (x-0)\,dy$$

$$= \int_2^3 \ln y\,dy$$

$$= [y\ln y - y]_2^3$$

$$= (3\ln 3-3)-(2\ln 2-2)$$

$$= \ln 27 - \ln 4 - 1$$

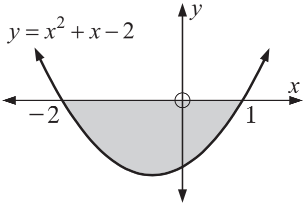
$$= \ln\left(\tfrac{27}{4}\right)-1 \text{ units}^2$$

$$\doteqdot 0.910 \text{ units}^2$$

3 **a** The curve cuts the x-axis when $y = 0$

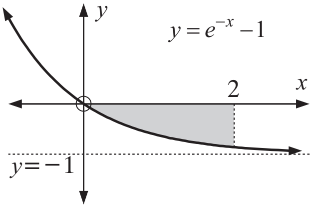
$$\therefore \quad x^2 + x - 2 = 0$$
$$\therefore \quad (x+2)(x-1) = 0$$
$$\therefore \quad x = -2 \text{ or } 1$$

i.e, x-intercepts are -2 and 1



$$\begin{aligned}\text{Area} &= \int_{-2}^{1} [0 - (x^2 + x - 2)]\, dx \\ &= \int_{-2}^{1} (-x^2 - x + 2)\, dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} \\ &= (-\tfrac{1}{3} - \tfrac{1}{2} + 2) - (\tfrac{8}{3} - 2 - 4) \\ &= 4\tfrac{1}{2} \text{ units}^2\end{aligned}$$

b The curve cuts the x-axis at (0, 0).

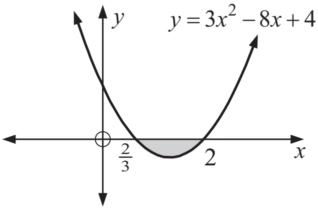


$$\begin{aligned}\text{Area} &= \int_{0}^{2} [0 - (e^{-x} - 1)]\, dx \\ &= \int_{0}^{2} (1 - e^{-x})\, dx \\ &= \left[x + e^{-x}\right]_{0}^{2} \\ &= \left(2 + \frac{1}{e^2}\right) - \left(0 + e^0\right) \\ &= 1 + \frac{1}{e^2} \qquad (\doteqdot 1.135 \text{ units}^2)\end{aligned}$$

c The curve cuts the x-axis when $y = 0$

$$\therefore \quad 3x^2 - 8x + 4 = 0$$
$$\therefore \quad (3x-2)(x-2) = 0$$
$$\therefore \quad x = 2 \text{ or } \tfrac{2}{3}$$

i.e, x-intercepts are 2 and $\frac{2}{3}$

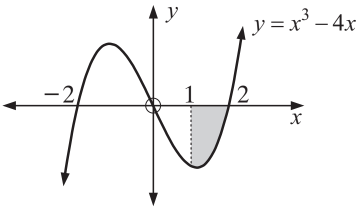


$$\begin{aligned}\text{Area} &= \int_{\frac{2}{3}}^{2} [0 - (3x^2 - 8x + 4)]\, dx \\ &= \int_{\frac{2}{3}}^{2} (-3x^2 + 8x - 4)\, dx \\ &= \left[-x^3 + 4x^2 - 4x\right]_{\frac{2}{3}}^{2} \\ &= (-8 + 16 - 8) - (-\tfrac{8}{27} + \tfrac{16}{3} - \tfrac{8}{3}) \\ &= 1\tfrac{5}{27} \text{ units}^2\end{aligned}$$

d The curve cuts the x-axis when $y = 0$

$$\therefore \quad x^3 - 4x = 0$$
$$\therefore \quad x(x^2 - 4) = 0$$
$$\therefore \quad x(x+2)(x-2) = 0$$

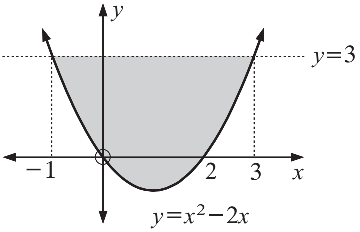
i.e., x-intercepts are 0 and ± 2



$$\begin{aligned}\text{Area} &= \int_{1}^{2} [0 - f(x)]\, dx \\ &= \int_{1}^{2} (-x^3 + 4x)\, dx \\ &= \left[-\frac{x^4}{4} + 2x^2\right]_{1}^{2} \\ &= (-4 + 8) - (-\tfrac{1}{4} + 2) \\ &= 2\tfrac{1}{4} \text{ units}^2\end{aligned}$$

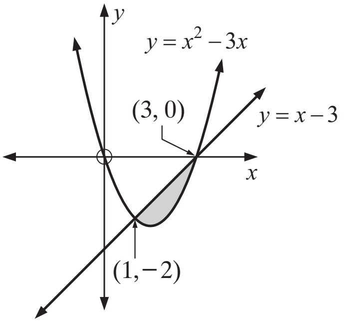
4 **a** $y = x^2 - 2x$ meets $y = 3$ where

$$\begin{aligned} x^2 - 2x &= 3 \\ \therefore \quad x^2 - 2x - 3 &= 0 \\ \therefore \quad (x-3)(x+1) &= 0 \\ \therefore \quad x &= 3 \quad \text{and} \quad x = -1 \end{aligned}$$



$$\begin{aligned} A &= \int_{-1}^{3} [3 - (x^2 - 2x)]\ dx \\ &= \int_{-1}^{3} (3 + 2x - x^2)\ dx \\ &= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} \\ &= (9 + 9 - 9) - (-3 + 1 + \tfrac{1}{3}) \\ &= 10\tfrac{2}{3} \text{ units}^2 \end{aligned}$$

b **i**



ii $y = x - 3$ meets $y = x^2 - 3x$

where $x = 3$ i.e., (3, 0)

and where $x = 1$ i.e., (1, −2)

Checking algebraically:

the graphs meet where $x - 3 = x^2 - 3x$

$$\begin{aligned} \therefore \quad x^2 - 3x - x + 3 &= 0 \\ \therefore \quad x^2 - 4x + 3 &= 0 \\ \therefore \quad (x-1)(x-3) &= 0 \\ \therefore \quad x &= 1 \text{ or } 3 \quad \checkmark \end{aligned}$$

iii

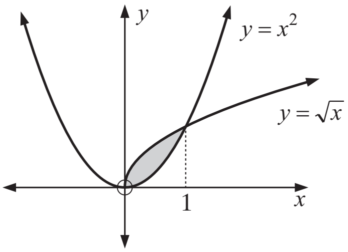
$$\begin{aligned} \text{Area} &= \int_{1}^{3} [(x-3) - (x^2 - 3x)]\ dx \\ &= \int_{1}^{3} (-3 + 4x - x^2)\ dx \\ &= \left[-3x + 2x^2 - \frac{x^3}{3}\right]_{1}^{3} \\ &= (-9 + 18 - 9) - (-3 + 2 - \tfrac{1}{3}) \\ &= 1\tfrac{1}{3} \text{ units}^2 \end{aligned}$$

c $y = \sqrt{x}$ meets $y = x^2$ where

$$\begin{aligned} \sqrt{x} &= x^2 \\ \therefore \quad x &= x^4 \\ \therefore \quad x^4 - x &= 0 \\ \therefore \quad x(x^3 - 1) &= 0 \\ \therefore \quad x(x-1)(x^2 + x + 1) &= 0 \\ \therefore \quad x &= 0 \text{ or } 1 \end{aligned}$$

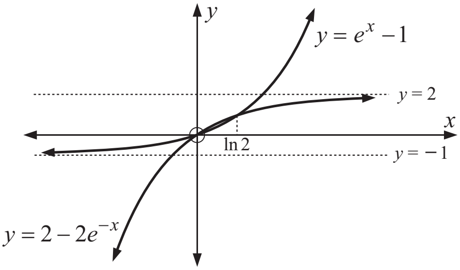
The factor $(x^2 + x + 1)$ has no real root since $\Delta = -3$ which is < 0.

$$\begin{aligned} \text{Area} &= \int_0^1 \left(\sqrt{x} - x^2\right)\ dx \\ &= \int_0^1 \left(x^{\frac{1}{2}} - x^2\right)\ dx \\ &= \left[\tfrac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3} \text{ unit}^2 \end{aligned}$$



d $y = e^x - 1$ meets $y = 2 - 2e^{-x}$ where

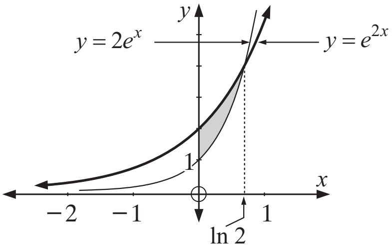
$$\begin{aligned} e^x - 1 &= 2 - 2e^{-x} \\ \therefore\ e^{2x} - e^x &= 2e^x - 2 \quad \{\times e^x\} \\ \therefore\ e^{2x} - 3e^x + 2 &= 0 \\ \therefore\ (e^x - 1)(e^x - 2) &= 0 \\ \therefore\ e^x &= 1 \text{ or } 2 \\ \therefore\ x &= 0 \text{ or } \ln 2 \end{aligned}$$



$$\begin{aligned} A &= \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)]\, dx \\ &= \int_0^{\ln 2} (3 - e^x - 2e^{-x})\, dx \\ &= \left[3x - e^x + 2e^{-x}\right]_0^{\ln 2} \\ &= (3\ln 2 - 2 + 1) - (0 - 1 + 2) \\ &= 3\ln 2 - 2 \\ &\doteqdot 0.0794 \text{ units}^2 \end{aligned}$$

e $y = 2e^x$ meets $y = e^{2x}$ where

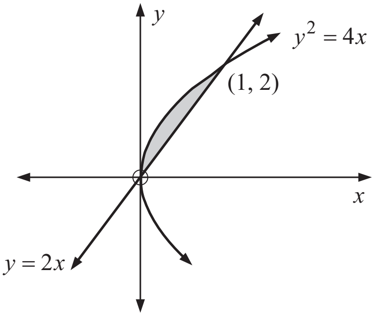
$$\begin{aligned} 2e^x &= e^{2x} \\ \therefore\ e^{2x} - 2e^x &= 0 \\ \therefore\ e^x(e^x - 2) &= 0 \\ \therefore\ e^x &= 2 \qquad \{\text{as } e^x > 0 \text{ for all } x\} \\ \therefore\ x &= \ln 2 \end{aligned}$$



$$\begin{aligned} A &= \int_0^{\ln 2} (2e^x - e^{2x})\, dx \\ &= \left[2e^x - \tfrac{1}{2}e^{2x}\right]_0^{\ln 2} \\ &= (4 - 2) - (2 - \tfrac{1}{2}) \\ &= \tfrac{1}{2} \text{ unit}^2 \end{aligned}$$

5 $y = 2x$ meets $y^2 = 4x$ where

$$\begin{aligned} (2x)^2 &= 4x \\ \therefore\ 4x^2 &= 4x \\ \therefore\ 4x^2 - 4x &= 0 \\ \therefore\ 4x(x - 1) &= 0 \\ \therefore\ x &= 0 \text{ or } 1 \end{aligned}$$



The upper part of $y^2 = 4x$

is $y = \sqrt{4x}$

i.e., $y = 2\sqrt{x}$

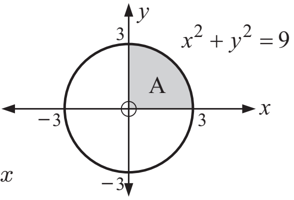
$$\begin{aligned} \text{Area} &= \int_0^1 (2\sqrt{x} - 2x)\, dx \\ &= \int_0^1 (2x^{\frac{1}{2}} - 2x)\, dx \\ &= \left[\tfrac{4}{3}x^{\frac{3}{2}} - x^2\right]_0^1 \\ &= \tfrac{4}{3} - 1 \\ &= \tfrac{1}{3} \text{ unit}^2 \end{aligned}$$

6 **a** Now $x^2+y^2=9 \quad \therefore \quad y^2 = 9-x^2$

$$y = \pm\sqrt{9-x^2}$$

In the upper half of the circle all y-values are $\geqslant 0$

$\therefore \quad y=+\sqrt{9-x^2}$ is the required equation.



b Now the shaded area above is A where $A=\int_0^3 \sqrt{9-x^2}\,dx$

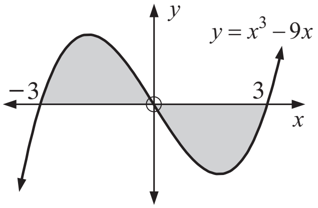
This is a quarter of the area of a circle radius 3 units $\therefore \quad A = \frac{1}{4}(\pi\times 3^2) = \frac{9}{4}\pi \doteqdot 7.069$ units^2

c The answer checks using technology.

7 **a**

$$\begin{aligned} f(x) &= x^3-9x \\ &= x(x^2-9) \\ &= x(x+3)(x-3) \end{aligned}$$

$\therefore \quad y=f(x)$ cuts the x-axis at $0, \pm 3$

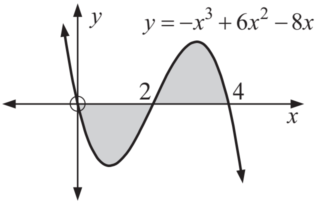


$$\begin{aligned} \text{Area} &= \int_{-3}^{0}(x^3-9x)\,dx + \int_0^3[0-(x^3-9x)]\,dx \\ &= \left[\frac{x^4}{4}-\frac{9x^2}{2}\right]_{-3}^{0} + \left[-\frac{x^4}{4}+\frac{9x^2}{2}\right]_0^3 \\ &= (0-[\tfrac{81}{4}-\tfrac{81}{2}]) + ([-\tfrac{81}{4}+\tfrac{81}{2}]-0) \\ &= 40\tfrac{1}{2} \text{ units}^2 \end{aligned}$$

b

$$\begin{aligned} f(x) &= -x(x-2)(x-4) \\ &= -x^3+6x^2-8x \end{aligned}$$

$\therefore \quad y=f(x)$ cuts the x-axis at 0, 2 and 4

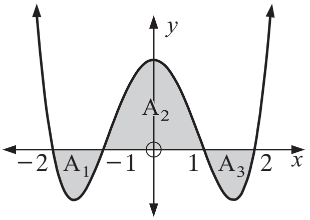


$$\begin{aligned} \text{Area} &= \int_0^2[0-(-x^3+6x^2-8x)]\,dx + \int_2^4[(-x^3+6x^2-8x)-0]\,dx \\ &= \int_0^2(x^3-6x^2+8x)\,dx + \int_2^4(-x^3+6x^2-8x)\,dx \\ &= \left[\frac{x^4}{4}-2x^3+4x^2\right]_0^2 + \left[-\frac{x^4}{4}+2x^3-4x^2\right]_2^4 \\ &= ([4-16+16]-0) + ([-64+128-64]-[-4+16-16]) \\ &= 8 \text{ units}^2 \end{aligned}$$

c

$$\begin{aligned} f(x) &= x^4-5x^2+4 \\ &= (x^2-1)(x^2-4) \\ &= (x+1)(x-1)(x+2)(x-2) \end{aligned}$$

$\therefore \quad y=f(x)$ cuts the x-axis at $\pm 1, \pm 2$



$$\begin{aligned} A_1 &= \int_{-2}^{-1}[0-(x^4-5x^2+4)]\,dx \\ &= \int_{-2}^{-1}(-x^4+5x^2-4)\,dx \\ &= \left[-\frac{x^5}{5}+\frac{5x^3}{3}-4x\right]_{-2}^{-1} \\ &= (\tfrac{1}{5}-\tfrac{5}{3}+4)-(\tfrac{32}{5}-\tfrac{40}{3}+8) \\ &= \tfrac{22}{15} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{-1}^{1}(x^4-5x^2+4)\,dx \\ &= \left[\frac{x^5}{5}-\frac{5x^3}{3}+4x\right]_{-1}^{1} \\ &= (\tfrac{1}{5}-\tfrac{5}{3}+4)-(-\tfrac{1}{5}+\tfrac{5}{3}-4) \\ &= \tfrac{76}{15} \text{ units}^2 \end{aligned}$$

Now by symmetry, $A_3 = A_1 \qquad \therefore \quad A = \frac{22}{15}+\frac{76}{15}+\frac{22}{15} = \frac{120}{15} = 8$ units^2

8 **a** **i** The graphs meet where

$$\begin{aligned} x^3-4x &= 3x+6 \\ \therefore\quad x^3-7x-6 &= 0 \\ \therefore\quad (x+1)(x^2-x-6) &= 0 \\ \therefore\quad (x+1)(x-3)(x+2) &= 0 \\ \therefore\quad x &= -2,\ -1 \text{ or } 3 \end{aligned}$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \multicolumn{1}{|r}{0} \end{array}$$

$$\begin{aligned} \therefore\quad \text{Area} &= \int_{-2}^{-1} \left(\left[x^3-4x\right]-[3x+6]\right)\,dx + \int_{-1}^{3} \left([3x+6]-\left[x^3-4x\right]\right)\,dx \\ &= \int_{-2}^{-1} \left(x^3-7x-6\right)\,dx + \int_{-1}^{3} \left(-x^3+7x+6\right)\,dx \end{aligned}$$

ii $\text{Area} = \int_{-2}^{3} \left|x^3-7x-6\right|\,dx$

b Using technology, area $= 32\frac{3}{4}$ units^2

9 **a** The graphs meet where

$$\begin{aligned} x^3-5x &= 2x^2-6 \\ \therefore\quad x^3-2x^2-5x+6 &= 0 \\ \therefore\quad (x-1)(x^2-x-6) &= 0 \\ \therefore\quad (x-1)(x-3)(x+2) &= 0 \\ \therefore\quad x &= -2,\ 1 \text{ or } 3 \end{aligned}$$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \multicolumn{1}{|r}{0} \end{array}$$

So, $\text{Area} = \int_{-2}^{3} \left|x^3-2x^2-5x+6\right|\,dx = 21\frac{1}{12}$ units^2 {technology}

b The graphs meet where

$$\begin{aligned} -x^3+3x^2+6x-8 &= 5x-5 \\ \therefore\quad x^3-3x^2-x+3 &= 0 \\ \therefore\quad (x-1)(x^2-2x-3) &= 0 \\ \therefore\quad (x-1)(x-3)(x+1) &= 0 \\ \therefore\quad x &= -1,\ 1 \text{ or } 3 \end{aligned}$$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & -1 & 3 \\ & 0 & 1 & -2 & -3 \\ \hline & 1 & -2 & -3 & \multicolumn{1}{|r}{0} \end{array}$$

So, $\text{Area} = \int_{-1}^{3} \left|x^3-3x^2-x+3\right|\,dx = 8$ units^2 {technology}

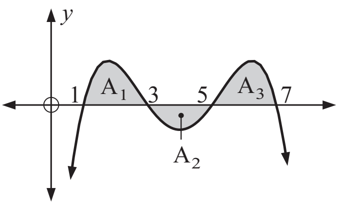
c $y=2x^3-3x^2+18$ meets $y=x^3+10x-6$

$$\begin{aligned} \text{where}\quad 2x^3-3x^2+18 &= x^3+10x-6 \\ \therefore\quad x^3-3x^2-10x+24 &= 0 \\ \therefore\quad (x-2)(x^2-x-12) &= 0 \\ \therefore\quad (x-2)(x-4)(x+3) &= 0 \\ \therefore\quad x &= -3,\ 2 \text{ or } 4 \end{aligned}$$

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & 0 & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & \multicolumn{1}{|r}{0} \end{array}$$

So, $\text{Area} = \int_{-3}^{4} \left|x^3-3x^2-10x+24\right|\,dx = 101\frac{3}{4}$ units^2 {technology}

10



a $\int_1^7 f(x)\,dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leqslant x \leqslant 7$. But $f(x)$ is not positive for $3 \leqslant x \leqslant 5$, so $\int_1^7 f(x)\,dx$ will give us

(Area enclosed in $1 \leqslant x \leqslant 3$)
− (Area enclosed in $3 \leqslant x \leqslant 5$)
+ (Area enclosed in $5 \leqslant x \leqslant 7$)

which is NOT the shaded area.

b

$$\begin{aligned} \text{shaded area} &= \int_1^3 f(x)\,dx + \int_3^5 [0-f(x)]\,dx + \int_5^7 f(x)\,dx \\ &= \int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx \end{aligned}$$

11

$$\text{Area } = \int_1^k \frac{1}{1+2x}\,dx = 0.2 \text{ units}^2$$

$$\therefore \quad \left[\tfrac{1}{2}\ln|1+2x|\right]_1^k = 0.2$$

$$\therefore \quad [\ln|1+2x|]_1^k = 0.4$$

$$\therefore \quad \ln|1+2k| - \ln 3 = 0.4$$

$$\therefore \quad \ln(1+2k) - \ln 3 = 0.4 \qquad \{\text{as } k>0, \text{then } \ 1+2k>0 \text{ also}\}$$

$$\therefore \quad \ln\left(\frac{1+2k}{3}\right) = 0.4$$

$$\therefore \quad \frac{1+2k}{3} = e^{0.4}$$

$$\therefore \quad 1+2k = 3e^{0.4}$$

$$\therefore \quad k = \frac{3e^{0.4}-1}{2} \doteqdot 1.7377$$

12 Area $= \int_0^b \sqrt{x}\,dx$

$$\therefore \quad \int_0^b x^{\frac{1}{2}}\,dx = 1$$

$$\therefore \quad \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^b = 1$$

$$\therefore \quad \tfrac{2}{3}b\sqrt{b} - 0 = 1$$

$$\therefore \quad b\sqrt{b} = \tfrac{3}{2}$$

$$\therefore \quad b^{\frac{3}{2}} = 1.5$$

$$\therefore \quad b = (1.5)^{\frac{2}{3}} \doteqdot 1.3104$$

13 $y = x^2$ meets $y = k$ where $x^2 = k$ $\therefore$ $x = \pm\sqrt{k}$

Now area $= \int_0^{\sqrt{k}} (k-x^2)\,dx$

$$\therefore \quad \int_0^{\sqrt{k}} (k-x^2)dx = 2.4$$

$$\therefore \quad \left[kx - \frac{x^3}{3}\right]_0^{\sqrt{k}} = 2.4$$

$$\therefore \quad k\sqrt{k} - \frac{k\sqrt{k}}{3} - 0 = 2.4$$

$$\therefore \quad \frac{2k\sqrt{k}}{3} = 2.4$$

$$\therefore \quad k^{\frac{3}{2}} = 3.6$$

$$\therefore \quad k = (3.6)^{\frac{2}{3}} \doteqdot 2.3489$$

14 By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(6a)$ units^2.

$$\therefore \quad \int_0^a (x^2+2)dx = 3a$$

$$\therefore \quad \left[\frac{x^3}{3} + 2x\right]_0^a = 3a$$

$$\therefore \quad \frac{a^3}{3} + 2a - 0 = 3a$$

$$\therefore \quad a^3 + 6a = 9a$$

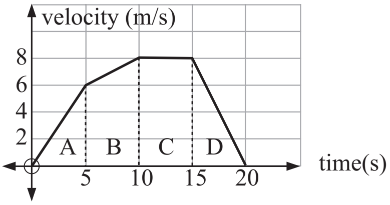
$$\therefore \quad a^3 - 3a = 0$$

$$\therefore \quad a(a^2-3) = 0$$

$$\therefore \quad a = 0 \text{ or } \pm\sqrt{3} \qquad \therefore \quad a = \sqrt{3} \quad \{\text{as } \ a > 0\}$$

EXERCISE 26D.1

1



Total distance travelled

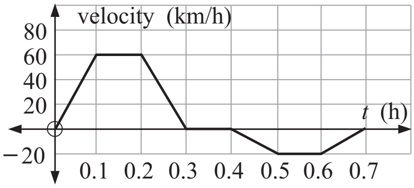
$= \text{area A} + \text{area B} + \text{area C} + \text{area D}$

$= \frac{1}{2}(5 \times 6) + \left(\dfrac{6+8}{2}\right)5 + 5 \times 8 + \frac{1}{2}(5 \times 8)$

$= 15 + 35 + 40 + 20$

$= 110$ m

2



a **i** The graph above the t-axis indicates that the velocity is positive and the car is travelling forwards.

ii The graph below the t-axis indicates that the velocity is negative and the car is travelling backwards.

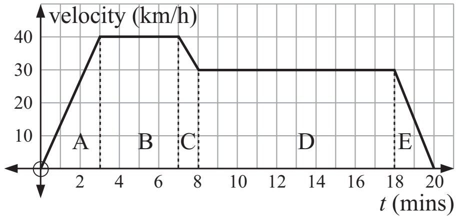
b Final displacement $= \text{area above the } t\text{-axis} - \text{area below the } t\text{-axis}$

$= \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right)60 - \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right)20$

$= 12 - 4$

$= 8$ km from the starting point in the positive direction

3



Total distance travelled

$= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E}$

$= \frac{1}{60}\left[\frac{1}{2}(3 \times 40) + (40 \times 4) + \left(\dfrac{40+30}{2}\right)1 + (10 \times 30) + \frac{1}{2}(2 \times 30)\right]$

$= \frac{1}{60}[60 + 160 + 35 + 300 + 30]$

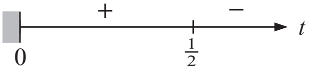
$= 9.75$ km

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km/h}

EXERCISE 26D.2

1 **a** $v(t) = 1 - 2t \text{ cms}^{-1}, \quad t \geqslant 0$

$v(t) = s'(t) = 1 - 2t$ which has sign diagram:



$\therefore$ a direction reversal occurs at $t = \frac{1}{2}$.

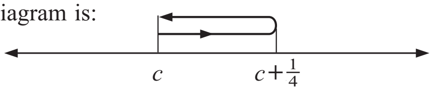
Now $s(t) = \int (1 - 2t)\,dt = t - \dfrac{2t^2}{2} + c = t - t^2 + c$

$\therefore \quad s(0) = c$

and $\quad s(\frac{1}{2}) = \frac{1}{4} + c$

and $\quad s(1) = c$

$\therefore$ motion diagram is:



$\therefore$ total distance travelled $= (c + \frac{1}{4} - c) + (c + \frac{1}{4} - c)$

$= \frac{1}{2}$ cm

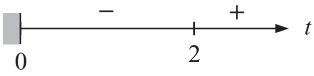
b Displacement
$= s(1) - s(0)$
$= c - c$
$= 0$ cm

or **a** total distance travelled
$= \int_0^1 |1 - 2t| \, dt$
$= 0.5$ cm

b displacement
$= \int_0^1 (1 - 2t) \, dt$
$= \left[t - t^2\right]_0^1$
$= 0 - 0$
$= 0$ cm

2 **a** $v(t) = t^2 - t - 2 \text{ cms}^{-1}, \quad t \geqslant 0$

$v(t) = s'(t) = t^2 - t - 2$
$= (t-2)(t+1)$ which has sign diagram:

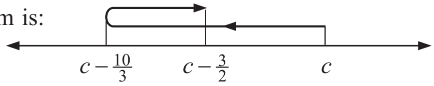


$\therefore$ a direction reversal occurs at $t = 2$.

Now $s(t) = \int (t^2 - t - 2) \, dt = \dfrac{t^3}{3} - \dfrac{t^2}{2} - 2t + c$

$s(0) = c$
$s(2) = c - \frac{10}{3}$
$s(3) = c - \frac{3}{2}$

$\therefore$ motion diagram is:



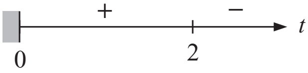
$\therefore$ total distance travelled $= \left(c - [c - \frac{10}{3}]\right) + \left(c - \frac{3}{2} - [c - \frac{10}{3}]\right)$
$= \frac{10}{3} - \frac{3}{2} + \frac{10}{3}$
$= \frac{31}{6}$
$= 5\frac{1}{6}$ cm

b Displacement $= s(3) - s(0)$
$= c - \frac{3}{2} - c$
$= -\frac{3}{2}$ cm

or **a** total distance travelled
$= \int_0^3 \left|t^2 - t - 2\right| \, dt$
$\doteqdot 5.17$ cm

b displacement
$= \int_0^3 (t^2 - t - 2) \, dt$
$= -1.5$ cm

3 $x'(t) = 16t - 4t^3 \text{ units s}^{-1}, \quad t \geqslant 0$
$= 4t(4 - t^2)$
$= 4t(2+t)(2-t)$ which has sign diagram:



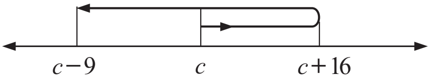
$\therefore$ a direction reversal occurs at $t = 2$.

Now $x(t) = \int (16t - 4t^3) \, dt = 8t^2 - t^4 + c$

a $0 \leqslant t \leqslant 3$

$x(0) = c$
$x(2) = 32 - 16 + c = c + 16$
$x(3) = 72 - 81 + c = c - 9$

$\therefore$ motion diagram is:

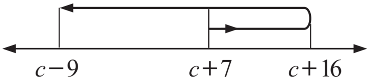


$\therefore$ total distance travelled $= (c + 16 - c) + (c + 16 - [c - 9])$
$= 41$ units

b $1 \leqslant t \leqslant 3$

$x(1) = 7 + c = c + 7$

$\therefore$ motion diagram is:



$\therefore$ total distance travelled $= (c + 16 - [c + 7]) + (c + 16 - [c - 9])$
$= 34$ units

or **a** total distance travelled
$= \int_0^3 \left|16t - 4t^3\right| \, dt$
$= 41$ units

b total distance travelled
$= \int_1^3 \left|16t - 4t^3\right| \, dt$
$= 34$ units

4 $v(t) = 50 - 10e^{-0.5t}\ \text{ms}^{-1} \quad t \geqslant 0$

a $v(0) = 50 - \dfrac{10}{e^0} = 50 - 10 = 40\ \text{ms}^{-1}$

b $v(3) = 50 - \dfrac{10}{e^{1.5}} \doteqdot 47.8\ \text{ms}^{-1}$

c The velocity reaches $45\ \text{ms}^{-1}$ when

$$\begin{aligned} 45 &= 50 - 10e^{-0.5t} \\ \therefore\quad 10e^{-\frac{t}{2}} &= 5 \\ \therefore\quad e^{-\frac{t}{2}} &= \tfrac{1}{2} \\ \therefore\quad e^{\frac{t}{2}} &= 2 \\ \therefore\quad \frac{t}{2} &= \ln 2 \\ \therefore\quad t &= 2\ln 2 \doteqdot 1.39 \text{ sec} \end{aligned}$$

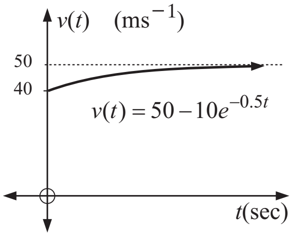
d $v(t) = 50 - \dfrac{10}{e^{\frac{t}{2}}}, \quad \therefore \quad$ as $\quad t \to \infty, \quad \dfrac{10}{e^{\frac{t}{2}}} \to {}^{+}0$ and so $\quad v(t) \to 50\ \text{ms}^{-1}$ (below)

e
$$\begin{aligned} a(t) &= v'(t) \\ &= -10e^{-0.5t}(-0.5) \\ &= 5e^{-0.5t}\ \text{ms}^{-2} \\ &= \frac{5}{e^{0.5t}}\ \text{ms}^{-2} \end{aligned}$$

$\therefore \quad a(t) > 0 \quad$ for all $t \quad \{e^x > 0$ for all $x\}$

i.e., the acceleration is always positive

f



g total distance travelled $= \int_0^3 \left|50 - 10e^{-0.5t}\right| dt$

$\doteqdot 134.5$ m

5 $\quad a(t) = \dfrac{t}{10} - 3\ \text{ms}^{-2} \quad \therefore \quad v(t) = \displaystyle\int \left(\frac{t}{10} - 3\right) dt = \frac{t^2}{20} - 3t + c$

But $\quad v(0) = 45 \quad \therefore \quad c = 45$

Now $\quad v(t) = \dfrac{t^2}{20} - 3t + 45$

$$= \frac{t^2 - 60t + 900}{20}$$

$$= \frac{(t-30)^2}{20}$$

Total distance travelled in the first minute

$$\begin{aligned} &= \int_0^{60} \left|\frac{(t-30)^2}{20}\right| dt \\ &= \tfrac{1}{20}\int_0^{60} \left|(t-30)^2\right| dt \\ &= \tfrac{1}{20}\int_0^{60} (t-30)^2\, dt \qquad \{\text{as } (t-30)^2 \geqslant 0\} \\ &= \tfrac{1}{20}\left[\frac{(t-30)^3}{3}\right]_0^{60} \\ &= \tfrac{1}{60}\left((30)^3 - (-30)^3\right) \\ &= \tfrac{1}{60}(30^3 + 30^3) \\ &= 900 \text{ m} \end{aligned}$$

6 $a(t) = 4e^{-\frac{t}{20}}\ \text{ms}^{-2} \quad \therefore \quad v(t) = \int 4e^{-\frac{t}{20}}\, dt = 4\, \dfrac{1}{-\frac{1}{20}}\, e^{-\frac{t}{20}} + c = -80e^{-\frac{t}{20}} + c$

Now $v(0) = 20\ \text{ms}^{-1} \quad \therefore \quad c = 100 \quad \therefore \quad v(t) = 100 - 80e^{-\frac{t}{20}}$

a as $t \to \infty$, $e^{-\frac{t}{20}} \to 0$ (above) $\therefore$ $v(t) \to 100$ (below)

i.e., the body approaches a limiting velocity of $100\ \text{ms}^{-1}$

b total distance travelled $= \int_0^{10} \left|100 - 80e^{-\frac{t}{20}}\right| dt \doteqdot 370.4$ m

EXERCISE 26E

1 The marginal cost is $C'(x)$ and

$$\begin{aligned} C'(x) &= 3.15 + 0.004x \ \ \text{\$/gadget} \\ \therefore \quad C(x) &= \int (3.15 + 0.004x)\, dx \\ &= 3.15x + 0.002x^2 + c \\ \text{But} \quad C(0) &= 450 \quad \therefore \quad c = 450 \\ \therefore \quad C(x) &= 3.15x + 0.002x^2 + 450 \ \ \text{dollars} \\ C(800) &= 3.15(800) + 0.002(800)^2 + 450 \\ &= \$4250 \end{aligned}$$

$\therefore$ total cost is \$4250

2 **a** The marginal profit is $P'(x)$ and

$$\begin{aligned} P'(x) &= 15 - 0.03x \ \ \text{\$/plate} \\ \therefore \quad P(x) &= \int (15 - 0.03x)\, dx \\ &= 15x - 0.015x^2 + c \\ \text{But} \quad P(0) &= -650 \quad \therefore \quad c = -650 \\ \therefore \quad P(x) &= 15x - 0.015x^2 - 650 \ \ \text{dollars} \end{aligned}$$

b The maximum profit occurs when $P'(x) = 0$ i.e., when

$$\begin{aligned} 15 - 0.03x &= 0 \\ \therefore \quad 0.03x &= 15 \\ \therefore \quad x &= \frac{15}{0.03} \\ \text{i.e.,} \quad x &= 500 \end{aligned}$$

and $P''(x) = -0.03 < 0$ $\therefore$ profit is at a maximum when $x = 500$

$$\begin{aligned} \therefore \quad \text{maximum profit} &= P(500) \\ &= 15(500) - 0.015(500)^2 - 650 \\ &= \$3100 \end{aligned}$$

c In order for a profit to be made $P(x)$ must be greater than 0

i.e., $15x - 0.015x^2 - 650 > 0$

Using technology to graph $P(x)$ and find x intercepts we get $x_1 = 45.39$ and $x_2 = 954.6$

$\therefore$ $46 \leqslant x \leqslant 954$ {We cannot produce part plates.}

3 $E(t) = 350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)$ calories/day

Total energy needs over the first week

$$\begin{aligned} &= \int_0^7 E(t)\, dt \\ &= \int_0^7 [350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)]\, dt \\ &= \left[\tfrac{1}{0.15} \times \frac{350(80 + 0.15t)^{1.8}}{1.8} - 9600t - 9t^2\right]_0^7 \\ &\doteqdot 14\,400 \text{ calories} \end{aligned}$$

4 $$\frac{dT}{dx} = \frac{-20}{x^{0.63}} = -20x^{-0.63} \quad \therefore \quad T = \int -20x^{-0.63}\ dx = \frac{-20x^{0.37}}{0.37} + c$$

But when $x = 3$, $T = 100$, $\therefore$ $\dfrac{-20(3^{0.37})}{0.37} + c = 100$

$$\therefore \quad c = 100 + \frac{20(3^{0.37})}{0.37} \doteqdot 181.1639$$

$$\text{i.e.,} \quad T = \frac{-20x^{0.37}}{0.37} + 181.1639$$

and when $x = 6$, $T = -104.8925 + 181.1639$

$$\doteqdot 76.27$$

$\therefore$ the outer surface temperature is about 76.3^oC

5 **a** $$\frac{d^2y}{dx^2} = -\tfrac{1}{10}(1-x)^2 \quad \text{and} \quad \frac{dy}{dx} = \int \frac{d^2y}{dx^2}\ dx$$

$$= \int -\tfrac{1}{10}(1-x)^2\ dx$$

$$= -\tfrac{1}{10}(\tfrac{1}{-1}) \times \frac{(1-x)^3}{3} + c$$

$$= \tfrac{1}{30}(1-x)^3 + c$$

But when $x = 0$, $\dfrac{dy}{dx} = 0$ {horizontal tangent}

$\therefore$ $\tfrac{1}{30}(1-0)^3 + c = 0$ and so $c = -\tfrac{1}{30}$

$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}$$

$$\therefore \quad y = \int [\tfrac{1}{30}(1-x)^3 - \tfrac{1}{30}]\ dx$$

$$= \tfrac{1}{30}\ (\tfrac{1}{-1})\ \frac{(1-x)^4}{4} - \tfrac{1}{30}x + d$$

$$= -\frac{(1-x)^4}{120} - \frac{x}{30} + d$$

Also, when $x = 0$, $y = 0$ $\therefore$ $-\tfrac{1}{120} - 0 + d = 0$ $\therefore$ $d = \tfrac{1}{120}$

and so $$y = \frac{1}{120} - \frac{(1-x)^4}{120} - \frac{x}{30}$$

b Maximum deflection occurs at the right hand end where $x \doteqdot 1$

and at $x \doteqdot 1$, $y \doteqdot \tfrac{1}{120} - 0 - \tfrac{1}{30} \doteqdot -0.025$ m

i.e., maximum deflection is about 2.5 cm.

6 **a** $$\frac{d^2y}{dx^2} = \tfrac{1}{100}\left(2x - \frac{x^2}{2}\right) = \tfrac{1}{50}x - \tfrac{1}{200}x^2$$

$$\therefore \quad \frac{dy}{dx} = \int (\tfrac{1}{50}x - \tfrac{1}{200}x^2)\ dx = \tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 + c$$

The sag, $y = \int (\tfrac{1}{100}x^2 - \tfrac{1}{600}x^3 + c)\ dx$

i.e., $y = \tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 + cx + d$

Now when $x = 0$, $y = 0$ $\therefore$ $0 - 0 + 0 + d = 0$

$$\therefore \quad d = 0$$

$$\therefore \quad y = \tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 + cx$$

Also when $x = 4$, $y = 0$ $\therefore$ $\frac{1}{300}\left(4^3\right) - \frac{1}{2400}\left(4^4\right) + 4c = 0$

$$\therefore \quad 4c = \tfrac{1}{2400}\left(4^4\right) - \tfrac{1}{300}\left(4^3\right)$$

$$\therefore \quad c = \tfrac{1}{2400}\left(4^3\right) - \tfrac{1}{300}\left(4^2\right)$$

$$\therefore \quad c = -\tfrac{2}{75}$$

$$\therefore \quad y = \left(\tfrac{1}{300}x^3 - \tfrac{1}{2400}x^4 - \tfrac{2}{75}x\right) \text{ m}$$

b The maximum sag occurs when $\dfrac{dy}{dx} = 0$ i.e., $\frac{1}{100}x^2 - \frac{1}{600}x^3 - \frac{2}{75} = 0$

$$6x^2 - x^3 - 16 = 0$$

Using technology the three solutions are: $x = -1.464$, 2 or 5.4641

But the maximum lies between 0 and 4, $\therefore$ maximum sag occurs when $x = 2$.

When $x = 2$, $y = \frac{1}{300}\left(2^3\right) - \frac{1}{2400}\left(2^4\right) - \frac{2}{75}(2)$

$$\doteqdot -0.033\,33 \text{ m}$$

$$\doteqdot -3.333 \text{ cm} \quad \therefore \quad \text{the maximum sag is} \doteqdot 3.33 \text{ cm}$$

c When 1 m from P, i.e., $x = 3$ m, $y = \frac{1}{300}\left(3^3\right) - \frac{1}{2400}\left(3^4\right) - \frac{2}{75}(3)$

$$= -0.023\,75 \text{ m}$$

$$= -2.375 \text{ cm} \quad \therefore \quad \text{the sag is } 2.375 \text{ cm}$$

d When 1 m from P, i.e., $x = 3$ m, $\dfrac{dy}{dx} = \frac{1}{100}\left(3^2\right) - \frac{1}{600}\left(3^3\right) - \frac{2}{75} \doteqdot 0.0183$

i.e., the angle θ, that the plank makes with the horizontal is such that $\tan\theta \doteqdot 0.0183$

i.e., $\theta = \tan^{-1}(0.0183) \doteqdot 1.05^o$

7 The cost per unit volume V is $\frac{1}{2}x^2 + 4$ \$/m^3

$\therefore$ $\dfrac{dC}{dV} = (\frac{1}{2}x^2 + 4)$ \$/m^3 and as $V = \pi r^2 x$, $\dfrac{dV}{dx} = \pi r^2$

Now $\dfrac{dC}{dx} = \dfrac{dC}{dV}\dfrac{dV}{dx}$ {Chain rule}

$$\therefore \quad \frac{dC}{dx} = (\tfrac{1}{2}x^2 + 4) \times \pi r^2$$

$\therefore$ $\dfrac{dC}{dx} = \pi r^2\left(\frac{1}{2}x^2 + 4\right)$ and the cost C, of digging a well $= \displaystyle\int \frac{dC}{dx}\,dx$

$$= \int[\pi r^2(\tfrac{1}{2}x^2 + 4)]\,dx$$

$$= \pi r^2\left(\frac{x^3}{6} + 4x\right) + c$$

$$\therefore \quad \text{cost of digging a well } h \text{ metres deep} = \pi r^2\left(\frac{h^3}{6} + 4h\right) + c$$

Now if the initial cost $= C_0$, $\pi r^2(\frac{0}{6} + 0) + c = C_0$ $\therefore$ $c = C_0$

$$\therefore \quad C(h) = \pi r^2\left(\frac{h^3 + 24h}{6}\right) + C_0$$

8 **a** The yield Y per unit area A is proportional to $\dfrac{1}{\sqrt{x+4}}$.

i.e., $\dfrac{dY}{dA} \propto \dfrac{1}{\sqrt{x+4}}$

$\therefore$ $\dfrac{dY}{dA} = k\left(\dfrac{1}{\sqrt{x+4}}\right) = \dfrac{k}{\sqrt{x+4}}$ where k is a constant

b Now the shaded area $A = \text{length} \times \text{width}$

$\therefore \quad A = (4-2p)x$

$\therefore \quad \dfrac{dA}{dx} = 4 - 2p$

Now $\dfrac{dY}{dx} = = \dfrac{dY}{dA}\,\dfrac{dA}{dx}$ {Chain rule}

$\therefore \quad \dfrac{dY}{dx} \quad \dfrac{k}{\sqrt{x+4}} \times (4-2p)$

i.e., $\dfrac{dY}{dx} = \dfrac{k(4-2p)}{\sqrt{x+4}}$

c $\dfrac{dY}{dx}$ is the instantaneous rate of change of the yield with respect to the distance x from the canal.

$$\therefore \quad \text{total yield} = Y = \int_0^p \frac{dY}{dx}dx = \int_0^p \frac{k(4-2p)}{\sqrt{x+4}}dx$$

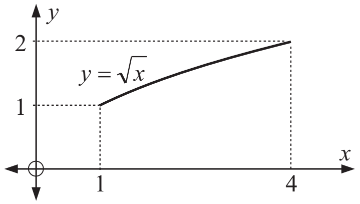
d
$$\begin{aligned} Y &= k(4-2p)\textstyle\int_0^p (x+4)^{-\frac{1}{2}}\,dx \quad \{\text{using } \mathbf{c}\} \\ &= k(4-2p) \times \left[\dfrac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^p \\ &= 2k(4-2p)\left[\sqrt{x+4}\right]_0^p \\ &= 4k(2-p)\left[\sqrt{p+4}-\sqrt{4}\right] \\ \therefore \quad Y &= 4k(2-p)\left(\sqrt{p+4}-2\right) \end{aligned}$$

e For yield to be a maximum we need to maximise Y. Using technology to graph Y and find the maximum, we find that the maximum occurs when $p \doteqdot 0.9735$ km

i.e., orchard is 0.974 km × 2.05 km

REVIEW SET 26A

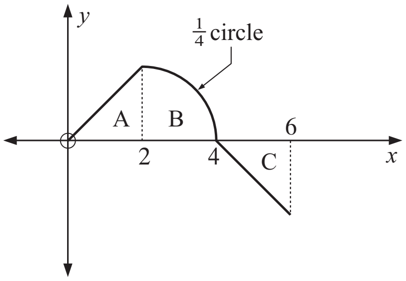
1 **a**



x	$\sqrt{x}$
1	1
1.5	1.2247
2	1.4142
2.5	1.5811
3	1.7321
3.5	1.8708
4	2

b
$$\begin{aligned} A_L &= 0.5\,(1 + 1.2247 + 1.4142 + 1.5811 + 1.7321 + 1.8708) \\ &= 0.5(8.8229) \\ &\doteqdot 4.41 \text{ units}^2 \end{aligned}$$

2



a
$$\begin{aligned} \textstyle\int_0^4 f(x)\,dx &= \text{area of triangle} + \text{area of } \tfrac{1}{4} \text{ circle} \\ &= \tfrac{1}{2}(2\times 2) + \tfrac{1}{4}\pi(2)^2 \\ &= 2 + \pi \end{aligned}$$

b
$$\begin{aligned} \textstyle\int_4^6 f(x)\,dx &= -\text{ area of triangle below } x\text{-axis} \\ &= -\tfrac{1}{2}(2\times 2) \\ &= -2 \end{aligned}$$

c
$$\begin{aligned} \textstyle\int_0^6 f(x)\,dx &= \textstyle\int_0^4 f(x)dx + \int_4^6 f(x)dx \\ &= (2+\pi) + (-2) \\ &= \pi \end{aligned}$$

3 **a** shaded area $= \int_a^b [f(x) - g(x)]\, dx + \int_b^c [g(x) - f(x)]\, dx + \int_c^d [f(x) - g(x)]\, dx$

b shaded area $= \int_a^d |f(x) - g(x)|\, dx$

4 $a(t) = 6t - 30 \text{ cms}^{-2} \quad \therefore \quad v(t) = \int (6t - 30)\, dt = 3t^2 - 30t + c$

But $v(0) = 27 \quad \therefore \quad c = 27 \quad \therefore \quad v(t) = 3t^2 - 30t + 27 \text{ cms}^{-1}$

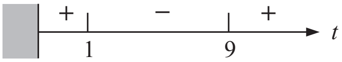
$\therefore \quad s(t) = \int (3t^2 - 30t + 27)\, dt = t^3 - 15t^2 + 27t + d$

But $s(0) = 0 \quad \therefore \quad d = 0 \quad \therefore \quad s(t) = t^3 - 15t^2 + 27t$

Now $v(t) = 3t^2 - 30t + 27$

$= 3(t^2 - 10t + 9)$

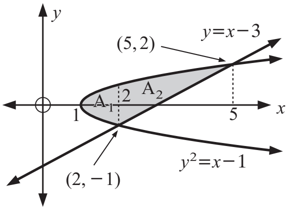
$= 3(t - 1)(t - 9)$ which has sign diagram:



The particle comes to rest for the second time at $t = 9$ seconds.

$\therefore$ total distance travelled $= \int_0^9 \left|3t^2 - 30t + 27\right| \text{ cm} = 269 \text{ cm}$

5 **a**



$y^2 = x - 1$ meets $y = x - 3$ where

$$x - 1 = (x - 3)^2$$
$$\therefore \quad x - 1 = x^2 - 6x + 9$$
$$\therefore \quad x^2 - 7x + 10 = 0$$
$$\therefore \quad (x - 5)(x - 2) = 0$$
$$\therefore \quad x = 2 \quad \text{or} \quad x = 5$$

$\therefore$ at $(5,\ 2)$ and $(2,\ -1)$

b Area $= A_1 + A_2$

$$= 2\int_1^2 (x-1)^{\frac{1}{2}}\, dx + \int_2^5 [(x-1)^{\frac{1}{2}} - (x-3)]\, dx$$

$$= 2\left[\tfrac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^2 + \left[\tfrac{2}{3}(x-1)^{\frac{3}{2}} - \frac{x^2}{2} + 3x\right]_2^5$$

$$= 2\left[\tfrac{2}{3} - 0\right] + \left[(\tfrac{2}{3}(8) - \tfrac{25}{2} + 15) - (\tfrac{2}{3} - 2 + 6)\right] = 4\tfrac{1}{2} \text{ units}^2$$

6 $y = k$ meets $y = x^2$ where $x^2 = k \quad \therefore \quad x = \pm\sqrt{k}$

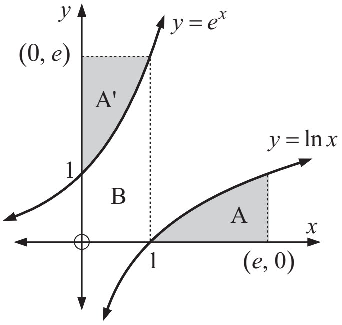
By symmetry, $\int_0^{\sqrt{k}} (k - x^2)\, dx = \frac{1}{2} \times 5\frac{1}{3} = \frac{1}{2} \times \frac{16}{3}$

$$\therefore \quad \left[kx - \frac{x^3}{3}\right]_0^{\sqrt{k}} = \tfrac{8}{3}$$

$$\therefore \quad k\sqrt{k} - \frac{k\sqrt{k}}{3} = \tfrac{8}{3}$$

$$\therefore \quad \tfrac{2}{3}k\sqrt{k} = \tfrac{8}{3}$$
$$\therefore \quad k\sqrt{k} = 4$$
$$\therefore \quad k^{\frac{3}{2}} = 4$$
$$\therefore \quad k = 4^{\frac{2}{3}} = \sqrt[3]{16}$$

7



$y = e^x$ and $y = \ln x$ are inverse functions,
i.e., they are symmetrical about $y = x$

$\therefore$ area A $=$ area A$'$

but area A$'$ + area B = area of rectangle

$\therefore$ area A + area B $= e \times 1 = e$

and as area A $= \int_1^e \ln x\, dx$

and area B $= \int_0^1 e^x\, dx$

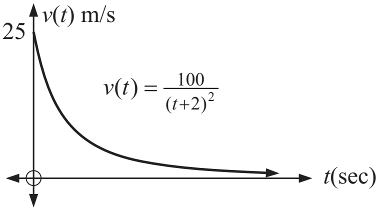
then $\int_1^e \ln x\, dx + \int_0^1 e^x\, dx = e$

8 $v(t) = \dfrac{100}{(t+2)^2} = 100(t+2)^{-2}$ ms^{-1}

a At $t = 0$, $v(0) = \dfrac{100}{2^2} = 25$ ms^{-1} At $t = 3$, $v(3) = \dfrac{100}{5^2} = 4$ ms^{-1}

b as $t \to +\infty$, $v(t) \to 0$ ms^{-1} (above)

c



d As $v(t)$ is always positive, the boat is always travelling forwards.

$$\begin{aligned} s(t) &= \int v(t)\,dt \\ &= \int 100(t+2)^{-2}\,dt \\ &= -100(t+2)^{-1} + c \\ &= \frac{-100}{t+2} + c \end{aligned}$$

$\therefore$ $s(0) = c - 50$ m

$\therefore$ when the boat has travelled 30 m,

$s(t) = c - 20$

$$\therefore\ c - 20 = \frac{-100}{t+2} + c$$

$$\therefore\ \frac{-100}{t+2} = -20$$

$\therefore$ $t + 2 = 5$

$\therefore$ $t = 3$ seconds

e
$$\begin{aligned} a(t) &= v'(t) \\ &= -200(t+2)^{-3} \text{ ms}^{-2} \\ &= \frac{-200}{(t+2)^3} \end{aligned}$$

f
$$\begin{aligned} \frac{dv}{dt} &= \frac{-200}{(t+2)^3} \\ &= -\tfrac{1}{5}\frac{1000}{(t+2)^3} \\ &= -\tfrac{1}{5}\left(\frac{100}{(t+2)^2}\right)^{\frac{3}{2}} \\ &= -\tfrac{1}{5}\,v^{\frac{3}{2}} \end{aligned}$$

$$\therefore\ \frac{dv}{dt} = -kv^{\frac{3}{2}} \quad \text{where} \quad k = \tfrac{1}{5}$$

9 $y = x^3$ meets $y = 7x^2 - 10x$

when $x^3 = 7x^2 - 10x$

i.e., $x^3 - 7x^2 + 10x = 0$

$\therefore$ $x(x^2 - 7x + 10) = 0$

$\therefore$ $x(x-2)(x-5) = 0$

$\therefore$ $x = 0$, 2 or 5

$\therefore$ total area $= \int_0^5 \left|x^3 - 7x^2 + 10x\right| dx$

$= 21\frac{1}{12}$ units^2

10 The area between $x = 0$ and $x = a$ is 2 units^2.

$\therefore$ $\int_0^a e^x\,dx = 2$

$\therefore$ $[e^x]_0^a = 2$

$\therefore$ $e^a - e^0 = 2$

$\therefore$ $e^a = 3$

$\therefore$ $a = \ln 3$

The area between $x = a = \ln 3$ and $x = b$ is 2 units^2.

$\therefore$ $\int_{\ln 3}^b e^x\,dx = 2$

$\therefore$ $[e^x]_{\ln 3}^b = 2$

$\therefore$ $e^b - e^{\ln 3} = 2$

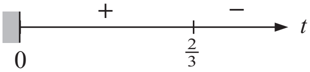
$\therefore$ $e^b - 3 = 2$

$\therefore$ $e^b = 5$

$\therefore$ $b = \ln 5$

REVIEW SET 26B

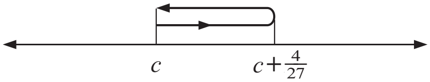
1 $v(t) = 2t - 3t^2 = t(2-3t)$ which has sign diagram:



$$\text{Now}\quad s(t) = \int (2t - 3t^2)\, dt$$
$$s(t) = t^2 - t^3 + c \text{ metres}$$

and so
$$s(0) = c$$
$$s(\tfrac{2}{3}) = \tfrac{4}{9} - \tfrac{8}{27} + c = c + \tfrac{4}{27}$$
$$s(1) = 1 - 1 + c = c$$

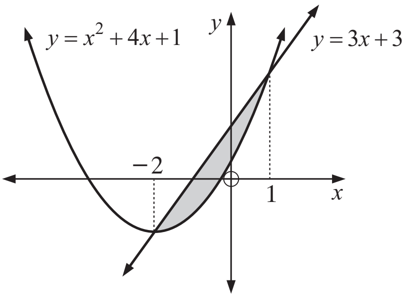
with motion diagram:



$\therefore$ total distance travelled $= (c + \frac{4}{27} - c) + (c + \frac{4}{27} - c) = \frac{8}{27}$ m $\doteqdot 29.6$ cm

or total distance travelled $= \int_0^1 \left|2t - 3t^2\right| dt$
$\doteqdot 0.296$ m

2 $y = x^2 + 4x + 1$ meets $y = 3x + 3$ where
$$x^2 + 4x + 1 = 3x + 3$$
$$\therefore\quad x^2 + x - 2 = 0$$
$$\therefore\quad (x+2)(x-1) = 0$$
$$\therefore\quad x = -2 \text{ or } 1$$



$$\therefore\quad \text{area} = \int_{-2}^{1} [(3x+3) - (x^2+4x+1)]\, dx$$
$$= \int_{-2}^{1} (-x^2 - x + 2)\, dx$$
$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$
$$= \left(-\tfrac{1}{3} - \tfrac{1}{2} + 2\right) - \left(\tfrac{8}{3} - 2 - 4\right)$$
$$= -\tfrac{1}{3} - \tfrac{1}{2} + 2 - \tfrac{8}{3} + 2 + 4$$
$$= 4\tfrac{1}{2} \text{ units}^2$$

3
$$\frac{dI}{dt} = -\frac{100}{t^2} \qquad t \geqslant 0.2 \text{ seconds}$$
$$\therefore\quad I(t) = \int \frac{dI}{dt}\, dt = \int -100t^{-2}\, dt = 100t^{-1} + c$$

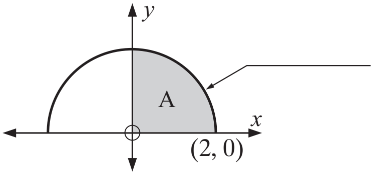
Now $I(2) = 150$ amps

$\therefore\ \dfrac{100}{2} + c = 150$ and so $c = 100$ $\quad\therefore\ I(t) = \left(\dfrac{100}{t} + 100\right)$ amps

a $I(20) = \dfrac{100}{20} + 100$
$= 105$ amps

b as $t \to +\infty$
$I(t) \to 100$ amps (above)

4



$\left\{ \begin{array}{l} y = \sqrt{4-x^2} \text{ is a} \\ \text{semi-circle above the } x\text{-axis} \\ \text{with centre 0 and radius 2} \end{array} \right.$

Now $\int_0^2 \sqrt{4-x^2}\, dx$ = shaded area = $\frac{1}{4}$ of the area of a circle of radius 2 units
$= \frac{1}{4}\pi(2^2)$
$= \pi$ units^2

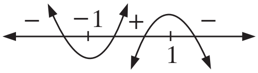
5 $\int_{-1}^{3} f(x)\, dx$ gives us the correct area only if $f(x)$ is positive on the interval $-1 \leqslant x \leqslant 3$.
But $f(x)$ is not positive for $1 \leqslant x \leqslant 3$ so $\int_{-1}^{3} f(x)\, dx$ does not provide the correct answer.

The shaded area below the x-axis is given by $\int_{1}^{3} [0 - f(x)]\, dx = -\int_{1}^{3} f(x)\, dx$

6 **a** $f(x) = \dfrac{x}{1+x^2}$ $\quad\therefore\quad$

$$\begin{aligned} f'(x) &= \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} \qquad \{\text{quotient rule}\} \\ &= \frac{1+x^2-2x^2}{(1+x^2)^2} \\ &= \frac{1-x^2}{(1+x^2)^2} \\ &= \frac{(1+x)(1-x)}{(1+x^2)^2} \end{aligned}$$

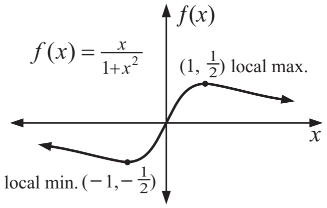
which has sign diagram:



$\therefore$ there is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(1, \frac{1}{2})$

b as $x \to \infty$ $f(x) \to 0$ (above) and
as $x \to -\infty$ $f(x) \to 0$ (below)

c



d

$$\begin{aligned} \text{Area} &= \int_{-2}^{0} \left[0 - \frac{x}{1+x^2}\right] dx \\ &= \int_{-2}^{0} \frac{-x}{1+x^2}\, dx \end{aligned}$$

Let $u = 1 + x^2$ $\quad\therefore\quad \dfrac{du}{dx} = 2x$

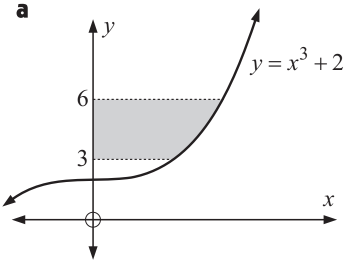
and when $x = 0$, $\quad u = 1$
when $x = -2$, $\quad u = 5$

$$\begin{aligned} \therefore \int_{-2}^{0} \frac{-x}{1+x^2}\, dx &= \int_{-2}^{0} \frac{1}{u}\,(-\tfrac{1}{2}\frac{du}{dx})\, dx \\ &= -\tfrac{1}{2}\int_{5}^{1} \frac{1}{u}\, du \\ &= -\tfrac{1}{2}\,[\ln|u|]_5^1 \\ &= -\tfrac{1}{2}(0 - \ln 5) \\ &= \tfrac{1}{2}\ln 5 \text{ units}^2 \quad (\doteqdot 0.805 \text{ units}^2) \end{aligned}$$

7 The coordinates of B are $(2, 4+k)$
$\therefore$ area rectangle OABC $= 2 \times (4+k)$
$= 8 + 2k$
$\therefore$ since the two shaded regions are equal in area, each area is $4 + k$ units^2.

$$\begin{aligned} \therefore \int_0^2 (x^2+k)\, dx &= 4+k \\ \therefore \left[\frac{x^3}{3} + kx\right]_0^2 &= 4+k \\ \therefore \tfrac{8}{3} + 2k &= 4+k \\ \therefore k &= 4 - \tfrac{8}{3} \\ \therefore k &= \tfrac{4}{3} \end{aligned}$$

8 **a**



b

$$\begin{aligned} y &= x^3 + 2 \\ \therefore\ x^3 &= y - 2 \\ \therefore\ x &= (y-2)^{\frac{1}{3}} \end{aligned}$$

c

$$\begin{aligned} \text{Area} &= \int_3^6 x\, dy \\ &= \int_3^6 (y-2)^{\frac{1}{3}}\, dy \\ &= \left[\tfrac{1}{1}\frac{(y-2)^{\frac{4}{3}}}{\frac{4}{3}}\right]_3^6 \\ &= \tfrac{3}{4}\left(4^{\frac{4}{3}} - 1^{\frac{4}{3}}\right) \\ &= \tfrac{3}{4}\left(4\sqrt[3]{4} - 1\right) \text{ units}^2 \end{aligned}$$

(or $\doteqdot 4.01$ units^2)

9 $y = 2x^3 - 9x$ meets $y = 3x^2 - 10$

when $2x^3 - 9x = 3x^2 - 10$

$\therefore \quad 2x^3 - 3x^2 - 9x + 10 = 0$

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -9 & 10 \\ & 0 & 2 & -1 & -10 \\ \hline & 2 & -1 & -10 & 0 \end{array}$$

$\therefore \quad (x-1)(2x^2 - x - 10) = 0$

$\therefore \quad (x-1)(2x-5)(x+2) = 0$

$\therefore \quad x = -2,\ 1$ or $\frac{5}{2}$

$\therefore$ total area

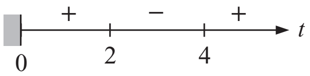
$= \int_{-2}^{\frac{5}{2}} \left|2x^3 - 3x^2 - 9x + 10\right| dx$

$\doteqdot 31.2$ units^2

REVIEW SET 26C

1 **a** $v(t) = t^2 - 6t + 8 \text{ ms}^{-1}, \quad t \geqslant 0$

$= (t-4)(t-2)$ which has sign diagram:



b Now $s(t) = \int (t^2 - 6t + 8)\, dt = \dfrac{t^3}{3} - 3t^2 + 8t + c$

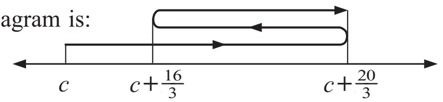
and so $s(0) = c$

$s(2) = c + 6\frac{2}{3}$

$s(4) = c + 5\frac{1}{3}$

$s(5) = c + 6\frac{2}{3}$

the motion diagram is:



The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

d total distance travelled $= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})]$

$= 9\frac{1}{3}$ m

2 If $x = \ln\left(\dfrac{y+3}{2}\right)$

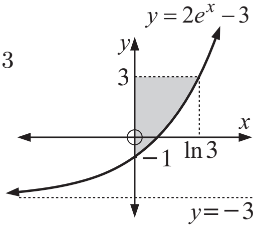
then $\dfrac{y+3}{2} = e^x \quad \therefore\ y = 2e^x - 3$

when $y = 3$, $2e^x - 3 = 3$

$\therefore\ 2e^x = 6$

$\therefore\ e^x = 3$

$\therefore\ x = \ln 3$



Area $= \int_0^{\ln 3} (3 - [2e^x - 3])\, dx$

$= \int_0^{\ln 3} (6 - 2e^x)\, dx$

$= [6x - 2e^x]_0^{\ln 3}$

$= \left(6\ln 3 - 2e^{\ln 3}\right) - (0 - 2)$

$= (6\ln 3 - 2 \times 3 + 2)$

$= 6\ln 3 - 4$ units^2

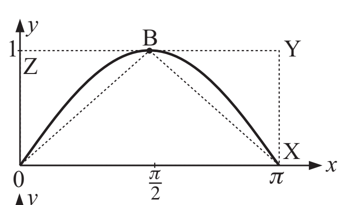
($\doteqdot 2.59$ units^2)

3 **a** From the graph,

area ΔOBX $<$ area under the curve $<$ area OXYZ

$\therefore \quad \frac{1}{2}\pi(1) < \int_0^{\pi} \sin x\, dx < \pi(1)$

i.e., $\frac{\pi}{2} < \int_0^{\pi} \sin x\, dx < \pi$



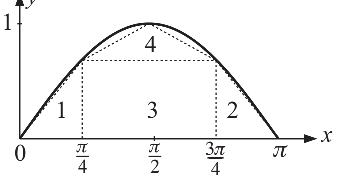
b If we partition the diagram into triangles as shown:

Area 1 $= \frac{1}{2}\left(\frac{\pi}{4}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{16}$

Area 2 $= \frac{\pi\sqrt{2}}{16}$

Area 3 $= \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$

Area 4 $= \frac{1}{2}\left(\frac{\pi}{2}\right)\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}\left(1 - \frac{\sqrt{2}}{2}\right)$



This total area is just less than the area under the arch

and total area $= \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{4} + \frac{\pi}{4}\left(1 - \frac{\sqrt{2}}{2}\right)$

$$\begin{aligned} &= \tfrac{\pi\sqrt{2}+\pi\sqrt{2}+4\pi\sqrt{2}+4\pi-2\pi\sqrt{2}}{16} \\ &= \tfrac{4\pi+4\pi\sqrt{2}}{16} \\ &= \tfrac{\pi+\pi\sqrt{2}}{4} \\ &= \tfrac{\pi}{4}(1+\sqrt{2}) \text{ units}^2 \end{aligned}$$

4 **a** A is the *upper half* of a circle centre (2, 0) and radius 2.

$$\begin{aligned} \therefore\quad (x-2)^2 + (y-0)^2 &= 2^2 \\ (x-2)^2 + y^2 &= 4 \\ y^2 &= 4 - (x-2)^2 \\ y^2 &= 4 - x^2 + 4x - 4 \\ y^2 &= 4x - x^2 \\ \therefore\quad y &= \pm\sqrt{4x-x^2} \end{aligned}$$

So, $y = \sqrt{4x-x^2}$

or $y = -\sqrt{4x-x^2}$

$\therefore$ required equation is $y_A = \sqrt{4x-x^2}$

b Now B is the *lower half* of a circle centre (5, 0) and radius 1.

$$\begin{aligned} \therefore\quad (x-5)^2 + (y-0)^2 &= 1^2 \\ (x-5)^2 + y^2 &= 1 \\ y^2 &= 1 - (x-5)^2 \\ y^2 &= 1 - x^2 + 10x - 25 \\ \therefore\quad y &= \pm\sqrt{10x-x^2-24} \end{aligned}$$

$\therefore\quad y_B = -\sqrt{10x-x^2-24}$

c $\int_0^4 y_A dx$

$= \frac{1}{2}\pi r^2$ where $r = 2$

$= \frac{1}{2}\pi(2)^2$

$= 2\pi$

$\int_4^6 y_B dx$

$= -\frac{1}{2}\pi r^2$ where $r = 1$

$= -\frac{1}{2}\pi(1)^2$

$= -\frac{\pi}{2}$

d $\int_0^6 f(x)dx$

$= \int_0^4 y_A dx + \int_4^6 y_B dx$

$= 2\pi + (-\frac{\pi}{2})$

$= \frac{3\pi}{2}$

5 The shaded area $= \int_0^2 ax(x-2)\ dx = 4$ units^2

$$\begin{aligned} \therefore\quad \int_0^2 (ax^2 - 2ax)\ dx &= 4 \\ \therefore\quad \left[\frac{ax^3}{3} - ax^2\right]_0^2 &= 4 \\ \therefore\quad \left[\frac{8a}{3} - 4a\right] - 0 &= 4 \end{aligned}$$

$$\begin{aligned} \therefore\quad \frac{8a}{3} - \frac{12a}{3} &= 4 \\ \therefore\quad -\frac{4a}{3} &= 4 \\ \therefore\quad a &= -3 \\ \therefore\quad y &= -3x(x-2) \end{aligned}$$

Suppose A has coordinates $(k,\ -3k(k-2))$

$\therefore$ slope OA $= \dfrac{-3k(k-2)-0}{k-0} = -3(k-2)$ $\quad\therefore$ equation of OA is $y = -3(k-2)x$

Now if OA divides the shaded area into equal areas,

$$\begin{aligned} \int_0^k [-3x(x-2) - (-3(k-2)x)]\ dx &= 2 \\ \therefore\quad \int_0^k (-3x^2 + 6x + 3kx - 6x)dx &= 2 \\ \therefore\quad \int_0^k (-3x^2 + 3kx)\ dx &= 2 \\ \therefore\quad \left[-x^3 + \frac{3kx^2}{2}\right]_0^k &= 2 \end{aligned}$$

$$\begin{aligned} \therefore\quad -k^3 + \frac{3k^3}{2} &= 2 \\ \therefore\quad \frac{k^3}{2} &= 2 \\ \therefore\quad k^3 &= 4 \\ \therefore\quad k &= \sqrt[3]{4} \end{aligned}$$

$\therefore$ the x-coordinate of A is $\sqrt[3]{4}$

6 The line $y = mx + c$ passes through $(-1, 0)$. $\therefore\ 0 = -m + c$ and so $c = m$

$\therefore$ the line is $y = cx + c$

The curve and the line meet when

$$\begin{aligned} cx + c &= -x^2 + 2x + 3 \\ \therefore\ x^2 + (c-2)x + (c-3) &= 0 \\ \therefore\ (x+1)(x+[c-3]) &= 0 \quad \{\text{as we know } x = -1 \text{ is a solution}\} \\ \therefore\ x &= -1 \text{ or } 3 - c \end{aligned}$$

If we let $a = 3 - c$, then the enclosed area

$$\begin{aligned} &= \int_{-1}^{a} [(-x^2 + 2x + 3) - (3-a)(x+1)]\, dx \\ &= \int_{-1}^{a} [-x^2 + (a-1)x + a]\, dx \\ &= \left[-\frac{x^3}{3} + \frac{(a-1)x^2}{2} + ax \right]_{-1}^{a} \\ &= \left(-\frac{a^3}{3} + \frac{(a-1)a^2}{2} + a^2 \right) - \left(\tfrac{1}{3} + \frac{(a-1)}{2} - a \right) \\ &= -\tfrac{1}{3}a^3 + \tfrac{1}{2}a^3 - \tfrac{1}{2}a^2 + a^2 - \tfrac{1}{3} - \tfrac{1}{2}a + \tfrac{1}{2} + a \\ &= \tfrac{1}{6}a^3 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a + \tfrac{1}{6} \end{aligned}$$

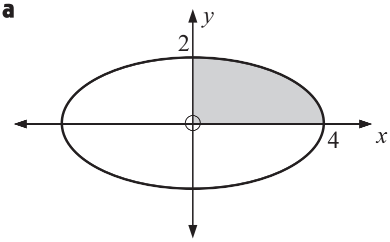
But this area is 4.5 units^2,

$$\begin{aligned} \therefore\ \tfrac{1}{6}a^3 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a + \tfrac{1}{6} &= \tfrac{9}{2} \\ \therefore\ a^3 + 3a^2 + 3a + 1 &= 27 \\ \therefore\ a^3 + 3a^2 + 3a - 26 &= 0 \\ \therefore\ (a-2)(a^2 + 5a + 13) &= 0 \quad \text{where the quadratic has } \Delta < 0, \\ & \qquad \therefore \text{ no real solutions} \\ \therefore\ a &= 2 \quad \text{and so} \quad c = 1 \quad (= m) \end{aligned}$$

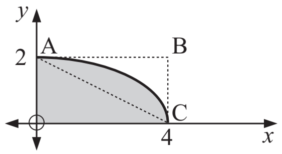
From technology, the only real solution to this equation is $c = 1$.

So $c = 1$, $m = 1$, and the line has equation $y = x + 1$.

7 **a**



b



Now area ΔAOC $<$ shaded area $<$ area ABCD

$$\begin{aligned} \therefore\ \tfrac{1}{2}(2 \times 4) &< \int_0^4 \tfrac{1}{2}\sqrt{16 - x^2}\, dx < 2 \times 4 \\ \therefore\ 4 &< \int_0^4 \tfrac{1}{2}\sqrt{16 - x^2}\, dx < 8 \\ \therefore\ 8 &< \int_0^4 \sqrt{16 - x^2}\, dx < 16 \quad \{(\times 2)\} \end{aligned}$$

8 The curves meet when

$$x^3 + x^2 + 2x + 6 = 7x^2 - x - 4$$

$$\therefore\ x^3 - 6x^2 + 3x + 10 = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & -6 & 3 & 10 \\ & 0 & -1 & 7 & -10 \\ \hline & 1 & -7 & 10 & 0 \end{array}$$

$$\begin{aligned} \therefore\ (x+1)(x^2 - 7x + 10) &= 0 \\ \therefore\ (x+1)(x-2)(x-5) &= 0 \\ \therefore\ x &= -1, 2 \text{ or } 5 \end{aligned}$$

$\therefore$ area enclosed

$$= \int_{-1}^{5} \left| x^3 - 6x^2 + 3x + 10 \right|\, dx$$

$$= 40\tfrac{1}{2} \text{ units}^2$$

Chapter 27

CIRCULAR FUNCTION INTEGRATION

EXERCISE 27A

1 **a**
$$\begin{aligned}&\int(3\sin x - 2)\;dx\\ &= -3\cos x - 2x + c\end{aligned}$$

b
$$\begin{aligned}&\int(4x - 2\cos x)\;dx\\ &= 2x^2 - 2\sin x + c\end{aligned}$$

c
$$\begin{aligned}&\int(2\sqrt{x} + \frac{4}{\cos^2 x})\;dx\\ &= \int(2x^{\frac{1}{2}} + 4\,\frac{1}{\cos^2 x})\;dx\\ &= \tfrac{4}{3}x^{\frac{3}{2}} + 4\tan x + c\end{aligned}$$

d
$$\begin{aligned}&\int(\frac{1}{\cos^2 x} + 2\sin x)\;dx\\ &= \tan x - 2\cos x + c\end{aligned}$$

e
$$\begin{aligned}&\int\left(\frac{x}{2} - \frac{1}{\cos^2 x}\right)\;dx\\ &= \frac{x^2}{4} - \tan x + c\end{aligned}$$

f
$$\begin{aligned}&\int(\sin x - 2\cos x + e^x)\;dx\\ &= -\cos x - 2\sin x + e^x + c\end{aligned}$$

2 **a**
$$\begin{aligned}&\int(\sqrt{x} + \tfrac{1}{2}\cos x)\;dx\\ &= \int(x^{\frac{1}{2}} + \tfrac{1}{2}\cos x)\;dx\\ &= \tfrac{2}{3}x^{\frac{3}{2}} + \tfrac{1}{2}\sin x + c\end{aligned}$$

b
$$\begin{aligned}&\int(\theta - \sin\theta)\;d\theta\\ &= \frac{\theta^2}{2} + \cos\theta + c\end{aligned}$$

c
$$\begin{aligned}&\int(t\sqrt{t} + \frac{2}{\cos^2 t})\;dt\\ &= \int(t^{\frac{3}{2}} + 2\sec^2 t)\;dt\\ &= \tfrac{2}{5}t^{\frac{5}{2}} + 2\tan t + c\end{aligned}$$

d
$$\begin{aligned}&\int(2e^t - 4\sin t)\;dt\\ &= 2e^t + 4\cos t + c\end{aligned}$$

e
$$\begin{aligned}&\int\left(3\cos t - \frac{1}{t}\right)\;dt\\ &= 3\sin t - \ln|t| + c\end{aligned}$$

f
$$\begin{aligned}&\int(3 - \frac{2}{\theta} + \frac{1}{\cos^2\theta})\;d\theta\\ &= 3\theta - 2\ln|\theta| + \tan\theta + c\end{aligned}$$

3 **a**
$$\frac{d}{dx}(e^x\sin x) = e^x\sin x + e^x\cos x \qquad \therefore\ \begin{aligned}\int e^x(\sin x + \cos x)\;dx &= \int(e^x\sin x + e^x\cos x)\;dx\\ &= e^x\sin x + c\end{aligned}$$

b
$$\begin{aligned}\frac{d}{dx}\left(e^{-x}\sin x\right) &= -e^{-x}\sin x + e^{-x}\cos x\\ &= \frac{\cos x - \sin x}{e^x}\end{aligned} \qquad \therefore\ \int\frac{\cos x - \sin x}{e^x}\;dx = e^{-x}\sin x + c$$

c
$$\begin{aligned}\frac{d}{dx}(x\cos x) &= \cos x + x(-\sin x)\\ &= \cos x - x\sin x\end{aligned}$$

$$\begin{aligned}\therefore\ \int(\cos x - x\sin x)\;dx &= x\cos x + c_1\\ \therefore\ \int\cos x\;dx - \int x\sin x\;dx &= x\cos x + c_1\\ \therefore\ \sin x - \int x\sin x\;dx &= x\cos x + c_1\\ \therefore\ \int x\sin x\;dx &= -x\cos x + \sin x + c\end{aligned}$$

d
$$\frac{1}{\cos x} = (\cos x)^{-1} \qquad \therefore\ \begin{aligned}\frac{d}{dx}(\frac{1}{\cos x}) &= -(\cos x)^{-2}(-\sin x)\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{\frac{\sin x}{\cos x}}{\cos x}\\ &= \frac{\tan x}{\cos x}\end{aligned} \qquad \therefore\ \int\frac{\tan x}{\sin x}\;dx = \frac{1}{\cos x} + c$$

4 **a**

$$\begin{aligned} f'(x) &= x^2 - 4\cos x \\ \therefore\quad f(x) &= \int (x^2 - 4\cos x)\ dx \\ &= \frac{x^3}{3} - 4\sin x + c \end{aligned}$$

But $f(0) = 3$

$$\begin{aligned} \therefore\quad 0 - 4\sin(0) + c &= 3 \\ \therefore\quad c &= 3 \end{aligned}$$

$$\therefore\quad f(x) = \frac{x^3}{3} - 4\sin x + 3$$

b

$$\begin{aligned} f'(x) &= 2\cos x - 3\sin x \\ \therefore\quad f(x) &= \int (2\cos x - 3\sin x)\ dx \\ &= 2\sin x + 3\cos x + c \end{aligned}$$

But $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$\begin{aligned} \therefore\quad 2\sin\tfrac{\pi}{4} + 3\cos\tfrac{\pi}{4} + c &= \tfrac{1}{\sqrt{2}} \\ \therefore\quad 2(\tfrac{1}{\sqrt{2}}) + 3(\tfrac{1}{\sqrt{2}}) + c &= \tfrac{1}{\sqrt{2}} \\ \therefore\quad c &= -\tfrac{4}{\sqrt{2}} \\ \therefore\quad c &= -2\sqrt{2} \end{aligned}$$

$$\therefore\quad f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$$

c

$$\begin{aligned} f'(x) &= \sqrt{x} - \frac{2}{\cos^2 x} \\ \therefore\quad f(x) &= \int (x^{\frac{1}{2}} - \frac{2}{\cos^2 x})\ dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x + c \end{aligned}$$

But $f(\pi) = 0$

$$\begin{aligned} \therefore\quad \tfrac{2}{3}\pi^{\frac{3}{2}} - 2\tan\pi + c &= 0 \\ \therefore\quad c &= -\tfrac{2}{3}\pi^{\frac{3}{2}} \end{aligned}$$

$$\therefore\quad f(x) = \tfrac{2}{3}x^{\frac{3}{2}} - 2\tan x - \tfrac{2}{3}\pi^{\frac{3}{2}}$$

EXERCISE 27B

1 **a**

$$\begin{aligned} &\int \sin(3x)\ dx \\ &= -\tfrac{1}{3}\cos(3x) + c \end{aligned}$$

b

$$\begin{aligned} &\int 2\cos(4x)\ dx \\ &= 2 \times \tfrac{1}{4}\sin(4x) + c \\ &= \tfrac{1}{2}\sin(4x) + c \end{aligned}$$

c

$$\begin{aligned} &\int \frac{1}{\cos^2(2x)}\ dx \\ &= \tfrac{1}{2}\tan(2x) + c \end{aligned}$$

d

$$\begin{aligned} &\int 3\cos\left(\tfrac{x}{2}\right)\ dx \\ &= 6\sin\left(\tfrac{x}{2}\right) + c \end{aligned}$$

e

$$\begin{aligned} &\int (3\sin(2x) - e^{-x})\ dx \\ &= -\tfrac{3}{2}\cos(2x) + e^{-x} + c \end{aligned}$$

f

$$\begin{aligned} &\int \left[e^{2x} - \frac{2}{\sec^2\left(\frac{x}{2}\right)}\right]\ dx \\ &= \tfrac{1}{2}e^{2x} - 2 \times 2\tan\left(\tfrac{x}{2}\right) + c \\ &= \tfrac{1}{2}e^{2x} - 4\tan\left(\tfrac{x}{2}\right) + c \end{aligned}$$

g

$$\begin{aligned} &\int 2\sin(2x + \tfrac{\pi}{6})\ dx \\ &= -\tfrac{2}{2}\cos(2x + \tfrac{\pi}{6}) + c \\ &= -\cos(2x + \tfrac{\pi}{6}) + c \end{aligned}$$

h

$$\begin{aligned} &\int -3\cos(\tfrac{\pi}{4} - x)\ dx \\ &= -3(-\sin(\tfrac{\pi}{4} - x)) + c \\ &= 3\sin(\tfrac{\pi}{4} - x) + c \end{aligned}$$

i

$$\begin{aligned} &\int \frac{4}{\cos^2\left(\frac{\pi}{3} - 2x\right)}\ dx \\ &= 4 \times (-\tfrac{1}{2})\tan(\tfrac{\pi}{3} - 2x) + c \\ &= -2\tan(\tfrac{\pi}{3} - 2x) + c \end{aligned}$$

j

$$\begin{aligned} &\int \cos(2x) + \sin(2x)\, dx \\ &= \tfrac{1}{2}\sin(2x) - \tfrac{1}{2}\cos(2x) + c \end{aligned}$$

k

$$\begin{aligned} &\int 2\sin(3x) + 5\cos(4x)\, dx \\ &= -\tfrac{2}{3}\cos(3x) + \tfrac{5}{4}\sin(4x) + c \end{aligned}$$

l

$$\begin{aligned} &\int \tfrac{1}{2}\cos(8x) - 3\sin x\, dx \\ &= \tfrac{1}{2}(\tfrac{1}{8})\sin(8x) + 3\cos x + c \\ &= \tfrac{1}{16}\sin(8x) + 3\cos x + c \end{aligned}$$

2 **a**

$$\begin{aligned}&\int \cos^2 x \; dx\\ &= \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right) dx\\ &= \tfrac{1}{2}x + \tfrac{1}{4}\sin(2x) + c\end{aligned}$$

b

$$\begin{aligned}&\int \sin^2 x \; dx\\ &= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right) dx\\ &= \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + c\end{aligned}$$

c

$$\begin{aligned}&\int (1 + \cos^2(2x)) \; dx\\ &= \int \left(1 + \tfrac{1}{2} + \tfrac{1}{2}\cos(4x)\right) dx\\ &= \int \left(\tfrac{3}{2} + \tfrac{1}{2}\cos(4x)\right) dx\\ &= \tfrac{3}{2}x + \tfrac{1}{8}\sin(4x) + c\end{aligned}$$

d

$$\begin{aligned}&\int (3 - \sin^2(3x)) \; dx\\ &= \int \left(3 - \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(6x)\right)\right) dx\\ &= \int \left(\tfrac{5}{2} + \tfrac{1}{2}\cos(6x)\right) dx\\ &= \tfrac{5}{2}x + \tfrac{1}{12}\sin(6x) + c\end{aligned}$$

e

$$\begin{aligned}&\int \tfrac{1}{2}\cos^2(4x) \; dx\\ &= \int \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(8x)\right) dx\\ &= \int \left(\tfrac{1}{4} + \tfrac{1}{4}\cos(8x)\right) dx\\ &= \tfrac{1}{4}x + \tfrac{1}{32}\sin(8x) + c\end{aligned}$$

f

$$\begin{aligned}&\int (1 + \cos x)^2 \; dx\\ &= \int (1 + 2\cos x + \cos^2 x) \; dx\\ &= \int \left(1 + 2\cos x + \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\right) dx\\ &= \tfrac{3}{2}x + 2\sin x + \tfrac{1}{4}\sin(2x) + c\end{aligned}$$

3

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta) \quad \therefore \quad \begin{aligned}\cos^4\theta &= \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\theta)\right)^2\\ &= \tfrac{1}{4} + \tfrac{1}{4}\cos^2(2\theta) + \tfrac{1}{2}\cos(2\theta)\\ &= \tfrac{1}{4} + \tfrac{1}{4}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4\theta)\right) + \tfrac{1}{2}\cos(2\theta)\\ &= \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8}\cos(4\theta) + \tfrac{1}{2}\cos(2\theta)\\ &= \tfrac{1}{8}\cos(4\theta) + \tfrac{1}{2}\cos(2\theta) + \tfrac{3}{8} \quad \text{as required}\end{aligned}$$

$$\therefore \quad \int \cos^4 x \; dx = \int \left(\tfrac{1}{8}\cos(4x) + \tfrac{1}{2}\cos(2x) + \tfrac{3}{8}\right) dx = \tfrac{1}{32}\sin(4x) + \tfrac{1}{4}\sin(2x) + \tfrac{3}{8}x + c$$

4 **a** Consider $\int \sin^4 x \cos x \; dx$

Let $u = \sin x, \quad \dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore \quad &\int \sin^4 x \cos x \; dx\\ &= \int u^4 \, \frac{du}{dx} \; dx\\ &= \int u^4 \; du\\ &= \frac{u^5}{5} + c\\ &= \tfrac{1}{5}\sin^5 x + c\end{aligned}$$

b Consider $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}} \; dx$

Let $u = \cos x, \quad \dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore \quad &\int \frac{\sin x}{\sqrt{\cos x}} \; dx\\ &= \int u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx\\ &= \int -u^{-\frac{1}{2}} \; du\\ &= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \quad \text{or} \quad -2\sqrt{\cos x} + c\end{aligned}$$

c Consider $\displaystyle\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx$

Let $u = \cos x, \quad \dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore \quad &\int \frac{\sin x}{\cos x} \; dx\\ &= \int \frac{1}{u}\left(-\frac{du}{dx}\right) dx\\ &= \int -\frac{1}{u} \; du\\ &= -\ln|u| + c\\ &= -\ln|\cos x| + c\end{aligned}$$

d Consider $\int \sqrt{\sin x}\cos x \; dx$

Let $u = \sin x, \quad \dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore \quad &\int \sqrt{\sin x}\cos x \; dx\\ &= \int u^{\frac{1}{2}} \, \frac{du}{dx} \; dx\\ &= \int u^{\frac{1}{2}} \; du\\ &= \tfrac{2}{3}u^{\frac{3}{2}} + c\\ &= \tfrac{2}{3}(\sin x)^{\frac{3}{2}} + c\end{aligned}$$

e Consider $\displaystyle\int \frac{\cos x}{(2+\sin x)^2}\,dx$

Let $u = 2+\sin x$, $\dfrac{du}{dx} = \cos x$

$$\therefore \int \frac{\cos x}{(2+\sin x)^2}\,dx$$
$$= \int u^{-2}\,\frac{du}{dx}\,dx$$
$$= \int u^{-2}\,du$$
$$= -u^{-1} + c$$
$$= \frac{-1}{2+\sin x} + c$$

f Consider $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore \int \frac{\sin x}{\cos^3 x}\,dx &= \int u^{-3}\left(-\frac{du}{dx}\right)dx\\ &= \int -u^{-3}\,du\\ &= \frac{-u^{-2}}{-2} + c\\ &= \tfrac{1}{2}u^{-2} + c\\ &= \tfrac{1}{2}(\cos x)^{-2} + c\\ &= \frac{1}{2\cos^2 x} + c\end{aligned}$$

g Consider $\displaystyle\int \frac{\sin x}{1-\cos x}\,dx$

Let $u = 1-\cos x$, $\dfrac{du}{dx} = \sin x$

$$\therefore \int \frac{\sin x}{1-\cos x}\,dx$$
$$= \int \frac{1}{u}\,\frac{du}{dx}\,dx$$
$$= \int \frac{1}{u}\,du$$
$$= \ln|u| + c$$
$$= \ln|1-\cos x| + c$$

h Consider $\displaystyle\int \frac{\cos(2x)}{\sin(2x)-3}\,dx$

Let $u = \sin(2x) - 3$, $\dfrac{du}{dx} = 2\cos(2x)$

$$\therefore \quad \cos(2x) = \tfrac{1}{2}\frac{du}{dx}$$

$$\therefore \int \frac{\cos(2x)}{\sin(2x)-3}\,dx$$
$$= \int \frac{1}{u}\left(\tfrac{1}{2}\frac{du}{dx}\right)dx$$
$$= \tfrac{1}{2}\int \frac{1}{u}\,du$$
$$= \tfrac{1}{2}\ln|u| + c$$
$$= \tfrac{1}{2}\ln|\sin(2x)-3| + c$$

5 **a**
$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x\cos x\,dx\\ &= \int (1-\sin^2 x)\cos x\,dx\end{aligned}$$

Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

$$\begin{aligned}\therefore \int \cos^3 x\,dx &= \int (1-u^2)\,\frac{du}{dx}\,dx\\ &= \int (1-u^2)\,du\\ &= u - \frac{u^3}{3} + c\\ &= \sin x - \frac{\sin^3 x}{3} + c\\ &= \sin x - \tfrac{1}{3}\sin^3 x + c\end{aligned}$$

b
$$\begin{aligned}&\int \sin^5 x\,dx\\ &= \int \sin^4 x\sin x\,dx\\ &= \int (1-\cos^2 x)^2\sin x\,dx\\ &= \int (1-2\cos^2 x+\cos^4 x)\sin x\,dx\end{aligned}$$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

$$\begin{aligned}\therefore \int \sin^5 x\,dx &= \int (1-2u^2+u^4)\left(-\frac{du}{dx}\right)dx\\ &= \int (-1+2u^2-u^4)\,du\\ &= -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + c\\ &= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + c\end{aligned}$$

c

$$\begin{aligned}&\int \sin^4 x\cos^3 x\ dx\\ =&\int \sin^4 x\cos^2 x\cos x\ dx\\ =&\int \sin^4 x(1-\sin^2 x)\cos x\ dx\\ =&\int (\sin^4 x-\sin^6 x)\cos x\ dx\end{aligned}$$

Let $u=\sin x,\quad \dfrac{du}{dx}=\cos x$

$$\begin{aligned}\therefore\quad \int \sin^4 x\cos^3 x\ dx &= \int (u^4-u^6)\ \frac{du}{dx}\,dx\\ &=\int (u^4-u^6)\ du\\ &=\tfrac{1}{5}u^5-\tfrac{1}{7}u^7+c\\ &=\tfrac{1}{5}\sin^5 x-\tfrac{1}{7}\sin^7 x+c\end{aligned}$$

6 **a** Let $u=\cos x,\quad \dfrac{du}{dx}=-\sin x$

$$\begin{aligned}\therefore\quad f(x)&=\int \sin x e^{\cos x}\ dx\\ &=\int e^{\cos x}\sin x\ dx\\ &=\int e^u\left(-\frac{du}{dx}\right)\ dx\\ &=-\int e^u\ du\\ &=-e^u+c\\ &=-e^{\cos x}+c\end{aligned}$$

b Let $u=\sin(2x),\quad \dfrac{du}{dx}=2\cos(2x)$

$$\begin{aligned}\therefore\quad f(x)&=\int \sin^3(2x)\cos(2x)\ dx\\ &=\int u^3\left(\tfrac{1}{2}\frac{du}{dx}\right)\ dx\\ &=\tfrac{1}{2}\int u^3\ du\\ &=\tfrac{1}{2}\times\frac{u^4}{4}+c\\ &=\tfrac{1}{8}\sin^4(2x)+c\end{aligned}$$

c Let $u=\sin x-\cos x,$

$$\frac{du}{dx}=\cos x+\sin x$$

$$\begin{aligned}\therefore\quad f(x)&=\int\frac{\sin x+\cos x}{\sin x-\cos x}\ dx\\ &=\int\frac{1}{u}\ \frac{du}{dx}\,dx\\ &=\int\frac{1}{u}\ du\\ &=\ln|u|+c\\ &=\ln|\sin x-\cos x|+c\end{aligned}$$

d Let $u=\tan x$

$$\therefore\quad \frac{du}{dx}=\frac{1}{\cos^2 x}$$

So, $\displaystyle\int\frac{e^{\tan x}}{\cos^2 x}\ dx$

$$\begin{aligned}&=\int e^u\left(\frac{du}{dx}\right)dx\\ &=\int e^u\,du\\ &=e^u+c\\ &=e^{\tan x}+c\end{aligned}$$

7 **a** $\displaystyle\int\cot x\,dx=\int\frac{\cos x}{\sin x}\,dx$

We let $u=\sin x,\quad$ so $\quad\dfrac{du}{dx}=\cos x$

$$\begin{aligned}\therefore\quad \int\cot x\,dx&=\int\frac{1}{u}\ \frac{du}{dx}\,dx\\ &=\int\frac{1}{u}\,du\\ &=\ln|u|+c\\ &=\ln|\sin x|+c\end{aligned}$$

b $\displaystyle\int\cot(3x)\,dx=\int\frac{\cos(3x)}{\sin(3x)}\,dx$

We let $u=\sin(3x),$

so $\dfrac{du}{dx}=3\cos(3x)$

$$\begin{aligned}\therefore\quad \int\cot(3x)\,dx&=\int\frac{\tfrac{1}{3}\times 3\cos(3x)}{\sin(3x)}\,dx\\ &=\tfrac{1}{3}\int\frac{1}{u}\ \frac{du}{dx}\,dx\\ &=\tfrac{1}{3}\int\frac{1}{u}\,du\\ &=\tfrac{1}{3}\ln|u|+c\\ &=\tfrac{1}{3}\ln|\sin(3x)|+c\end{aligned}$$

c We let $u = \cot x$

$$\text{so}\quad \frac{du}{dx} = -\csc^2 x$$

$$\therefore\quad \int \csc^2 x\,dx = \int \left(-\frac{du}{dx}\right) dx$$

$$= -\int du$$

$$= -u + c$$

$$= -\cot x + c$$

d We let $u = \sec x$

$$\text{so}\quad \frac{du}{dx} = \sec x \tan x$$

$$\therefore\quad \int \sec x \tan x\,dx = \int \frac{du}{dx}\,dx$$

$$= \int du$$

$$= u + c$$

$$= \sec x + c$$

e We let $u = \csc x$,

$$\text{so}\quad \frac{du}{dx} = -\csc x \cot x$$

$$\therefore\quad \int \csc x \cot x\,dx = \int \left(-\frac{du}{dx}\right) dx$$

$$= -\int du$$

$$= -u + c$$

$$= -\csc x + c$$

f We let $u = \sec(3x)$,

$$\text{so}\quad \frac{du}{dx} = 3\sec(3x)\tan(3x)$$

$$\therefore\quad \int \tan(3x)\sec(3x)\,dx$$

$$= \int \tfrac{1}{3} \times 3\sec(3x)\tan(3x)\,dx$$

$$= \tfrac{1}{3}\int \frac{du}{dx}\,dx$$

$$= \tfrac{1}{3}\int du$$

$$= \tfrac{1}{3}u + c$$

$$= \tfrac{1}{3}\sec(3x) + c$$

g We let $u = \csc\left(\frac{x}{2}\right)$,

$$\text{so}\quad \frac{du}{dx} = -\tfrac{1}{2}\csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)$$

$$\therefore\quad \int \csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)dx = \int (-2) \times \left(-\tfrac{1}{2}\csc\left(\tfrac{x}{2}\right)\cot\left(\tfrac{x}{2}\right)\right) dx$$

$$= -2\int \frac{du}{dx}\,dx$$

$$= -2\int du$$

$$= -2u + c$$

$$= -2\csc\left(\tfrac{x}{2}\right) + c$$

h $\int \sec^3 x\,\sin x\,dx = \int (\cos x)^{-3}\sin x\,dx$

We let $u = \cos x$

$$\text{so}\quad \frac{du}{dx} = -\sin x$$

$$\therefore\quad \int \sec^3 x \sin x\,dx = \int u^{-3}\left(-\frac{du}{dx}\right) dx$$

$$= -\int u^{-3}\,du$$

$$= \tfrac{1}{2}u^{-2} + c$$

$$= \frac{1}{2\cos^2 x} + c$$

i We let $u = \cot x$

$$\text{so}\quad \frac{du}{dx} = -\csc^2 x$$

$$\therefore\quad \int \frac{\csc^2 x}{\sqrt{\cot x}}\,dx = \int \frac{1}{\sqrt{u}}\left(-\frac{du}{dx}\right) dx$$

$$= -\int u^{-\frac{1}{2}}\,du$$

$$= -2u^{\frac{1}{2}} + c$$

$$= -2\sqrt{\cot x} + c$$

EXERCISE 27C

1 **a**
$$\begin{aligned}&\int_0^{\frac{\pi}{6}} \cos x\ dx\\ &= [\sin x]_0^{\frac{\pi}{6}}\\ &= \sin\tfrac{\pi}{6} - \sin 0\\ &= \tfrac{1}{2}\end{aligned}$$

b
$$\begin{aligned}&\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\ dx\\ &= [-\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\\ &= -\cos\tfrac{\pi}{2} + \cos\tfrac{\pi}{3}\\ &= \tfrac{1}{2}\end{aligned}$$

c
$$\begin{aligned}&\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\ dx\\ &= [\tan x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}\\ &= \tan\tfrac{\pi}{3} - \tan\tfrac{\pi}{4}\\ &= \sqrt{3} - 1\end{aligned}$$

d
$$\begin{aligned}&\int_0^{\frac{\pi}{6}} \sin(3x)\ dx\\ &= \left[-\tfrac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{6}}\\ &= -\tfrac{1}{3}[\cos\tfrac{\pi}{2} - \cos 0]\\ &= -\tfrac{1}{3}[0-1]\\ &= \tfrac{1}{3}\end{aligned}$$

e
$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \cos^2 x\ dx\\ &= \int_0^{\frac{\pi}{4}} (\tfrac{1}{2} + \tfrac{1}{2}\cos(2x))\ dx\\ &= \left[\frac{x}{2} + \tfrac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{4}}\\ &= [\tfrac{\pi}{8} + \tfrac{1}{4}\sin\tfrac{\pi}{2}] - 0\\ &= \tfrac{\pi}{8} + \tfrac{1}{4}\end{aligned}$$

f
$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \sin^2 x\ dx\\ &= \int_0^{\frac{\pi}{2}} (\tfrac{1}{2} - \tfrac{1}{2}\cos(2x))\ dx\\ &= \left[\frac{x}{2} - \tfrac{1}{4}\sin(2x)\right]_0^{\frac{\pi}{2}}\\ &= [\tfrac{\pi}{4} - \tfrac{1}{4}\sin\pi] - 0\\ &= \tfrac{\pi}{4}\end{aligned}$$

2 **a** Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned}\therefore \quad \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\ dx &= \int_1^{\frac{1}{2}} u^{-\frac{1}{2}}\left(-\frac{du}{dx}\right)\ dx\\ &= \int_{\frac{1}{2}}^1 u^{-\frac{1}{2}}\ du\\ &= \left[2u^{\frac{1}{2}}\right]_{\frac{1}{2}}^1\\ &= 2\sqrt{1} - 2\sqrt{\tfrac{1}{2}}\\ &= 2 - \sqrt{2}\end{aligned}$$

b Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

When $x = 0$, $u = \sin 0 = 0$

When $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned}\therefore \quad \int_0^{\frac{\pi}{6}} \sin^2 x\cos x\ dx &= \int_0^{\frac{1}{2}} u^2\ \frac{du}{dx}\ dx\\ &= \int_0^{\frac{1}{2}} u^2\ du\\ &= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}}\\ &= \tfrac{1}{3}(\tfrac{1}{2})^3 \quad \text{or} \quad \tfrac{1}{24}\end{aligned}$$

c Let $u = \cos x \ \therefore\ \dfrac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan x\ dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\ dx\\ &= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\left(-\frac{du}{dx}\right)\ dx\\ &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\ du\\ &= [\ln|u|]_{\frac{1}{\sqrt{2}}}^1\\ &= \ln 1 - \ln\tfrac{1}{\sqrt{2}}\\ &= \ln\sqrt{2}\\ &= \tfrac{1}{2}\ln 2\end{aligned}$$

d Let $u = \sin x$, $\dfrac{du}{dx} = \cos x$

When $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

When $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$$\begin{aligned}\therefore \quad &\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x}\ dx\\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\ dx\\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\ \frac{du}{dx}\ dx\\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\ du\\ &= [\ln|u|]_{\frac{1}{2}}^1\\ &= \ln 1 - \ln\tfrac{1}{2}\\ &= 0 + \ln 2 \qquad \{\ln\left(\tfrac{1}{a}\right) = -\ln a\}\\ &= \ln 2\end{aligned}$$

e Let $u = 1 - \sin x, \quad \dfrac{du}{dx} = -\cos x$

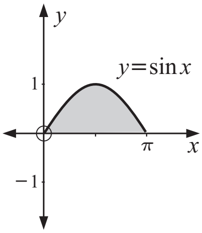
When $x = 0, \quad u = 1 - \sin 0 = 1$

When $x = \frac{\pi}{6}, \quad u = 1 - \sin \frac{\pi}{6} = \frac{1}{2}$

$$\therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{1 - \sin x}\, dx = \int_1^{\frac{1}{2}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx = \int_{\frac{1}{2}}^{1} \frac{1}{u}\, du = [\ln|u|]_{\frac{1}{2}}^{1} = \ln 1 - \ln \tfrac{1}{2} = \ln 2$$

EXERCISE 27D

1 **a**

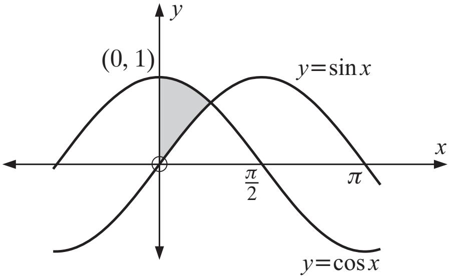


$$A = \textstyle\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi} = [-\cos \pi + \cos 0] = -(-1) + 1 = 2 \text{ units}^2$$

b We first note that $\sin^2 x \geqslant 0$ always, so the function never drops below the x-axis.

$$\therefore \quad A = \textstyle\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} (\frac{1}{2} - \frac{1}{2}\cos(2x))\, dx = \left[\dfrac{x}{2} - \frac{1}{4}\sin(2x)\right]_0^{\pi} = \left[\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)\right] - [0 - \frac{1}{4}\sin 0] = \frac{\pi}{2} \text{ units}^2$$

2



The curves $y = \cos x$ and $y = \sin x$ meet when $x = \frac{\pi}{4}$.

$$\therefore \quad A = \textstyle\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\frac{\pi}{4}} = (\sin\frac{\pi}{4} + \cos\frac{\pi}{4}) - (\sin 0 + \cos 0) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1) = (\sqrt{2} - 1) \text{ units}^2$$

3 **a** $y = \tan x$

A has y-coordinate 1 and lies on the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{2}]$.

$\therefore \quad \tan x = 1 \quad \therefore \quad x = \frac{\pi}{4}$ i.e., A is at $(\frac{\pi}{4}, 1)$

b $\tan x = \dfrac{\sin x}{\cos x}$

Let $u = \cos x, \quad \dfrac{du}{dx} = -\sin x$

When $x = 0, \quad u = \cos 0 = 1$

When $x = \frac{\pi}{4}, \quad u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \quad \text{area} = \textstyle\int_0^{\frac{\pi}{4}} \tan x\, dx = \displaystyle\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u}\, du = [\ln|u|]_{\frac{1}{\sqrt{2}}}^{1} = \ln 1 - \ln\frac{1}{\sqrt{2}} = \ln\sqrt{2} = \tfrac{1}{2}\ln 2 \text{ units}^2$$

4 **a** $y = \sin(2x)$ is the curve C_1
$y = \sin x$ is the curve C_2

b The curves meet when

$$\begin{aligned} \sin(2x) &= \sin x \\ \therefore\quad 2\sin x\cos x - \sin x &= 0 \\ \therefore\quad \sin x(2\cos x - 1) &= 0 \\ \text{i.e.,}\quad \sin x = 0 \text{ or } \cos x &= \tfrac{1}{2} \\ \therefore\quad x \text{ coordinate of A} &= \tfrac{\pi}{3} \\ \therefore\quad \text{A is at } (\tfrac{\pi}{3}, \tfrac{\sqrt{3}}{2}) \end{aligned}$$

c

$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x)\; dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x))\; dx \\ &= \left[-\tfrac{1}{2}\cos(2x) + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \tfrac{1}{2}\cos(2x)\right]_{\frac{\pi}{3}}^{\pi} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right) + \left(-\cos\pi + \tfrac{1}{2}\cos 2\pi\right) \\ &\qquad - \left(-\cos\tfrac{\pi}{3} + \tfrac{1}{2}\cos\tfrac{2\pi}{3}\right) \\ &= (\tfrac{1}{4} + \tfrac{1}{2}) - (-\tfrac{1}{2} + 1) + (1 + \tfrac{1}{2}) - (-\tfrac{1}{2} - \tfrac{1}{4}) \\ &= 2\tfrac{1}{2} \text{ units}^2 \end{aligned}$$

5 **a** $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1

b

When $x = 0,\ y = \cos 0 = 1.$	$\therefore$ A is at (0, 1)
When $x = \frac{\pi}{4},\ y = \cos(\frac{\pi}{2}) = 0.$	$\therefore$ B is at $(\frac{\pi}{4}, 0)$
Also, $\cos^2(\frac{\pi}{2}) = 0,$	$\therefore$ C is at $(\frac{\pi}{2}, 0)$
When $x = \frac{3\pi}{4},\ y = \cos(\frac{3\pi}{2}) = 0,$	$\therefore$ D is at $(\frac{3\pi}{4}, 0)$
$\cos(2\pi) = \cos^2\pi = 1\ \therefore\ y = \cos(2x)$ and $y = \cos^2 x = 1$ when $x = \pi$	$\therefore$ E is at $(\pi, 1)$

c

$$\begin{aligned} A &= \int_0^{\pi} (\cos^2 x - \cos(2x))\; dx \\ &= \int_0^{\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) - \cos(2x\,)\; dx \\ &= \int_0^{\pi} \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\; dx \\ &= \left[\frac{x}{2} - \tfrac{1}{4}\sin(2x)\right]_0^{\pi} = (\tfrac{\pi}{2} - 0) - (0 - 0) = \tfrac{\pi}{2} \text{ units}^2 \end{aligned}$$

REVIEW SET 27A

1 **a**

$$\begin{aligned} &\int \sin^7 x\cos x\, dx \\ &= \int u^7 \left(\frac{du}{dx}\right) dx \quad \text{on letting} \quad u = \sin x \\ &= \int u^7\, du \\ &= \frac{u^8}{8} + c \\ &= \tfrac{1}{8}\sin^8 x + c \end{aligned}$$

b Consider $\int \tan(2x)\; dx = \displaystyle\int \frac{\sin(2x)}{\cos(2x)}\, dx$

Let $u = \cos(2x),\quad \dfrac{du}{dx} = -2\sin(2x)$

$$\therefore\quad \sin(2x) = -\tfrac{1}{2}\frac{du}{dx}$$

$$\begin{aligned} \therefore\quad \int \tan(2x)\; dx &= \int \frac{1}{u}\left(-\tfrac{1}{2}\frac{du}{dx}\right) dx \\ &= -\tfrac{1}{2}\int \frac{1}{u}\, du \\ &= -\tfrac{1}{2}\ln|u| + c \\ &= -\tfrac{1}{2}\ln|\cos(2x)| + c \end{aligned}$$

c Consider $\int e^{\sin x}\cos x\; dx$

Let $u = \sin x,\quad \dfrac{du}{dx} = \cos x$

$$\begin{aligned} \therefore\quad \int e^{\sin x}\cos x\; dx &= \int e^u \frac{du}{dx}\, dx \\ &= \int e^u\; du \\ &= e^u + c \\ &= e^{\sin x} + c \end{aligned}$$

2 If $y = x\tan x$ then

$$\frac{dy}{dx} = 1\tan x + x\left(\frac{1}{\cos^2 x}\right)$$

$$= \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}$$

$$\therefore \int \frac{\sin x}{\cos x}\,dx + \int \frac{x}{\cos^2 x}\,dx = x\tan x + c_1$$

$$\therefore \int \frac{1}{u}\left(-\frac{du}{dx}\right)dx + \int \frac{x}{\cos^2 x}\,dx = x\tan x + c_1$$

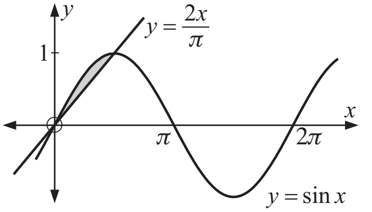
$$\{\text{letting } u = \cos x, \ \therefore \ \frac{du}{dx} = -\sin x\}$$

$$\therefore \ -\int \frac{1}{u}\,du + \int \frac{x}{\cos^2 x}\,dx = x\tan x + c_1$$

$$\therefore \int \frac{x}{\cos^2 x}\,dx = x\tan x + \ln|u| + c$$

$$= x\tan x + \ln|\cos x| + c$$

3 Using technology, the graphs meet when $x = 0$ and $x = \frac{\pi}{2}$.



$\therefore$ enclosed area

$$= \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2x}{\pi}\right) dx$$

$$= \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\frac{\pi}{2}}$$

$$= (0 - \tfrac{\pi}{4}) - (-1 - 0)$$

$$= (1 - \tfrac{\pi}{4}) \text{ units}^2$$

4 **a**

$$\int_0^{\frac{\pi}{3}} \cos^2\left(\tfrac{x}{2}\right) dx = \int_0^{\frac{\pi}{3}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos x\right) dx$$

$$= \left[\tfrac{x}{2} + \tfrac{1}{2}\sin x\right]_0^{\frac{\pi}{3}}$$

$$= \tfrac{\pi}{6} + \tfrac{1}{2}\sin\tfrac{\pi}{3} - 0 - \tfrac{1}{2}\sin 0$$

$$= \tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4}$$

b

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{4}} \tan x\,dx = \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u}\left(-\frac{du}{dx}\right)dx$$

$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u}\,du$$

$$= [\ln|u|]_{\frac{1}{\sqrt{2}}}^1$$

$$= \ln 1 - \ln\tfrac{1}{\sqrt{2}}$$

$$= \ln\sqrt{2}$$

$$= \tfrac{1}{2}\ln 2$$

5

$$y = \ln\left(\frac{1}{\cos x}\right)$$

$$= -\ln(\cos x)$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\cos x} \times (-\sin x)$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$$\therefore \int \tan x\,dx = \ln\left(\frac{1}{\cos x}\right) + c$$

$$= -\ln(\cos x) + c$$

$$\{\text{since } \frac{1}{\cos x} > 0\}$$

or $\int \tan x\,dx = \ln(\sec x) + c$

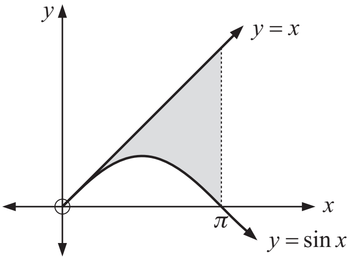
6 Let $u = \sin\theta \quad \therefore \quad \dfrac{du}{d\theta} = \cos\theta$

$u(\frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$

$u(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\frac{du}{d\theta}\,d\theta$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\,du$$
$$= [\ln|u|]_{\frac{1}{2}}^{1}$$
$$= \ln 1 - \ln\left(\tfrac{1}{2}\right)$$
$$= \ln 2$$

7



Required area

$= \text{area of } \Delta - \text{area under sine curve}$

$= \frac{1}{2}\pi \times \pi - \int_0^{\pi} \sin x\,dx$

$= \dfrac{\pi^2}{2} - [-\cos x]_0^{\pi}$

$= \dfrac{\pi^2}{2} - [-\cos\pi + \cos 0]$

$= \dfrac{\pi^2}{2} - 2 \text{ units}^2$

REVIEW SET 27B

1 **a** $\displaystyle\int 4\sin^2\left(\tfrac{x}{2}\right)\,dx$

$= \displaystyle\int 2\left(2\sin^2\left(\tfrac{x}{2}\right)\right)\,dx$

Now $\cos(2\theta) = 1 - 2\sin^2\theta$

$\therefore \quad 2\sin^2\theta = 1 - \cos(2\theta)$

$\therefore \quad \displaystyle\int 4\sin^2\left(\tfrac{x}{2}\right)\,dx = 2\int (1 - \cos x)\,dx$

$= 2[x - \sin x] + c$

$= 2x - 2\sin x + c$

b $\displaystyle\int (2 - \cos x)^2\,dx$

$= \displaystyle\int \left(4 - 4\cos x + \cos^2 x\right)dx$

$= \displaystyle\int 4 - 4\cos x + \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)dx$

$= \frac{9}{2}x - 4\sin x + \frac{1}{2}\left(\frac{1}{2}\right)\sin(2x)\,dx$

$= \frac{9}{2}x - 4\sin x + \frac{1}{4}\sin(2x) + c$

2 If $u = \cos x, \quad \dfrac{du}{dx} = -\sin x \quad \therefore$

$$\int \frac{\sin x}{\cos^4 x}\,dx$$
$$= \int u^{-4}\left(-\frac{du}{dx}\right)dx$$
$$= -\int u^{-4}\,du$$
$$= \frac{u^{-3}}{3} + c$$
$$= \frac{1}{3\cos^3 x} + c$$

3 $\dfrac{d}{dx}(\sin(x^2))$

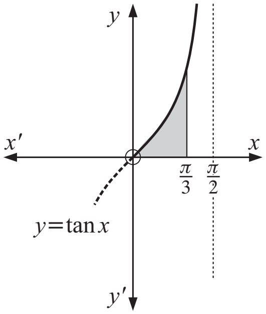
$= 2x\cos(x^2)$

$\therefore \quad \int x\cos(x^2)\,dx$

$= \frac{1}{2}\int 2x\cos(x^2)\,dx$

$= \frac{1}{2}\sin(x^2) + c$

4



Now $\tan x \geqslant 0$ for $0 \leqslant x \leqslant \frac{\pi}{3}$

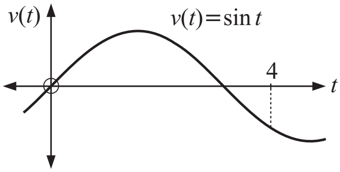
$$\therefore \quad A = \int_0^{\frac{\pi}{3}} \tan x \; dx$$
$$= \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx$$
$$= \int_1^{\frac{1}{2}} \frac{1}{u}\left(-\frac{du}{dx}\right) dx$$
$$= \int_{\frac{1}{2}}^{1} \frac{1}{u}\, du \;=\; [\ln|u|]_{\frac{1}{2}}^{1}$$
$$= \ln 1 - \ln \tfrac{1}{2}$$
$$= \ln 2 \text{ units}^2$$

Let $u = \cos x$, $\dfrac{du}{dx} = -\sin x$

When $x = \frac{\pi}{3}$, $u = \cos\frac{\pi}{3} = \frac{1}{2}$

When $x = 0$, $u = \cos 0 = 1$

5



Since $v(t) = \sin t \text{ ms}^{-1}$, $v(t) = 0$ when $t = \pi$ seconds.

We may therefore see from the graph that the particle reverses its direction at time $t = \pi$.

$\therefore$ the distance travelled is
$$D = \int_0^{\pi} \sin t \; dt + \int_{\pi}^{4} (-\sin t) \; dt$$
$$= [-\cos t]_0^{\pi} + [\cos t]_{\pi}^{4}$$
$$= -\cos\pi + \cos 0 + \cos 4 - \cos\pi$$
$$= 3 + \cos 4$$
$$\doteqdot 2.35 \text{ m}$$

6 **a**

$$\int_0^{\frac{\pi}{6}} \sin^2\left(\tfrac{x}{2}\right) \; dx$$
$$= \int_0^{\frac{\pi}{6}} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos x\right) \; dx$$
$$= \left[\frac{x}{2} - \tfrac{1}{2}\sin x\right]_0^{\frac{\pi}{6}}$$
$$= \left(\tfrac{\pi}{12} - \tfrac{1}{4}\right) - (0 - 0)$$
$$= \tfrac{\pi}{12} - \tfrac{1}{4}$$

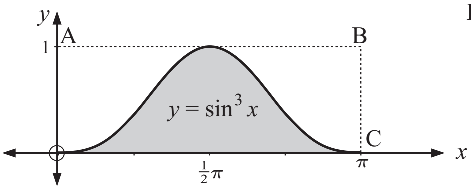
b Let $u = \tan x$, $\dfrac{du}{dx} = \sec^2 x = \dfrac{1}{\cos^2 x}$

When $x = \frac{\pi}{4}$, $u = \tan\frac{\pi}{4} = 1$

When $x = \frac{\pi}{3}$, $u = \tan\frac{\pi}{3} = \sqrt{3}$

$$\therefore \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x \tan x} \; dx$$
$$= \int_1^{\sqrt{3}} \frac{1}{u}\,\frac{du}{dx} \; dx$$
$$= \int_1^{\sqrt{3}} \frac{1}{u} \; du \;=\; [\ln|u|]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1$$
$$= \tfrac{1}{2}\ln 3$$

7 Consider the graph of $y = \sin^3 x$, $0 \leqslant x \leqslant \pi$



Now $\int_0^{\pi} \sin^3 x \; dx =$ shaded area

But the shaded area $<$ area of rectangle ABCO

i.e., $\int_0^{\pi} \sin^3 x \; dx < \pi$

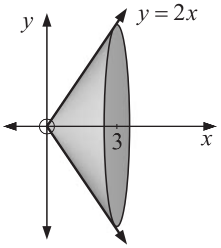
$\therefore \quad \int_0^{\pi} \sin^3 x \; dx < 4$

Chapter 28

VOLUMES OF REVOLUTION

EXERCISE 28A.1

1 **a**



Volume

$= \pi \int_0^3 (2x)^2 \, dx$

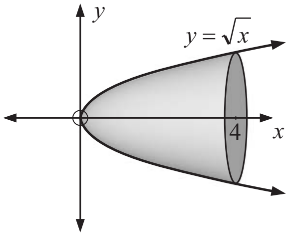
$= 4\pi \int_0^3 x^2 \, dx$

$= 4\pi \left[\frac{x^3}{3}\right]_0^3$

$= 4\pi(9 - 0)$

$= 36\pi$ units^3

b



Volume

$= \pi \int_0^4 (\sqrt{x})^2 \, dx$

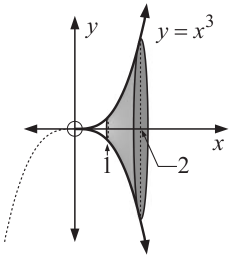
$= \pi \int_0^4 x \, dx$

$= \pi \left[\frac{x^2}{2}\right]_0^4$

$= \pi(8 - 0)$

$= 8\pi$ units^3

c



Volume

$= \pi \int_1^2 (x^3)^2 \, dx$

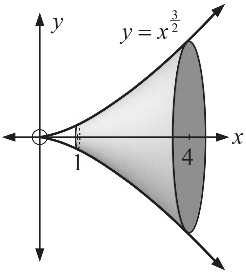
$= \pi \int_1^2 x^6 \, dx$

$= \pi \left[\frac{x^7}{7}\right]_1^2$

$= \pi \left(\frac{128}{7} - \frac{1}{7}\right)$

$= \frac{127\pi}{7}$ units^3

d



Volume

$= \pi \int_1^4 \left(x^{\frac{3}{2}}\right)^2 \, dx$

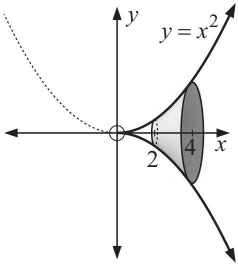
$= \pi \int_1^4 x^3 \, dx$

$= \pi \left[\frac{x^4}{4}\right]_1^4$

$= \pi \left(\frac{256}{4} - \frac{1}{4}\right)$

$= \frac{255\pi}{4}$ units^3

e



Volume

$= \pi \int_2^4 (x^2)^2 \, dx$

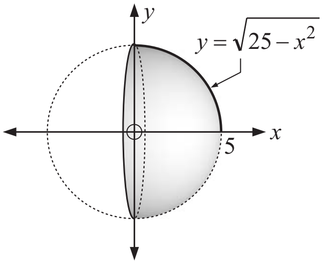
$= \pi \int_2^4 x^4 \, dx$

$= \pi \left[\frac{x^5}{5}\right]_2^4$

$= \pi \left[\frac{1024}{5} - \frac{32}{5}\right]$

$= \frac{992\pi}{5}$ units^3

f



Volume

$= \pi \int_0^5 (25 - x^2) \, dx$

$= \pi \left[25x - \frac{x^3}{3}\right]_0^5$

$= \pi \left[\left(125 - \frac{125}{3}\right) - 0\right]$

$= \pi \left(\frac{2}{3}\right) 125$

$= \frac{250\pi}{3}$ units^3

2 **a**

$$\begin{aligned} V &= \pi\int_0^6 \left(\frac{x}{2}+4\right)^2 dx \\ &= \pi\int_0^6 \left(\tfrac{1}{4}x^2+4x+16\right) dx \\ &= \pi\left[\frac{x^3}{12}+\frac{4x^2}{2}+16x\right]_0^6 \\ &= \pi(18+72+96)-0 \\ &= 186\pi \text{ units}^3 \end{aligned}$$

b

$$\begin{aligned} V &= \pi\int_1^2 (x^2+3)^2\, dx \\ &= \pi\int_1^2 \left(x^4+6x^2+9\right) dx \\ &= \pi\left[\frac{x^5}{5}+\frac{6x^3}{3}+9x\right]_1^2 \\ &= \pi\left(\left\{\tfrac{32}{5}+16+18\right\}-\left\{\tfrac{1}{5}+2+9\right\}\right) \\ &= \pi\left(\tfrac{146}{5}\right) \\ &= \tfrac{146\pi}{5} \text{ units}^3 \end{aligned}$$

c

$$\begin{aligned} V &= \pi\int_0^4 (e^x)^2\, dx \\ &= \pi\int_0^4 e^{2x}\, dx \\ &= \pi\left[\tfrac{1}{2}e^{2x}\right]_0^4 \\ &= \pi\left(\tfrac{1}{2}e^8-\tfrac{1}{2}\right) \\ &= \tfrac{\pi}{2}(e^8-1) \ \text{ units}^3 \end{aligned}$$

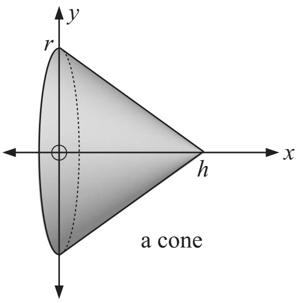
3 **a**

$$\begin{aligned} \text{Volume} &= \pi\int_5^8 y^2\, dx \\ &= \pi\int_5^8 (64-x^2)dx \\ &= \pi\left[64x-\frac{x^3}{3}\right]_5^8 \\ &= \pi\left(\left\{512-\tfrac{512}{3}\right\}-\left\{320-\tfrac{125}{3}\right\}\right) \\ &= 63\pi \text{ units}^3 \end{aligned}$$

b

$$\begin{aligned} &63\pi \text{ cm}^3 \\ &\doteqdot 198 \text{ cm}^3 \end{aligned}$$

4 **a** a cone of base r and height h



b

$$\begin{aligned} \text{slope} &= \frac{r-0}{0-h} \\ &= -\frac{r}{h} \end{aligned}$$

$\therefore$ equation is $y = -\dfrac{r}{h}\,x + r$

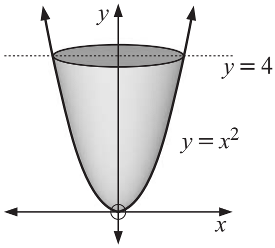
c

$$\begin{aligned} V &= \pi\int_0^h \left(\frac{-r}{h}x+r\right)^2 dx \\ &= \pi r^2\int_0^h \left(-\frac{x}{h}+1\right)^2 dx \\ &= \pi r^2\int_0^h \left(\frac{x^2}{h^2}-\frac{2x}{h}+1\right) dx \\ &= \pi r^2\left[\frac{x^3}{3h^2}-\frac{2x^2}{2h}+x\right]_0^h \\ &= \pi r^2\left(\left\{\frac{h}{3}-h+h\right\}-0\right) \\ &= \tfrac{1}{3}\pi r^2 h \text{ units}^3 \end{aligned}$$

5 **a** a sphere of radius r

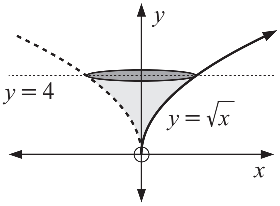
b
$$\begin{aligned} V &= \pi \int_{-r}^{r} y^2\, dx \\ &= 2\pi \int_0^r (r^2 - x^2)\, dx \\ &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= 2\pi(r^3 - \frac{r^3}{3} - 0) \\ &= 2\pi \times \tfrac{2}{3} r^3 \\ &= \tfrac{4}{3}\pi r^3 \text{ units}^3 \end{aligned}$$

6 **a**



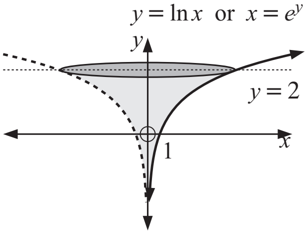
Volume
$$\begin{aligned} &= \pi \textstyle\int_0^4 x^2\, dy \\ &= \pi \textstyle\int_0^4 y\, dy \\ &= \pi \left[\frac{y^2}{2} \right]_0^4 \\ &= \pi(8 - 0) \\ &= 8\pi \text{ units}^3 \end{aligned}$$

b



Volume
$$\begin{aligned} &= \pi \textstyle\int_0^4 x^2\, dy \\ &= \pi \textstyle\int_0^4 y^4\, dy \\ &= \pi \left[\frac{y^5}{5} \right]_0^4 \\ &= \pi \left(\tfrac{4^5}{5} - 0 \right) \\ &= \frac{1024\pi}{5} \text{ units}^3 \end{aligned}$$

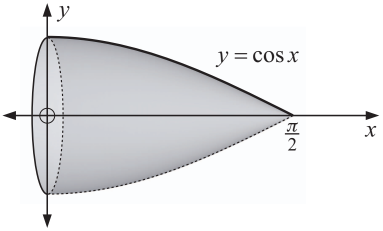
c



Volume
$$\begin{aligned} &= \pi \textstyle\int_0^2 x^2\, dy \\ &= \pi \textstyle\int_0^2 (e^y)^2\, dy \\ &= \pi \textstyle\int_0^2 e^{2y}\, dy \\ &= \pi \left[\tfrac{1}{2} e^{2y} \right]_0^2 \\ &= \pi(\frac{e^4}{2} - \tfrac{1}{2}) \\ &= \tfrac{\pi}{2}(e^4 - 1) \text{ units}^3 \end{aligned}$$

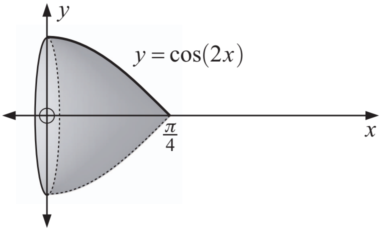
EXERCISE 28A.2

1 **a**



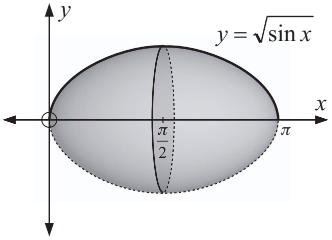
$$\begin{aligned} \text{Volume } &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2\, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \\ &= \pi \int_0^{\frac{\pi}{2}} \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)\, dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x) \right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\tfrac{\pi}{4} + \tfrac{1}{4}\sin\pi - 0 \right] \\ &= \tfrac{\pi^2}{4} \text{ units}^3 \end{aligned}$$

b



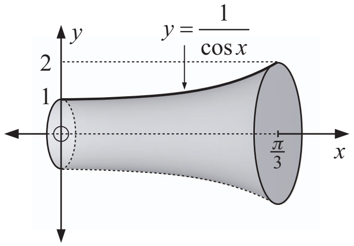
$$\begin{aligned} \text{Volume } &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x)\, dx \\ &= \pi \int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(4x) \right) dx \\ &= \pi \left[\tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x) \right]_0^{\frac{\pi}{4}} \\ &= \pi \left[\tfrac{\pi}{8} + \tfrac{1}{8}\sin\pi - 0 \right] \\ &= \tfrac{\pi^2}{8} \text{ units}^3 \end{aligned}$$

c



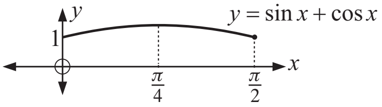
$$\begin{aligned}\text{Volume} &= \pi\int_0^{\pi} \sin x\,dx\\ &= \pi\left[-\cos x\right]_0^{\pi}\\ &= \pi\left[-\cos\pi - -\cos 0\right]\\ &= \pi(2)\\ &= 2\pi \text{ units}^3\end{aligned}$$

d



$$\begin{aligned}\text{Volume} &= \pi\int_0^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx\\ &= \pi\left[\tan x\right]_0^{\frac{\pi}{3}}\\ &= \pi\left(\tan\tfrac{\pi}{3} - \tan 0\right)\\ &= \pi\times(\sqrt{3}-0)\\ &= \pi\sqrt{3} \text{ units}^3\end{aligned}$$

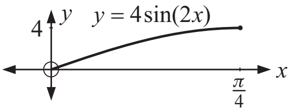
2 **a**



b Volume

$$\begin{aligned}&= \pi\int_0^{\frac{\pi}{4}} (\sin x+\cos x)^2\,dx\\ &= \pi\int_0^{\frac{\pi}{4}} \left(\sin^2 x + 2\sin x\cos x + \cos^2 x\right)dx\\ &= \pi\int_0^{\frac{\pi}{4}} (1+\sin(2x))\,dx\\ &= \pi\left[x-\tfrac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}\\ &= \pi\left(\left\{\tfrac{\pi}{4}-\tfrac{1}{2}\cos\left(\tfrac{\pi}{2}\right)\right\}-\left\{0-\tfrac{1}{2}(1)\right\}\right)\\ &= \pi\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right)\text{ units}^3\end{aligned}$$

3 **a**



b Volume

$$\begin{aligned}&= \pi\int_0^{\frac{\pi}{4}} (4\sin(2x))^2\,dx\\ &= 16\pi\int_0^{\frac{\pi}{4}} \sin^2(2x)\,dx\\ &= 16\pi\int_0^{\frac{\pi}{4}} \left(\tfrac{1}{2}-\tfrac{1}{2}\cos(4x)\right)dx\\ &= 16\pi\left[\frac{x}{2}-\tfrac{1}{2}\left(\tfrac{1}{4}\right)\sin(4x)\right]_0^{\frac{\pi}{4}}\\ &= 16\pi\left\{\tfrac{\pi}{8}-\tfrac{1}{8}\sin\pi-\left(0-\tfrac{1}{8}\sin 0\right)\right\}\\ &= 2\pi^2\text{ units}^3\end{aligned}$$

EXERCISE 28B

1 **a** The graphs meet where

$$\begin{aligned}4-x^2 &= 3\\ \therefore\quad x^2 &= 1\\ \therefore\quad x &= \pm 1\end{aligned}$$

$\therefore$ A is $(-1, 3)$
and B is $(1, 3)$

b

$$\begin{aligned}V &= 2\pi\int_0^1 \left((4-x^2)^2-3^2\right)dx\\ &= 2\pi\int_0^1 \left(16-8x^2+x^4-9\right)dx\\ &= 2\pi\int_0^1 \left(x^4-8x^2+7\right)dx\\ &= 2\pi\left[\frac{x^5}{5}-\frac{8x^3}{3}+7x\right]_0^1\\ &= 2\pi\left\{\tfrac{1}{5}-\tfrac{8}{3}+7-0\right\}\\ &= \tfrac{136\pi}{15}\text{ units}^3\end{aligned}$$

2 **a** The graphs meet where

$$\begin{aligned} e^{\frac{x}{2}} &= e \\ \therefore\quad e^{\frac{x}{2}} &= e^1 \\ \therefore\quad \frac{x}{2} &= 1 \\ \therefore\quad x &= 2 \end{aligned}$$

i.e., at A(2, e)

b

$$\begin{aligned} V &= \pi \int_0^2 \left(e^2 - \left(e^{\frac{x}{2}}\right)^2\right) dx \\ &= \pi \int_0^2 \left(e^2 - e^x\right) dx \\ &= \pi \left[e^2 x - e^x\right]_0^2 \\ &= \pi \left[2e^2 - e^2 - (0-1)\right] \\ &= \pi \left[e^2 + 1\right] \text{ units}^3 \end{aligned}$$

3 **a** The graphs meet where

$$\begin{aligned} x &= \frac{1}{x} \\ \therefore\quad x^2 &= 1 \\ \therefore\quad x &= \pm 1 \\ \therefore\quad x &= 1 \quad \{\text{as } x > 0\} \end{aligned}$$

$\therefore$ A is (1, 1)

b

$$\begin{aligned} V &= \pi \int_1^2 \left(x^2 - \left(\frac{1}{x}\right)^2\right) dx \\ &= \pi \int_1^2 \left(x^2 - x^{-2}\right) dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1}\right]_1^2 \\ &= \pi \left\{\left(\tfrac{8}{3} + \tfrac{1}{2}\right) - \left(\tfrac{1}{3} + 1\right)\right\} \\ &= \tfrac{11\pi}{6} \text{ units}^3 \end{aligned}$$

4 **a** The curves meet where

$$\begin{aligned} \sqrt{x-4} &= 1 \\ \therefore\quad x - 4 &= 1 \\ \therefore\quad x &= 5 \end{aligned}$$

$\therefore$ A is (5, 1)

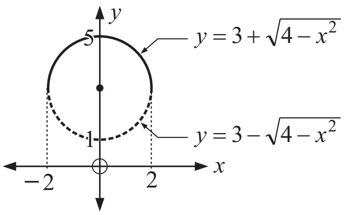
b

$$\begin{aligned} V &= \pi \int_5^8 \left(\left(\sqrt{x-4}\right)^2 - 1^2\right) dx \\ &= \pi \int_5^8 (x - 4 - 1)\, dx \\ &= \pi \int_5^8 (x - 5)\, dx \\ &= \pi \left[\frac{x^2}{2} - 5x\right]_5^8 \\ &= \pi \left\{(32 - 40) - \left(\tfrac{25}{2} - 25\right)\right\} \\ &= \pi\left(\tfrac{9}{2}\right) = \tfrac{9\pi}{2} \text{ units}^3 \end{aligned}$$

5 **a**

$$\begin{aligned} x^2 + (y-3)^2 &= 4 \\ \therefore\quad (y-3)^2 &= 4 - x^2 \\ \therefore\quad y - 3 &= \pm\sqrt{4-x^2} \\ \therefore\quad y &= 3 \pm \sqrt{4-x^2} \end{aligned}$$

b

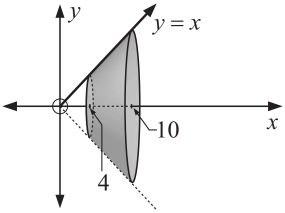


c

$$\begin{aligned} V &= \pi \int_{-2}^2 \left\{\left(3 + \sqrt{4-x^2}\right)^2 - \left(3 - \sqrt{4-x^2}\right)^2\right\} dx \\ &= 2\pi \int_0^2 \left\{\left(3 + \sqrt{4-x^2}\right)^2 - \left(3 - \sqrt{4-x^2}\right)^2\right\} dx \\ &= 2\pi \int_0^2 \left\{\left(9 + 6\sqrt{4-x^2} + 4 - x^2\right) - \left(9 - 6\sqrt{4-x^2} + 4 - x^2\right)\right\} dx \\ &= 2\pi \int_0^2 12\sqrt{4-x^2}\, dx \\ &= 24\pi \int_0^2 \sqrt{4-x^2}\, dx = 24\pi^2 \text{ units}^3 \quad (\doteqdot 236.9 \text{ units}^3) \qquad \{\text{using technology}\} \end{aligned}$$

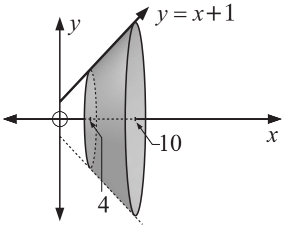
REVIEW SET 28

1 **a**



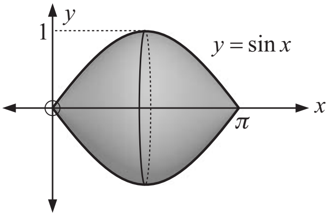
$$\begin{aligned} V &= \pi\int_4^{10} x^2\,dx \\ &= \pi\left[\frac{x^3}{3}\right]_4^{10} \\ &= \pi\left(\tfrac{1000}{3}-\tfrac{64}{3}\right) \\ &= \tfrac{936\pi}{3} \\ &= 312\pi \text{ units}^3 \end{aligned}$$

b



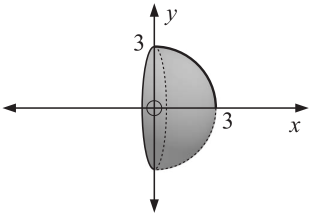
$$\begin{aligned} V &= \pi\int_4^{10} (x+1)^2\,dx \\ &= \pi\left[\left(\tfrac{1}{1}\right)\frac{(x+1)^3}{3}\right]_4^{10} \\ &= \pi\left(\tfrac{11^3}{3}-\tfrac{5^3}{3}\right) \\ &= \tfrac{1206\pi}{3} = 402\pi \text{ units}^3 \end{aligned}$$

c



$$\begin{aligned} V &= \pi\int_0^{\pi} \sin^2 x\,dx \\ &= \pi\int_0^{\pi} \left(\tfrac{1}{2}-\tfrac{1}{2}\cos(2x)\right)\,dx \\ &= \pi\left[\tfrac{1}{2}x-\tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^{\pi} \\ &= \pi\left[\tfrac{1}{2}\pi-\tfrac{1}{4}\sin 2\pi-0\right] \\ &= \tfrac{\pi^2}{2} \text{ units}^3 \end{aligned}$$

d



$$\begin{aligned} V &= \pi\int_0^{3} (9-x^2)\,dx \\ &= \pi\left[9x-\frac{x^3}{3}\right]_0^{3} \\ &= \pi\left\{27-\tfrac{27}{3}-0\right\} \\ &= 18\pi \text{ units}^3 \end{aligned}$$

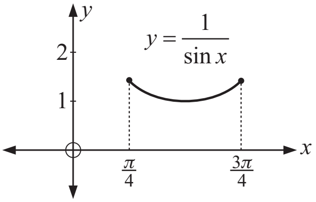
2 **a** $y=\cos(2x)$ meets the x-axis where

$$\begin{aligned} 2x &= \tfrac{\pi}{2} \\ \therefore\quad x &= \tfrac{\pi}{4} \end{aligned}$$

so,
$$\begin{aligned} V &= \pi\int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x)\,dx \\ &= \pi\int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\tfrac{1}{2}+\tfrac{1}{2}\cos(4x)\right)\,dx \\ &= \pi\left[\tfrac{1}{2}x+\tfrac{1}{8}\sin(4x)\right]_{\frac{\pi}{16}}^{\frac{\pi}{4}} \\ &= \pi\left\{\left(\tfrac{\pi}{8}+\tfrac{1}{8}\sin\pi\right)-\left(\tfrac{\pi}{32}+\tfrac{1}{8}\sin\left(\tfrac{\pi}{4}\right)\right)\right\} \\ &= \pi\left\{\tfrac{\pi}{8}-\tfrac{\pi}{32}-\tfrac{1}{8}\left(\tfrac{1}{\sqrt{2}}\right)\right\} \\ &= \pi\left\{\tfrac{3\pi}{32}-\tfrac{1}{8\sqrt{2}}\right\} \text{ units}^3 \end{aligned}$$

b
$$\begin{aligned} V &= \pi\int_0^2 (e^{-x}+4)^2\,dx \\ &= \pi\int_0^2 \left(e^{-2x}+8e^{-x}+16\right)\,dx \\ &= \pi\left[\tfrac{1}{-2}e^{-2x}+\tfrac{8}{-1}e^{-x}+16x\right]_0^2 \\ &= \pi\left\{\left(-\tfrac{1}{2}e^{-4}-8e^{-2}+32\right)-\left(-\tfrac{1}{2}-8\right)\right\} \\ &= \pi\left\{\frac{81}{2}-\frac{1}{2e^4}-\frac{8}{e^2}\right\}\text{ units}^3 \qquad (\doteqdot 123.8\text{ units}^3) \end{aligned}$$

3



$$\begin{aligned} V &= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{1}{\sin x}\right)^2 dx \\ &= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x\,dx \\ &= \pi\left[-\cot x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= \pi\left\{-\cot\left(\tfrac{3\pi}{4}\right)--\cot\left(\tfrac{\pi}{4}\right)\right\} \\ &= \pi\left\{-(-1)+1\right\} \\ &= 2\pi\text{ units}^3 \end{aligned}$$

4 They meet where $x^2 = 4$

$\therefore\ x = \pm 2$

but $x > 0$ $\therefore\ x = 2$

i.e., at (2, 4)

$$\begin{aligned} V &= \pi\int_0^2 \left(4^2-(x^2)^2\right)\,dx \\ &= \pi\int_0^2 \left(16-x^4\right)\,dx \\ &= \pi\left[16x-\frac{x^5}{5}\right]_0^2 \\ &= \pi\left(32-\tfrac{32}{5}-0\right) \\ &= \frac{128\pi}{5}\text{ units}^3 \end{aligned}$$

5 They meet where $\sin x = \cos x$

$\therefore\ \dfrac{\sin x}{\cos x} = 1$

$\therefore\ \tan x = 1$

$\therefore\ x = \frac{\pi}{4}$

$$\begin{aligned} V &= \pi\int_0^{\frac{\pi}{4}} (\cos^2 x-\sin^2 x)\,dx \\ &= \pi\int_0^{\frac{\pi}{4}} \cos(2x)\,dx \\ &= \pi\left[\tfrac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} \\ &= \pi\left(\tfrac{1}{2}\sin\left(\tfrac{\pi}{2}\right)-\tfrac{1}{2}\sin 0\right) \\ &= \pi\left(\tfrac{1}{2}(1)-0\right) \\ &= \tfrac{\pi}{2}\text{ units}^3 \end{aligned}$$

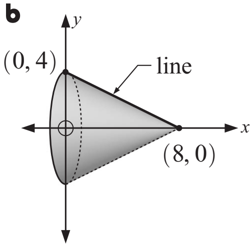
6 **a** $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 4^2 \times 8$

$= \frac{1}{3}\pi \times 128$

$= \frac{128\pi}{3}$ units^3

b



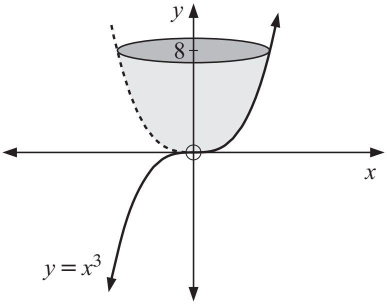
slope $= \frac{0-4}{8-0} = -\frac{1}{2}$

$\therefore$ line has equation

$y = -\frac{1}{2}x + 4$

$$\therefore\quad V = \pi \int_0^8 \left(-\tfrac{1}{2}x + 4\right)^2 dx$$
$$= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx$$
$$= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x\right]_0^8$$
$$= \pi \left\{\tfrac{128}{3} - 128 + 128 - 0\right\}$$
$$= \tfrac{128\pi}{3} \text{ units}^3$$

7



$$\text{Volume} = \pi \int_0^8 x^2 dy$$
$$= \pi \int_0^8 y^{\frac{2}{3}}\, dy$$
$$= \pi \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^8$$
$$= \tfrac{3\pi}{5}\left(8^{\frac{5}{3}} - 0^{\frac{5}{3}}\right)$$
$$= \tfrac{3\pi}{5} \times (2^3)^{\frac{5}{3}}$$
$$= \tfrac{3\pi}{5} \times 2^5$$
$$= \tfrac{96\pi}{5} \text{ units}^3$$

$x^3 = y$

$\therefore\quad (x^3)^{\frac{2}{3}} = y^{\frac{2}{3}}$

i.e., $x^2 = y^{\frac{2}{3}}$

Chapter 29

FURTHER INTEGRATION AND DIFFERENTIAL EQUATIONS

EXERCISE 29A

1 **a**
$$\int \frac{4}{\sqrt{1-x^2}}\,dx$$
$$= 4\int \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= 4\arcsin\left(\tfrac{x}{1}\right)+c$$
$$= 4\arcsin(x)+c$$

b
$$\int \frac{3}{\sqrt{4-x^2}}\,dx$$
$$= 3\int \frac{1}{\sqrt{4-x^2}}\,dx$$
$$= 3\arcsin\left(\tfrac{x}{2}\right)+c$$

c
$$\int \frac{1}{x^2+16}\,dx$$
$$= \tfrac{1}{4}\arctan\left(\tfrac{x}{4}\right)+c$$

d
$$\int \frac{1}{4x^2+1}\,dx$$
$$= \tfrac{1}{4}\int \frac{1}{x^2+(\frac{1}{2})^2}\,dx$$
$$= \tfrac{1}{4}\left(\frac{1}{\frac{1}{2}}\right)\arctan\left(\frac{x}{\frac{1}{2}}\right)+c$$
$$= \tfrac{1}{2}\arctan(2x)+c$$

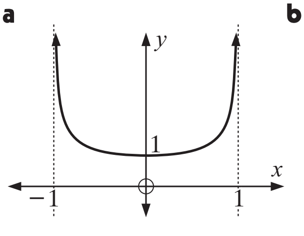
e
$$\int \frac{1}{\sqrt{1-4x^2}}\,dx$$
$$= \tfrac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx$$
$$= \tfrac{1}{2}\arcsin\left(\frac{x}{\frac{1}{2}}\right)+c$$
$$= \tfrac{1}{2}\arcsin(2x)+c$$

f
$$\int \frac{2}{\sqrt{4-9x^2}}\,dx$$
$$= 2\int \frac{1}{\sqrt{4-9x^2}}\,dx$$
$$= \tfrac{2}{3}\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,dx$$
$$= \tfrac{2}{3}\arcsin\left(\tfrac{3x}{2}\right)+c$$

g
$$\int \frac{1}{4+2x^2}\,dx$$
$$= \tfrac{1}{2}\int \frac{1}{2+x^2}\,dx$$
$$= \tfrac{1}{2}\left(\frac{1}{\sqrt{2}}\right)\arctan\left(\frac{x}{\sqrt{2}}\right)+c$$
$$= \frac{1}{2\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+c$$

h
$$\int \frac{5}{9+4x^2}\,dx$$
$$= \tfrac{5}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$$
$$= \tfrac{5}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{x}{\frac{3}{2}}\right)+c$$
$$= \tfrac{5}{6}\arctan\left(\tfrac{2x}{3}\right)+c$$

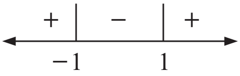
2 **a**



b **i** If $f(x) = \dfrac{1}{\sqrt{1-x^2}}$ then
$$f(-x) = \frac{1}{\sqrt{1-(-x)^2}}$$
$$= \frac{1}{\sqrt{1-x^2}}$$
$$= f(x) \quad \text{for all } x$$

$\therefore$ $f(x)$ is an even function
and so is symmetric about the y-axis

ii $f(x)$ is defined when $1-x^2 > 0$
$$\therefore\ x^2-1 < 0$$
$$\therefore\ (x+1)(x-1) < 0$$



$\therefore$ $x \in \,]-1, 1[$

c Area $= \displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$
$$= [\arcsin(x)]_0^{\frac{1}{2}}$$
$$= \arcsin(\tfrac{1}{2}) - \arcsin(0)$$
$$= \tfrac{\pi}{6} - 0$$
$$= \tfrac{\pi}{6} \text{ units}^2$$

EXERCISE 29B

1 **a** Let $u = x - 3$

so $\dfrac{du}{dx} = 1$ and $x = u + 3$

$$\therefore \int x\sqrt{x-3}\,dx$$
$$= \int (u+3)\sqrt{u}\,\frac{du}{dx}\,dx$$
$$= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du$$
$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \tfrac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + c$$
$$= \tfrac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c$$

c Let $u = 3 - x^2$ so $\dfrac{du}{dx} = -2x$

$$\therefore \int x^3\sqrt{3-x^2}\,dx$$
$$= \int x^2\sqrt{3-x^2}\,x\,dx$$
$$= \int x^2\sqrt{3-x^2}\left(-\tfrac{1}{2}\frac{du}{dx}\right)dx$$
$$= -\tfrac{1}{2}\int (3-u)\sqrt{u}\,du$$
$$= -\tfrac{1}{2}\int \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$
$$= -\tfrac{1}{2}\left[\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right] + c$$
$$= -\tfrac{1}{2}\left[2u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right] + c$$
$$= -u^{\frac{3}{2}} + \tfrac{1}{5}u^{\frac{5}{2}} + c$$
$$= -\left(3-x^2\right)^{\frac{3}{2}} + \tfrac{1}{5}\left(3-x^2\right)^{\frac{5}{2}} + c$$

b Let $u = x + 1$

so $\dfrac{du}{dx} = 1$ and $x = u - 1$

$$\therefore \int x^2\sqrt{x+1}\,dx$$
$$= \int (u-1)^2\sqrt{u}\,\frac{du}{dx}\,dx$$
$$= \int (u^2 - 2u + 1)u^{\frac{1}{2}}\,du$$
$$= \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$
$$= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \tfrac{2}{7}u^{\frac{7}{2}} - \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{2}{7}(x+1)^{\frac{7}{2}} - \tfrac{4}{5}(x+1)^{\frac{5}{2}} + \tfrac{2}{3}(x+1)^{\frac{3}{2}} + c$$

d Let $u = t^2 + 2$ so $\dfrac{du}{dt} = 2t$

$$\therefore \int t^3\sqrt{t^2+2}\,dt$$
$$= \int t^2\sqrt{t+2}\,t\,dt$$
$$= \int t^2\sqrt{t^2+2}\left(\tfrac{1}{2}\frac{du}{dt}\right)dt$$
$$= \tfrac{1}{2}\int (u-2)\sqrt{u}\,du$$
$$= \tfrac{1}{2}\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$$
$$= \tfrac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$
$$= \tfrac{1}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{1}{5}\left(t^2+2\right)^{\frac{5}{2}} - \tfrac{2}{3}\left(t^2+2\right)^{\frac{3}{2}} + c$$

e Let $u = \sqrt{x-1}$

so $\dfrac{du}{dx} = \tfrac{1}{2}(x-1)^{-\frac{1}{2}}(1)$

$= \dfrac{1}{2\sqrt{x-1}}$

and $x = u^2 + 1$

$$\therefore \int \frac{\sqrt{x-1}}{x}\,dx = \int \frac{x-1}{x}\,\frac{1}{\sqrt{x-1}}\,dx$$
$$= \int \frac{u^2}{u^2+1}\left(2\,\frac{du}{dx}\right)dx$$
$$= \int \frac{2u^2}{u^2+1}\,du$$
$$= \int \frac{2(u^2+1)-2}{u^2+1}\,du$$
$$= \int 2 - \frac{2}{u^2+1}\,du$$
$$= 2u - 2\arctan u + c$$
$$= 2\sqrt{x-1} - 2\arctan\sqrt{x-1} + c$$

2 **a** Let $u = x - 1$ so $\dfrac{du}{dx} = 1$

When $x = 4$, $u = 3$
and when $x = 3$, $u = 2$

$$\therefore \int_3^4 x\sqrt{x-1}\,dx$$
$$= \int_2^3 (u+1)\sqrt{u}\,du$$
$$= \int_2^3 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$
$$= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^3$$
$$= \left(\tfrac{2}{5}(3)^{\frac{5}{2}} + \tfrac{2}{3}(3)^{\frac{3}{2}}\right) - \left(\tfrac{2}{5}(2)^{\frac{5}{2}} + \tfrac{2}{3}(2)^{\frac{3}{2}}\right)$$
$$= \tfrac{2}{5}\left(9\sqrt{3}\right) + \tfrac{2}{3}\left(3\sqrt{3}\right) - \tfrac{2}{5}\left(4\sqrt{2}\right) - \tfrac{2}{3}\left(2\sqrt{2}\right)$$
$$= \left(\tfrac{18}{5} + 2\right)\sqrt{3} - \left(\tfrac{8}{5} + \tfrac{4}{3}\right)\sqrt{2}$$
$$= \tfrac{28\sqrt{3}}{5} - \tfrac{44\sqrt{2}}{15}$$

b Let $u = x + 6$ so $\dfrac{du}{dx} = 1$

When $x = 3$, $u = 9$
and when $x = 0$, $u = 6$

$$\therefore \int_0^3 x\sqrt{x+6}\,dx$$
$$= \int_6^9 (u-6)\sqrt{u}\,du$$
$$= \int_6^9 \left(u^{\frac{3}{2}} - 6u^{\frac{1}{2}}\right) du$$
$$= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{6u^{\frac{3}{2}}}{\frac{3}{2}}\right]_6^9$$
$$= \left(\tfrac{2}{5}(9)^{\frac{5}{2}} - 4(9)^{\frac{3}{2}}\right) - \left(\tfrac{2}{5}(6)^{\frac{5}{2}} - 4(6)^{\frac{3}{2}}\right)$$
$$= \tfrac{2}{5}\left(3^5\right) - 4\left(3^3\right) - \tfrac{2}{5} \times 36\sqrt{6} + 4 \times 6\sqrt{6}$$
$$= \tfrac{486}{5} - 108 - \tfrac{72}{5}\sqrt{6} + 24\sqrt{6}$$
$$= -\tfrac{54}{5} + \tfrac{48}{5}\sqrt{6}$$
$$= \tfrac{1}{5}(48\sqrt{6} - 54)$$

c Let $u = x - 2$ so $\dfrac{du}{dx} = 1$

When $x = 5$, $u = 3$
and when $x = 2$, $u = 0$

$$\therefore \int_2^5 x^2\sqrt{x-2}\,dx$$
$$= \int_0^3 (u+2)^2\sqrt{u}\,du$$
$$= \int_0^3 \left(u^2 + 4u + 4\right)\sqrt{u}\,du$$
$$= \int_0^3 \left(u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right) du$$
$$= \left[\frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{4u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{4u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$
$$= \tfrac{2}{7}(3)^{\frac{7}{2}} + \tfrac{8}{5}(3)^{\frac{5}{2}} + \tfrac{8}{3}(3)^{\frac{3}{2}} - 0$$
$$= \tfrac{2}{7}\left(27\sqrt{3}\right) + \tfrac{8}{5}\left(9\sqrt{3}\right) + \tfrac{8}{3}\left(3\sqrt{3}\right)$$
$$= \tfrac{1054}{35}\sqrt{3}$$

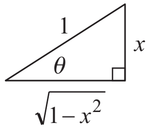
3 **a** Let $x = 3\tan\theta$ so $\dfrac{dx}{d\theta} = 3\sec^2\theta$

$$\therefore \int \frac{x^2}{9+x^2}\,dx$$
$$= \int \frac{9\tan^2\theta}{9+9\tan^2\theta}\,3\sec^2\theta\,d\theta$$
$$= 3\int \frac{\tan^2\theta}{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta \quad \{\text{as } \sec^2\theta = 1 + \tan^2\theta\}$$
$$= 3\int (\sec^2\theta - 1)\,d\theta$$
$$= 3\tan\theta - 3\theta + c$$
$$= x - 3\arctan\left(\tfrac{x}{3}\right) + c$$

b Let $x = \sin\theta$ so $\dfrac{dx}{d\theta} = \cos\theta$

$$\therefore \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\,\cos\theta\,d\theta$$
$$= \int \frac{\sin^2\theta}{\cos\theta}\,\cos\theta\,d\theta$$
$$= \int \sin^2\theta\,d\theta$$
$$= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta$$
$$= \tfrac{1}{2}\theta - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin 2\theta + c$$
$$= \tfrac{1}{2}\arcsin x - \tfrac{1}{2}\sin\theta\cos\theta + c$$
$$= \tfrac{1}{2}\arcsin x - \tfrac{1}{2}\,x\sqrt{1-x^2} + c$$

$\{$since $\cos\theta = \sqrt{1-\sin^2\theta}\}$



c $\displaystyle\int \frac{2x}{x^2+9}\,dx$

$$= \ln\left|x^2+9\right| + c \quad \left\{\text{form } \int \frac{f'(x)}{f(x)}\,dx\right\}$$

$$= \ln(x^2+9) + c \quad \{\text{as } x^2+9>0 \text{ for all } x\}$$

e Let $x = 2\sec\theta$ so $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$$\therefore \int \frac{\sqrt{x^2-4}}{x}\,dx$$

$$= \int \frac{\sqrt{4\sec^2\theta-4}}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= \tfrac{2}{2}\int \frac{\sqrt{\sec^2\theta-1}}{\cancel{\sec\theta}} \times 2\cancel{\sec\theta}\tan\theta\,d\theta$$

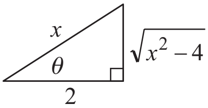
$$= 2\int \sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta$$

$$= 2\int \tan\theta\tan\theta\,d\theta \quad \{\sec^2\theta-1=\tan^2\theta\}$$

$$= 2\int \tan^2\theta\,d\theta$$

$$= 2\int (\sec^2\theta-1)\,d\theta$$

$$= 2\tan\theta - 2\theta + c$$



$$= 2\,\frac{\sqrt{x^2-4}}{2} - 2\arccos\left(\tfrac{2}{x}\right) + c$$

$$= \sqrt{x^2-4} - 2\arccos\left(\tfrac{2}{x}\right) + c$$

g $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$$= \tfrac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$$

Let $x = \tfrac{3}{2}\sin\theta$, so $\dfrac{dx}{d\theta} = \tfrac{3}{2}\cos\theta$

$\therefore$ the integral

$$= \tfrac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-\frac{9}{4}\sin^2\theta}}\,\tfrac{3}{2}\cos\theta\,d\theta$$

$$= \tfrac{1}{2}\int \frac{1}{\frac{3}{2}\sqrt{1-\sin^2\theta}} \times \tfrac{3}{2}\cos\theta\,d\theta$$

$$= \tfrac{1}{2}\int \frac{\cos\theta}{\cos\theta}\,d\theta$$

$$= \tfrac{1}{2}\int 1\,d\theta$$

$$= \tfrac{1}{2}\theta + c$$

$$= \tfrac{1}{2}\arcsin\left(\tfrac{2x}{3}\right) + c \quad \{\text{since } \sin\theta = \tfrac{2x}{3}\}$$

d Let $u = \ln x$ so $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore \int \frac{4\ln x}{x\,(1+[\ln x]^2)}\,dx$$

$$= \int \frac{4u}{1+u^2}\,\frac{du}{dx}\,dx$$

$$= 2\int \frac{2u}{1+u^2}\,du$$

$$= 2\ln\left|1+u^2\right| + c \quad \left\{\text{form } \int \frac{f'(x)}{f(x)}\,dx\right\}$$

$$= 2\ln(1+u^2) + c \quad \{1+u^2>0\}$$

$$= 2\ln\left(1+[\ln x]^2\right) + c$$

f $\int \sin x\cos 2x\,dx$

$$= \int \sin x(2\cos^2 x-1)\,dx$$

$$= 2\int \cos^2 x\sin x\,dx - \int \sin x\,dx$$

$$= -2\int [\cos x]^2(-\sin x)\,dx - \int \sin x\,dx$$

$$= -2\frac{[\cos x]^3}{3} - (-\cos x) + c$$

$$= -\tfrac{2}{3}\cos^3 x + \cos x + c$$

$$= \cos x - \tfrac{2}{3}\cos^3 x + c$$

h $\displaystyle\int \frac{x^3}{1+x^2}\,dx$

$$= \int \frac{x(1+x^2)-x}{1+x^2}\,dx$$

$$= \int \left(x - \frac{x}{1+x^2}\right)dx$$

$$= \int \left(x - \tfrac{1}{2}\left(\frac{2x}{1+x^2}\right)\right)dx$$

$$= \frac{x^2}{2} - \tfrac{1}{2}\ln\left|1+x^2\right| + c$$

$$= \frac{x^2}{2} - \tfrac{1}{2}\ln(1+x^2) + c$$

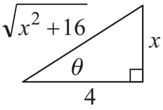
$$\{\text{as } 1+x^2>0\}$$

i Let $u = \ln x$ so $\dfrac{du}{dx} = \dfrac{1}{x}$

$$\therefore \int \frac{1}{x\,(9+4[\ln x]^2)}\,dx$$
$$= \int \frac{1}{9+4u^2}\,\frac{du}{dx}\,dx$$
$$= \int \frac{1}{9+4u^2}\,du$$
$$= \tfrac{1}{4}\int \frac{1}{u^2+\frac{9}{4}}\,du$$
$$= \tfrac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{u}{\frac{3}{2}}\right)+c$$
$$= \tfrac{1}{6}\arctan\left(\tfrac{2u}{3}\right)+c$$
$$= \tfrac{1}{6}\arctan\left(\tfrac{2\ln x}{3}\right)+c$$

j Let $x = 4\tan\theta$ so $\dfrac{dx}{d\theta} = 4\sec^2\theta$

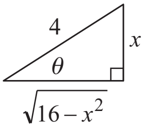
$$\therefore \int \frac{1}{x(x^2+16)}\,dx$$
$$= \int \frac{1}{4\tan\theta(16\tan^2\theta+16)} \times 4\sec^2\theta\ d\theta$$
$$= \int \frac{1}{4\tan\theta \times 16\sec^2\theta} \times 4\sec^2\theta\ d\theta$$
$$= \tfrac{1}{16}\int \frac{1}{\tan\theta}\,d\theta$$



$$= \tfrac{1}{16}\int \frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \tfrac{1}{16}\ln|\sin\theta|+c \qquad \left\{\text{form } \frac{f'(\theta)}{f(\theta)}\right\}$$
$$= \tfrac{1}{16}\ln\left|\frac{x}{\sqrt{x^2+16}}\right|+c$$
$$= \tfrac{1}{16}\ln\left(\frac{|x|}{\sqrt{x^2+16}}\right)+c$$

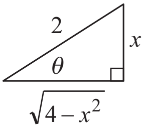
k Let $x = 4\sin\theta$ so $\dfrac{dx}{d\theta} = 4\cos\theta$

$$\therefore \int \frac{1}{x^2\sqrt{16-x^2}}\,dx$$
$$= \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}\,4\cos\theta\ d\theta$$
$$= \int \frac{1}{16\sin^2\theta \times 4\cos\theta}\,4\cos\theta\ d\theta$$
$$= \tfrac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta$$
$$= \tfrac{1}{16}\int \operatorname{cosec}^2\theta\ d\theta$$
$$= \tfrac{1}{16}(-\cot\theta)+c$$
$$= -\tfrac{1}{16}\cot\theta+c$$
$$= -\tfrac{1}{16}\,\frac{\sqrt{16-x^2}}{x}+c$$
$$= -\frac{\sqrt{16-x^2}}{16x}+c$$



l Let $x = 2\sin\theta$ so $\dfrac{dx}{d\theta} = 2\cos\theta$

$$\therefore \int x^2\sqrt{4-x^2}\ dx$$
$$= \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\ d\theta$$
$$= \int 4\sin^2\theta\ 2\cos\theta\ 2\cos\theta\ d\theta$$
$$= 4\int 4\sin^2\theta\cos^2\theta\ d\theta$$
$$= 4\int \sin^2(2\theta)\ d\theta$$
$$= 4\int \left(\tfrac{1}{2}-\tfrac{1}{2}\cos(4\theta)\right)\,d\theta$$
$$= 2\theta - 2(\tfrac{1}{4})\sin(4\theta)$$
$$= 2\theta - \tfrac{1}{2}\sin(4\theta)+c$$



Now $\sin\theta = \dfrac{x}{2}$, $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

$$\therefore \sin 2\theta = 2\left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2} = \frac{x\sqrt{4-x^2}}{2}$$

and $\cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{4-x^2}{4} - \dfrac{x^2}{4}$

$$= \frac{4-2x^2}{4}$$

$$\sin 4\theta = 2\left(\frac{x\sqrt{4-x^2}}{2}\right)\left(\frac{4-2x^2}{4}\right)$$
$$= \frac{x\sqrt{4-x^2}(2-x^2)}{2}$$

$$\therefore \int x^2\sqrt{4-x^2}\ dx$$
$$= 2\arcsin\left(\tfrac{x}{2}\right) - \tfrac{1}{4}x\sqrt{4-x^2}(2-x^2)+c$$

EXERCISE 29C

1 **a** $\int xe^x\,dx$ has $u = x \quad v' = e^x$
$u' = 1 \quad v = e^x$

$$\begin{aligned}\therefore\ \int xe^x\,dx &= xe^x - \int 1e^x\,dx\\ &= xe^x - e^x + c\end{aligned}$$

b $\int x\sin x\,dx$ has $u = x \quad v' = \sin x$
$u' = 1 \quad v = -\cos x$

$$\begin{aligned}\therefore\ &\int x\sin x\,dx\\ &= x(-\cos x) - \int 1(-\cos x)\,dx\\ &= -x\cos x + \int \cos x\,dx\\ &= -x\cos x + \sin x + c\end{aligned}$$

c $\int x^2\ln x\,dx$ has $u = \ln x \quad v' = x^2$
$u' = \dfrac{1}{x} \quad v = \dfrac{x^3}{3}$

$$\begin{aligned}\therefore\ &\int x^2\ln x\,dx\\ &= \ln x\left(\frac{x^3}{3}\right) - \int \frac{1}{x}\,\frac{x^3}{3}\,dx\\ &= \frac{x^3\ln x}{3} - \tfrac{1}{3}\int x^2\,dx\\ &= \frac{x^3\ln x}{3} - \tfrac{1}{3}\,\frac{x^3}{3} + c\\ &= \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3 + c\end{aligned}$$

d $\int x\sin 3x\,dx$ has $u = x \quad v' = \sin 3x$
$u' = 1 \quad v = -\frac{1}{3}\cos 3x$

$$\begin{aligned}\therefore\ &\int x\sin 3x\,dx\\ &= x(-\tfrac{1}{3}\cos 3x) - \int\left(-\tfrac{1}{3}\cos 3x\right)dx\\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}(\tfrac{1}{3})\sin 3x + c\\ &= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + c\end{aligned}$$

e $\int x\cos 2x\,dx$ has $u = x \quad v' = \cos 2x$
$u' = 1 \quad v = \frac{1}{2}\sin 2x$

$$\begin{aligned}\therefore\ &\int x\cos 2x\,dx\\ &= x(\tfrac{1}{2}\sin 2x) - \int \tfrac{1}{2}\sin 2x\,dx\\ &= \tfrac{1}{2}x\sin 2x - \tfrac{1}{2}(-\tfrac{1}{2})\cos 2x + c\\ &= \tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x + c\end{aligned}$$

f $\int x\sec^2 x\,dx$ has $u = x \quad v' = \sec^2 x$
$u' = 1 \quad v = \tan x$

$$\begin{aligned}\therefore\ &\int x\sec^2 x\,dx\\ &= x\tan x - \int \tan x\,dx\\ &= x\tan x - \int \frac{\sin x}{\cos x}\,dx\\ &= x\tan x + \int \frac{-\sin x}{\cos x}\,dx \quad \left\{\text{form } \int \frac{f'(x)}{f(x)}\,dx\right\}\\ &= x\tan x + \ln|\cos x| + c\end{aligned}$$

g $\int \ln x\,dx = \int 1\times\ln x\,dx$,
which has $u = \ln x \quad v' = 1$
$u' = \dfrac{1}{x} \quad v = x$

$$\begin{aligned}\therefore\ &\int \ln x\,dx\\ &= x\ln x - \int\left(\frac{1}{x}\right)x\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + c\end{aligned}$$

h $\int(\ln x)^2\,dx$
$= \int(\ln x)(\ln x)\,dx$
which has $u = \ln x \quad v' = \ln x$
$u' = \dfrac{1}{x} \quad v = x\ln x - x$
{using **g**}

$$\begin{aligned}\therefore\ &\int(\ln x)^2\,dx\\ &= \ln x(x\ln x - x) - \int \frac{1}{x}(x\ln x - x)\,dx\\ &= x(\ln x)^2 - x\ln x - \int(\ln x - 1)\,dx\\ &= x(\ln x)^2 - x\ln x - [x\ln x - x] + x + c\\ &= x(\ln x)^2 - 2x\ln x + 2x + c\end{aligned}$$

i $\int \arctan x\,dx = \int 1\arctan x\,dx$
which has $u = \arctan x \quad v' = 1$
$u' = \dfrac{1}{x^2+1} \quad v = x$

$$\begin{aligned}\therefore\ &\int \arctan x\,dx\\ &= x\arctan x - \int \frac{x}{x^2+1}\,dx\\ &= x\arctan x - \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx\\ &= x\arctan x - \tfrac{1}{2}\ln\left|x^2+1\right| + c\\ &= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c \quad \{\text{as } x^2+1 > 0\}\end{aligned}$$

2 **a** $\int x^2e^{-x}\,dx \qquad \left\{\begin{array}{ll} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x}\end{array}\right\}$

$= -x^2e^{-x} - \int 2x(-e^{-x})\,dx$

$= -x^2e^{-x} + 2\int xe^{-x}\,dx \qquad \left\{\begin{array}{ll} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x}\end{array}\right\}$

$= -x^2e^{-x} + 2\left[x(-e^{-x}) - \int -e^{-x}\,dx\right]$

$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$

b $\int e^x\cos x\,dx \qquad \left\{\begin{array}{ll} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x\end{array}\right\}$

$= e^x\sin x - \int e^x\sin x\,dx \qquad \left\{\begin{array}{ll} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x\end{array}\right\}$

$= e^x\sin x - \left[-e^x\cos x - \int e^x(-\cos x)\,dx\right]$

$= e^x\sin x + e^x\cos x - \int e^x\cos x\,dx + c_1$

$\therefore\ \ 2\int e^x\cos x\,dx = e^x(\sin x + \cos x) + c_1$

$\therefore\ \ \int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) + c$

c $\int e^{-x}\sin x\,dx \qquad \left\{\begin{array}{ll} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x\end{array}\right\}$

$= -e^{-x}\cos x - \int -e^{-x}(-\cos x)\,dx$

$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx \qquad \left\{\begin{array}{ll} u = e^{-x} & v' = \cos x \\ u' = -e^{-x} & v = \sin x\end{array}\right\}$

$= -e^{-x}\cos x - \left[e^{-x}\sin x - \int -e^{-x}\sin x\,dx\right]$

$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx + c_1$

$\therefore\ \ 2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x) + c_1$

$\therefore\ \ \int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + c$

d $\int x^2\sin x\,dx \qquad \left\{\begin{array}{ll} u = x^2 & v' = \sin x \\ u' = 2x & v = -\cos x\end{array}\right\}$

$= -x^2\cos x - \int -2x\cos x\,dx$

$= -x^2\cos x + \int 2x\cos x\,dx \qquad \left\{\begin{array}{ll} u = 2x & v' = \cos x \\ u' = 2 & v = \sin x\end{array}\right\}$

$= -x^2\cos x + \left[2x\sin x - \int 2\sin x\,dx\right]$

$= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx + c$

$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$

$= -x^2\cos x + 2x\sin x + 2\cos x + c$

3 **a** $\int u^2e^u\,du \qquad \left\{\begin{array}{ll} a = u^2 & b' = e^u \\ a' = 2u & b = e^u\end{array}\right\}$

$= u^2e^u - \int 2ue^u\,du$

$= u^2e^u - 2\int ue^u\,du \qquad \left\{\begin{array}{ll} a = u & b' = e^u \\ a' = 1 & b = e^u\end{array}\right\}$

$= u^2e^u - 2\left[ue^u - \int e^u\,du\right] + c$

$= u^2e^u - 2ue^u + 2e^u + c$

b Let $u = \ln x$ so $\dfrac{du}{dx} = \dfrac{1}{x} = \dfrac{1}{e^u}$

$\therefore\ \ \int(\ln x)^2\,dx$

$= \int u^2e^u\,du$

$= u^2e^u - 2ue^u + 2e^u + c \quad$ {using **a**}

$= (\ln x)^2e^{\ln x} - 2\ln xe^{\ln x} + 2e^{\ln x} + c$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

4 **a**

$$\begin{aligned}&\int u\sin u\,du\\&= -u\cos u - \int -\cos u\;du\\&= -u\cos u + \sin u + c\end{aligned}$$

$$\left\{\begin{array}{llll} a = u & & b' = \sin u \\ a' = 1 & & b = -\cos u\end{array}\right\}$$

b If $u^2 = 2x$, $2u\,\dfrac{du}{dx} = 2$

$$\therefore\ \frac{du}{dx} = \frac{1}{u}$$

$$\begin{aligned}\therefore\ &\int \sin\sqrt{2x}\;dx\\&= \int \sin u(u\;du)\\&= \int u\sin u\;du\\&= -u\cos u + \sin u + c\\&= -\sqrt{2x}\cos\sqrt{2x} + \sin\sqrt{2x} + c\end{aligned}$$

5

$$\begin{aligned}u^2 &= 3x\\ \therefore\ 2u\,\frac{du}{dx} &= 3\\ \therefore\ \frac{du}{dx} &= \frac{3}{2u}\end{aligned}$$

$$\begin{aligned}&\int \cos\sqrt{3x}\;dx\\&= \int \cos u\left(\tfrac{2u}{3}\right)\,du\\&= \tfrac{2}{3}\int u\cos u\;du\\&= \tfrac{2}{3}\left[u\sin u - \int \sin u\;du\right]\\&= \tfrac{2}{3}u\sin u - \tfrac{2}{3}(-\cos u) + c\\&= \tfrac{2}{3}\sqrt{3x}\sin\sqrt{3x} + \tfrac{2}{3}\cos\sqrt{3x} + c\end{aligned}$$

$$\left\{\begin{array}{llll} a = u & & b' = \cos u \\ a' = 1 & & b = \sin u\end{array}\right\}$$

EXERCISE 29D.1

1 **a**

$$\begin{aligned}\frac{dy}{dx} &= 5y\\ \therefore\ \frac{1}{y}\frac{dy}{dx} &= 5\\ \therefore\ \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int 5\;dx\\ \therefore\ \int \frac{1}{y}\;dy &= \int 5\;dx\\ \therefore\ \ln|y| &= 5x + c\\ \therefore\ y &= Ae^{5x}\end{aligned}$$

b

$$\begin{aligned}\frac{dM}{dt} &= -2M\\ \therefore\ \frac{1}{M}\frac{dM}{dt} &= -2\\ \therefore\ \int \frac{1}{M}\frac{dM}{dt}\,dt &= \int -2\;dt\\ \therefore\ \int \frac{1}{M}\;dM &= \int -2\;dt\\ \therefore\ \ln|M| &= -2t + c\\ \therefore\ M &= Ae^{-2t}\end{aligned}$$

c

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{y}\\ \therefore\ y\,\frac{dy}{dx} &= 2\\ \therefore\ \int y\,\frac{dy}{dx}\;dx &= \int 2\;dx\\ \therefore\ \int y\;dy &= \int 2\;dx\\ \therefore\ \tfrac{1}{2}y^2 &= 2x + a\\ \therefore\ x &= \frac{y^2}{4} + c\end{aligned}$$

d

$$\begin{aligned}\frac{dP}{dt} &= 3\sqrt{P}\\ \therefore\ \frac{1}{\sqrt{P}}\frac{dP}{dt} &= 3\\ \therefore\ \int \frac{1}{\sqrt{P}}\frac{dP}{dt}\,dt &= \int 3\;dt\\ \therefore\ \int \frac{1}{\sqrt{P}}\;dP &= \int 3\;dt\\ \therefore\ 2\sqrt{P} &= 3t + a\\ \text{i.e.,}\quad \sqrt{P} &= \tfrac{3}{2}t + c\end{aligned}$$

e

$$\begin{aligned}\frac{dQ}{dt} &= 2Q + 3\\ \therefore\ \frac{1}{2Q+3}\,\frac{dQ}{dt} &= 1\\ \therefore\ \int \frac{1}{2Q+3}\,\frac{dQ}{dt}\,dt &= \int 1\,dt\end{aligned}$$

$$\begin{aligned}\therefore\ \int \frac{1}{2Q+3}\,dQ &= \int 1\,dt\\ \therefore\ \tfrac{1}{2}\ln|2Q+3| &= t + a\\ \ln|2Q+3| &= 2t + 2a\\ \therefore\ 2Q + 3 &= ce^{2t}\\ \therefore\ Q &= Ae^{2t} - \tfrac{3}{2}\end{aligned}$$

f

$$\begin{aligned}\frac{dQ}{dt} &= \frac{1}{2Q+3}\\ \therefore\quad (2Q+3)\frac{dQ}{dt} &= 1\\ \therefore\quad \textstyle\int(2Q+3)\frac{dQ}{dt}\,dt &= \textstyle\int 1\,dt\end{aligned}$$

$$\begin{aligned}\therefore\quad \textstyle\int(2Q+3)\ dQ &= \textstyle\int 1\,dt\\ \therefore\quad Q^2+3Q &= t+a\\ \therefore\quad t &= Q^2+3Q+c\end{aligned}$$

2 **a**

$$\begin{aligned}\frac{dy}{dx} &= 4y\\ \therefore\quad \frac{1}{y}\frac{dy}{dx} &= 4\\ \therefore\quad \int\frac{1}{y}\frac{dy}{dx}\,dx &= \textstyle\int 4\ dx\\ \therefore\quad \int\frac{1}{y}\ dy &= \textstyle\int 4\ dx\\ \therefore\quad \ln|y| &= 4x+c\\ \therefore\quad y &= Ae^{4x}\\ \text{But}\quad y &= 10\quad\text{when}\quad x=0,\\ \text{so}\quad A &= 10\\ \therefore\quad y &= 10e^{4x}\end{aligned}$$

b

$$\begin{aligned}\frac{dM}{dt} &= -3M\\ \therefore\quad \frac{1}{M}\frac{dM}{dt} &= -3\\ \therefore\quad \int\frac{1}{M}\frac{dM}{dt}\,dt &= \textstyle\int -3\ dt\\ \therefore\quad \int\frac{1}{M}\ dM &= \textstyle\int -3\ dt\\ \therefore\quad \ln|M| &= -3t+c\\ \therefore\quad M &= Ae^{-3t}\\ \text{But}\quad M(0) &= 20,\\ \text{so}\quad A &= 20\\ \therefore\quad M &= 20e^{-3t}\end{aligned}$$

c

$$\begin{aligned}\frac{dy}{dt} &= \frac{\sqrt{y}}{3}\\ \therefore\quad \frac{1}{\sqrt{y}}\frac{dy}{dt} &= \tfrac{1}{3}\\ \therefore\quad \int\frac{1}{\sqrt{y}}\frac{dy}{dt}\,dt &= \textstyle\int \frac{1}{3}\ dt\\ \therefore\quad \int\frac{1}{\sqrt{y}}\ dy &= \textstyle\int \frac{1}{3}\ dt\\ \therefore\quad 2\sqrt{y} &= \tfrac{1}{3}t+c\\ \therefore\quad 6\sqrt{y} &= t+3c\\ \text{But}\quad y &= 9\quad\text{when}\quad t=24,\\ \text{so}\quad 18 &= 24+3c\\ \therefore\quad 3c &= -6\\ \therefore\quad 6\sqrt{y} &= t-6\\ \therefore\quad \sqrt{y} &= \tfrac{1}{6}t-1\end{aligned}$$

d

$$\begin{aligned}\frac{dP}{dn} &= 2P+3\\ \therefore\quad \frac{1}{2P+3}\frac{dP}{dn} &= 1\\ \therefore\quad \int\frac{1}{2P+3}\frac{dP}{dn}\,dn &= \textstyle\int 1\,dn\\ \therefore\quad \int\frac{1}{2P+3}\ dP &= \textstyle\int dn\\ \therefore\quad \tfrac{1}{2}\ln|2P+3| &= n+c\\ \therefore\quad \ln|2P+3| &= 2n+2c\\ \therefore\quad 2P+3 &= Ae^{2n}\\ \therefore\quad 2P &= Ae^{2n}-3\\ \therefore\quad P &= Be^{2n}-\tfrac{3}{2}\\ \text{But}\quad P &= 2\quad\text{when}\quad n=0,\\ \text{so}\quad 2 &= B-\tfrac{3}{2}\\ \therefore\quad B &= \tfrac{7}{2}\\ \therefore\quad P &= \tfrac{7}{2}e^{2n}-\tfrac{3}{2}\end{aligned}$$

e

$$\begin{aligned}\frac{dy}{dx} &= k\sqrt{y}\\ \therefore\quad \frac{1}{\sqrt{y}}\frac{dy}{dx} &= k\\ \therefore\quad \int\frac{1}{\sqrt{y}}\frac{dy}{dx}\,dx &= \textstyle\int k\ dx\\ \therefore\quad \int\frac{1}{\sqrt{y}}\,dy &= \textstyle\int k\ dx\\ \therefore\quad 2\sqrt{y} &= kx+c\end{aligned}$$

$$\begin{aligned}\text{Now}\quad y(4) &= 1,\\ \text{so}\quad 4k+c &= 2\\ \text{and}\quad y(5) &= 4,\\ \text{so}\quad 5k+c &= 4\\ \therefore\quad k &= 2\\ \text{and}\quad c &= -6\\ \therefore\quad 2\sqrt{y} &= 2x-6\\ \therefore\quad y &= (x-3)^2\end{aligned}$$

3

$$\begin{aligned} \frac{dy}{dx} &= 2y \\ \therefore \quad \frac{1}{y}\frac{dy}{dx} &= 2 \\ \therefore \quad \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int 2\ dx \end{aligned}$$

$$\begin{aligned} \therefore \quad \int \frac{1}{y}\ dy &= \int 2\ dx \\ \therefore \quad \ln|y| &= 2x+c \\ \therefore \quad y &= Ae^{2x} \end{aligned}$$

i.e., the curve is an exponential function

4 **a**

$$\begin{aligned} \frac{dp}{dt} &= -\tfrac{1}{2}p \\ \therefore \quad \frac{1}{p}\frac{dp}{dt} &= -\tfrac{1}{2} \\ \therefore \quad \int \frac{1}{p}\frac{dp}{dt}\,dt &= \int -\tfrac{1}{2}\ dt \\ \therefore \quad \int \frac{1}{p}\ dp &= \int -\tfrac{1}{2}\ dt \\ \therefore \quad \ln|p| &= -\tfrac{1}{2}t+c \\ \therefore \quad p &= Ae^{-\frac{1}{2}t} \end{aligned}$$

$$\begin{aligned} \text{when}\quad t=0,\quad p &= 10, \\ \text{so}\quad A &= 10 \\ \therefore \quad p &= 10e^{-\frac{1}{2}t} \\ \therefore \quad \text{when}\quad t=2,\quad p &= \tfrac{10}{e} \end{aligned}$$

b

$$\begin{aligned} \frac{dM}{dr} &= 8-2M \\ \therefore \quad \frac{1}{8-2M}\frac{dM}{dr} &= 1 \\ \therefore \quad \int \frac{1}{8-2M}\frac{dM}{dr}\,dr &= \int 1\,dr \\ \therefore \quad \int \frac{1}{8-2M}\ dM &= \int 1\,dr \\ \therefore \quad r &= -\tfrac{1}{2}\ln|8-2M|+c \end{aligned}$$

$$\begin{aligned} \text{But when}\quad r=0,\quad M &= 2 \\ \therefore \quad -\tfrac{1}{2}\ln|8-4|+c &= 0 \\ \therefore \quad c &= \tfrac{1}{2}\ln 4 = \ln 4^{\frac{1}{2}} = \ln 2 \\ \text{Hence when}\quad M &= 3.5, \\ r &= -\tfrac{1}{2}\ln|8-7|+\ln 2 \\ \therefore \quad r &= -\tfrac{1}{2}\ln 1+\ln 2 = \ln 2 \end{aligned}$$

5

$$\begin{aligned} \frac{ds}{dt}+ks &= 0 \\ \therefore \quad \frac{ds}{dt} &= -ks \\ \therefore \quad \frac{1}{s}\frac{ds}{dt} &= -k \\ \therefore \quad \int \frac{1}{s}\frac{ds}{dt}\,dt &= \int -k\ dt \\ \therefore \quad \frac{1}{s}\ ds &= \int -k\ dt \end{aligned}$$

$$\begin{aligned} \therefore \quad \ln|s| &= -kt+c \\ \therefore \quad s &= Ae^{-kt} \quad ......\ (*) \end{aligned}$$

Now $s(0) = 50$, so $A = 50$ and $s(3) = 20$,

$$\begin{aligned} \text{so}\quad 50e^{-3k} &= 20 \\ \therefore \quad e^{-3k} &= \tfrac{20}{50} = 0.4 \end{aligned}$$

But from $(*)$, $s = 50(e^{-3k})^{\frac{t}{3}}$,

$\therefore \quad s = 50\,(0.4)^{\frac{t}{3}}$ as required

EXERCISE 29D.2

1 **a**

$$\begin{aligned} xy' &= 3y \\ \therefore \quad x\frac{dy}{dx} &= 3y \\ \therefore \quad \frac{1}{y}\frac{dy}{dx} &= \frac{3}{x} \\ \therefore \quad \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int \frac{3}{x}\ dx \\ \therefore \quad \int \frac{1}{y}\ dy &= \int \frac{3}{x}\ dx \\ \therefore \quad \ln|y| &= 3\ln|x|+c \\ &= \ln\left|x^3\right|+c \\ \therefore \quad y &= Ax^3 \end{aligned}$$

b

$$\begin{aligned} xy &= 4y' \\ \therefore \quad xy &= 4\frac{dy}{dx} \\ \therefore \quad x &= \frac{4}{y}\frac{dy}{dx} \\ \therefore \quad \int x\,dx &= \int \frac{4}{y}\frac{dy}{dx}\,dx \\ \therefore \quad \int x\,dx &= \int \frac{4}{y}\,dy \\ \therefore \quad \frac{x^2}{2} &= 4\ln|y|+c \\ \therefore \quad \frac{x^2}{8}+d &= \ln|y| \\ \therefore \quad y &= Ae^{\frac{x^2}{8}} \end{aligned}$$

c

$$\begin{aligned} y' &= ye^x \\ \therefore \quad \frac{dy}{dx} &= ye^x \\ \therefore \quad \frac{1}{y}\frac{dy}{dx} &= e^x \\ \therefore \quad \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int e^x\ dx \\ \therefore \quad \int \frac{1}{y}\ dy &= \int e^x\ dx \\ \therefore \quad \ln|y| &= e^x+c \\ \therefore \quad y &= Ae^{e^x} \end{aligned}$$

d

$$\begin{aligned} y' &= xe^{y} \\ \therefore\quad \frac{dy}{dx} &= xe^{y} \\ \therefore\quad e^{-y}\frac{dy}{dx} &= x \\ \therefore\quad \int e^{-y}\frac{dy}{dx}\,dx &= \int x\,dx \\ \therefore\quad \int e^{-y}\,dy &= \int x\,dx \\ \therefore\quad -e^{-y} &= \frac{x^2}{2}+d \\ \therefore\quad e^{-y} &= -\frac{x^2}{2}-d \\ \therefore\quad -y &= \ln(-\frac{x^2}{2}+c) \\ \therefore\quad y &= -\ln(-\frac{x^2}{2}+c) \end{aligned}$$

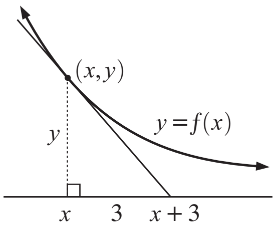
2

$$\begin{aligned} \frac{dz}{dr} &= z+zr^2 = z(1+r^2) \\ \therefore\quad \frac{1}{z}\frac{dz}{dr} &= 1+r^2 \\ \therefore\quad \int \frac{1}{z}\frac{dz}{dr}\,dr &= \int(1+r^2)\,dr \\ \therefore\quad \int \frac{1}{z}\,dz &= \int(1+r^2)\,dr \\ \therefore\quad \ln|z| &= r+\frac{r^3}{3}+c \\ \text{But}\quad z(0) &= 1, \\ \text{so}\quad \ln 1 &= 0+0+c \\ \therefore\quad c &= 0 \\ \therefore\quad \ln|z| &= r+\frac{r^3}{3} \\ \therefore\quad z &= e^{r+\frac{r^3}{3}} \end{aligned}$$

3

$$\begin{aligned} \frac{dy}{dx} &= -2xy \\ \therefore\quad \frac{1}{y}\frac{dy}{dx} &= -2x \\ \therefore\quad \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int -2x\,dx \\ \therefore\quad \int \frac{1}{y}\,dy &= \int -2x\,dx \\ \therefore\quad \ln|y| &= -x^2+c \\ \therefore\quad y &= Ae^{-x^2} \\ \text{But}\quad y(0) &= 1, \\ \text{so}\quad A &= 1 \\ \therefore\quad y &= e^{-x^2} \end{aligned}$$

4



$$\begin{aligned} \frac{dy}{dx} &= -\frac{y}{3} \\ \therefore\quad \frac{1}{y}\frac{dy}{dx} &= -\tfrac{1}{3} \\ \therefore\quad \int \frac{1}{y}\frac{dy}{dx}\,dx &= \int -\tfrac{1}{3}\,dx \\ \therefore\quad \int \frac{1}{y}\,dy &= \int -\tfrac{1}{3}\,dx \\ \therefore\quad \ln|y| &= -\tfrac{1}{3}x+c \\ \therefore\quad y &= Ae^{-\frac{1}{3}x} \\ \text{But}\quad y(0) &= 2, \\ \text{so}\quad 2 &= Ae^{0} \\ \therefore\quad A &= 2 \\ \text{So,}\quad y &= 2e^{-\frac{x}{3}} \end{aligned}$$

5

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y} \\ \therefore\quad y\frac{dy}{dx} &= x \\ \therefore\quad \int y\frac{dy}{dx}\,dx &= \int x\,dx \\ \therefore\quad \int y\,dy &= \int x\,dx \\ \therefore\quad \frac{y^2}{2} &= \frac{x^2}{2}+c \\ \text{i.e.,}\quad x^2-y^2 &= -2c \end{aligned}$$

The curve passes through (5, -4),

$$\begin{aligned} \text{so}\quad 5^2-(-4)^2 &= -2c \\ \text{i.e.,}\quad -2c &= 9 \\ \therefore\quad x^2-y^2 &= 9 \end{aligned}$$

If $(a, 3)$ lies on the curve,

$$\begin{aligned} \text{then}\quad a^2-3^2 &= 9 \\ \therefore\quad a^2 &= 18 \\ \therefore\quad a &= \pm 3\sqrt{2} \end{aligned}$$

6 **a**

$$\begin{aligned} \frac{dy}{dx} &= y^2(1+x) \\ \therefore\quad \frac{1}{y^2}\frac{dy}{dx} &= 1+x \\ \therefore\quad \int \frac{1}{y^2}\frac{dy}{dx}\,dx &= \int(1+x)\,dx \\ \therefore\quad \int y^{-2}\,dy &= \int(1+x)\,dx \\ \therefore\quad -\frac{1}{y} &= x+\tfrac{1}{2}x^2+c \\ \therefore\quad y &= \frac{1}{-x-\frac{1}{2}x^2-c} \end{aligned}$$

But (1, 2) lies on the curve,

$$\text{so} \quad \frac{1}{-1-\frac{1}{2}-c} = 2$$

$$\therefore \quad \tfrac{1}{2} = -\tfrac{3}{2} - c$$

$$\therefore \quad c = -2$$

$$\therefore \quad y = \frac{1}{2-x-\frac{1}{2}x^2}$$

$$\text{i.e.,} \quad y = \frac{-2}{x^2+2x-4}$$

b When $x^2+2x-4=0$,

$$x = \frac{-2 \pm \sqrt{4-4(1)(-4)}}{2}$$

$$\therefore \quad x = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

$\therefore$ VAs are $x=-1+\sqrt{5}$, $x=-1-\sqrt{5}$

HA is $y=0$ $\{$as $|x| \to \infty$, $y \to 0\}$

7 **a** At any point (x, y), the product of the slopes of the curves is $\left(-\dfrac{x}{y}\right)\left(\dfrac{y}{x}\right) = -1$

$\therefore$ the solution curves are always at right angles.

b For $\dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\therefore \quad \int y\frac{dy}{dx}\,dx = \int -x\,dx$$

$$\therefore \quad \int y\,dy = \int -x\,dx$$

$$\therefore \quad \frac{y^2}{2} = -\frac{x^2}{2} + a$$

$$\text{i.e.,} \quad x^2+y^2 = c$$

This represents a circle with centre (0, 0) and radius $\sqrt{c}$ ...... (1)

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \quad \int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\therefore \quad \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\therefore \quad \ln|y| = \ln|x| + a$$

$$\therefore \quad \ln|y| - \ln|x| = a$$

$$\therefore \quad \ln\left|\frac{y}{x}\right| = a$$

$$\therefore \quad \frac{y}{x} = \pm e^a = k, \quad \text{say}$$

$$\therefore \quad y = kx \quad \text{...... (2)}$$

This represents a line with slope k that passes through (0, 0).

Any line passing through the centre of a circle will include a diameter of the circle and therefore be normal to the circle. This is why the solution curves in **a** were at right angles.

8 acceleration $a \propto$ velocity v

$$\therefore \quad \frac{dv}{dt} = kv \quad \text{for some } k$$

$$\therefore \quad \frac{1}{v}\frac{dv}{dt} = k$$

$$\therefore \quad \int \frac{1}{v}\frac{dv}{dt}\,dt = \int k\,dt$$

$$\therefore \quad \int \frac{1}{v}\,dv = \int k\,dt$$

$$\therefore \quad \ln|v| = kt + c$$

when $t=0$, $v=4$

so $c = \ln 4$

Also, when $t=4$, $v=6$

$$\therefore \quad 4k + \ln 4 = \ln 6$$

$$\therefore \quad 4k = \ln \tfrac{6}{4}$$

$$\therefore \quad k = \tfrac{1}{4}\ln\tfrac{3}{2}$$

$$\therefore \quad \ln|v| = \frac{t}{4}\ln\tfrac{3}{2} + \ln 4$$

$$= \ln\left[\left(\tfrac{3}{2}\right)^{\frac{t}{4}} 4\right]$$

$$\therefore \quad v = 4\left(\tfrac{3}{2}\right)^{\frac{t}{4}} \text{ ms}^{-1}$$

and when $t = 5$

$$v = 4(\tfrac{3}{2})^{\frac{5}{4}}$$

i.e., $v \doteqdot 6.64 \text{ ms}^{-1}$

9

$$\begin{aligned}\frac{dw}{dt} &\propto w\\ \therefore\ \frac{dw}{dt} &= kw \quad \text{for some } k\\ \therefore\ \frac{1}{w}\frac{dw}{dt} &= k\\ \therefore\ \int \frac{1}{w}\frac{dw}{dt}\,dt &= \int k\,dt\\ \therefore\ \int \frac{1}{w}\,dw &= \int k\,dt\\ \therefore\ \ln w &= kt + c \quad (\text{since}\quad w \geqslant 0)\\ \therefore\ w &= Ae^{kt}\end{aligned}$$

If there was weight w_0 of sugar at time 0,

$$\begin{aligned}\text{then}\quad A &= w_0\\ \text{i.e.,}\quad w &= w_0e^{kt}\\ \text{When}\quad t=10,\quad w &= 0.2w_0,\\ \text{so}\quad w_0e^{10k} &= 0.2w_0\\ \therefore\ e^{10k} &= 0.2\\ \text{But}\quad w &= w_0\left(e^{10k}\right)^{\frac{t}{10}}\\ \text{so}\quad w &= w_0 \times 0.2^{\frac{t}{10}}\\ \therefore\ \text{when}\quad t=30,\quad w &= w_0 \times 0.2^3\\ \therefore\ w &= 0.008w_0\end{aligned}$$

i.e., 0.8% is remaining

10

$$\begin{aligned}\frac{dI}{dt} &= -kI\\ \therefore\ \frac{1}{I}\frac{dI}{dt} &= -k\\ \therefore\ \int \frac{1}{I}\frac{dI}{dt}\,dt &= -\int k\,dt\\ \therefore\ \int \frac{1}{I}\,dI &= -\int k\,dt\\ \therefore\ \ln|I| &= -kt + c\end{aligned}$$

If $I(0) = I_0$, then $c = I_0$

$$\begin{aligned}\therefore\ \ln I &= -kt + \ln I_0\\ \therefore\ I &= I_0e^{-kt}\end{aligned}$$

$$\begin{aligned}\text{When}\quad t=1,\quad I &= 0.1I_0,\\ \text{so}\quad I_0e^{-k} &= 0.1I_0\\ \therefore\ e^{-k} &= 0.1\\ \therefore\ I &= I_0 0.1^t\\ \therefore\ \text{when}\quad I &= 0.001I_0,\\ I_0 0.1^t &= 0.001I_0\\ \text{i.e.,}\quad t &= 3 \text{ seconds}\end{aligned}$$

11 **a**

$$\begin{aligned}\frac{dv}{dt} &= g - 4v\\ \therefore\ \frac{1}{g-4v}\frac{dv}{dt} &= 1\\ \therefore\ \int \frac{1}{g-4v}\frac{dv}{dt}\,dt &= \int 1\,dt\\ \therefore\ \int \frac{1}{g-4v}\,dv &= \int 1\,dt\\ \therefore\ -\tfrac{1}{4}\ln|g-4v| &= t + c\\ \therefore\ \ln|g-4v| &= -4t - 4c\\ \therefore\ g - 4v &= Ae^{-4t}\\ \text{i.e.,}\quad 4v &= g - Ae^{-4t}\end{aligned}$$

Since the metal is released from rest,

$$\begin{aligned}v(0) &= 0\\ \therefore\ g - A &= 0\\ \therefore\ A &= g\\ \therefore\ 4v &= g - ge^{-4t}\\ \therefore\ v &= \frac{g}{4}(1-e^{-4t}) \quad \text{as required}\end{aligned}$$

as $t \to \infty$, $e^{-4t} \to 0$ and $\therefore\ v(t) \to \frac{g}{4}$ ms^{-1}

$\therefore$ there is a limiting velocity of $\frac{g}{4}$ ms^{-1}

b $v = \frac{g}{10}$ when

$$\begin{aligned}\frac{g}{10} &= \frac{g}{4}(1-e^{-4t})\\ \therefore\ 1 - e^{-4t} &= \tfrac{4}{10}\\ \therefore\ e^{-4t} &= \tfrac{3}{5}\\ \therefore\ -4t &= \ln\tfrac{3}{5}\\ \therefore\ t &= -\tfrac{1}{4}\ln\tfrac{3}{5} \doteqdot 0.128 \text{ sec}\end{aligned}$$

12 **a** Suppose V_0 is the initial volume of water in the lake, and $V(t)$ is the total amount of water that has evaporated at time t. Then the volume of water remaining in the lake is $(V_0 - V)$.

Hence $\dfrac{dV}{dt} \propto (V_0 - V)$ i.e., $\dfrac{dV}{dt} = k(V_0 - V)$

b
$$\frac{1}{V_0 - V}\frac{dV}{dt} = k$$
$$\therefore \int \frac{1}{V_0 - V}\frac{dV}{dt}\,dt = \int k\,dt$$
$$\therefore \int \frac{1}{V_0 - V}\,dV = \int k\,dt$$
$$\therefore -\ln|V_0 - V| = kt + c$$
$$\therefore \ln|V_0 - V| = -kt - c$$
$$\therefore V_0 - V = Ae^{-kt}$$

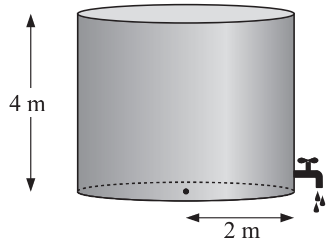
But $V(0) = 0$, so $A = V_0$

After 20 days, $V = \frac{1}{2}V_0$
$$\text{so } V_0 - \tfrac{1}{2}V_0 = V_0 e^{-20k}$$
$$\therefore e^{-20k} = \tfrac{1}{2}$$
$$\therefore V_0 - V = V_0\left(e^{-20k}\right)^{\frac{t}{20}} = V_0\left(\tfrac{1}{2}\right)^{\frac{t}{20}}$$
$$\therefore V = V_0\left(1 - 0.5^{\frac{t}{20}}\right)$$

Hence after 50 days without rain,
$V = V_0(1 - 0.5^{2.5}) \doteqdot 0.8232V_0$

$\therefore$ 82.3% has evaporated and 17.7% remains.

13 Let V be the volume of water in the tank and h be the water depth.



$$\text{Then } \frac{dV}{dt} \propto \sqrt{h},$$
$$\text{i.e., } \frac{dV}{dt} = k\sqrt{h} \quad \text{for some } k$$

We also know that $V = \pi r^2 h$

where $r = 2$ m is the radius of the tank
$$\text{Hence } V = 4\pi h$$
$$\text{and } \frac{dV}{dh} = 4\pi$$
$$\text{Now } \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \quad \{\text{Chain Rule}\}$$
$$\text{so } k\sqrt{h} = 4\pi \times \frac{dh}{dt}$$
$$\therefore \frac{1}{\sqrt{h}}\frac{dh}{dt} = \frac{k}{4\pi}$$
$$\therefore \int \frac{1}{\sqrt{h}}\frac{dh}{dt}\,dt = \int \frac{k}{4\pi}\,dt$$
$$\therefore \int \frac{1}{\sqrt{h}}\,dh = \int \frac{k}{4\pi}\,dt$$
$$\therefore 2\sqrt{h} = \frac{k}{4\pi}t + c$$

Initially the tank is full,
$$\text{so } h(0) = 4$$
$$\therefore 4 = 0 + c$$
$$\text{i.e., } 2\sqrt{h} = \frac{k}{4\pi}t + 4$$

Also, $h(2) = 1$
$$\text{so } 2 = \frac{k}{4\pi} \times 2 + 4$$
$$\therefore \frac{k}{4\pi} = -1$$
$$\therefore k = -4\pi$$
$$\therefore 2\sqrt{h} = 4 - t$$

$\therefore$ the tank is empty when $t = 4$

i.e., it takes 4 hours to empty.

14 **a**
$$V = \tfrac{1}{3}\pi h^2(3r - h)$$
$$= \pi h^2 r - \tfrac{1}{3}\pi h^3$$
$$\therefore \text{ for fixed } r, \frac{dV}{dh} = 2\pi hr - \pi h^2$$
$$\text{Now } \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \quad \{\text{Chain Rule}\}$$
$$\therefore \text{ since } \frac{dV}{dt} = -r^2,$$
$$-r^2 = (2\pi hr - \pi h^2)\frac{dh}{dt}$$
$$\therefore \frac{dh}{dt} = \frac{r^2}{\pi h^2 - 2\pi hr}$$

b

$$(\pi h^2 - 2\pi hr)\frac{dh}{dt} = r^2$$

$$\therefore \int(\pi h^2 - 2\pi hr)\frac{dh}{dt}\,dt = \int r^2\,dt$$

$$\therefore \int(\pi h^2 - 2\pi hr)\,dh = \int r^2\,dt$$

$$\therefore \tfrac{1}{3}\pi h^3 - \pi h^2 r = r^2 t + c$$

If $r = 10$ cm and the bowl is initially full,

i.e., $h = 10$ cm,

then $\frac{1000}{3}\pi - 1000\pi = c$

i.e., $c = -\frac{2000}{3}\pi$

$$\therefore 100t = \tfrac{1}{3}\pi h^3 - 10\pi h^2 + \tfrac{2000}{3}\pi$$

$$\therefore t = \tfrac{\pi}{300}h^3 - \tfrac{\pi}{10}h^2 + \tfrac{2000}{300}\pi$$

$$\therefore t = \tfrac{\pi}{300}[h^3 - 30h^2 + 2000]$$

$$t = \tfrac{1375\pi}{300} \doteqdot 14.4 \text{ hours}$$

15

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\therefore \frac{dT}{dt} = k(T - T_m) \quad \text{for some } k$$

$$\therefore \frac{1}{T - T_m}\frac{dT}{dt} = k$$

$$\therefore \int \frac{1}{T - T_m}\,dT = \int k\,dt$$

$$\therefore \ln|T - T_m| = kt + c$$

$$\therefore T - T_m = Ae^{kt}$$

Given $T_m = 5^o$C

and that $T = 100^o$C when $t = 0$,

$$Ae^0 = 100 - 5$$

$$\therefore A = 95$$

Also, since $T(1) = 80$,

$$95e^k = 80 - 5 = 75$$

$$\therefore e^k = \tfrac{15}{19}$$

$$\therefore T - 5 = 95(\tfrac{15}{19})^t$$

i.e., $T = 5 + 95(\frac{15}{19})^t$

Now $T = 10^o$C

when $10 = 5 + 95(\frac{15}{19})^t$

$$\therefore (\tfrac{15}{19})^t = \tfrac{5}{95} = \tfrac{1}{19}$$

$$\therefore t\ln\tfrac{15}{19} = \ln\tfrac{1}{19}$$

$$\therefore t = \frac{\ln\frac{1}{19}}{\ln\frac{15}{19}}$$

$$\therefore t \doteqdot 12.5 \text{ minutes}$$

16 From question **15**, a general solution to Newton's Law of Cooling is $T - T_m = Ae^{kt}$.

Assume $t = 0$ to be 6 am, so $T(0) = 13$.

Then since $T_m = 5^o$C

$$Ae^0 = 13 - 5 = 8$$

$$\therefore A = 8$$

$$\therefore T = 5 + 8e^{kt}$$

Also, $T(3) = 9$,

so $8e^{3k} = 4$

$$\therefore e^{3k} = \tfrac{1}{2}$$

$$\therefore T = 5 + 8(e^{3k})^{\frac{t}{3}} = 5 + 8(\tfrac{1}{2})^{\frac{t}{3}}$$

$\therefore$ the temperature was 37^oC when

$$5 + 8\left(\tfrac{1}{2}\right)^{\frac{t}{3}} = 37$$

$$\therefore \left(\tfrac{1}{2}\right)^{\frac{t}{3}} = \tfrac{32}{8} = 4$$

$$\therefore \frac{t}{3} = -2$$

$$\therefore t = -6 \text{ hours}$$

i.e., time of death was 6 hours before 6 am,

i.e., midnight.

17 **a** $z = r\cos\theta + ir\sin\theta$

Treating r and i as constants,

$$\begin{aligned} \frac{dz}{d\theta} &= r(-\sin\theta) + ir\cos\theta \\ &= i(r\cos\theta + ir\sin\theta) \\ &= iz \end{aligned}$$

b

$$\begin{aligned} \text{Since}\quad \frac{dz}{d\theta} &= iz \\ \text{then}\quad \frac{1}{z}\frac{dz}{d\theta} &= i \\ \therefore\quad \int \frac{1}{z}\frac{dz}{d\theta}\,d\theta &= \int i\,d\theta \\ \therefore\quad \int \frac{1}{z}\,dz &= \int i\,d\theta \\ \therefore\quad \ln|z| &= i\theta + c \\ \therefore\quad z &= Ae^{i\theta} \end{aligned}$$

But when $\theta = 0$, $z = r$

$$\therefore\quad z = re^{i\theta}$$

c

$$\begin{aligned} \text{Thus,}\quad r\cos\theta + ir\sin\theta &= re^{i\theta} \\ \therefore\quad \cos\theta + i\sin\theta &= e^{i\theta} \\ \therefore\quad \text{cis }\theta &= e^{i\theta} \end{aligned}$$

d **i**

$$\begin{aligned} \text{If}\quad z &= |z|\text{cis }\theta \quad\text{and}\quad w = |w|\text{cis }\phi \\ \text{then}\quad z &= |z|\,e^{i\theta} \quad\text{and}\quad w = |w|\,e^{i\phi} \\ \text{So,}\quad zw &= |z|\,|w|\,e^{i\theta+i\phi} \\ \text{i.e.,}\quad zw &= |z|\,|w|\,e^{i(\theta+\phi)} \\ \text{So,}\quad |zw| &= |z|\,|w| \\ \text{and}\quad \arg(zw) &= \theta+\phi \\ &= \arg z + \arg w \end{aligned}$$

ii

$$\begin{aligned} \frac{\text{cis }\theta}{\text{cis }\phi} &= \frac{e^{i\theta}}{e^{i\phi}} \\ &= e^{i\theta - i\phi} \\ &= e^{i(\theta-\phi)} \\ &= \text{cis }(\theta-\phi) \end{aligned}$$

iii

$$\begin{aligned} \text{Using}\quad e^{i\theta} &= \cos\theta + i\sin\theta \\ \text{and letting}\quad \theta &= \pi, \\ e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 + i(0) \\ &= -1 \\ \therefore\quad e^{i\pi} + 1 &= 0 \end{aligned}$$

EXERCISE 29D.3

1 $\dfrac{dy}{dx} - 2xe^x = y$ so $\dfrac{dy}{dx} = 2xe^x + y$

Now letting $y = ue^x$, $\dfrac{dy}{dx} = \dfrac{du}{dx}e^x + ue^x$

$$\begin{aligned} \therefore\quad 2xe^x + ue^x &= \frac{du}{dx}e^x + ue^x \\ \therefore\quad \frac{du}{dx}e^x &= 2xe^x \\ \therefore\quad \frac{du}{dx} &= 2x \end{aligned}$$

$$\begin{aligned} \therefore\quad \int \frac{du}{dx}\,dx &= \int 2x\,dx \\ \therefore\quad \int 1\,du &= \int 2x\,dx \\ \therefore\quad u &= x^2 + c \\ \therefore\quad \frac{y}{e^x} &= x^2 + c \\ \therefore\quad y &= e^x(x^2+c) \end{aligned}$$

2 If $y = ue^x$, then $\dfrac{dy}{dx} = \dfrac{du}{dx}e^x + ue^x$

$$\begin{aligned} \text{But}\quad \left(\frac{dy}{dx}\right)^2 &= y^2 + 2e^x y + e^{2x} \\ \therefore\quad \left[\frac{du}{dx}e^x + ue^x\right]^2 &= u^2e^{2x} + 2e^x(ue^x) + e^{2x} \\ \therefore\quad \left[\frac{du}{dx}e^x\right]^2 + 2ue^{2x}\frac{du}{dx} + u^2e^{2x} &= u^2e^{2x} + 2ue^{2x} + e^{2x} \end{aligned}$$

$$\begin{aligned} \therefore\quad \left(\frac{du}{dx}\right)^2 e^{2x} + 2ue^{2x}\left(\frac{du}{dx}\right) &= 2ue^{2x} + e^{2x} \\ \therefore\quad \left(\frac{du}{dx}\right)^2 + 2u\frac{du}{dx} &= 2u + 1 \\ \therefore\quad \left(\frac{du}{dx}\right)^2 + 2u\frac{du}{dx} - 2u - 1 &= 0 \\ \therefore\quad \left(\frac{du}{dx} - 1\right)\left(\frac{du}{dx} + 2u + 1\right) &= 0 \\ \therefore\quad \frac{du}{dx} &= 1 \text{ or } -2u-1 \end{aligned}$$

If $\frac{du}{dx} = 1$ then

$$\begin{aligned} \int \frac{du}{dx}\,dx &= \int 1\,dx \\ \therefore \int 1\,du &= x + c \\ \therefore u &= x + c \\ \text{i.e., } \frac{y}{e^x} &= x + c \\ \therefore y &= e^x(x+c) \end{aligned}$$

or, if $\frac{du}{dx} = -2u - 1$ then

$$\begin{aligned} \int \frac{1}{2u+1}\frac{du}{dx}\,dx &= \int -1\,dx \\ \therefore \int \frac{1}{2u+1}\,du &= \int -1\,dx \\ \therefore \tfrac{1}{2}\ln|2u+1| &= -x + c \\ \therefore 2u + 1 &= Ae^{-2x} \\ \text{i.e., } \frac{2y}{e^x} + 1 &= Ae^{-2x} \\ \therefore 2y + e^x &= Ae^{-x} \\ \therefore y &= \frac{Ae^{-x} - e^x}{2} \end{aligned}$$

Hence $y = e^x(x+c)$ or $y = \dfrac{Ae^{-x} - e^x}{2}$

3

$$\begin{aligned} 4xy\frac{dy}{dx} &= -x^2 - y^2, \quad x > 0 \end{aligned}$$

Let $y = ux$ so $\dfrac{dy}{dx} = \dfrac{du}{dx}x + u$

$$\begin{aligned} \therefore 4x(ux)\left[\frac{du}{dx}x + u\right] &= -x^2 - u^2x^2 \\ \therefore 4ux^2\left[\frac{du}{dx}x + u\right] &= -x^2 - u^2x^2 \\ \therefore 4u\left[\frac{du}{dx}x + u\right] &= -1 - u^2 \\ \therefore x\frac{du}{dx} + u &= \frac{-u^2-1}{4u} \\ \therefore x\frac{du}{dx} &= \frac{-u^2-1-4u^2}{4u} \\ \therefore x\frac{du}{dx} &= \frac{-5u^2-1}{4u} \\ \therefore \frac{4u}{5u^2+1}\frac{du}{dx} &= -\frac{1}{x} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{4u}{5u^2+1}\frac{du}{dx}\,dx &= \int -\frac{1}{x}\,dx \\ \therefore \int \frac{4u}{5u^2+1}\,du &= -\int \frac{1}{x}\,dx \\ \therefore \tfrac{4}{10}\int \frac{10u}{5u^2+1}\,du &= -\int \frac{1}{x}\,dx \\ \therefore \tfrac{2}{5}\ln\left|5u^2+1\right| &= -\ln|x| + c \\ \therefore \ln\left|5u^2+1\right| &= -\tfrac{5}{2}\ln|x| + c \\ \therefore \frac{5y^2}{x^2} + 1 &= Ax^{-\frac{5}{2}} \\ \therefore \frac{x^2+5y^2}{x^2} &= Ax^{-\frac{5}{2}} \\ \therefore \sqrt{x}\left(x^2+5y^2\right) &= A \\ \therefore x\left(x^2+5y^2\right)^2 &= k \end{aligned}$$

4

$$x\frac{dy}{dx} - y = 4x^2y$$

Let $y = ux$, so $\dfrac{dy}{dx} = \dfrac{du}{dx}x + u$

$$\begin{aligned} \therefore x\left[\frac{du}{dx}x + u\right] - ux &= 4x^2(ux) \\ \therefore \frac{du}{dx}x^2 + ux - ux &= 4ux^3 \\ \therefore \frac{du}{dx}x^2 &= 4ux^3 \\ \therefore \frac{du}{dx} &= 4ux \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{u}\frac{du}{dx} &= 4x \\ \therefore \int \frac{1}{u}\frac{du}{dx}\,dx &= \int 4x\,dx \\ \therefore \int \frac{1}{u}\,du &= \int 4x\,dx \\ \therefore \ln|u| &= \frac{4x^2}{2} + c \\ \therefore u &= Ae^{2x^2} \end{aligned}$$

But $y = ux$, so $\dfrac{y}{x} = Ae^{2x^2}$

$$\therefore y = Axe^{2x^2}$$

REVIEW SET 29A

1 Let $u = 4 - x$ so $\dfrac{du}{dx} = -1$

$$\begin{aligned}
\therefore\ & \textstyle\int x^2\sqrt{4-x}\,dx \\
&= \int (4-u)^2\sqrt{u}\left(-\frac{du}{dx}\right) dx \\
&= -\int (16-8u+u^2)u^{\frac{1}{2}}\,du \\
&= -\int \left(16u^{\frac{1}{2}} - 8u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du \\
&= -\left[\frac{16u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{8u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{7}{2}}}{\frac{7}{2}}\right] + c \\
&= -\tfrac{32}{3}u^{\frac{3}{2}} + \tfrac{16}{5}u^{\frac{5}{2}} - \tfrac{2}{7}u^{\frac{7}{2}} + c \\
&= -\tfrac{32}{3}(4-x)^{\frac{3}{2}} + \tfrac{16}{5}(4-x)^{\frac{5}{2}} - \tfrac{2}{7}(4-x)^{\frac{7}{2}} + c
\end{aligned}$$

2

$$\begin{aligned}
& \textstyle\int \arctan x\,dx \\
&= \textstyle\int 1 \arctan x\,dx \qquad \left\{\begin{array}{ll} u = \arctan x & v' = 1 \\ u' = \dfrac{1}{x^2+1} & v = x \end{array}\right\} \\
&= x\arctan x - \int \frac{1}{x^2+1}(x)\,dx \\
&= x\arctan x - \tfrac{1}{2}\int \frac{2x}{x^2+1}\,dx \\
&= x\arctan x - \tfrac{1}{2}\ln\left|x^2+1\right| + c \\
&= x\arctan x - \tfrac{1}{2}\ln(x^2+1) + c \qquad \{x^2+1>0\}
\end{aligned}$$

3 **a**

$$\begin{aligned}
& \textstyle\int e^{-x}\cos x\,dx \qquad \left\{\begin{array}{ll} u = e^{-x} & v' = \cos x \\ u' = -e^{-x} & v = \sin x \end{array}\right\} \\
&= e^{-x}\sin x - \textstyle\int -e^{-x}\sin x\,dx \\
&= e^{-x}\sin x + \textstyle\int e^{-x}\sin x\,dx \qquad \left\{\begin{array}{ll} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x \end{array}\right\} \\
&= e^{-x}\sin x + e^{-x}(-\cos x) - \textstyle\int(-e^{-x})(-\cos x)\,dx \\
&= e^{-x}\sin x - e^{-x}\cos x - \textstyle\int e^{-x}\cos x\,dx + c_1 \\
\therefore\ & 2\textstyle\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + c_1 \\
\therefore\ & \textstyle\int e^{-x}\cos x\,dx = \tfrac{1}{2}e^{-x}(\sin x - \cos x) + c
\end{aligned}$$

b

$$\begin{aligned}
& \textstyle\int x^2e^x\,dx \qquad \left\{\begin{array}{ll} u = x^2 & v' = e^x \\ u' = 2x & v = e^x \end{array}\right\} \\
&= x^2e^x - \textstyle\int 2xe^x\,dx \qquad \left\{\begin{array}{ll} u = 2x & v' = e^x \\ u' = 2 & v = e^x \end{array}\right\} \\
&= x^2e^x - \left[2xe^x - \textstyle\int 2e^x\,dx\right] \\
&= x^2e^x - 2xe^x + 2\textstyle\int e^x\,dx \\
&= x^2e^x - 2xe^x + 2e^x + c \\
&= e^x(x^2-2x+2) + c
\end{aligned}$$

c Let $u = 9 - x^2$ so $\dfrac{du}{dx} = -2x$

$$\begin{aligned}
\therefore\ & \int \frac{x^3}{\sqrt{9-x^2}}\,dx \\
&= \int \frac{x^2}{\sqrt{9-x^2}}\,x\,dx \\
&= \int \frac{1}{\sqrt{u}}(9-u)\left(-\tfrac{1}{2}\frac{du}{dx}\right) dx \\
&= -\tfrac{1}{2}\int \frac{9-u}{\sqrt{u}}\,du \\
&= -\tfrac{1}{2}\int \left(9u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du \\
&= -\tfrac{1}{2}\left[\frac{9u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c \\
&= -9u^{\frac{1}{2}} + \tfrac{1}{3}u^{\frac{3}{2}} + c \\
&= -9\sqrt{9-x^2} + \tfrac{1}{3}\left(9-x^2\right)^{\frac{3}{2}} + c
\end{aligned}$$

4

$$\begin{aligned}
y' &= -\frac{2e^x}{y} \\
\therefore\ y\frac{dy}{dx} &= -2e^x \\
\therefore\ \textstyle\int y\,dy &= \textstyle\int -2e^x\,dx \\
\therefore\ \frac{y^2}{2} &= -2e^x + c
\end{aligned}$$

$$\begin{aligned}
\text{But}\ y(0) &= 4, \\
\text{so}\ \tfrac{16}{2} &= -2e^0 + c \\
\therefore\ c &= 10 \\
\therefore\ \frac{y^2}{2} &= -2e^x + 10 \\
\therefore\ y^2 &= 20 - 4e^x
\end{aligned}$$

5

$$L\frac{dI}{dt} = E - RI$$

Using $R = 4$, $L = 0.2$ and $E = 20$,

$$\tfrac{1}{5}\frac{dI}{dt} = 20 - 4I = 4(5-I)$$

$$\therefore \quad \frac{1}{5-I}\frac{dI}{dt} = 20$$

$$\therefore \quad \int \frac{1}{5-I}\frac{dI}{dt}\,dt = \int 20\,dt$$

$$\therefore \quad \int \frac{1}{5-I}\,dI = \int 20\,dt$$

$$\therefore \quad -\ln|5-I| = 20t + c$$

But $I(0) = 0$, so $c = -\ln 5$

$$\therefore \quad 20t = \ln 5 - \ln|5-I|$$

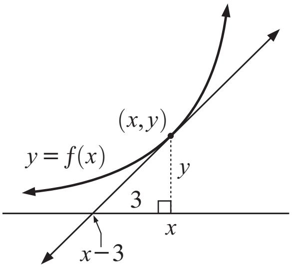
$$\therefore \quad \therefore \quad t = \tfrac{1}{20}\ln\left|\frac{5}{5-I}\right|$$

The current is therefore 0.5 amps at time

$$t = \tfrac{1}{20}\ln(\tfrac{5}{4.5})$$

$$t \doteqdot 0.005\,27 \text{ sec}$$

6



$$\frac{dy}{dx} = \frac{y}{3}$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \tfrac{1}{3}$$

$$\therefore \quad \int \frac{1}{y}\frac{dy}{dx}\,dx = \int \tfrac{1}{3}\,dx$$

$$\therefore \quad \int \frac{1}{y}\,dy = \int \tfrac{1}{3}\,dx$$

$$\therefore \quad \ln|y| = \tfrac{1}{3}x + c$$

$$\therefore \quad y = Ae^{\frac{1}{3}x}$$

But when $x = 0$, $y = 3$

$$\therefore \quad 3 = Ae^0$$

$$\therefore \quad A = 3$$

So, $y = 3e^{\frac{1}{3}x}$

7 Letting $y = ux$, $\dfrac{dy}{dx} = \dfrac{du}{dx}x + u$

So, $2xy\dfrac{dy}{dx} = x^2 + y^2$ becomes

$$2x(ux)\left[\frac{du}{dx}\,x + u\right] = x^2 + (ux)^2$$

$$\therefore \quad 2ux^2\left[x\frac{du}{dx} + u\right] = x^2 + u^2x^2$$

$$\therefore \quad 2u\left[x\frac{du}{dx} + u\right] = 1 + u^2 \quad \{\div \text{ each term by } x^2\}$$

$$\therefore \quad x\frac{du}{dx} + u = \frac{1+u^2}{2u}$$

$$\therefore \quad x\frac{du}{dx} = \frac{1-u^2}{2u}$$

$$\therefore \quad \frac{2u}{1-u^2}\frac{du}{dx} = \frac{1}{x}$$

$$\therefore \quad \int \frac{2u}{1-u^2}\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\therefore \quad \int \frac{2u}{1-u^2}\,du = \int \frac{1}{x}\,dx$$

$$\therefore \quad -\int \frac{-2u}{1-u^2}\,du = \int \frac{1}{x}\,dx$$

$$\therefore \quad -\ln\left|1-u^2\right| = \ln|x| + c$$

$$\therefore \quad \ln\left|x(1-u^2)\right| = -c$$

$$\therefore \quad x(1-u^2) = A$$

$$\therefore \quad x\left(1 - \frac{y^2}{x^2}\right) = A$$

$$\therefore \quad x\left(\frac{x^2-y^2}{x^2}\right) = A$$

i.e., $x^2 - y^2 = Ax$

8 **a**

$$\begin{aligned} &\text{cis }\theta \text{ cis }\phi \\ &= e^{i\theta} \times e^{i\phi} \\ &= e^{i\theta + i\phi} \\ &= e^{i(\theta+\phi)} \\ &= \text{cis }(\theta+\phi) \end{aligned}$$

b

$$\begin{aligned} &(\text{cis }\theta)^n \\ &= \left(e^{i\theta}\right)^n \\ &= e^{i\theta n} \\ &= e^{i(n\theta)} \\ &= \text{cis } n\theta \end{aligned}$$

REVIEW SET 29B

1 **a**

$$\int \frac{5}{\sqrt{9-x^2}}\,dx$$
$$= 5\int \frac{1}{\sqrt{3^2-x^2}}\,dx$$
$$= 5\arcsin\left(\tfrac{x}{3}\right) + c$$

b

$$\int \frac{1}{9+4x^2}\,dx$$
$$= \tfrac{1}{4}\int \frac{1}{\frac{9}{4}+x^2}\,dx$$
$$= \tfrac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\arctan\left(\frac{x}{\frac{3}{2}}\right) + c$$
$$= \tfrac{1}{6}\arctan\left(\tfrac{2x}{3}\right) + c$$

c Let $u = x - 5$ so $\dfrac{du}{dx} = 1$

When $x = 10$, $u = 5$
and when $x = 7$, $u = 2$

$$\therefore \int_7^{10} x\sqrt{x-5}\,dx$$
$$= \int_2^5 (u+5)\sqrt{u}\,du$$
$$= \int_2^5 \left(u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du$$
$$= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{5u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^5$$
$$= \tfrac{2}{5}\left(5^{\frac{5}{2}}\right) + \tfrac{10}{3}\left(5^{\frac{3}{2}}\right) - \left[\tfrac{2}{5}\left(2^{\frac{5}{2}}\right) + \tfrac{10}{3}\left(2^{\frac{3}{2}}\right)\right]$$
$$= \tfrac{2}{5}\left(25\sqrt{5}\right) + \tfrac{10}{3}\left(5\sqrt{5}\right) - \tfrac{2}{5}\left(4\sqrt{2}\right) - \tfrac{10}{3}\left(2\sqrt{2}\right)$$
$$= 10\sqrt{5} + \tfrac{50}{3}\sqrt{5} - \tfrac{8}{5}\sqrt{2} - \tfrac{20}{3}\sqrt{2}$$
$$= \tfrac{80}{3}\sqrt{5} - \tfrac{124}{15}\sqrt{2}$$

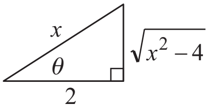
2 **a** $\int x\cos x\,dx$

has $u = x \quad v' = \cos x$
$\quad\; u' = 1 \quad v = \sin x$

$$\therefore \int x\cos x\,dx$$
$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x - (-\cos x) + c$$
$$= x\sin x + \cos x + c$$

b Let $x = 2\sec\theta$ so $\dfrac{dx}{d\theta} = 2\sec\theta\tan\theta$

$$\therefore \int \frac{\sqrt{x^2-4}}{x}\,dx$$
$$= \int \frac{\sqrt{4\sec^2 - 4}}{2\sec\theta} \times 2\sec\theta\tan\theta\,d\theta$$
$$= \int \sqrt{4(\sec^2\theta - 1)}\,\tan\theta\,d\theta$$
$$= \int 2\tan\theta\tan\theta\,d\theta$$
$$= 2\int \tan^2\theta\,d\theta$$
$$= 2\int (\sec^2\theta - 1)\,d\theta$$
$$= 2[\tan\theta - \theta] + c$$
$$= 2\tan\theta - 2\theta + c$$
$$= 2\left(\frac{\sqrt{x^2-4}}{2}\right) - 2\arccos\left(\tfrac{2}{x}\right) + c$$
$$= \sqrt{x^2-4} - 2\arccos\left(\tfrac{2}{x}\right) + c$$



3

$$\frac{dy}{dx} = \frac{1}{y+2}$$
$$\therefore (y+2)\frac{dy}{dx} = 1$$
$$\therefore \int (y+2)\,dy = \int 1\,dx$$
$$\therefore \frac{y^2}{2} + 2y = x + c$$

But when $x = 0$, $y = 0$
$\therefore c = 0$

$$\therefore \frac{y^2}{2} + 2y = x$$
$$\therefore y^2 + 4y = 2x$$
$$\therefore (y+2)^2 = 2x + 4$$
$$\therefore y + 2 = \pm\sqrt{2x+4}$$

However, since $y = 0$ when $x = 0$,

$$y + 2 = +\sqrt{2x+4}$$
$$\therefore y = \sqrt{2x+4} - 2$$

4

$$\frac{dy}{dx} = \frac{2x}{\cos y}$$

$$\therefore \quad \cos y \frac{dy}{dx} = 2x$$

$$\therefore \quad \int \cos y \; dy = \int 2x \; dx$$

$$\therefore \quad \sin y = x^2 + c$$

But $y(1) = \frac{\pi}{2}$

$$\therefore \quad \sin(\tfrac{\pi}{2}) = 1 + c$$

$$\therefore \quad c = 0$$

$$\therefore \quad \sin y = x^2$$

i.e., $y = \arcsin(x^2)$ as required

5 **a** Since the uniformly inclined plane has highest point P and a 5 m horizontal base,

it has slope $\dfrac{dy}{dx} = -\dfrac{y}{5}$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = -\tfrac{1}{5}$$

$$\therefore \quad \int \frac{1}{y}\frac{dy}{dx}\, dx = \int -\tfrac{1}{5} \; dx$$

$$\therefore \quad \int \frac{1}{y} \; dy = \int -\tfrac{1}{5} \; dx$$

$$\therefore \quad \ln|y| = -\tfrac{1}{5}x + c$$

$$\therefore \quad y = Ae^{-\frac{1}{5}x}$$

Now when $x = 0, \quad y = 10,$

so $A = 10$

$\therefore \quad y = 10e^{-\frac{1}{5}x}$ as required.

b

$x = 30$ m at L

$$\therefore \quad y = 10e^{-\frac{30}{5}}$$

$$= 10e^{-6}$$

$$\doteqdot 0.0248 \text{ m}$$

c Since $\dfrac{dy}{dx} = -\dfrac{y}{5},$

slope at H $= -\frac{10}{5} = -2$

and slope at L $\doteqdot -\frac{0.024\,79}{5}$

$$\doteqdot -0.004\,96$$

6 **a** Letting $N(t)$ be the number of bacteria at time t,

$\dfrac{dN}{dt} = kN$ for some k

b Using **a**, $\dfrac{1}{N}\dfrac{dN}{dt} = k$

$$\therefore \quad \int \frac{1}{N} \; dN = \int k \; dt$$

$$\therefore \quad \ln|N| = kt + c$$

$$\therefore \quad N = Ae^{kt}$$

The initial population was $N(0) = 10^5$, $\quad \therefore \quad A = 10^5$

i.e., $N = 10^5 e^{kt}$

Since the population doubles every 37 minutes,

$$10^5 e^{k(t+37)} = 2 \times 10^5 e^{kt}$$

$$\therefore \quad e^{kt}e^{37k} = 2e^{kt}$$

$$\therefore \quad (e^k)^{37} = 2$$

$$\therefore \quad e^k = 2^{\frac{1}{37}}$$

$$\therefore \quad N = 10^5 (2^{\frac{1}{37}})^t$$

i.e., $N = 10^5 \times 2^{\frac{t}{37}}$

So after 4 hours, or 240 mins, the population is $N(240) = 10^5 \times 2^{\frac{240}{37}}$

$$\doteqdot 8.97 \times 10^6$$

7 **a**

$$\frac{dy}{dx} = (y-1)^2(2+x)$$

$$\therefore\quad (y-1)^{-2}\frac{dy}{dx} = 2+x$$

$$\therefore\quad \int (y-1)^{-2}\frac{dy}{dx}\,dx = \int(2+x)\,dx$$

$$\therefore\quad \int (y-1)^{-2}\,dy = \int(2+x)\,dx$$

$$\therefore\quad -(y-1)^{-1} = 2x+\frac{x^2}{2}+c$$

But the curve passes through $(-1, 2)$

$$\therefore\quad -2+\tfrac{1}{2}+c = -1$$

$$\therefore\quad c = \tfrac{1}{2}$$

$$\therefore\quad -\frac{1}{y-1} = 2x+\frac{x^2}{2}+\tfrac{1}{2}$$

$$\therefore\quad y-1 = -\frac{1}{\frac{x^2}{2}+2x+\frac{1}{2}}$$

$$= -\frac{2}{x^2+4x+1}$$

$$\therefore\quad y = 1-\frac{2}{x^2+4x+1}$$

b As $x\to\pm\infty$, $y\to 1$.

$\therefore$ $y=1$ is a horizontal asymptote.

The function is undefined when $x^2+4x+1 = 0$

i.e., $x = \dfrac{-4\pm\sqrt{16-4}}{2} = -2\pm\sqrt{3}$

$\therefore$ vertical asymptotes are $x=-2+\sqrt{3}$ and $x=-2-\sqrt{3}$

8 If V is the volume of water in the tank, then

$$\frac{dV}{dt} \propto V$$

$$\therefore\quad \frac{1}{V}\frac{dV}{dt} = k \quad \text{for some } k$$

$$\therefore\quad \int\frac{1}{V}\,dV = \int k\,dt$$

$$\therefore\quad \ln|V| = kt+c$$

$$\text{i.e.,}\quad V = Ae^{kt}$$

At time zero, the tank is full, and we call this volume V_0.

Then $A = V_0$,

and $V = V_0e^{kt}$

But after 20 minutes, the tank is half full

$$\therefore\quad V_0e^{20k} = \tfrac{1}{2}V_0$$

$$\therefore\quad (e^k)^{20} = \tfrac{1}{2}$$

$$\text{i.e.,}\quad e^k = (\tfrac{1}{2})^{\frac{1}{20}}$$

$$\therefore\quad V = V_0\left(\tfrac{1}{2}\right)^{\frac{t}{20}}$$

So, after one hour, the volume present is $V(60) = V_0(\frac{1}{2})^3 = \frac{1}{8}V_0$

Hence $\frac{1}{8}$ of the water remains.

Chapter 30

STATISTICAL DISTRIBUTIONS

EXERCISE 30A

1 **a** The quantity of fat in a lamb chop is a continuous random variable.

b The mark out of 50 for the Geography test is a discrete random variable.

c The weight of seventeen year-old students is a continuous random variable.

d The volume of water is a cup of coffee is a continuous random variable.

e The number of trout in a lake is a discrete random variable.

f The number of hairs on a cat is a discrete random variable.

g The length of hairs on a horse is a continuous random variable.

h The height of a sky-scraper is a continuous random variable.

2 **a** **i** The random variable is the height of water in the rain gauge.

ii $0 \leqslant x \leqslant 200$ mm

iii The variable is a continuous random variable.

b **i** The random variable is the stopping distance.

ii $0 \leqslant x \leqslant 50$ m

iii The variable is a continuous random variable.

c **i** The random variable could be the time taken for the switch to fail or it could be the number of times that the switch is turned on or off before it fails.

ii $0 \leqslant x \leqslant 10\,000$ hours if we are measuring time, or $1 \leqslant x \leqslant 2000$ if we are counting on/off.

iii The variable is a continuous random variable in the first case and a discrete random variable in the second.

3 **a** Since x is the number of weighing devices that are accurate, $x = 0$, 1, 2, 3 or 4.

b

		YYNN		
		YNYN		
	YYYN	YNNY	NNNY	
	YYNY	NNYY	NNYN	
	YNYY	NYNY	NYNN	
YYYY	NYYY	NYYN	YNNN	NNNN
$(x = 4)$	$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

c **i** If two are accurate then $x = 2$

ii If at least two are accurate then 2, 3 or 4 are accurate i.e., $x = 2$, 3 or 4

4 **a** If 3 coins are tossed then the number of heads x can be 0, 1, 2 or 3.

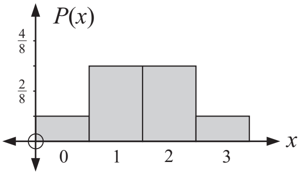
b Suppose H represents heads, T represents tails.

	HHT	TTH	
	HTH	THT	
HHH	THH	HTT	TTT
$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

c $\mathrm{P}(x = 0) = \frac{1}{8}$ $\mathrm{P}(x = 1) = \frac{3}{8}$

$\mathrm{P}(x = 2) = \frac{3}{8}$ $\mathrm{P}(x = 3) = \frac{1}{8}$

d



EXERCISE 30B

1 **a** $$\sum_{x=0}^{2} P(x) = 1$$
$$\therefore \quad 0.3 + k + 0.5 = 1$$
$$\therefore \quad k = 0.2$$

b $$\sum_{x=0}^{3} P(x) = 1$$
$$\therefore \quad k + 2k + 3k + k = 1$$
$$\therefore \quad 7k = 1$$
$$\therefore \quad k = \tfrac{1}{7}$$

2 **a** Since this is a probability distribution $\sum P(i) = 1$

$$\therefore \quad a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$$
$$\therefore \quad a + 0.4512 = 1$$
$$\therefore \quad a = 0.5488$$

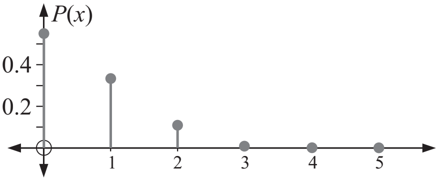
$P(0)$ is the probability that Jason does not hit a home run in a game.

b $P(2) = 0.1088$ (from table)

c $$P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.00$$
$$= 0.4512$$

This represents the probability that Jason hits *at least one* home run in a game.

d



3 **a** The probabilities all lie in $0 \leqslant P(i) \leqslant 1$, $\therefore$ OK

Sum of probabilities $\sum P(i) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$

$\therefore$ sum of probabilities $\neq 1$

$\therefore$ this is not a valid probability distribution.

b Notice that $P(5) = -0.2$, $\therefore$ not all of the probabilities lie in $0 \leqslant P(i) \leqslant 1$

$\therefore$ this is not a valid probability distribution.

4 **a** The random variable represents the number of hits that Sally has in each game.

b $$0.7 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1 \qquad \{\text{since } \textstyle\sum p(i) = 1\}$$
$$\therefore \quad k + 0.77 = 1$$
$$\therefore \quad k = 0.23$$

c **i** $$\begin{aligned} & P(x \geqslant 2) \\ &= P(x = 2 \text{ or } x = 3 \text{ or } x = 4 \text{ or } x = 5) \\ &= P(2) + P(3) + P(4) + P(5) \\ &= 0.23 + 0.46 + 0.08 + 0.02 \\ &= 0.79 \end{aligned}$$

ii $$\begin{aligned} & P(1 \leqslant x \leqslant 3) \\ &= P(1) + P(2) + P(3) \\ &= 0.14 + 0.23 + 0.46 \\ &= 0.83 \end{aligned}$$

5 **a** Rolling a die twice, sample space:

	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
roll 1	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		1	2	3	4	5	6

roll 2

b $\text{P}(0) = 0$ $\quad \text{P}(1) = 0$

$\text{P}(2) = \frac{1}{36}$ $\quad \text{P}(3) = \frac{2}{36}$

$\text{P}(4) = \frac{3}{36}$ $\quad \text{P}(5) = \frac{4}{36}$

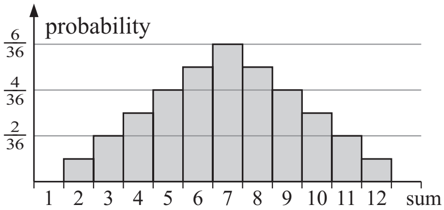
$\text{P}(6) = \frac{5}{36}$ $\quad \text{P}(7) = \frac{6}{36}$

$\text{P}(8) = \frac{5}{36}$ $\quad \text{P}(9) = \frac{4}{36}$

$\text{P}(10) = \frac{3}{36}$ $\quad \text{P}(11) = \frac{2}{36}$

$\text{P}(12) = \frac{1}{36}$

c



6 **a** $P(x) = k(x+2), \quad x = 1, 2, 3$

$\therefore \quad P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$

and since this is a probability distribution, $\sum P(i) = 3k + 4k + 5k$

and $12k = 1 \quad \{\text{as } \sum P(i) = 1\}$

$\therefore \quad k = \frac{1}{12}$

b $P(x) = \dfrac{k}{x+1} \qquad x = 0, 1, 2, 3$

$\therefore \quad P(0) = k, \qquad P(1) = \dfrac{k}{2}, \qquad P(2) = \dfrac{k}{3}, \qquad P(3) = \dfrac{k}{4}$

Now $k + \dfrac{k}{2} + \dfrac{k}{3} + \dfrac{k}{4} = 1 \qquad \{\text{as } \sum P(i) = 1\}$

$$\therefore \quad \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \quad \frac{25k}{12} = 1$$

$$\therefore \quad k = \tfrac{12}{25}$$

7 **a** $P(x) = k\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \qquad x = 0, 1, 2, 3, 4$

$P(0) = k\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = \dfrac{16k}{81} \qquad P(1) = k\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = \dfrac{8k}{81} \qquad P(2) = k\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \dfrac{4k}{81}$

$P(3) = k\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = \dfrac{2k}{81} \qquad P(4) = k\left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \dfrac{k}{81}$

b As $\sum P(i) = 1$,

$$\therefore \quad \frac{16k}{81} + \frac{8k}{81} + \frac{4k}{81} + \frac{2k}{81} + \frac{k}{81} = 1$$

$$\therefore \quad \frac{31k}{81} = 1$$

$$\therefore \quad k = \tfrac{81}{31}$$

$$\therefore \quad k \doteqdot 2.6130$$

$\therefore \quad P(x \geqslant 2) = P(2) + P(3) + P(4)$

$= \dfrac{4k}{81} + \dfrac{2k}{81} + \dfrac{k}{81}$

$= \dfrac{7k}{81}$

$= \dfrac{7 \times 2.6130}{81}$

$\doteqdot 0.2258$

8 **a** P(no faulty component)

$= \text{P}(x = 0)$

$= P(0)$

$= C_0^{10}(0.04)^0(0.96)^{10-0}$

$= C_0^{10}(0.96)^{10}$

$= (0.96)^{10}$

$\doteqdot 0.665$

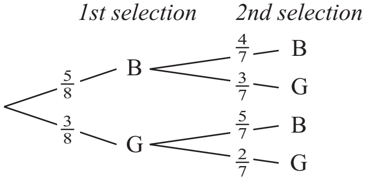
b P(at least one faulty component)

$= 1 - \text{P(none are faulty)}$

$= 1 - 0.6648$

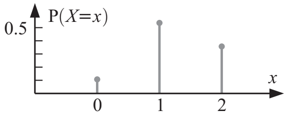
$\doteqdot 0.335$

9 **a**

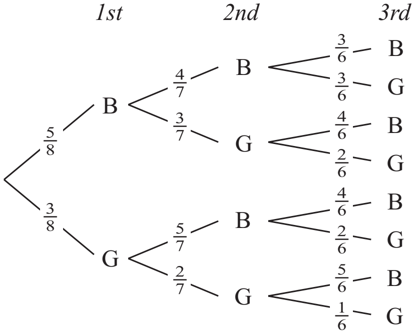


Event	x	*Probability*
BB	2	$\frac{5}{8}\times\frac{4}{7}=\frac{20}{56}$
BG	1	$\frac{5}{8}\times\frac{3}{7}=\frac{15}{56}$
GB	1	$\frac{3}{8}\times\frac{5}{7}=\frac{15}{56}$
GG	0	$\frac{3}{8}\times\frac{2}{7}=\frac{6}{56}$

x	0	1	2
$P(X=x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

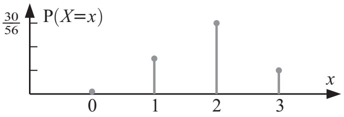


b



Event	x	*Probability*
BBB	3	$\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}=\frac{10}{56}$
BBG	2	$\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}=\frac{10}{56}$
BGB	2	$\frac{5}{8}\times\frac{3}{7}\times\frac{4}{6}=\frac{10}{56}$
BGG	1	$\frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}=\frac{5}{56}$
GBB	2	$\frac{3}{8}\times\frac{5}{7}\times\frac{4}{6}=\frac{10}{56}$
GBG	1	$\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}=\frac{5}{56}$
GGB	1	$\frac{3}{8}\times\frac{2}{7}\times\frac{5}{6}=\frac{5}{56}$
GGG	0	$\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}=\frac{1}{56}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$



10 **a**

Die 2

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die 1	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

36 sample points

b $\mathrm{P}(D=7)=\frac{6}{36}=\frac{1}{6}$

c

x	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{P}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

d

$$\begin{aligned}&\mathrm{P}(D\geqslant 8 \mid D\geqslant 6)\\&=\frac{\mathrm{P}(D\geqslant 8\ \cap\ D\geqslant 6)}{\mathrm{P}(D\geqslant 6)}\\&=\frac{\mathrm{P}(D\geqslant 8)}{\mathrm{P}(D\geqslant 6)}\\&=\tfrac{15}{36}\div\tfrac{26}{36}\\&=\tfrac{15}{26}\end{aligned}$$

11 $\mathrm{P}(X=x)=\dfrac{(0.2)^x e^{-0.2}}{x!}$

a **i**

$$\begin{aligned}&\mathrm{P}(X=0)\\&=\frac{(0.2)^0 e^{-0.2}}{0!}\\&=0.8187\,.....\\&\doteqdot 0.819\end{aligned}$$

ii

$$\begin{aligned}&\mathrm{P}(X=1)\\&=\frac{(0.2)^1 e^{-0.2}}{1!}\\&=0.1637\,.....\\&\doteqdot 0.164\end{aligned}$$

iii

$$\begin{aligned}&\mathrm{P}(X=2)\\&=\frac{(0.2)^2 e^{-0.2}}{2!}\\&=0.01637\,.....\\&\doteqdot 0.0164\end{aligned}$$

b

$$\begin{aligned}&\mathrm{P}(X\geqslant 3)\\&=1-\ \mathrm{P}(X\leqslant 2)\\&=1-\ \mathrm{P}(X=0\ \text{ or }\ X=1\ \text{ or }\ X=2)\\&\doteqdot 1-(0.8187+0.1637+0.0164)\\&\doteqdot 0.0012\end{aligned}$$

EXERCISE 30C

1 P(rain) $= 0.28$ $\therefore$ would expect rain on 0.28×365.25 days a year
i.e., $\doteqdot 102$ days.

2 **a** P(HHH)
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$

b For 200 tosses, we expect $200 \times \frac{1}{8}$ to be '3 heads'
i.e., 25 to be '3 heads'

3 P(double) $=$ P(1, 1 or 2, 2 or 3, 3 or 4, 4 or 5, 5 or 6, 6)
$= \frac{6}{36}$ {6 of the possible 36 outcomes}
$= \frac{1}{6}$

$\therefore$ when rolling 180 times we expect $180 \times \frac{1}{6} = 30$ doubles.

4 **a** **i** P(wins \$10)
$=$ P(rolls a 6)
$= \frac{1}{6}$

ii P(wins \$4)
$=$ P(rolls 4 or 5)
$= \frac{2}{6}$ (or $\frac{1}{3}$)

iii P(wins \$1)
$=$ P(rolls 1, 2 or 3)
$= \frac{3}{6}$ (or $\frac{1}{2}$)

b **i** Expectation
$= \frac{2}{6} \times \$4$
$\doteqdot \$1.33$

ii Expectation
$= \frac{3}{6} \times \$1$
$= \$0.50$

iii Expectation
$= \frac{1}{6} \times \$10 + \frac{2}{6} \times \$4 + \frac{3}{6} \times \$1$
$= \frac{1}{6}(\$21)$
$= \$3.50$

c It costs \$4 to play and the expected return is \$3.50.
$\therefore$ you expect to lose \$0.50 per game.

d So, over 100 games you expect to lose $100 \times \$0.50 = \$50$.

5 Expect to see snow falling on $\frac{3}{7} \times 5 \times 7$ days $= 15$ days

6 **a** $165 + 87 + 48 = 300$

i P(votes A)
$\doteqdot \frac{165}{300} = 0.55$

ii P(votes B)
$\doteqdot \frac{87}{300} = 0.29$

iii P(votes C)
$\doteqdot \frac{48}{300} = 0.16$

b **i** Expect $7500 \times 0.55 = 4125$ to vote A **ii** Expect $7500 \times 0.29 = 2175$ to vote B
iii Expect $7500 \times 0.16 = 1200$ to vote C

7 **a** Expect to win $\frac{1}{6} \times \$1 + \frac{1}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$4 + \frac{1}{6} \times \$5 + \frac{1}{6} \times \$6$
$= \frac{1}{6} \times \$21$
$= \$3.50$

b No, as on each occasion he would expect to lose \$0.50 (on average).

8

result	*win*
H	\$2
T	−\$1

For playing *once*,
expect to win $\frac{1}{2} \times \$2 + \frac{1}{2} \times (-\$1) = \$0.50$
$\therefore$ for 3 games, expect to win \$1.50.

9

result	*win*
HH	\$10
HT or TH	\$3
TT	−\$5

a Expectation $= \frac{1}{4} \times \$10 + \frac{2}{4} \times \$3 + \frac{1}{4} \times (-\$5) = \$2.75$

b Expected win per game (payout) $= \$2.75$
$\therefore$ organiser would charge $\$2.75 + \$1.00$
$= \$3.75$ to play each game.

EXERCISE 30D

1

x_i	0	1	2	3	4	5	>5
$P(x_i)$	0.54	0.26	0.15	k	0.01	0.01	0.00

a
$$\begin{aligned} 0.54+0.26+0.15+k+0.01+0.01 &= 1 \\ \therefore\ \ k+0.97 &= 1 \\ \therefore\ \ k &= 0.03 \end{aligned}$$

b
$$\begin{aligned} \mu &= \sum x_i p_i \\ &= 0\times 0.54 + 1\times 0.26 + \ldots\ldots + 5\times 0.01 \\ &= 0.26+0.30+0.09+0.04+0.05 \\ &= 0.74 \end{aligned}$$
i.e., over a long period the mean number of deaths per dozen crayfish is 0.74.

c
$$\begin{aligned} \sigma &= \sqrt{\sum (x_i-\mu)^2 p_i} \\ &= \sqrt{(0-0.74)^2\times 0.54 + (1-0.74)^2\times 0.26 + \ldots\ldots + (5-0.74)^2\times 0.01} \\ &\doteqdot 0.9962 \end{aligned}$$

2 $P(x) = \dfrac{x^2+x}{20}$ for $x = 1, 2, 3$

x_i	1	2	3
$P(x_i)=p_i$	$\frac{2}{20}=0.1$	$\frac{6}{20}=0.3$	$\frac{12}{20}=0.6$

$$\begin{aligned} \mu &= \sum x_i p_i \\ &= 1\times 0.1 + 2\times 0.3 + 3\times 0.6 \\ &= 2.5 \end{aligned}$$

and
$$\begin{aligned} \sigma &= \sqrt{\sum (x_i-\mu)^2 p_i} \\ &= \sqrt{(1-2.5)^2\times 0.1 + (2-2.5)^2\times 0.3 + (3-2.5)^2\times 0} \\ &\doteqdot 0.6708 \end{aligned}$$

3 **a** $P(x) = C^3_x\,(0.4)^x(0.6)^{3-x}$ for $x = 0,\ 1,\ 2,\ 3$

$$\begin{aligned} \therefore\ \ P(0) &= C^3_0\,(0.4)^0(0.6)^3 \\ &= (0.6)^3 \\ &= 0.216 \end{aligned}$$

$$\begin{aligned} P(1) &= C^3_1\,(0.4)^1(0.6)^2 \\ &= 3(0.4)(0.6)^2 \\ &= 0.432 \end{aligned}$$

$$\begin{aligned} P(2) &= C^3_2(0.4)^2(0.6)^1 \\ &= 3(0.16)(0.6) \\ &= 0.288 \end{aligned}$$

$$\begin{aligned} P(3) &= C^3_3(0.4)^3(0.6)^0 \\ &= 1(0.4)^3 \\ &= 0.064 \end{aligned}$$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

b
$$\mu = \sum x_i p_i = 0(0.216)+1(0.432)+2(0.288)+3(0.064) = 1.2$$
$$\begin{aligned} \sigma &= \sqrt{\sum (x_i-\mu)^2 p_i} \\ &= \sqrt{(0-1.2)^2(0.216)+(1-1.2)^2(0.432)+(2-1.2)^2\times 0.288+(3-1.2)^2\times 0.064} \\ &\doteqdot 0.8485 \end{aligned}$$

4
$$\begin{aligned} \sigma &= \sqrt{\sum (x_i-\mu)^2 p_i} \\ \therefore\ \ \sigma^2 &= \sum (x_i-\mu)^2 p_i \\ &= (x_i-\mu)^2 p_1 + (x_2-\mu)^2 p_2 + \ldots\ldots + (x_n-\mu)^2 p_n \\ &= (x_1^2-2x_1\mu+\mu^2)p_1 + (x_2^2-2x_2\mu+\mu^2)p_2 + \ldots\ldots + (x_n^2-2x_n\mu+\mu^3)p_n \\ &= (x_1^2p_1+x_2^2p_2+x_3^2p_3+\ldots\ldots+x_n^2p_n) - 2\mu(x_1p_1+x_2p_2+\ldots\ldots+x_np_n) \\ &\qquad + \mu^2(p_1+p_2+p_3+\ldots\ldots+p_n) \end{aligned}$$

Now $p_1 + p_2 + \ldots\ldots + p_n = 1$

$$\begin{aligned}\therefore\quad \sigma^2 &= \sum {x_i}^2 p_i - 2\mu\left(\sum x_i p_i\right) + \mu^2(1)\\ &= \sum {x_i}^2 p_i - 2\mu(\mu) + \mu^2 \quad \{\text{since } \sum x_i^2 p_i = \mu\}\\ &= \sum x_i^2 p_i - \mu^2\end{aligned}$$

5 **a**

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

b

$$\begin{aligned}\mu &= \sum x_i p_i\\ &= 1(0.1) + 2(0.2) + \ldots\ldots + 5(0.1)\\ &= 0.1 + 0.4 + 1.2 + 0.8 + 0.5\\ &= 3\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i}\\ &= \sqrt{\sum x_i^2 p_i - \mu^2}\\ &= \sqrt{1^2(0.1) + 2^2(0.2) + \ldots\ldots + 5^2(0.1) - (3.0)^2}\\ &= \sqrt{0.1 + 0.8 + 3.6 + 3.2 + 2.5 - 9}\\ &= \sqrt{1.2}\\ &\doteqdot 1.095\end{aligned}$$

c **i**

$$\begin{aligned}&\mathrm{P}(\mu - \sigma < x < \mu + \sigma)\\ &= \mathrm{P}(3 - 1.095 < x < 3 + 1.095)\\ &= \mathrm{P}(1.905 < x < 4.095)\\ &\doteqdot \mathrm{P}(x = 2,\ 3,\ 4)\\ &\doteqdot 0.2 + 0.4 + 0.2\\ &\doteqdot 0.8\end{aligned}$$

ii

$$\begin{aligned}&\mathrm{P}(\mu - 2\sigma < x < \mu + 2\sigma)\\ &= \mathrm{P}(3 - 2.19 < x < 3 + 2.19)\\ &= \mathrm{P}(0.81 < x < 5.19)\\ &\doteqdot \mathrm{P}(x = 1,\ 2,\ 3,\ 4 \text{ or } 5)\\ &\doteqdot 0.1 + 0.2 + 0.4 + 0.2 + 0.1\\ &\doteqdot 1.0\end{aligned}$$

6 Let x be the payout values, then $x = \$20\,000$, $\$8000$, or $\$0$

$\therefore$ the probability distribution is

x_i	20 000	8000	0
$P(x_i) = p_i$	0.0025	0.03	0.9675

The expectation is $\mu = \sum x_i p_i = 20\,000(0.0025) + 8000(0.03) + 0(0.9675)$

$= \$290$

i.e., the company expects to pay out \$290 on average in the long run

$\therefore$ the company should charge $\$290 + \$100 = \$390$

7

Die 2

		1	2	3	4	5	6
	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
Die 1	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6

a

m_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b

$$\begin{aligned}\mu &= \sum m_i p_i\\ &= 1\left(\tfrac{1}{36}\right) + 2\left(\tfrac{3}{36}\right) + 3\left(\tfrac{5}{36}\right) + \ldots\ldots + 6\left(\tfrac{11}{36}\right)\\ &= \tfrac{1}{36} + \tfrac{6}{36} + \tfrac{15}{36} + \tfrac{28}{36} + \tfrac{45}{36} + \tfrac{66}{36}\\ &= \tfrac{161}{36}\\ &\doteqdot 4.472\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum {m_i}^2 p_i - \mu^2}\\ &= \sqrt{1^2\left(\tfrac{1}{36}\right) + 2^2\left(\tfrac{3}{36}\right) + \ldots\ldots + 6^2\left(\tfrac{11}{36}\right) - (4.4722)^2}\\ &= \sqrt{1.971\,65}\\ &\doteqdot 1.404\end{aligned}$$

8 Examples are:

(1) Tossing one coin, where X is the number of 'heads' resulting. $X = 0$ or 1

x	0	1
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$

(2) Rolling one die, where X is the number on the uppermost face. $X = 1, 2, 3, 4, 5$ or 6

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

EXERCISE 30E.1

1 **a** mean of X
$$\begin{aligned} &= E(X) \\ &= \sum x p_x \\ &= 2(0.3) + 3(0.3) + 4(0.2) + 5(0.1) + 6(0.1) \\ &= 3.4 \end{aligned}$$

b $E(X^2) = \sum x^2 p_x = 4(0.3) + 9(0.3) + 16(0.2) + 25(0.1) + 36(0.1) = 13.2$

Now
$$\begin{aligned} \text{Var}(X) &= E(X^2) - \{E(X)\}^2 \\ &= 13.2 - (3.4)^2 \\ &= 1.64 \end{aligned}$$

c $\sigma = \sqrt{\text{Var } X} \doteqdot 1.28$

2 **a**
$$\begin{aligned} \sum p_x &= 1 \\ \therefore\ 0.2 + k + 0.4 + 0.1 &= 1 \\ \therefore\ k &= 0.3 \end{aligned}$$

b
$$\begin{aligned} E(X) &= \sum x p_x \\ &= 5(0.2) + 6(0.3) + 7(0.4) + 8(0.1) \\ &= 6.4 \end{aligned}$$

c
$$\begin{aligned} \text{Var }(X) &= \sum x^2 p_x - \{E(X)\}^2 \\ &= 25(0.2) + 36(0.3) + 49(0.4) + 64(0.1) - 6.4^2 \\ &= 0.84 \end{aligned}$$

3 **a**
$$\begin{aligned} &E(X) \\ &= \sum x p_x \\ &= 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) \\ &= 2 \end{aligned}$$

b
$$\begin{aligned} &E(X^2) \\ &= \sum x^2 p_x \\ &= 1(0.4) + 4(0.3) + 9(0.2) + 16(0.1) \\ &= 5 \end{aligned}$$

c
$$\begin{aligned} &\text{Var }(X) \\ &= E(X^2) - \{E(X)\}^2 \\ &= 5 - 2^2 \\ &= 1 \end{aligned}$$

d
$$\begin{aligned} \sigma &= \sqrt{\text{Var }(X)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

e
$$\begin{aligned} &E(X+1) \\ &= E(X) + E(1) \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

f
$$\begin{aligned} &\text{Var }(X+1) \\ &= E\left((X+1)^2\right) - \{E(X+1)\}^2 \\ &= E\left(X^2 + 2X + 1\right) - 3^2 \\ &= E(X^2) + 2E(X) + E(1) - 9 \\ &= 5 + 2(2) + 1 - 9 \\ &= 1 \end{aligned}$$

g
$$\begin{aligned} &E(2X^2 + 3X - 7) \\ &= 2E(X^2) + 3E(X) - E(7) \\ &= 2(5) + 3(2) - 7 \\ &= 9 \end{aligned}$$

4 $E(X) = 2.8$
$$\begin{aligned} \therefore\ 1(0.2) + 2a + 3(0.3) + 4b &= 2.8 \\ \therefore\ 0.2 + 2a + 0.9 + 4b &= 2.8 \\ \therefore\ 2a + 4b &= 1.7 \quad \text{......} \ (1) \end{aligned}$$

$$\begin{aligned}\text{Now}\quad \text{Var}(X) &= E(X^2)-\{E(X)\}^2\\ &= 1(0.2)+4a+9(0.3)+16b-(2.8)^2\\ \therefore\quad 0.2+4a+2.7+16b-2.8^2 &= 1.26\\ \therefore\quad 4a+16b &= 6.2\\ \therefore\quad 2a+8b &= 3.1\quad \text{...... (2)}\end{aligned}$$

Solving (1) and (2) gives $1.7-4b = 3.1-8b \quad \therefore\ 4b = 1.4 \quad \therefore\ b = 0.35$

$$\begin{aligned}\text{and in (1),}\quad 2a+1.4 &= 1.7\\ \therefore\quad 2a &= 0.3\\ \therefore\quad a &= 0.15 \qquad \text{i.e.,}\quad a=0.15,\quad b=0.35\end{aligned}$$

5 **a** $P(X=0)=a(0)=0$

$P(X=1)=a(-7)=-7a$

$P(X=2)=a(-12)=-12a$

$P(X=3)=a(-15)=-15a$

$P(X=4)=a(-16)=-16a$

$P(X=5)=a(-15)=-15a$

$P(X=6)=a(-12)=-12a$

$P(X=7)=a(-7)=-7a$

$P(X=8)=a(0)=0$

$$\begin{aligned}\therefore\quad 2(-7a-12a-15a)-16a &= 1\\ \therefore\quad a(-84) &= 1\\ \therefore\quad a &= -\tfrac{1}{84}\end{aligned}$$

b $E(X)$

$=\sum xp_x$

$=1\left(\frac{7}{84}\right)+2\left(\frac{12}{84}\right)+3\left(\frac{15}{84}\right)+4\left(\frac{16}{84}\right)+5\left(\frac{15}{84}\right)+6\left(\frac{12}{84}\right)+7\left(\frac{7}{84}\right)$

$=\frac{336}{84}$

$=4$

c $E(X^2)$

$=\sum x^2p_x$

$=1\left(\frac{7}{84}\right)+4\left(\frac{12}{84}\right)+9\left(\frac{15}{84}\right)+16\left(\frac{16}{84}\right)+25\left(\frac{15}{84}\right)+36\left(\frac{12}{84}\right)+49\left(\frac{7}{84}\right)$

$=\frac{1596}{84}$

$=19$

$$\begin{aligned}\therefore\quad \text{Var}(X) &= E(X^2)-\{E(X)\}^2\\ &= 19-4^2\\ &= 3\\ \therefore\quad \sigma &= \sqrt{\text{Var}(X)}=\sqrt{3}\end{aligned}$$

6 **a** $\left(\frac{1}{2}+\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4+4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1+6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4$

$= \frac{1}{16}+\frac{4}{16}+\frac{6}{16}+\frac{4}{16}+\frac{1}{16}$ i.e., binomial

So, the probability distribution for X, the number of heads occurring is

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b **i** $E(X)=\sum xp_x = 0\left(\frac{1}{16}\right)+1\left(\frac{4}{16}\right)+2\left(\frac{6}{16}\right)+3\left(\frac{4}{16}\right)+4\left(\frac{1}{16}\right)=2$

$\therefore$ mean $=2$

ii $E(X^2)=\sum x^2p_x = 0\left(\frac{1}{16}\right)+1\left(\frac{4}{16}\right)+4\left(\frac{6}{16}\right)+9\left(\frac{4}{16}\right)+16\left(\frac{1}{16}\right)=5$

$\sigma=\sqrt{\text{Var}\,(X)}=\sqrt{E(X^2)-\{E(X)\}^2}=\sqrt{5-2^2}=1$

7 P(0 bitter, 3 not bitter)

$$= \frac{C_0^2 C_3^8}{C_3^{10}}$$

$$= \tfrac{8\times7\times6}{3\times2\times1} \times \tfrac{3\times2\times1}{10\times9\times8}$$

$$= \tfrac{42}{90}$$

$$= \tfrac{7}{15}$$

P(1 bitter, 2 not)

$$= \frac{C_1^2 C_2^8}{C_3^{10}}$$

$$= 2 \times \tfrac{8\times7}{2\times1} \times \tfrac{3\times2\times1}{10\times9\times8}$$

$$= \tfrac{7}{15}$$

P(2 bitter, 1 not)

$$= \frac{C_2^2 C_1^8}{C_3^{10}}$$

$$= 8 \times \tfrac{3\times2\times1}{10\times9\times8}$$

$$= \tfrac{6}{90}$$

$$= \tfrac{1}{15}$$

x	0	1	2
$P(x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

a $E(X)$

$$= 0\left(\tfrac{7}{15}\right) + 1\left(\tfrac{7}{15}\right) + 2\left(\tfrac{1}{15}\right)$$

$$= \tfrac{9}{15}$$

$$= 0.6$$

$\therefore$ mean $= 0.6$ bitter almonds

b $E(X^2) = 0^2\left(\tfrac{7}{15}\right) + 1^2\left(\tfrac{7}{15}\right) + 2^2\left(\tfrac{1}{15}\right) = \tfrac{11}{15}$

$$\begin{aligned}\therefore \quad \text{Var}(X) &= E(X^2) - \{E(X)\}^2 \\ &= \tfrac{11}{15} - \left(\tfrac{3}{5}\right)^2 \\ &= \tfrac{11}{15} - \tfrac{9}{25} \\ &\doteqdot 0.3733 \quad \text{and so} \quad \sigma \doteqdot 0.611\end{aligned}$$

EXERCISE 30E.2

1 X has mean 6 and standard deviation 2.

$$\begin{aligned}E(Y) &= E(2X+5) \\ &= 2E(X) + E(5) \\ &= 2\times6+5 \\ &= 17\end{aligned}$$

$\therefore$ mean of Y distribution is 17

$$\begin{aligned}&\text{Var}(2X+5) \\ &= 2^2\ \text{Var}\ (X) \\ &= 4\times 2^2 \\ &= 16\end{aligned}$$

$\therefore$ standard deviation of Y distribution is $\sqrt{16} = 4$

2 **a**
$$\begin{aligned}E(aX+b) &= E(aX) + E(b) && \{\text{using } E(A+B) = E(A) + E(B)\} \\ &= aE(X) + E(b) && \{\text{using } E(kX) = kE(X)\} \\ &= aE(X) + b && \{\text{using } E(k) = k, \ k \text{ a constant}\}\end{aligned}$$

b **i**
$$\begin{aligned}&E(3X+4) \\ &= 3E(X)+4 \\ &= 3(3)+4 \\ &= 13\end{aligned}$$

ii
$$\begin{aligned}&E(-2X+1) \\ &= -2E(X)+1 \\ &= -2(3)+1 \\ &= -5\end{aligned}$$

iii
$$\begin{aligned}&E\left(\frac{4X-2}{3}\right) \\ &= E\left(\tfrac{4}{3}X - \tfrac{2}{3}\right) \\ &= \tfrac{4}{3}E(X) - \tfrac{2}{3} \\ &= \tfrac{4}{3}(3) - \tfrac{2}{3} \\ &= 3\tfrac{1}{3}\end{aligned}$$

3 X has mean 5 and standard deviation 2.

a **i** $E(Y) = E(2X+3) = 2E(X) + 3 = 2\times5+3 = 13$

ii $\text{Var}(Y) = \text{Var}(2X+3) = 2^2\ \text{Var}(X) = 4\times2^2 = 16$

b **i** $E(Y) = E(-2X+3) = -2E(X)+3 = -2\times5+3 = -7$

ii $\text{Var}(Y) = \text{Var}(-2X+3) = (-2)^2\ \text{Var}\ (X) = 4\times2^2 = 16$

c $Y = \dfrac{X-5}{2} = \tfrac{1}{2}X - \tfrac{5}{2}$

i $E(Y) = E\left(\tfrac{1}{2}X - \tfrac{5}{2}\right) = \tfrac{1}{2}E(X) - \tfrac{5}{2} = \tfrac{1}{2}\times5 - \tfrac{5}{2} = 0$

ii $\text{Var}(Y) = \text{Var}\left(\tfrac{1}{2}X - \tfrac{5}{2}\right) = \left(\tfrac{1}{2}\right)^2\ \text{Var}(X) = \tfrac{1}{4}\times2^2 = 1$

4 $Y = 2X+3$

a
$$\begin{aligned}&E(Y) \\ &= E(2X+3) \\ &= 2E(X)+3\end{aligned}$$

b
$$\begin{aligned}&E(Y^2) \\ &= E(4X^2+12X+9) \\ &= 4E(X^2)+12E(X)+9\end{aligned}$$

c $\text{Var}(Y)$
$= E(Y^2) - \{E(Y)\}^2$
$= 4E(X^2) + 12E(X) + 9 - \left[4\{E(X)\}^2 + 12E(X) + 9\right]$
$= 4E(X^2) - 4\{E(X)\}^2$

5 $V(aX+b) = E\left((aX+b)^2\right) - \{E(aX+b)\}^2$
$= E\left(a^2X^2 + 2abX + b^2\right) - \{aE(X)+b\}^2$
$= a^2E(X^2) + 2abE(X) + b^2 - \left[a^2\{E(X)\}^2 + 2abE(X) + b^2\right]$
$= a^2E(X^2) + 2abE(X) + b^2 - a^2\{E(X)\}^2 - 2abE(X) - b^2$
$= a^2\left(E(X^2) - \{E(X)\}^2\right)$
$= a^2 \text{ Var }(X)$

EXERCISE 30F

1 **a** The binomial distribution applies, as tossing a coin has one of two possible outcomes ($X = 0$ for a tail and $X = 1$ for a head) and each toss is independent of every other toss.

b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.

c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.

d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.

e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials.

2 X is the random variable for the number working night-shift.
$\therefore$ $X = 0, 1, 2, 3, 4, 5, 6, 7$ and X is Bin (7, 0.35).

a P($X = 3$)
= binompdf (7, 0.35, 3)
$\doteqdot$ 0.268

b P($X < 4$)
= P($X \leqslant 3$)
= binomcdf (7, 0.35, 3)
$\doteqdot$ 0.800

c P(at least 4 work night-shift)
= P($X \geqslant 4$)
= 1 − P($X \leqslant 3$)
$\doteqdot$ 1 − 0.800
$\doteqdot$ 0.200

3 X is the number of faulty items.
$\therefore$ $X = 0, 1, 2, 3, ..., 12$ and X is Bin (12, 0.06).

a P($X = 0$)
= binompdf (12, 0.06, 0)
$\doteqdot$ 0.476

b P(at most one is faulty)
= P($X \leqslant 1$)
= binomcdf(12, 0.06, 1)
$\doteqdot$ 0.840

c P(at least 2 are faulty)
= P($X \geqslant 2$)
= 1 − P($X \leqslant 1$)
$\doteqdot$ 0.160 {from **b**}

d P(less than 4 are faulty)
= P($X < 4$)
= 1 − P($X \leqslant 3$)
= binomcdf (12, 0.06, 3)
= 0.996

4 X is the random variable for the number of times in a week when the bus is on time.

$\therefore$ $X = 0, 1, 2, 3, 4, 5, 6$ or 7 and X is Bin (7, 0.6) {late 2 in 5, on time 3 in 5}

a $\text{P}(X = 7)$
$= \text{binompdf}\,(7, 0.6, 7)$
$\doteqdot 0.0280$

b P(on time only on Monday)
$= 0.6 \times (0.4)^6$
$= 0.002\,46$

c $\text{P}(X = 6)$
$= \text{binompdf}\,(7, 0.6, 6)$
$\doteqdot 0.131$

d $\text{P}(X \geqslant 4)$
$= 1 - \text{P}(X \leqslant 3)$
$= 1 - \text{binomcdf}\,(7, 0.6, 3)$
$\doteqdot 0.710$

5 X is the random variable for the number with flu

$\therefore$ $X = 0, 1, 2, 3, ..., 25$ and X is Bin (25, 0.3)

a $\text{P}(X \geqslant 2)$
$= 1 - \text{P}(X \leqslant 1)$
$= 1 - \text{binomcdf}\,(25, 0.3, 1)$
$\doteqdot 0.998$

b P(test cancelled)
$= \text{P}(X \geqslant 6)$ {20% of 25 = 5}
$= 1 - \text{P}(X \leqslant 5)$
$= 1 - \text{binomcdf}\,(25, 0.3, 5)$
$\doteqdot 0.8065$

EXERCISE 30G

1 x is distributed Bin(6, p)

a If $p = 0.5$, x is distributed Bin(6, 0.5)

i $\mu = np = 6 \times 0.5 = 3$ and $\sigma = \sqrt{npq} = \sqrt{6 \times 0.5 \times 0.5} \doteqdot 1.225$

ii $\text{P}(x = 0) = C_0^6(0.5)^0(0.5)^6 = 0.0156$

$\text{P}(x = 1) = C_1^6(0.5)^1(0.5)^5 = 0.0938$

$\text{P}(x = 2) = C_2^6(0.5)^2(0.5)^4 = 0.2344$

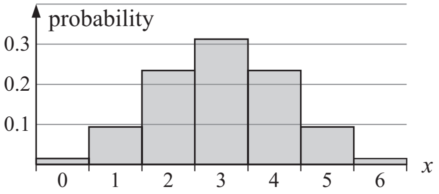
$\text{P}(x = 3) = C_3^6(0.5)^3(0.5)^3 = 0.3125$

$\text{P}(x = 4) = C_4^6(0.5)^4(0.5)^2 = 0.2344$

$\text{P}(x = 5) = C_5^6(0.5)^5(0.5)^1 = 0.0938$

$\text{P}(x = 6) = C_6^6(0.5)^6(0.5)^0 = 0.0156$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156



iii The distribution is bell-shaped.

b If $p = 0.2$, x is distributed Bin(6, 0.2)

i $\mu = np = 6 \times 0.2 = 1.2$ and $\sigma = \sqrt{npq} = \sqrt{6 \times 0.2 \times 0.8} \doteqdot 0.980$

ii $\text{P}(x = 0) = C_0^6(0.2)^0(0.8)^6 = 0.2621$

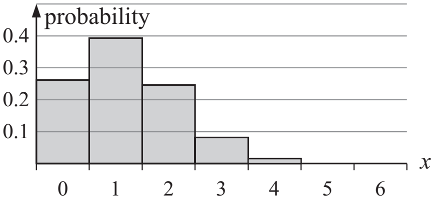
$\text{P}(x = 1) = C_1^6(0.2)^1(0.8)^5 = 0.3932$

$\text{P}(x = 2) = C_2^6(0.2)^2(0.8)^4 = 0.2458$

$\text{P}(x = 3) = C_3^6(0.2)^3(0.8)^3 = 0.0819$

$$\begin{aligned}&\text{P}(x=4)\\&=C_4^6(0.2)^4(0.8)^2\\&=0.0154\end{aligned}\qquad\begin{aligned}&\text{P}(x=5)\\&=C_5^6(0.2)^5(0.8)^1\\&=0.0015\end{aligned}\qquad\begin{aligned}&\text{P}(x=6)\\&=C_6^6(0.2)^6(0.8)^0\\&=0.0001\end{aligned}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001



iii The distribution is skewed to the right, i.e., positively skewed.

c If $p=0.8$, x is distributed Bin(6, 0.8)

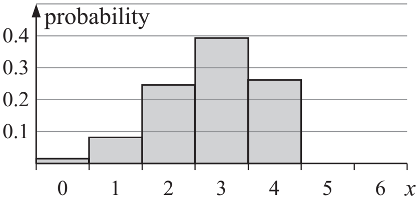
i $$\begin{aligned}\mu&=np\\&=6\times 0.8\\&=4.8\end{aligned}\qquad\begin{aligned}\sigma&=\sqrt{npq}\\&=\sqrt{6\times 0.8\times 0.2}\\&\doteqdot 0.980\end{aligned}$$

ii $$\begin{aligned}&\text{P}(x=0)\\&=C_0^6(0.8)^0(0.2)^6\\&=0.0001\end{aligned}\qquad\begin{aligned}&\text{P}(x=1)\\&=C_1^6(0.8)^1(0.2)^5\\&=0.0015\end{aligned}\qquad\begin{aligned}&\text{P}(x=2)\\&=C_2^6(0.8)^2(0.2)^4\\&=0.0154\end{aligned}\qquad\begin{aligned}&\text{P}(x=3)\\&=C_3^6(0.8)^3(0.2)^3\\&=0.0819\end{aligned}$$

$$\begin{aligned}&\text{P}(x=4)\\&=C_4^6(0.8)^4(0.2)^2\\&=0.2458\end{aligned}\qquad\begin{aligned}&\text{P}(x=5)\\&=C_5^6(0.8)^5(0.2)^1\\&=0.3932\end{aligned}\qquad\begin{aligned}&\text{P}(x=6)\\&=C_6^6(0.8)^6(0.2)^0\\&=0.2621\end{aligned}$$

Notice that this is the reverse of **b**.

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii This distribution is the exact reflection of **b**. It is skewed to the left or negatively skewed.

2 Number of tosses, $n=10$

X is the number of heads obtained. $\therefore$ x is distributed Bin(10, 0.5)

$$\begin{aligned}\mu&=np\\&=10\times 0.5\\&=5\end{aligned}\qquad\begin{aligned}\sigma&=\sqrt{npq}\\&=\sqrt{10\times 0.5\times 0.5}\\&\doteqdot 1.58\end{aligned}$$

3 **a** x is distributed Bin(3, p)

$$\begin{aligned}\text{P}(x=0)&=C_0^3\,p^0q^3\\&=C_0^3p^0(1-p)^3\\&=(1-p)^3\end{aligned}\qquad\begin{aligned}\text{P}(x=1)&=C_1^3\,p^1q^2\\&=3p(1-p)^2\end{aligned}\qquad\begin{aligned}\text{P}(x=2)&=C_2^3\,p^2(q)^1\\&=3p^2(1-p)\end{aligned}$$

$$\begin{aligned}\text{P}(x=3)&=C_3^3\,p^3q^0\\&=p^3\end{aligned}$$

x_i	0	1	2	3
$P(x_i)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

b
$$\begin{aligned}\mu &= \sum x_i p_i\\ &= 0(1-p)^3 + 1\times 3p(1-p)^2 + 2\times 3(1-p) + 3p^3\\ &= 3p(1-p)^2 + 6p^2(1-p) + 3p^3\\ &= 3p(1-2p+p^2) + 6p^2 - 6p^3 + 3p^3\\ &= 3p - 6p^2 + 3p^3 + 6p^2 - 6p^3 + 3p^3\\ &= 3p \quad \text{as required}\end{aligned}$$

c
$$\begin{aligned}\sigma^2 &= \sum x_i^2 p_i - \mu^2\\ &= 0^2\times(1-p)^3 \;+\; 1^2\times 3p(1-p)^2 \;+\; 2^2\times 3p^2(1-p) \;+\; 3^2p^3 - (3p)^2\\ &= 3p(1-p)^2 \;+\; 12p^2(1-p) \;+\; 9p^2(p-1)\\ &= (1-p)\left[3p(1-p) \;+\; 12p^2 \;-\; 9p^2\right]\\ &= (1-p)\left[3p - 3p^2 \;+\; 3p^2\right]\\ &= 3p(1-p)\\ &= 3pq\end{aligned}$$
$$\therefore \quad \sigma = \sqrt{3pq} \quad \text{as required}$$

4 X is the number of defective bolts in the sample.

X is distributed Bin(30, 0.04)

$$\begin{aligned}\mu &= np\\ &= 30\times 0.04\\ &= 1.2\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sigma &= \sqrt{npq}\\ &= \sqrt{30\times 0.04\times 0.96}\\ &\doteqdot 1.07\end{aligned}$$

5 X is the number of groups that do not arrive.

X is distributed Bin(5, 0.13)

$$\begin{aligned}\mu &= np\\ &= 5\times 0.13\\ &= 0.65\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sigma &= \sqrt{npq}\\ &= \sqrt{5\times 0.13\times 0.87}\\ &= 0.752\end{aligned}$$

EXERCISE 30H

1 **a** $\overline{x} = \dfrac{\sum fx}{\sum f} = \dfrac{0+18+24+18+12+0+6}{52} = \dfrac{78}{52} = 1.5$

b Using $m = 1.5$, $p_x = \dfrac{(1.5)^x e^{-1.5}}{x!}$

$\therefore\ p_0 = e^{-1.5}$
$\therefore\ p_0 \doteqdot 0.223\,13$
$\therefore\ 52p_0 \doteqdot 11.6$

$\therefore\ p_1 = \dfrac{1.5e^{-1.5}}{1}$
$\therefore\ p_1 \doteqdot 0.3347$
$\therefore\ 52p_1 \doteqdot 17.4$

$\therefore\ p_2 = \dfrac{(1.5)^2e^{-1.5}}{2!}$
$\therefore\ p_2 \doteqdot 0.2510$
$\therefore\ 52p_2 \doteqdot 13.0$

$\therefore\ p_3 = \dfrac{(1.5)^3e^{-1.5}}{3!}$
$\therefore\ p_3 \doteqdot 0.1255$
$\therefore\ 52p_3 \doteqdot 6.5$

$\therefore\ p_4 = \dfrac{(1.5)^4e^{-1.5}}{4!}$
$\therefore\ p_4 \doteqdot 0.0471$
$\therefore\ 52p_4 \doteqdot 2.4$

$\therefore\ p_5 = \dfrac{(1.5)^5e^{-1.5}}{5!}$
$\therefore\ p_5 \doteqdot 0.0144$
$\therefore\ 52p_5 \doteqdot 0.7$

$\therefore\ p_6 = \dfrac{(1.5)^6e^{-1.5}}{6!}$
$\therefore\ p_6 \doteqdot 0.0035$
$\therefore\ 52p_6 \doteqdot 0.2$

Comparison:

x	0	1	2	3	4	5	6
f	12	18	12	6	3	0	1
$52p_x$	11.6	17.4	13.0	6.5	2.4	0.7	0.2

2 **a** Standard deviation $= 2.67$

i $\therefore\ \text{Var}(X) = 2.67^2 \doteqdot 7.13$

ii Since $\text{Var}(X) = m,$

$$m \doteqdot 7.13$$

$$\therefore\ p_x \doteqdot \frac{(7.13)^x e^{-7.13}}{x!}$$

where $x = 0, 1, 2, 3, 4, 5, ...$

b **i** $P(X = 2)$

$= \textit{poissonpdf}\,(7.13, 2)$

$\doteqdot 0.0204$

ii $P(X \leqslant 3)$

$=$ poissoncdf $(7.13, 3)$

$\doteqdot 0.0752$

iii $P(X \geqslant 5)$

$= 1 - P(X \leqslant 4)$

$= 1-$ poissoncdf $(7.13, 4)$

$\doteqdot 0.839$

iv $P(X \geqslant 3 \mid x \geqslant 1)$

$$= \frac{P(X \geqslant 3 \cap X \geqslant 1)}{P(X \geqslant 1)}$$

$$= \frac{P(X \geqslant 3)}{P(X \geqslant 1)}$$

$$= \frac{1 - P(X \leqslant 2)}{1 - P(X = 0)}$$

$$= \frac{1 - \text{ poissoncdf } (7.13, 2)}{1 - \text{ poissonpdf } (7.13, 0)}$$

$\doteqdot 0.974$

3 **a**

$$\overline{x} = \frac{1 \times 156 + 2 \times 132 + 3 \times 75 + 4 \times 33 + 5 \times 9 + 6 \times 3 + 7 \times 1}{91 + 156 + 132 + 75 + 33 + 9 + 3 + 1}$$

$$= \frac{847}{500}$$

$$\doteqdot 1.694$$

b Using $m = 1.694$, $p_x = \dfrac{(1.694)^x e^{-1.694}}{x!}$ where $x = 0, 1, 2, 3, 4, ...$

$500p_0 = 500 \times 1.694^0 \times e^{-1.694} \times \frac{1}{0!} \doteqdot 91.9$

$500p_1 = 500 \times 1.694^1 \times e^{-1.694} \times \frac{1}{1!} \doteqdot 155.7$

$500p_2 = 500 \times 1.694^2 \times e^{-1.694} \times \frac{1}{2!} \doteqdot 131.8$

$500p_3 = 500 \times 1.694^3 \times e^{-1.694} \times \frac{1}{3!} \doteqdot 74.4$

$500p_4 = 500 \times 1.694^4 \times e^{-1.694} \times \frac{1}{4!} \doteqdot 31.5$

$500p_5 = 500 \times 1.694^5 \times e^{-1.694} \times \frac{1}{5!} \doteqdot 10.7$

$500p_6 = 500 \times 1.694^6 \times e^{-1.694} \times \frac{1}{6!} \doteqdot 3.0$

$500p_7 = 500 \times 1.694^7 \times e^{-1.694} \times \frac{1}{7!} \doteqdot 0.7$

Comparison:

x	0	1	2	3	4	5	6	7
f	91	156	132	75	33	9	3	1
$500p_x$	92	156	132	74	32	11	3	1

The fit is excellent.

c $\quad \text{Var}(X)$

$= E(X^2) - \{E(X)\}^2$

$= \sum x^2 p_x - (1.694)^2$

$= 1 \times \frac{156}{500} + 4 \times \frac{132}{500} + 9 \times \frac{75}{500} + 16 \times \frac{33}{500} + 25 \times \frac{9}{500} + 36 \times \frac{3}{500} + 49 \times \frac{1}{500} - (1.694)^2$

$= 1.6683$

$\therefore \quad s \doteqdot 1.29 \quad$ and $\quad \sqrt{m} = \sqrt{1.694} \doteqdot 1.30 \qquad \therefore \quad s$ is very close to $\sqrt{m}$

4 $p_x = \dfrac{3^x e^{-3}}{x!}$ where $x = 0, 1, 2, 3, 4, 5, \ldots\ldots$

a $\quad P(X = 0)$

$=$ poissonpdf (3, 0)

$\doteqdot 0.0498$

b $\quad$ P(at least 3 are rented)

$=$ P($X = 3$ or 4)

$=$ poissonpdf (3, 3) $+$ poissonpdf (3, 4)

$\doteqdot 0.392$

c $\quad$ P(some requests are refused)

$= P(X \geqslant 5)$

$= 1 - P(X \leqslant 4)$

$= 1-$ poissoncdf (3, 4)

$\doteqdot 0.185$

d $\quad \text{P}(X = 4 \mid X \geqslant 2)$

$$= \frac{\text{P}(X = 4 \cap X \geqslant 2)}{\text{P}(X \geqslant 2)}$$

$$= \frac{\text{P}(X = 4)}{1 - \text{P}(X \leqslant 1)}$$

$$= \frac{\text{poissonpdf (3, 4)}}{1 - \text{ poissoncdf (3, 1)}} \doteqdot 0.210$$

5 $P(X = x) = \dfrac{m^x e^{-m}}{x!}$ where $x = 0, 1, 2, 3, 4, \ldots\ldots$

a If $\text{P}(X = 1) + \text{P}(X = 2) = \text{P}(X = 3)$, then

$$\frac{me^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} = \frac{m^3 e^{-m}}{3!}$$

$$\therefore \quad m + \frac{m^2}{2} = \frac{m^3}{6} \qquad \{\text{dividing each term by } e^{-m}\}$$

$$\therefore \quad 6m + 3m^2 = m^3$$

$$\therefore \quad m(m^2 - 3m - 6) = 0 \quad \text{where} \quad m \neq 0$$

$$\therefore \quad m^2 - 3m - 6 = 0$$

$$\therefore \quad m = \frac{3 \pm \sqrt{9 - 4(1)(-6)}}{2} = \frac{3 \pm \sqrt{33}}{2}$$

But $m > 0$, so $m = \dfrac{3 + \sqrt{33}}{2}$

b **i** $\quad \text{P}(X \geqslant 3)$

$= 1 - \text{P}(X \leqslant 2)$

$= 1-$ poissoncdf (2.7, 2)

$\doteqdot 0.506$

ii $\quad \text{P}(X \leqslant 4 \mid X \geqslant 2)$

$$= \frac{\text{P}(X \leqslant 4 \cap X \geqslant 2)}{\text{P}(X \geqslant 2)}$$

$$= \frac{\text{P}(X = 2, 3 \text{ or } 4)}{\text{P}(X \geqslant 2)}$$

$$= \frac{\text{P}(X \leqslant 4) - \text{P}(X \leqslant 1)}{1 - \text{P}(X \leqslant 1)}$$

$$= \frac{\text{poissoncdf (2.7, 4)} - \text{poissoncdf (2.7, 1)}}{1 - \text{ poissoncdf (2.7, 1)}}$$

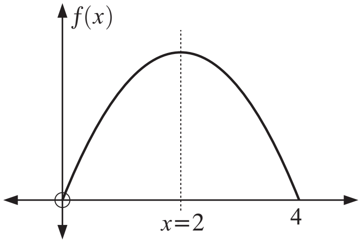
$\doteqdot 0.818$

EXERCISE 30I

1 **a**

$$\int_0^4 ax(x-4)dx = 1$$
$$\therefore\quad a\int_0^4 (x^2-4x)dx = 1$$
$$\therefore\quad a\left[\frac{x^3}{3}-\frac{4x^2}{2}\right]_0^4 = 1$$
$$\therefore\quad a\left(\tfrac{64}{3}-32\right) = 1$$
$$\therefore\quad a\left(\tfrac{-32}{3}\right) = 1$$
$$\therefore\quad a = -\tfrac{3}{32}$$

b $f(x) = -\frac{3}{32}x(x-4),\quad 0 \leqslant x \leqslant 4$



c **i**

$$\mu = \int_0^4 x\,f(x)\,dx$$
$$= \int_0^4 -\tfrac{3}{32}x^2(x-4)\,dx$$
$$= -\tfrac{3}{32}\int_0^4 x^2(x-4)\,dx$$
$$= 2 \qquad \{\text{technology}\}$$

ii mode = 2
{symmetry of graph}

iv

$$\int_0^4 x^2\,f(x)\,dx$$
$$= \int_0^4 -\tfrac{3}{32}x^3(x-4)\,dx$$
$$= 4.8 \quad \{\text{technology}\}$$
$$\therefore\quad \text{Var}(X)$$
$$= 4.8 - 2^2$$
$$= 0.8$$

iii

$$\text{If}\quad \int_0^m -\tfrac{3}{32}x(x-4) = \tfrac{1}{2}$$
$$\text{then}\quad \int_0^m (x^2-4x)dx = -\tfrac{16}{3}$$
$$\therefore\quad \left[\frac{x^3}{3}-\frac{4x^2}{2}\right]_0^m = -\tfrac{16}{3}$$
$$\therefore\quad \frac{m^3}{3} - 2m^2 - 0 = -\tfrac{16}{3}$$
$$\therefore\quad m^3 - 6m^2 = -16$$
$$\therefore\quad m^3 - 6m^2 + 16 = 0$$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 0 & 16 \\ & 0 & 2 & -8 & -16 \\ \hline & 1 & -4 & -8 & \multicolumn{1}{|r}{0} \end{array}$$

$$\therefore\quad m = 2 \quad\text{or}\quad \frac{4 \pm \sqrt{16+32}}{2}$$
$$\text{i.e.,}\quad m = 2 \quad\text{or}\quad 2 \pm 2\sqrt{3}$$
$$\therefore\quad m = 2 \quad \{\text{as}\quad 0 < m < 4\}$$

i.e., the median is 2.

2 **a**

$$\int_0^b -0.2x(x-b)\,dx = 1$$
$$\therefore\quad -0.2\int_0^b (x^2-bx)\,dx = 1$$
$$\therefore\quad \left[\frac{x^3}{3}-\frac{bx^2}{2}\right]_0^b = -5$$
$$\therefore\quad \frac{b^3}{3}-\frac{b^3}{2} - 0 = -5$$
$$\therefore\quad 2b^3 - 3b^3 = -30$$
$$\therefore\quad -b^3 = -30$$
$$\therefore\quad b^3 = 30$$
$$\therefore\quad b = \sqrt[3]{30}$$

b **i**

$$\mu = \int_0^{\sqrt[3]{30}} -0.2x^2(x-\sqrt[3]{30})\,dx$$
$$= 1.5536.....\quad \{\text{technology}\}$$
$$\doteqdot 1.55$$

ii

$$\int_0^{\sqrt[3]{30}} x^2\,f(x)\,dx$$
$$= \int_0^{\sqrt[3]{30}} -0.2x^3(x-\sqrt[3]{30})\,dx$$
$$= 2.8964.....\quad \{\text{technology}\}$$
$$\therefore\quad \text{Var}(X) \doteqdot 2.8964 - \mu^2$$
$$\doteqdot 0.483$$

3 **a**

$$\begin{aligned}\int_0^3 ke^{-x}\,dx &= 1\\ \therefore\ k\int_0^3 e^{-x}\,dx &= 1\\ \therefore\ k\left[\frac{e^{-x}}{-1}\right]_0^3 &= 1\\ \therefore\ k(-e^{-3}-(-1)) &= 1\\ \therefore\ k(1-e^{-3}) &= 1\\ \therefore\ k &\doteqdot 1.0524\end{aligned}$$

b If m is the median then

$$\begin{aligned}\int_0^m ke^{-x}\,dx &= \tfrac{1}{2}\\ \therefore\ \int_0^m e^{-x}\,dx &= \frac{1}{2k}\\ \therefore\ \left[\frac{e^{-x}}{-1}\right]_0^m &= \frac{1}{2k}\\ \therefore\ -e^{-m}-(-1) &= \frac{1}{2k}\\ \therefore\ e^{-m} &\doteqdot 1-\tfrac{1}{2(1.0524)}\\ \therefore\ e^{-m} &\doteqdot 0.52489\\ \therefore\ -m &\doteqdot \ln(0.52489)\\ \therefore\ m &\doteqdot 0.645\end{aligned}$$

4 **a**

$$\begin{aligned}\int_0^5 kx^2(x-6)\,dx &= 1\\ \therefore\ k\int_0^5 (x^3-6x^2)dx &= 1\\ \therefore\ k\left[\frac{x^4}{4}-\frac{6x^3}{3}\right]_0^5 &= 1\\ \therefore\ k\left(\tfrac{625}{4}-250\right) &= 1\\ \therefore\ k\left(\tfrac{-375}{4}\right) &= 1\\ \therefore\ k &= -\tfrac{4}{375}\end{aligned}$$

b

$$\begin{aligned}f(x) &= -\tfrac{4}{375}x^2(x-6)\\ &= -\tfrac{4}{375}(x^3-6x^2)\\ \therefore\ f'(x) &= -\tfrac{4}{375}(3x^2-12x)\end{aligned}$$

which is 0 when $3x(x-4) = 0$

i.e., $x = 0$ or 4

$\therefore$ the mode is 4.

c If m is the median,

$$\begin{aligned}\int_0^m \tfrac{-4}{375}\,x^2(x-6)\,dx &= \tfrac{1}{2}\\ \therefore\ \int_0^m (x^3-6x^2)\,dx &= -\tfrac{375}{8}\\ \therefore\ \left[\frac{x^4}{4}-\frac{6x^3}{3}\right]_0^m &= -\tfrac{375}{8}\\ \therefore\ \frac{m^4}{4}-2m^3 &= -\tfrac{375}{8}\\ \therefore\ 2m^4-16m^3+375 &= 0\\ \therefore\ m &\doteqdot 3.455\end{aligned}$$

{technology}

d

$$\begin{aligned}\mu &= \int_0^5 x\,f(x)\,dx\\ &= \int_0^5 \tfrac{-4}{375}\,x^3(x-6)\,dx\\ &= 3\tfrac{1}{3}\quad \{\text{technology}\}\end{aligned}$$

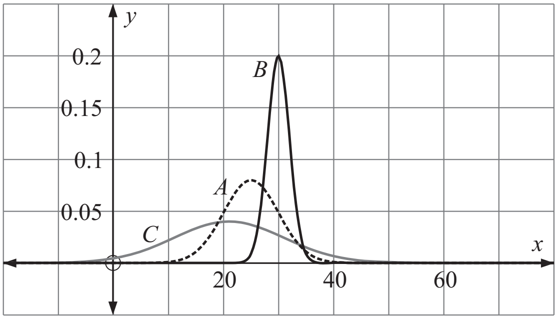
e

$$\begin{aligned}E(X^2) &= \int_0^5 x^2\,f(x)\,dx\\ &= \int_0^5 \tfrac{-4}{375}\,x^4(x-6)\,dx\\ &= 12\tfrac{2}{9}\end{aligned}$$

$$\begin{aligned}\therefore\ \text{Var}(X) &= 12\tfrac{2}{9}-\left(3\tfrac{1}{3}\right)^2\\ &= 1\tfrac{1}{9}\end{aligned}$$

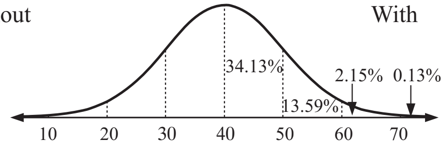
EXERCISE 30J.1

1

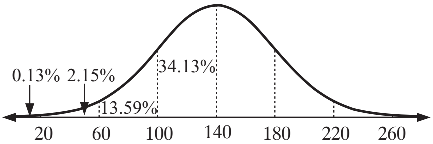


2 **a/b/c/d** The mean volume (or life time, weight, diameter etc) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

3 Without



With



a P(without and < 50)
$\doteqdot 50\% + 34.13\%$
$\doteqdot 84.1\%$

b P(with and < 60)
$\doteqdot 0.13\% + 2.15\%$
$\doteqdot 2.28\%$
$\doteqdot 2.3\%$

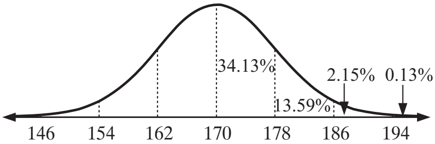
c P(with and $20 \leqslant x \leqslant 60$)
$\doteqdot 2.15\%$

P(without and $20 \leqslant x \leqslant 60$)
$\doteqdot 2(34.13\% + 13.59\%)$
$\doteqdot 95.4\%$

d P(with and $x \geqslant 60$)
$\doteqdot 13.59\% + 34.13\% + 50\%$
$\doteqdot 97.7\%$

P(without and $x \geqslant 60$)
$\doteqdot 2.15\% + 0.13\%$
$\doteqdot 2.3\%$

4



a P$(162 < x < 170)$
$\doteqdot 34.1\%$

b P$(170 < x < 186)$
$\doteqdot 34.13\% + 13.59\%$
$\doteqdot 47.7\%$

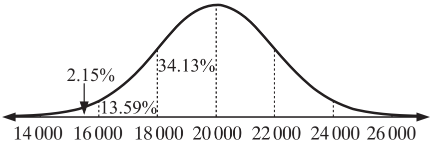
c P$(178 < x < 186)$
$\doteqdot 13.59\%$
$\doteqdot 0.136$

d P$(x < 162)$
$\doteqdot 1 - (0.5 + 0.3413)$
$\doteqdot 0.159$

e P$(x < 154)$
$\doteqdot 0.0215 + 0.0013$
$\doteqdot 0.0228$

f P$(x > 162)$
$\doteqdot 1 - 0.159$ {using **d**}
$\doteqdot 0.841$

5



a P$(x < 18\,000)$
$\doteqdot 1 - 0.5 - 0.3413$
$\doteqdot 0.1587$
$\therefore$ expect $260 \times 0.1587 = 41$ days

b P$(x > 16\,000)$
$\doteqdot 0.1359 + 0.3413 + 0.5$
$\doteqdot 0.9772$
$\therefore$ expect $260 \times 0.9772 = 254$ days

c P$(18\,000 \leqslant x \leqslant 24\,000)$
$\doteqdot 0.3413 \times 2 + 0.1359$
$\doteqdot 0.8185$
$\therefore$ expect $0.8185 \times 260 = 213$ days

EXERCISE 30J.2 Using Technology

1 **a** 0.341 **b** 0.383 **c** 0.106

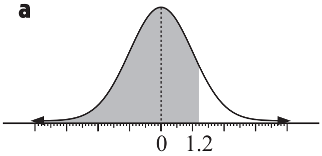
2 **a** 0.341 **b** 0.264 **c** 0.212 **d** 0.945 **e** 0.579 **f** 0.383

EXERCISE 30K.1

1 **a**
$$E\left(\frac{X-\mu}{\sigma}\right) = E\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma} = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = 0$$

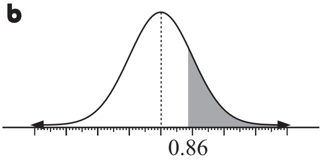
b
$$\text{Var}\left(\frac{X-\mu}{\sigma}\right) = \text{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right) = \left(\frac{1}{\sigma}\right)^2 \text{Var}(X) = \frac{1}{\sigma^2} \times \sigma^2 = 1$$

2 **a**



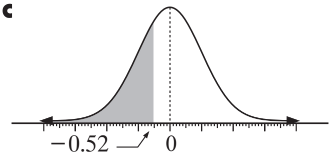
$$\begin{aligned} &\mathrm{P}(z \leqslant 1.2) \\ &= 0.8849 \\ &\doteqdot 0.885 \end{aligned}$$

b



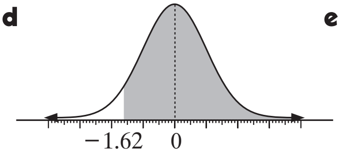
$$\begin{aligned} &\mathrm{P}(z \geqslant 0.86) \\ &= 1 - \mathrm{P}(z \leqslant 0.86) \\ &= 1 - 0.8051 \\ &\doteqdot 0.195 \end{aligned}$$

c



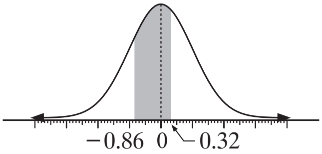
$$\begin{aligned} &\mathrm{P}(z \leqslant -0.52) \\ &\doteqdot 0.3015 \end{aligned}$$

d



$$\begin{aligned} &\mathrm{P}(z \geqslant -1.62) \\ &= 1 - \mathrm{P}(z \leqslant -1.62) \\ &\doteqdot 1 - 0.0526 \\ &\doteqdot 0.947 \end{aligned}$$

e



$$\begin{aligned} &\mathrm{P}(-0.86 < z < 0.32) \\ &= \mathrm{P}(z < 0.32) - \mathrm{P}(z < -0.86) \\ &= 0.6255 - 0.1949 \\ &\doteqdot 0.431 \end{aligned}$$

3 **a**

$$\begin{aligned} &\mathrm{P}(z \geqslant 0.837) \\ &= 1 - \mathrm{P}(z \leqslant 0.837) \\ &\doteqdot 1 - \text{normalcdf}\,(-\text{E99}, 0.837) \\ &\doteqdot 0.201 \end{aligned}$$

b

$$\begin{aligned} &\mathrm{P}(z \leqslant 0.0614) \\ &= \text{normalcdf}\,(-\text{E99}, 0.0614) \\ &\doteqdot 0.524 \end{aligned}$$

c

$$\begin{aligned} &\mathrm{P}(z \geqslant -0.876) \\ &= 1 - \mathrm{P}(z \leqslant -0.876) \\ &\doteqdot 0.809 \end{aligned}$$

d

$$\begin{aligned} &\mathrm{P}(-0.3862 \leqslant z \leqslant 0.2506) \\ &= \text{normalcdf}\,(-0.3862, 0.2506) \\ &\doteqdot 0.249 \end{aligned}$$

e

$$\begin{aligned} &\mathrm{P}(-2.367 \leqslant z \leqslant -0.6503) \\ &= \text{normalcdf}\,(-2.367, -0.6503) \\ &\doteqdot 0.249 \end{aligned}$$

4 **a**

$$\begin{aligned} &\mathrm{P}(-0.5 < z < 0.5) \\ &= \text{normalcdf}\,(-0.5, 0.5) \\ &\doteqdot 0.383 \end{aligned}$$

b

$$\begin{aligned} &\mathrm{P}(-1.960 < z < 1.960) \\ &= \text{normalcdf}\,(-1.96, 1.96) \\ &\doteqdot 0.950 \end{aligned}$$

5 **a**

$$\begin{aligned} \mathrm{P}(z \leqslant a) &= 0.95 \\ \therefore\ a &\doteqdot 1.645 \end{aligned}$$

{searching in tables}

b

$$\begin{aligned} \mathrm{P}(z \geqslant a) &= 0.90 \\ \therefore\ 1 - \mathrm{P}(z \leqslant a) &= 0.90 \\ \therefore\ \mathrm{P}(z \leqslant a) &= 0.1 \\ \therefore\ a &\doteqdot -1.28 - \tfrac{3}{18}(0.01) \\ \therefore\ a &\doteqdot -1.282 \end{aligned}$$

6 **a**

For Physics, $Z = \dfrac{83 - 78}{10.8} \doteqdot 0.463$ For Chemistry, $Z = \dfrac{77 - 72}{11.6} \doteqdot 0.431$

For Maths, $Z = \dfrac{84 - 74}{10.1} \doteqdot 0.990$ For German, $Z = \dfrac{91 - 86}{9.6} \doteqdot 0.521$

For Biology, $Z = \dfrac{72 - 62}{12.2} \doteqdot 0.820$

b Maths, Biology, German, Physics, Chemistry

7 Z-score for algebra $= \dfrac{56-50.2}{15.8} \doteqdot 0.3671$ $\quad$ Z-score for geometry $= \dfrac{x-58.7}{18.7}$

$$\therefore \text{ we need to solve } \frac{x-58.7}{18.7} = 0.3671$$
$$\therefore \quad x - 58.7 = 6.9$$
$$\therefore \quad x = 65.6 \quad \text{i.e., a result of } 65.6\%$$

EXERCISE 30K.2

1 X is normal with mean 70, standard deviation 4.

a $\text{P}(x \geqslant 74)$
$= \text{normalcdf}\,(74, \text{E99}, 70, 4)$
$\doteqdot 0.159$

b $\text{P}(x \leqslant 68)$
$= \text{normalcdf}\,(-\text{E99}, 68, 70, 4)$
$\doteqdot 0.309$

c $\text{P}(60.6 \leqslant x \leqslant 68.4)$
$= \text{normalcdf}\,(60.6, 68.4, 70, 4)$
$\doteqdot 0.335$

2 X is normal with mean 58.3 and standard deviation 8.96.

a $\text{P}(x \geqslant 61.8)$
$= \text{normalcdf}\,(61.8, \text{E99}, 58.3, 8.96)$
$\doteqdot 0.348$

b $\text{P}(x \leqslant 54.2)$
$= \text{normalcdf}\,(-\text{E99}, 54.2, 58.3, 8.96)$
$\doteqdot 0.324$

c $\text{P}(50.67 \leqslant x \leqslant 68.92)$
$= \text{normalcdf}\,(50.67, 68.92, 58.3, 8.96)$
$\doteqdot 0.685$

3 L is normal with mean 50.2 mm and standard deviation 0.93 mm.

a $\text{P}(l \geqslant 50)$
$= \text{normalcdf}\,(50, \text{E99}, 50.2, 0.93)$
$\doteqdot 0.585$

b $\text{P}(l \leqslant 51)$
$= \text{normalcdf}\,(-\text{E99}, 51, 50.2, 0.93)$
$= 0.805$

c $\text{P}(49 \leqslant l \leqslant 50.5)$
$= \text{normalcdf}\,(49, 50.5, 50.2, 0.93)$
$\doteqdot 0.528$

EXERCISE 30K.3

1 **a** $\text{P}(z \leqslant k) = 0.81$
$\therefore \quad k \doteqdot 0.87 + \frac{22}{28}(0.01)$
$\therefore \quad k \doteqdot 0.878$

b $\text{P}(z \leqslant k) = 0.58$
$\therefore \quad k \doteqdot 0.20 + \frac{7}{39}(0.01)$
$\therefore \quad k \doteqdot 0.202$

c $\text{P}(z \leqslant k) = 0.17$
$\therefore \quad k \doteqdot -0.96 + \frac{15}{26}(0.01)$
$\therefore \quad k \doteqdot -0.954$

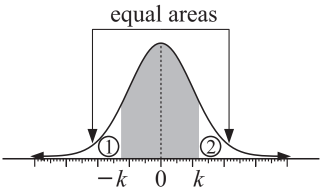
2 **a** $\text{P}(z \leqslant k) = 0.384$
$\therefore \quad k \doteqdot \text{invNorm}\,(0.384)$
$\therefore \quad k \doteqdot -0.295$

b $\text{P}(z \leqslant k) = 0.878$
$\therefore \quad k \doteqdot \text{invNorm}\,(0.878)$
$\therefore \quad k \doteqdot 1.165$

c $\text{P}(z \leqslant k) = 0.1384$
$\therefore \quad k \doteqdot \text{invNorm}\,(0.1384)$
$\therefore \quad k \doteqdot -1.088$

3 **a**

$$\begin{aligned}&\mathrm{P}(-k \leqslant z \leqslant k)\\ &= \mathrm{P}(z \leqslant k) - \mathrm{P}(z \leqslant -k)\\ &= \mathrm{P}(z \leqslant k) - \mathrm{P}(z \geqslant k) \quad \{\text{as area 1 = area 2}\}\\ &= \mathrm{P}(z \leqslant k) - [1 - \mathrm{P}(z \leqslant k)]\\ &= \mathrm{P}(z \leqslant k) - 1 + \mathrm{P}(z \leqslant k)\\ &= 2\mathrm{P}(z \leqslant k) - 1\end{aligned}$$



b **i**

$$\begin{aligned}\mathrm{P}(-k \leqslant z \leqslant k) &= 0.238\\ \therefore\ 2\mathrm{P}(z \leqslant k) - 1 &= 0.238\\ \therefore\ 2\mathrm{P}(z \leqslant k) &= 1.238\\ \therefore\ \mathrm{P}(z \leqslant k) &= 0.619\\ \therefore\ k &= \text{invNorm } (0.619)\\ \therefore\ k &\doteqdot 0.303\end{aligned}$$

ii

$$\begin{aligned}\mathrm{P}(-k \leqslant z \leqslant k) &= 0.7004\\ \therefore\ 2\mathrm{P}(z \leqslant k) - 1 &= 0.7004\\ \therefore\ 2\mathrm{P}(z \leqslant k) &= 1.7004\\ \therefore\ \mathrm{P}(z \leqslant k) &= 0.8502\\ \therefore\ k &= \text{invNorm } (0.8502)\\ \therefore\ k &\doteqdot 1.037\end{aligned}$$

4 **a**

$$\begin{aligned}&\mathrm{P}(X \leqslant k) = 0.9\\ &\therefore\ k = \text{inv Norm } (0.9, 56, \sqrt{18})\\ &\therefore\ k \doteqdot 61.4\end{aligned}$$

b

$$\begin{aligned}\mathrm{P}(X \geqslant k) &= 0.8\\ \therefore\ \mathrm{P}(X \leqslant k) &= 0.2\\ \therefore\ k &= \text{invNorm } (0.2, 38.7, \sqrt{8.8})\\ \therefore\ k &\doteqdot 36.2\end{aligned}$$

EXERCISE 30L

1 Let X be the length (in cm) of a bolt.
Then X is normally distributed with $\mu = 19.8$ and $\sigma = 0.3$.

$$\begin{aligned}\mathrm{P}(19.7 < x < 20) &= \text{normalcdf } (19.7, 20, 19.8, 0.3)\\ &\doteqdot 0.378\end{aligned}$$

2 Let X be the money collected (in \$)
Then X is normally distributed with $\mu = 40$ and $\sigma = 6$.

a

$$\begin{aligned}&\mathrm{P}(30.00 < x < 50.00)\\ &= \text{normalcdf } (30, 50, 40, 6)\\ &\doteqdot 0.904\\ &\doteqdot 90.4\%\end{aligned}$$

b

$$\begin{aligned}&\mathrm{P}(x \geqslant 50)\\ &= \text{normalcdf } (50, \text{E}99, 40, 6)\\ &\doteqdot 0.0478\\ &\doteqdot 4.78\%\end{aligned}$$

3 Let X be the result of the Physics test.
Then X is normally distributed with $\mu = 46$ and $\sigma = 25$.

$$\begin{aligned}\text{We need to find } k \text{ such that } \mathrm{P}(x \geqslant k) &= 0.07\\ \text{i.e., } 1 - \mathrm{P}(x \leqslant k) &= 0.07\\ \text{i.e., } \mathrm{P}(x \leqslant k) &= 0.93\\ \therefore\ k &= \text{invNorm } (0.93, 46, 25)\\ \therefore\ k &= 82.894......\\ \therefore\ k &\doteqdot 83 \quad \{k \text{ assumed to be an integer}\}\end{aligned}$$

i.e., lowest score would be 83 to get an A.

4 Let X be the length of an eel (in cm)
Then X is normally distributed with $\mu = 41$ and $\sigma = \sqrt{11}$

a

$$\begin{aligned}&\mathrm{P}(x \geqslant 50)\\ &= \text{normalcdf } (50, \text{E}99, 41, \sqrt{11})\\ &\doteqdot 0.003\,33\end{aligned}$$

b

$$\begin{aligned}&\mathrm{P}(40 \leqslant x \leqslant 50)\\ &= \text{normalcdf } (40, 50, 41, \sqrt{11})\\ &\doteqdot 0.615\\ &\doteqdot 61.5\%\end{aligned}$$

c

$$\begin{aligned}&\mathrm{P}(x \geqslant 45)\\ &= \text{normalcdf } (45, \text{E}99, 41, \sqrt{11})\\ &\doteqdot 0.114 \qquad \therefore\ \text{we expect } 200 \times 0.114 \doteqdot 23 \text{ eels}\end{aligned}$$

5

$$
\begin{aligned}
\mathrm{P}(X \geqslant 35) &= 0.32\\
\therefore\ \mathrm{P}(X \leqslant 35) &= 0.68\\
\therefore\ \mathrm{P}\left(\frac{X-\mu}{\sigma} \leqslant \frac{35-\mu}{\sigma}\right) &= 0.68\\
\therefore\ \mathrm{P}\left(Z \leqslant \frac{35-\mu}{\sigma}\right) &= 0.68\\
\therefore\ \frac{35-\mu}{\sigma} &= \text{invNorm }(0.68)\\
\therefore\ \frac{35-\mu}{\sigma} &\doteqdot 0.4677\\
\therefore\ 35-\mu &\doteqdot 0.4677\sigma \text{ ...... (1)}
\end{aligned}
$$

$$
\begin{aligned}
P\,(X \leqslant 8) &= 0.26\\
\therefore\ P\left(\frac{X-\mu}{\sigma} \leqslant \frac{8-\mu}{\sigma}\right) &= 0.26\\
\therefore\ P\left(Z \leqslant \frac{8-\mu}{\sigma}\right) &= 0.26\\
\therefore\ \frac{8-\mu}{\sigma} &= \text{invNorm }(0.26)\\
\therefore\ 8-\mu &\doteqdot -0.6433\sigma \text{ ...... (2)}
\end{aligned}
$$

Solving (1) and (2), $35 - 0.4677\sigma = 8 + 0.6433\mu$

$$
\begin{aligned}
\therefore\ 27 &= 1.111\sigma\\
\therefore\ \sigma &\doteqdot 24.3 \text{ and } \mu = 35 - 0.4677 \times 24.3\\
&\text{i.e., } \mu \doteqdot 23.6
\end{aligned}
$$

i.e., $\mu \doteqdot 23.6$ and $\sigma \doteqdot 24.3$

6 **a** Let the mean be μ and standard deviation be σ.

$$
\begin{aligned}
\text{Then }\ \mathrm{P}(x \geqslant 80) &= 0.1\\
\therefore\ \mathrm{P}(x \leqslant 80) &= 0.9\\
\therefore\ \mathrm{P}\left(\frac{x-\mu}{\sigma} \leqslant \frac{80-\mu}{\sigma}\right) &= 0.9\\
\therefore\ \mathrm{P}\left(z \leqslant \frac{80-\mu}{\sigma}\right) &= 0.9\\
\therefore\ \frac{80-\mu}{\sigma} &= \text{invNorm }(0.9)\\
\therefore\ 80-\mu &\doteqdot 1.2816\sigma \text{ ...... (1)}
\end{aligned}
$$

$$
\begin{aligned}
\text{and }\ \mathrm{P}(x \leqslant 30) &= 0.15\\
\therefore\ \mathrm{P}\left(\frac{x-\mu}{\sigma} \leqslant \frac{30-\mu}{\sigma}\right) &= 0.15\\
\therefore\ \mathrm{P}\left(z \leqslant \frac{30-\mu}{\sigma}\right) &= 0.15\\
\therefore\ \frac{30-\mu}{\sigma} &= \text{invNorm }(0.15)\\
\therefore\ 30-\mu &\doteqdot -1.0364\sigma \text{ ...... (2)}
\end{aligned}
$$

From (1) and (2) $(80-\mu)-(30-\mu) \doteqdot 1.2816\sigma + 1.0364\sigma$

$$
\begin{aligned}
50 &\doteqdot 2.318\sigma\\
\therefore\ \sigma &\doteqdot \frac{50}{2.318} \doteqdot 21.57\\
\text{and in (1) }\ 80-\mu &\doteqdot 1.2816 \times 21.57 \doteqdot 27.6\\
\therefore\ \mu &\doteqdot 52.4\\
\text{i.e., }\ \mu &\doteqdot 52.4 \text{ and } \sigma \doteqdot 21.6
\end{aligned}
$$

b X is the result of the Maths Exam.

X is normally distributed with mean μ and standard deviation σ.

Then $\mathrm{P}(x \geqslant 80) = 0.1$ and $\mathrm{P}(x \leqslant 30) = 0.15$

So, from **a** $\mu \doteqdot 52.4$ and $\sigma \doteqdot 21.6$

If part marks can be given,

$\mathrm{P}(x > 50)$

$=$ normalcdf (50, E99, 52.4, 21.6)

$\doteqdot 0.544$

$\doteqdot 54.4\%$

or For integer marks only,

$\mathrm{P}(x \geqslant 51)$

$=$ normalcdf (51, E99, 52.4, 21.6)

$\doteqdot 0.526$

$\doteqdot 52.6\%$

7 **a** Let the mean be μ and standard deviation be σ and X be the diameter (in cm).

$$\begin{aligned} \therefore\quad \mathrm{P}(x < 1.94) &= 0.02 \quad \text{and} \\ \therefore\quad \mathrm{P}\left(\frac{x-\mu}{\sigma} < \frac{1.94-\mu}{\sigma}\right) &= 0.02 \\ \therefore\quad \mathrm{P}\left(z < \frac{1.94-\mu}{\sigma}\right) &= 0.02 \\ \therefore\quad \frac{1.94-\mu}{\sigma} &= \text{invNorm}\ (0.02) \\ \therefore\quad 1.94-\mu &\doteq -2.054\sigma \ ......\ (1) \end{aligned}$$

$$\begin{aligned} \mathrm{P}(x > 2.06) &= 0.03 \\ \therefore\quad \mathrm{P}\left(\frac{x-\mu}{\sigma} > \frac{2.06-\mu}{\sigma}\right) &= 0.03 \\ \therefore\quad \mathrm{P}\left(z > \frac{2.06-\mu}{\sigma}\right) &= 0.03 \\ \text{i.e.,}\quad \mathrm{P}\left(z < \frac{2.06-\mu}{\sigma}\right) &= 0.97 \\ \frac{2.06-\mu}{\sigma} &= \text{invNorm}\ (0.97) \\ 2.06-\mu &\doteq 1.881\sigma \ .....\ (2) \end{aligned}$$

$$\begin{aligned} \text{From (1) and (2)}\quad (2.06-\mu)-(1.94-\mu) &= 1.881\sigma + 2.054\sigma \\ \therefore\quad 3.935\sigma &= 0.12 \\ \therefore\quad \sigma &\doteq 0.0305 \end{aligned}$$

$$\begin{aligned} \text{and in (1)}\quad 1.94-\mu &= -2.054 \times 0.0305 \doteq -0.0626 \\ \therefore\quad \mu &\doteq 2.00 \end{aligned}$$

$$\text{i.e.,}\quad \mu \doteq 2.00 \quad \text{and} \quad \sigma \doteq 0.0305$$

b This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$

$$\begin{aligned} \therefore\quad & \mathrm{P}(\text{less than 2 will operate}) \\ &= \mathrm{P}(x \leqslant 1) \\ &= \text{binomcdf}\ (20,\ 0.05,\ 1) \\ &\doteq 0.736 \end{aligned}$$

REVIEW SET 30A

1 **a** $P(x) = \dfrac{a}{x^2+1}$ for $a = 0, 1, 2, 3$

x_i	0	1	2	3
$P(x_i)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

$$\begin{aligned} \text{Now}\quad a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} &= 1 \qquad \{\text{as } \textstyle\sum P(x_i) = 1\} \\ \therefore\quad 10a + 5a + 2a + a &= 10 \qquad \{\times \text{ each term by } 10\} \\ \therefore\quad 18a &= 10 \\ \therefore\quad a &= \tfrac{5}{9} \end{aligned}$$

b
$$\begin{aligned} \mathrm{P}(x \geqslant 1) &= \mathrm{P}(x = 1,\ x = 2 \text{ or } x = 3) \\ &= \mathrm{P}(x = 1) + \mathrm{P}(x = 2) + \mathrm{P}(x = 3) \\ &= \tfrac{5}{18} + \tfrac{1}{9} + \tfrac{5}{90} \\ &= \tfrac{4}{9} \end{aligned}$$

or
$$\begin{aligned} \mathrm{P}(x \geqslant 1) &= 1 - \mathrm{P}(x < 1) \\ &= 1 - \mathrm{P}(x = 0) \\ &= 1 - \tfrac{5}{9} \\ &= \tfrac{4}{9} \end{aligned}$$

2 **a**
$$\begin{aligned} P(x) &= C^4_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{4-x} \\ \therefore\quad P(0) &= C^4_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 = 0.0625 \\ P(1) &= C^4_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^3 = 0.25 \end{aligned}$$

$$\begin{aligned} P(2) &= C^4_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^2 = 0.375 \\ P(3) &= C^4_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^1 = 0.25 \\ P(4) &= C^4_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^0 = 0.0625 \end{aligned}$$

i.e.,

x_i	0	1	2	3	4
$P(x_i)$	0.0625	0.25	0.375	0.25	0.0625

b $$\begin{aligned}\mu &= \sum x_i P(x_i)\\ &= 0 \times 0.0625 + 1 \times 0.25 + 2 \times 0.375 + 3 \times 0.25 + 4 \times 0.0625\\ &= 2\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 P(x_i)}\\ &= \sqrt{(-2)^2(0.0625) + (-1)^2(0.25) + 0^2(0.375) + 1^2(0.25) + 2^2(0.0625)}\\ &= 1\end{aligned}$$

3 X is the number of defectives. Then X is Bin (10, 0.18). $X = 0, 1, 2, 3, ..., 10.$

a $$\begin{aligned}&\text{P}(X = 1)\\ &= \text{binompdf (10, 0.18, 1)}\\ &\doteqdot 0.302\end{aligned}$$

b $$\begin{aligned}&\text{P}(X = 2)\\ &= \text{binompdf (10, 0.18, 2)}\\ &\doteqdot 0.298\end{aligned}$$

c $$\begin{aligned}\text{P}(X \geqslant 2) &= 1 - \text{P}(X \leqslant 1)\\ &= 1 - \text{binomcdf (10, 0.18, 1)}\\ &\doteqdot 0.561\end{aligned}$$

4 Let X be the number of defective toothbrushes.

X is distributed Bin(120, 0.04), $n = 120$, $p = 0.04$, $q = 0.96$,

a $$\begin{aligned}\mu &= np\\ &= 120 \times 0.04\\ &= 4.8\end{aligned}$$

b $$\begin{aligned}\sigma &= \sqrt{npq}\\ &= \sqrt{120 \times 0.4 \times 0.6}\\ &\doteqdot 2.15\end{aligned}$$

5

Result	*Pays*
1, 3, 5	$2
2	$3
4	$6
6	$9

a $$\begin{aligned}\text{Expected return} &= \tfrac{3}{6} \times \$2 + \tfrac{1}{6} \times \$3 + \tfrac{1}{6} \times \$6 + \tfrac{1}{6} \times \$9\\ &= \tfrac{1}{6}(\$24)\\ &= \$4\end{aligned}$$

b For a $5 amount to play the game the club expects a $1 return/game
$\therefore$ for 75 people, the return expected is $75.

6 If X is the arm length random variable (in cm) then X is normally distributed with $\mu = 64$ and $\sigma = 4$.

a i $$\begin{aligned}&\text{P}(60 < x < 72)\\ &= \text{normalcdf (60, 72, 64, 4)}\\ &\doteqdot 0.8186\\ &\doteqdot 81.9\%\end{aligned}$$

ii $$\begin{aligned}&\text{P}(X > 60)\\ &= \text{normalcdf (60, E99, 64, 4)}\\ &\doteqdot 0.841\\ &\doteqdot 84.1\%\end{aligned}$$

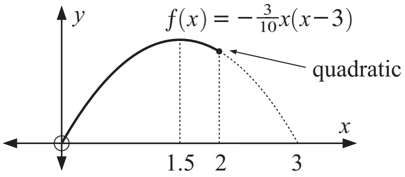
b $$\begin{aligned}\text{P}(56 < x < 68) &= \text{normalcdf (56, 68, 64, 4)}\\ &\doteqdot 0.819\\ &\doteqdot 81.9\%\end{aligned}$$

7 X is the rod length (in mm)
X is normally distributed
with mean μ and $\sigma = 3$

$$\begin{aligned}\text{Now}\quad \text{P}(x < 25) &= 0.02\\ \therefore\quad \text{P}\left(\frac{x-\mu}{3} < \frac{25-\mu}{3}\right) &= 0.02\\ \therefore\quad \text{P}\left(z < \frac{25-\mu}{3}\right) &= 0.02\\ \therefore\quad \frac{25-\mu}{3} &= \text{invNorm (0.02)}\\ \therefore\quad \frac{25-\mu}{3} &\doteqdot -2.0537\\ \therefore\quad 25 - \mu &\doteqdot -6.161\\ \therefore\quad \mu &\doteqdot 31.2\end{aligned}$$

8 **a**
$$\begin{aligned} \int_0^2 ax(x-3) &= 1 \\ \therefore\quad a\int_0^2 (x^2-3x) &= 1 \\ \therefore\quad a\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^2 &= 1 \\ \therefore\quad a\left[\tfrac{8}{3}-6\right] &= 1 \\ \therefore\quad a\left(-\tfrac{10}{3}\right) &= 1 \\ \therefore\quad a &= -\tfrac{3}{10} \end{aligned}$$

b



c **i**
$$\begin{aligned} \mu &= \int_0^2 x\,f(x)\,dx \\ &= \int_0^2 -\tfrac{3}{10}x^2(x-3)\,dx \\ &= 1.2 \qquad \{\text{technology}\} \end{aligned}$$

ii mode $= 1.5$ {at max. value of $f(x)$}

iii If the median is m, then
$$\begin{aligned} \int_0^m f(x)\,dx &= \tfrac{1}{2} \\ \int_0^m -0.3x(x-3) &= \tfrac{1}{2} \\ \therefore\quad \int_0^m (x^2-3x)\,dx &= -\tfrac{5}{3} \\ \left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^m &= -\tfrac{5}{3} \\ \therefore\quad \frac{m^3}{3}-\frac{3m^2}{2}+\frac{5}{3} &= 0 \\ \therefore\quad 2m^3-9m^2+10 &= 0 \\ \therefore\quad m &\doteqdot 1.24 \end{aligned}$$

iv
$$\begin{aligned} E(X^2) &= \int_0^2 x^2\,f(x)\,dx \\ &= \int_0^2 -0.3x^3(x-3)\,dx \\ &= 1.2 \end{aligned}$$
$$\begin{aligned} \therefore\quad \text{Var}(X) &= E(X^2)-\{E(X)\}^2 \\ &= 1.68-(1.2)^2 \\ &\doteqdot 0.24 \end{aligned}$$

v
$$\begin{aligned} &P(1 \leqslant x \leqslant 2) \\ &= \int_1^2 -0.3x(x-3)\,dx \\ &= 0.65 \end{aligned}$$

REVIEW SET 30B

1 **a** $P(x_i) = k\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$

$P(0) = k\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^3 = \dfrac{k}{64}$ $\qquad$ $P(2) = k\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1 = \dfrac{9k}{64}$

$P(1) = k\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2 = \dfrac{3k}{64}$ $\qquad$ $P(3) = k\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^0 = \dfrac{27k}{64}$

x_i	0	1	2	3
$P(x_i)$	$\frac{k}{64}$	$\frac{3k}{64}$	$\frac{9k}{64}$	$\frac{27k}{64}$

Now $\dfrac{k}{64}+\dfrac{3k}{64}+\dfrac{9k}{64}+\dfrac{27k}{64} = 1$ $\{$as $\sum p(x_i) = 1\}$

$\therefore\quad \dfrac{40k}{64} = 1$

$\therefore\quad k = 1.6$

b
$$\begin{aligned} \text{P}(x \geqslant 1) = 1-\text{P}(x=0) \quad &= 1-\frac{k}{64} \\ &= 1-\frac{1.6}{64} \\ &= 0.975 \end{aligned}$$

2 X is the number of hits, then $X = 0, 1, 2, 3, 4$ and X is Bin (4, 0.96)

a P$(X = 4)$
= binompdf (4, 0.96, 4)
$\doteqdot$ 0.849

b P$(X = 0)$
= binompdf (4, 0.96, 0)
$\doteqdot$ 0.000

c P$(X \geqslant 3)$
= 1− P$(X \leqslant 2)$
= 1− binomcdf (4, 0.96, 2)
$\doteqdot$ 0.991

d P$(X = 1)$
= binompdf (4, 0.96, 1)
$\doteqdot$ 0.000 246

3 X is the contents of the container (in mL).
X is normally distributed with $\mu = 377$ and $\sigma = 4.2$

a **i** $\text{P}(X < 368.6)$
$= \text{normalcdf}\,(-\text{E}99, 368.6, 377, 4.2)$
$\doteqdot 0.0228$
$\doteqdot 2.28\%$

ii $\text{P}(372.8 < X < 389.6)$
$= \text{normalcdf}\,(372.8, 389.6, 377, 4.2)$
$\doteqdot 0.840$
$\doteqdot 84.0\%$

b $\text{P}(364.4 < X < 381.2) = \text{normalcdf}\,(364.4, 381.2, 377, 4.2)$
$\doteqdot 0.840$

4 X is the life of a battery (in weeks)
X is normally distributed with $\mu = 33.2$, $\sigma = 2.8$

a $\text{P}(X \geqslant 35) = \text{normalcdf}\,(35, \text{E}99, 33.2, 2.8)$
$\doteqdot 0.260$

b We need to find k such that $\text{P}(X \geqslant k) = 0.08$
i.e., $\text{P}(X \leqslant k) = 0.92$
$\therefore\ k = \text{invNorm}\,(0.92, 33.2, 2.8)$
$\therefore\ k \doteqdot 37.134$

So, the manufacturer can expect that no more than 8% will fail for a maximum of 37 weeks and 1 day, i.e., 260 days.

5

x_i	0	1	2	3	4
$P(x_i)$	0.10	0.30	0.45	0.10	k

a If this is a probability distribution then $\sum p(x_i) = 1$
$\therefore\ 1 = 0.1 + 0.3 + 0.45 + 0.1 + k$
$\therefore\ k = 1 - 0.95$
$\therefore\ k = 0.05$

b
$$\begin{aligned}\mu &= \textstyle\sum x_i p_i \\ &= 0(0.1) + 1(0.3) + 2(0.45) + 3(0.1) + 4(0.05) \\ &= 0 + 0.3 + 0.9 + 0.3 + 0.2 \\ &= 1.7\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \textstyle\sum x_i^2 p_i - (\mu)^2 \\ &= 0^2(0.1) + 1^2(0.3) + 2^2(0.45) + 3^2(0.1) + 4^2(0.05) - (1.7)^2 \\ &= 0.3 + 1.8 + 0.9 + 0.8 - 2.89 \\ &= 0.91 \qquad \therefore\ \sigma = \sqrt{0.91} \doteqdot 0.954\end{aligned}$$

6 Let X denote the number of cases of netballers needing knee surgery.
x is distributed $\text{Bin}(487, 0.0132)$, $n = 487$, $p = 0.0132$, $q = 0.9868$

$$\therefore\ \mu = np = 487 \times 0.0132 \doteqdot 6.43$$

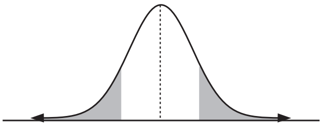
$$\sigma = \sqrt{npq} = \sqrt{487 \times 0.0132 \times 0.9868} \doteqdot 2.52$$

7 Let X denote the mass of a Coffin Bay Oyster. X is distributed normally with a mean of 38.6 and a standard deviation of 6.3.

a
$$\text{P}(38.6 - a \leqslant x \leqslant 38.6 + a) = 0.6826$$
$$\text{P}\left(\frac{38.6 - a - 38.6}{6.3} \leqslant \frac{x - 38.6}{6.3} \leqslant \frac{38.6 + a - 38.6}{6.3}\right) = 0.6826$$

$$\text{P}\left(-\frac{a}{6.3} \leqslant z \leqslant \frac{a}{6.3}\right) = 0.6826$$

$$\therefore \text{ by symmetry, } \text{P}\left(z \leqslant -\frac{a}{6.3}\right) = \frac{1-0.6826}{2} = 0.1587$$



$$\therefore -\frac{a}{6.3} = \text{invNorm } (0.1587)$$

$$\therefore -\frac{a}{6.3} \doteqdot -0.9998 \quad \text{and} \quad \therefore a \doteqdot 6.3 \text{ gms}$$

b

$$\begin{aligned} \text{P}(x \geqslant b) &= 0.8413 \\ \therefore \text{P}(x \leqslant b) &= 0.1587 \\ \text{P}\left(\frac{x-38.6}{6.3} \leqslant \frac{b-38.6}{6.3}\right) &= 0.1587 \\ \text{i.e., } \text{P}\left(z \leqslant \frac{b-38.6}{6.3}\right) &= 0.1587 \\ \therefore \frac{b-38.6}{6.3} &\doteqdot \text{invNorm } (0.1587) \\ \therefore b - 38.6 &\doteqdot 6.3 \times -0.9998 \\ \therefore b &\doteqdot -6.298 + 38.6 \\ \therefore b &\doteqdot 32.3 \text{ gms} \end{aligned}$$

8 $f(x) = ax^2(2-x)$ for $0 < x < 2$

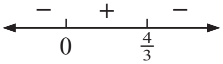
a Since $f(x)$ is a probability distribution function the area under the curve is 1

$$\begin{aligned} \text{i.e., } \int_0^2 ax^2(2-x)\,dx &= 1 \\ \therefore a\int_0^2 \left(2x^2 - x^3\right) dx &= 1 \\ \therefore a\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 &= 1 \\ \therefore a\left(\tfrac{16}{3} - \tfrac{16}{4}\right) - 0 &= 1 \\ \therefore a\left(\tfrac{4}{3}\right) &= 1 \\ \therefore a &= \tfrac{3}{4} \end{aligned}$$

b The mode is the most frequently occurring score, i.e., the value of x when $f(x)$ is a maximum.

$$\begin{aligned} f(x) &= \tfrac{3}{4}\left(2x^2 - x^3\right) \\ f'(x) &= \tfrac{3}{4}\left(4x - 3x^2\right) \\ &= \tfrac{3}{4}x(4-3x) \end{aligned}$$

which has sign diagram:



$\therefore$ is a maximum when $x = \frac{4}{3}$

$\therefore$ the mode is $\frac{4}{3}$.

c If the median is m, then,

$$\begin{aligned} \int_0^m \tfrac{3}{4}x^2(2-x)\,dx &= \tfrac{1}{2} \\ \tfrac{3}{4}\int_0^m \left(2x^2 - x^3\right) dx &= \tfrac{1}{2} \\ \therefore \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^m &= \tfrac{2}{3} \\ \therefore \frac{2m^3}{3} - \frac{m^4}{4} &= \tfrac{2}{3} \\ \therefore m &\doteqdot 1.2285 \end{aligned}$$

{using technology}

i.e., the median is approximately 1.23.

d

$$P(0.6 < x < 1.2) = \int_{0.6}^{1.2} \tfrac{3}{4}x^2(2-x)\,dx = 0.3915$$

{using technology}